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# Solutions for planar Kirchhoff-Schrödinger-Poisson systems with general nonlinearities

Rui Niu[1] and Hefan Wang[2*]

*Correspondence:
wanghefan0058@163.com
[2]College of Civil and Architectural Engineering, Heilongjiang Institute of Technology, Harbin, 150050, P.R. China
Full list of author information is available at the end of the article

**Abstract**

In this paper, we study the following Kirchhoff-type Schrödinger-Poisson systems in $\mathbb{R}^2$:

$$\begin{cases} -(a+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x)\Delta u + V(x)u + \mu\phi u = f(u), & x\in\mathbb{R}^2,\\ \Delta\phi = u^2, & x\in\mathbb{R}^2, \end{cases}$$

where $a,b>0$, $V\in\mathcal{C}(\mathbb{R}^2,\mathbb{R})$ and $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$. By using variational methods combined with some inequality techniques, we obtain the existence of the least energy solution, the mountain pass solution, and the ground state solutions for the above systems under some general conditions for the nonlinearities. Our results extend and improve the main results in [Chen, Shi, Tang, Discrete Contin. Dyn. Syst. 39 (2019) 5867–5889].

**MSC:** 35B33; 35J20; 35J61

**Keywords:** Kirchhoff-type problems; Schrödinger-Poisson systems; Variational methods; Existence of solutions

## 1 Introduction and main results

In this paper, we consider existence results for solutions of the following 2-D Schrödinger-Poisson systems of Kirchhoff type:

$$\begin{cases} -(a+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x)\Delta u + V(x)u + \mu\phi u = f(u), & x\in\mathbb{R}^2,\\ \Delta\phi = u^2, & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $a,b>0$, $\mu>0$, $V:\mathbb{R}^2\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ are continuous functions. Furthermore, we impose the following assumptions for $V$ and $f$:

$(V)$ $V\in\mathcal{C}^1(\mathbb{R}^2,[0,\infty))$ and $V(x)\geq\inf_{x\in\mathbb{R}^2}V(x)>0$;

$(V_1)$ $6V(x)+(\nabla V(x),x)\geq 0$;

$(V_2)$ $2(1+t^4)V(x)-4t^2V(t^{-1}x)+(1-t^4)(\nabla V(x),x)\geq 0$, for every $t\geq 0$, $x\in\mathbb{R}^2\backslash\{0\}$;

$(F_1)$ $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, and there exist constants $C > 0$, $p \in (5,\infty)$ such that

$$\left|f(u)\right| \le C\left(1+|u|^{p-1}\right), \quad \forall u \in \mathbb{R};$$

$(F_2)$ $f(u) = o(|u|)$ as $u \to 0$;
$(F_3)$ $\lim_{|u|\to\infty} \frac{F(u)}{|u|^5} = \infty$;
$(F_4)$ there exist constants $\alpha, C > 0$ and $q > 1$ such that

$$f(u)u \ge 5F(u), \quad \forall u \in \mathbb{R},$$

and

$$\left|\frac{f(u)}{u}\right| \ge \alpha \quad \Rightarrow \quad \left|\frac{f(u)}{u}\right|^q \le C\left[f(u)u - 5F(u)\right];$$

$(F_5)$ the function $\frac{f(u)u-F(u)}{u^3}$ is nondecreasing on $(-\infty,0)\cup(0,\infty)$.

It is easy to verify that there are simple examples of functions $V$ and $f$ satisfying the above hypotheses:

$$V(x) \equiv \text{Const}, \qquad f(u) = |u|^{p-2}u, \quad p \in (5,\infty).$$

The direct motivation for our work was inspired by [3]. More precisely, Chen et al. [3] concerned the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta\phi = u^2, & x \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

The authors showed that problem (1.2) admits a nontrivial mountain pass solution and a Nehari-Pohozaev type ground state solution. Based on the work of [3], we consider the problem (1.2) with the Kirchhoff term in the present paper. On one hand, this consideration is mainly from an interest in mathematics itself. Actually, if $a = 1$, $b = 0$, $V(x) \equiv 1$, problem (1.1) is equivalent to (1.2). When $b \neq 0$, there are two nonlocal terms $\|\nabla u\|_2^4$ and $\int_{\mathbb{R}^2} \phi_u u^2 \,\mathrm{d}x$ in this problem, and we have to compare these two nonlocal terms when we prove the boundedness of the Cerami sequence. In addition, the existence of the nonlocal term $\|\nabla u\|_2^4$ makes it difficult to prove that the corresponding functional of problem (1.1) satisfies the mountain pass geometry.

On the other hand, Kirchhoff-type problems have intrigued many researchers since they have many applications, for example, in physics and biological systems. More specifically, Kirchhoff established a model given by

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where $\rho$, $p_0$, $h$, $E$, $L$ are constants, which contain some physical meanings. In fact, (1.3) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Note that the presence of the nonlocal Kirchhoff term makes (1.3) no longer a pointwise identity. Moreover, Schrödinger-Poisson system was introduced by *Benci* and *Fortunato* in [1] as a physical model describing solitary waves for nonlinear Schrödinger type equations coupled with a Poisson equation. The nonlocal term $\phi u$ represents the interaction with the electric field. For more details on the physical background and existence results related to Kirchhoff-Schrödinger-Poisson systems, we refer to [2, 4, 9–11, 14–20, 22, 23] and the references therein.

In the present paper, we study the existence of solutions for system (1.1). For this, by applying the reduction argument introduced in [6], we first simplify the system (1.1) to the following equation

$$-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u+\phi_u u=f(u)\quad\text{in }\mathbb{R}^2,\tag{1.4}$$

where

$$\phi_u(x)=\left(\Gamma_2*u^2\right)(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|u^2(y)\,\mathrm{d}y,$$

$\Gamma_N$ is given by

$$\Gamma_N(z)=\begin{cases}\frac{1}{2\pi}\log|z|, & N=2,\\ \frac{1}{N(2-N)\omega_N}|z|^{2-N}, & N\geq 3,\end{cases}$$

where $\omega_N$ is the volume of the unit $N$-ball. That is, the solution of (1.4) is also a solution of (1.1). One can easily get the corresponding functional as follows:

$$\begin{aligned}I(u)=&\frac{a}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x\\&+\frac{\mu}{4}\int_{\mathbb{R}^2}\phi_u u^2\,\mathrm{d}x-\int_{\mathbb{R}^2}F(u)\,\mathrm{d}x.\end{aligned}\tag{1.5}$$

Let $H^1(\mathbb{R}^2)$ denote the Sobolev space endowed with the standard scalar product and norm

$$(u,v)=\int_{\mathbb{R}^2}(\nabla u\nabla v+uv)\,\mathrm{d}x,\qquad \|u\|_{H^1}:=\left(\int_{\mathbb{R}^2}\left(|\nabla u|^2+u^2\right)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Define

$$\|u\|^2:=\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x,\qquad |u\|_*^2:=\int_{\mathbb{R}^2}\log\left(1+|x|\right)u^2(x)\,\mathrm{d}x.$$

Denote by $E=\{u\in H^1(\mathbb{R}^2):\|u\|_*<+\infty,\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x<+\infty\}$ the Hilbert space endowed with the norm

$$\|u\|_E:=\left(\|u\|^2+\|u\|_*^2\right)^{\frac{1}{2}}.$$

It is obvious that $\|u\|$ is equivalent to the standard norm $\|u\|_{H^1}$ under the assumption $(V)$. It is standard to verify that $I\in\mathcal{C}^1(E,\mathbb{R})$ and the critical points of $I$ correspond to the weak solutions of problem (1.1). It is worth pointing out that the methods for the three-dimensional situation is not often easily adapted to the two-dimensional one, because the kernel $\Gamma_2(x)$ is sign-changing and is bounded neither from above nor from below, and the corresponding functional $I$ is not well defined on $H^1(\mathbb{R}^2)$.

Analogously to [7, Lemma 2.4], the Pohozaev funtional of (1.4) can be defined as follows:

$$\begin{aligned} P(u) = {} & \frac{1}{2}\int_{\mathbb{R}^2}\bigl[2V(x) + \bigl(\nabla V(x), x\bigr)\bigr]u^2\,\mathrm{d}x + \mu\int_{\mathbb{R}^2}\phi_u u^2\,\mathrm{d}x + \frac{\mu}{8\pi}\|u\|_2^4 \\ & - 2\int_{\mathbb{R}^2}F(u)\,\mathrm{d}x. \end{aligned} \tag{1.6}$$

It is well-known that $P(u) = 0$ when $u$ is the solution of (1.1). Now, we define

$$\begin{aligned} J(u) &= 2\bigl\langle I'(u), u\bigr\rangle - P(u) \\ &= 2a\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + 2b\biggl(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\biggr)^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}\bigl[2V(x) - \bigl(\nabla V(x), x\bigr)\bigr]u^2\,\mathrm{d}x + \mu\int_{\mathbb{R}^2}\phi_u u^2\,\mathrm{d}x \\ &\quad - \frac{\mu}{8\pi}\|u\|_2^4 - 2\int_{\mathbb{R}^2}\bigl[f(u)u - F(u)\bigr]\,\mathrm{d}x. \end{aligned} \tag{1.7}$$

And we define the following Nehari-Pohozaev manifold of the functional $I$:

$$\mathcal{M} := \bigl\{u \in E\backslash\{0\} : J(u) = 0\bigr\}. \tag{1.8}$$

It is clear that $\mathcal{M}$ contains any nontrivial solution of (1.1). Specially, if the solution $\hat{u}$ of (1.1) satisfies $I(\hat{u}) = \inf_{u\in\mathcal{M}} I(u)$, $\hat{u}$ is a ground state solution. Meanwhile, we call a solution $\hat{u}$ of (1.1) to be a least energy solution if $I(u)$ is the smallest among all nontrivial solutions of (1.1). In addition, we call a solution $\hat{u}$ is a mountain pass type solution when $I(\hat{u}) = \beta$, here

$$\beta = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I\bigl(\eta(t)\bigr), \qquad \Gamma = \bigl\{\eta \in C\bigl([0,1], E\bigr) : \eta(0) = 0, I\bigl(\eta(1)\bigr) < 0\bigr\}.$$

Now, we can demonstrate the first main result as follows.

**Theorem 1.1** *Assume that $(V)$, $(V_1)$ and $(F_1)$–$(F_4)$ hold. There exists $\mu^* > 0$ such that for $0 < \mu \le \mu^*$, problem (1.1) has a nontrivial least energy solution in E, and problem (1.1) has a solution of mountain pass type in E with positive energy.*

Here is the second main result.

**Theorem 1.2** *Suppose that V satisfies $(V)$, $(V_2)$ and f satisfies $(F_1)$–$(F_3)$, $(F_5)$. There exists $\mu^{**} > 0$ such that for $\mu \ge \mu^{**}$, problem (1.1) has a ground state solution in E.*

*Remark* 1.1 If $V(x) \equiv 1$ and $b = 0$ in (1.1), Theorems 1.1–1.2 reduces to the main results of [3]. In this sense, we extend and improve the related results of [3].

Here we sketch the main approaches in this paper. In detail, we first establish a Cerami sequence for the corresponding functional by using the minimax principle. And then, with the help of the Gagliardo-Nirenberg inequality, the Hardy-Littlewood-Sobolev inequality, we verify the boundedness of the Cerami sequence. Finally, motivated by [5], we get the existence of the lowest energy solution and the mountain pass solution. Moreover, by constructing some key inequalities, we prove the existence of ground state solutions for problem (1.1).

This paper is organized as follows. Section 2 gives the preliminaries and variational framework. Section 3 and Sect. 4 present the proofs of Theorem 1.1 and 1.2, exactly, Sect. 3 illustrates the existence of mountain pass type solutions and the lowest energy solutions for problem (1.1), Sect. 4 explains the existence of ground state solutions for problem (1.1).

Finally, we state some notations used in this paper: $L^q(\mathbb{R}^2)$ denotes the Lebesgue space equipped with the norm $\|u\|_q = (\int_{\mathbb{R}^2} |u|^q \,\mathrm{d}x)^{1/q}$, $2 \le q < +\infty$; $B_r(z)$ denotes the open ball centered at $z$ with radius $r > 0$; $C$, $\bar{C}$ and $\hat{C}$ denote possibly different positive constants in different places.

## 2 Preliminaries and variational framework

Firstly, we define the symmetric bilinear forms as follows:

$$
\begin{aligned}
(w,z) &\mapsto \mathcal{A}_1(w,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\bigl(1+|x-y|\bigr)w(x)z(y)\,\mathrm{d}x\,\mathrm{d}y,\\
(w,z) &\mapsto \mathcal{A}_2(w,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)w(x)z(y)\,\mathrm{d}x\,\mathrm{d}y,\\
(w,z) &\mapsto \mathcal{A}_0(w,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\bigl(|x-y|\bigr)w(x)z(y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

one can easily get that $\mathcal{A}_0(w,z) = \mathcal{A}_1(w,z) - \mathcal{A}_2(w,z)$. Actually, the definition aforementioned is restricted to measurable functions $w, z : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. It follows from the Hardy-Littlewood-Sobolev inequality [12] that

$$
\bigl|\mathcal{A}_2(w,z)\bigr| \le \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}\bigl|w(x)z(y)\bigr|\,\mathrm{d}x\,\mathrm{d}y \le C\|w\|_{4/3}\|z\|_{4/3}, \tag{2.1}
$$

where $C > 0$ is a constant. Based on the aforementioned functionals, we define the functionals $I_0, I_1, I_2$ as follows:

$$
\begin{aligned}
&I_1 : H^1(\mathbb{R}^2) \to [0,\infty], && I_1(u) = \mathcal{A}_1(u^2,u^2),\\
&I_2 : L^{\frac{8}{3}}(\mathbb{R}^2) \to [0,\infty], && I_2(u) = \mathcal{A}_2(u^2,u^2),\\
&I_0 : H^1(\mathbb{R}^2) \to \mathbb{R}\cup\{\infty\}, && I_0(u) = \mathcal{A}_1(u^2,u^2).
\end{aligned}
$$

Here $I_2$ only takes finite values on $L^{\frac{8}{3}}(\mathbb{R}^2)$. In fact, from (2.1), we have

$$
\bigl|I_2(u)\bigr| \le C\|u\|_{8/3}^4, \quad \forall u \in L^{\frac{8}{3}}(\mathbb{R}^2). \tag{2.2}
$$

By the definition of $E$, one can get that $E$ is compactly embedded in $L^s(\mathbb{R}^2)$ for $s \in [2,\infty)$. Note that

$$
\log\bigl(1+|x-y|\bigr) \le \log\bigl(1+|x|+|y|\bigr) \le \log\bigl(1+|x|\bigr) + \log\bigl(1+|y|\bigr),
$$

one has

$$\begin{aligned}\left|\mathcal{A}_1(wz,uv)\right| &\le \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bigl[\log\bigl(1+|x|\bigr)+\log\bigl(1+|y|\bigr)\bigr]\left|w(x)z(x)\right|\left|u(y)v(y)\right|\,\mathrm{d}x\,\mathrm{d}y\\ &\le \|w\|_*\|z\|_*\|u\|_2\|v\|_2+\|w\|_2\|z\|_2\|u\|_*\|v\|_*,\quad \forall w,z,u,v\in E.\end{aligned}\tag{2.3}$$

Similar to [5, Lemma 2.2], for $i = 0, 1, 2$, we conclude that $I_i$ is of class $\mathcal{C}^1$ on $E$, and

$$\langle I_i'(u),v\rangle = 4\mathcal{A}_i\bigl(u^2,v\bigr),\quad \forall u,v\in E.\tag{2.4}$$

From $(F_1)$, $(F_2)$ and (2.4), we deduce that $I\in\mathcal{C}^1(E,\mathbb{R})$, and

$$\begin{aligned}I(u) &= \frac{a}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x\\ &\quad+\frac{\mu}{4}\bigl[I_1(u)-I_2(u)\bigr]-\int_{\mathbb{R}^2}F(u)\,\mathrm{d}x,\end{aligned}\tag{2.5}$$

$$\begin{aligned}\langle I'(u),v\rangle &= a\int_{\mathbb{R}^2}\nabla u\nabla v\,\mathrm{d}x+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\int_{\mathbb{R}^2}\nabla u\nabla v\,\mathrm{d}x+\int_{\mathbb{R}^2}V(x)uv\,\mathrm{d}x\\ &\quad+\mu\bigl[\mathcal{A}_1\bigl(u^2,uv\bigr)-\mathcal{A}_2\bigl(u^2,uv\bigr)\bigr]-\int_{\mathbb{R}^2}f(u)u\,\mathrm{d}x,\end{aligned}\tag{2.6}$$

$$\begin{aligned}J(u) &= 2a\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x+2b\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^2}\bigl[2V(x)-\bigl(\nabla V(x),x\bigr)\bigr]u^2\,\mathrm{d}x+\mu\bigl[I_1(u)-I_2(u)\bigr]\\ &\quad-\frac{\mu}{8\pi}\|u\|_2^4-2\int_{\mathbb{R}^2}\bigl[f(u)u-F(u)\bigr]\,\mathrm{d}x.\end{aligned}\tag{2.7}$$

Now, we introduce the general minimax principle, which will be used later.

**Lemma 2.1** ([8, Proposition 2.8]) *Let $X$ be a Banach space, and $G_0$ be the closed subspace of the metric space $G$, $\Gamma_0\subset\mathcal{C}(G_0,X)$. Now define*

$$\bar{\Gamma} := \bigl\{\gamma\in\mathcal{C}(G,X):\gamma|_{G_0}\in\Gamma_0\bigr\}.$$

*If $\Phi\in\mathcal{C}^1(X,\mathbb{R})$ satisfies*

$$\bar{c} := \sup_{\gamma_0\in\Gamma_0}\sup_{u\in G_0}\Phi\bigl(\gamma_0(u)\bigr) < c := \inf_{\gamma\in\Gamma}\sup_{u\in G}\Phi\bigl(\gamma(u)\bigr) < \infty,$$

*then, for $\varepsilon\in(0,(c-\bar{c})/2)$, $\xi>0$ and $\gamma\in\bar{\Gamma}$ such that*

$$\sup_G \Phi\circ\gamma \le c+\varepsilon,$$

*there is $u\in X$ such that*

(i) $c-2\varepsilon\le\Phi(u)\le c+2\varepsilon$;
(ii) $\mathrm{dist}(u,\gamma(G))\le 2\xi$;
(iii) $\|\Phi'(u)\|\le\frac{8\varepsilon}{\xi}$.

Motivated by [7, Lemma 3.2], we illustrate that the functional $I$ has a Cerami sequence here.

**Lemma 2.2** *Assume that* $(V)$ *and* $(F_1)$ – $(F_3)$ *hold. Then there is a sequence* $\{u_n\} \subset E$ *satisfying*

$$I(u_n) \to c > 0, \qquad \|I'(u_n)\|_{E^*}\bigl(1 + \|u_n\|_E\bigr) \to 0, \qquad J(u_n) \to 0, \tag{2.8}$$

*where*

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\bigl(\gamma(t)\bigr), \qquad \Gamma := \bigl\{\gamma \in \mathcal{C}\bigl([0,1],E\bigr) : \gamma(0) = 0, I\bigl(\gamma(1)\bigr) < 0\bigr\}.$$

*Proof* We verify that $0 < c < \infty$ firstly. Define $u_t := u(tx)$ for $t > 0$ here and in the sequel. One can deduce that

$$\begin{aligned}
I_0\bigl(t^2u_t\bigr) &= \frac{t^4}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bigl(\log|tx-ty| - \log t\bigr)u^2(tx)u^2(ty)\,\mathrm{d}(tx)\,\mathrm{d}(ty) \\
&= \frac{t^4}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bigl(\log|x-y| - \log t\bigr)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= t^4I_0(u) - \frac{t^4\log t}{2\pi}\|u\|_2^4, \quad \forall t > 0.
\end{aligned}$$

Then,

$$\begin{aligned}
I\bigl(t^2u_t\bigr) &= \frac{a}{2}t^4\|\nabla u\|_2^2 + \frac{b}{4}t^8\|\nabla u\|_2^4 + \frac{t^2}{2}\int_{\mathbb{R}^2}V\bigl(t^{-1}x\bigr)u^2\,\mathrm{d}x + \frac{\mu t^4}{4}I_0(u) \\
&\quad - \frac{\mu t^4\log t}{8\pi}\|u\|_2^4 - \frac{1}{t^2}\int_{\mathbb{R}^2}F\bigl(t^2u\bigr)\,\mathrm{d}x, \quad \forall t > 0.
\end{aligned} \tag{2.9}$$

By $(F_1)$–$(F_3)$ and (2.9), we have

$$\lim_{t\to 0} I\bigl(t^2u_t\bigr) = 0, \qquad \sup_{t>0} I\bigl(t^2u_t\bigr) < \infty, \qquad I\bigl(t^2u_t\bigr) = -\infty \quad \text{as } t \to +\infty.$$

Thus, choosing $T > 0$ large such that $I(T^2u_T) < 0$. For $t \in [0,1]$, set $\gamma_T(t) = (tT)^2u_{tT}$, then, $\gamma_T \in \mathcal{C}([0,1],E)$ satisfies $\gamma_T(0) = 0$, $I(\gamma_T(1)) < 0$ and $\max_{t\in[0,1]} I(\gamma_T(t)) < \infty$. Hence, $\Gamma \neq 0$, $c < \infty$.

From $(F_1)$ and $(F_2)$, for every $\varepsilon > 0$, there is $C(\varepsilon) > 0$ satisfying

$$f(u)u \le \varepsilon u^2 + C(\varepsilon)|u|^p, \qquad F(u) \le \varepsilon u^2 + C(\varepsilon)|u|^p, \quad \forall u \in \mathbb{R}. \tag{2.10}$$

Fix $\varepsilon = a/4$, by (2.2), (2.5), (2.10) and Sobolev imbedding inequality, one has

$$\begin{aligned}
I(u) &\ge \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x - \frac{C\mu}{4}\|u\|_{\frac{8}{3}}^4 \\
&\quad - \frac{a}{4}\|u\|_2^2 - C\|u\|_p^p \\
&\ge \min\{a,1\}\frac{1}{4}\|u\|^2 - C\mu\|u\|^4 - C\|u\|^p, \forall u \in E.
\end{aligned}$$

One can easily get that there exist constants $\hat{\rho} > 0$ and $d > 0$ satisfying

$$I(u) \geq 0, \quad \forall \|u\| \leq \hat{\rho} \quad \text{and} \quad I(u) \geq d, \quad \forall \|u\| = \hat{\rho}. \tag{2.11}$$

For $\gamma \in \Gamma$, note that $\gamma(0) = 0$, $I(\gamma(1)) < 0$, by (2.11), one has $\|\gamma(1)\| > \hat{\rho}$. Noticing that $\gamma(t)$ is a continuous function, using the intermediate value theorem, there is $\hat{t} \in (0,1)$ such that $\|\gamma(\hat{t})\| = \hat{\rho}$. Consequently,

$$\sup_{t\in[0,1]} I\big(\gamma(t)\big) \geq I\big(\gamma(\hat{t})\big) \geq d > 0, \quad \forall \gamma \in \Gamma,$$

which implies

$$0 < d \leq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big) = c < \infty. \tag{2.12}$$

Let $\hat{E}$ be a Banach space endowed with the product norm

$$\big\|(s,v)\big\|_{\hat{E}} := \big(|s|^2 + \|v\|_E^2\big)^{\frac{1}{2}}.$$

Now, define the map

$$g : \hat{E} := \mathbb{R} \times E \to E, \qquad g(s,v)(x) := e^{2s} v\big(e^s x\big) \quad \text{for } s \in \mathbb{R}, v \in E, x \in \mathbb{R}^2,$$

Here we consider the functional

$$\begin{aligned}
\psi(s,v) &= I\big(g(s,v)\big) \\
&= \frac{a}{2}\int_{\mathbb{R}^2} \big|\nabla g(s,v)\big|^2 \,\mathrm{d}x + \frac{b}{4}\bigg(\int_{\mathbb{R}^2} \big|\nabla g(s,v)\big|^2 \,\mathrm{d}x\bigg)^2 + \frac{1}{2}\int_{\mathbb{R}^2} V(x)\big|g(s,v)\big|^2 \,\mathrm{d}x \\
&\quad + \frac{\mu}{4}\big[I_1\big(g(s,v)\big) - I_2\big(g(s,v)\big)\big] - \int_{\mathbb{R}^2} F\big(g(s,v)\big) \,\mathrm{d}x \\
&= \frac{a}{2}e^{4s}\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x + \frac{b}{4}e^{8s}\bigg(\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x\bigg)^2 \\
&\quad + \frac{e^{2s}}{2}\int_{\mathbb{R}^2} V\big(e^{-s}x\big)v^2 \,\mathrm{d}x + \frac{\mu e^{4s}}{4}\big[I_1(v) - I_2(v)\big] \\
&\quad - \frac{\mu s e^{4s}}{8\pi}\bigg(\int_{\mathbb{R}^2} v^2 \,\mathrm{d}\bigg)^2 - \frac{1}{e^{2s}}\int_{\mathbb{R}^2} F\big(e^{2s}v\big) \,\mathrm{d}x.
\end{aligned}$$

Then,

$$\begin{aligned}
\partial_s\psi(s,v) &= 2ae^{4s}\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x + 2be^{8s}\bigg(\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x\bigg)^2 + e^{2s}\int_{\mathbb{R}^2} V\big(e^{-s}x\big)v^2 \,\mathrm{d}x \\
&\quad - \frac{e^{2s}}{2}\int_{\mathbb{R}^2} \big(\nabla V\big(e^{-s}x\big), \big(e^{-s}x\big)\big)v^2 \,\mathrm{d}x + \mu e^{4s}\big[I_1(v) - I_2(v)\big] \\
&\quad - \mu\bigg(\frac{se^{4s}}{2\pi} + \frac{e^{4s}}{8\pi}\bigg)\bigg(\int_{\mathbb{R}^2} v^2 \,\mathrm{d}x\bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{e^{2s}}\int_{\mathbb{R}^2}F\big(e^{2s}v\big)\,\mathrm{d}x-\frac{2}{e^{2s}}\int_{\mathbb{R}^2}f\big(e^{2s}v\big)e^{2s}v\,\mathrm{d}x\\
={}&2a\big\|\nabla g(s,v)\big\|_2^2+2b\big\|\nabla g(s,v)\big\|_2^4+\int_{\mathbb{R}^2}V(x)\big|g(s,v)\big|^2\,\mathrm{d}x\\
&-\frac{1}{2}\int_{\mathbb{R}^2}\big(\nabla V(x),x\big)\big|g(s,v)\big|^2\,\mathrm{d}x\\
&+\mu\big[I_1\big(g(s,v)\big)-I_2\big(g(s,v)\big)\big]-\frac{\mu}{8\pi}\bigg(\int_{\mathbb{R}^2}\big|g(s,v)\big|^2\,\mathrm{d}x\bigg)^2\\
&-2\int_{\mathbb{R}^2}\big[f\big(g(s,v)\big)g(s,v)-F\big(g(s,v)\big)\big]\,\mathrm{d}x\\
={}&J\big(g(s,v)\big),\quad\forall s\in\mathbb{R},v\in E.
\end{aligned}
\tag{2.13}
$$

This shows that $\psi$ is of class $\mathcal{C}^1$ on $\hat{E}$. Furthermore, for $s\in\mathbb{R}, v, w\in E$, we have

$$
\partial_v\psi(s,v)w=I'\big(g(s,v)\big)g(s,v),
\tag{2.14}
$$

since for every $s\in\mathbb{R}$, the map $v\mapsto g(s,v)$ is linear. Next, we define

$$
\hat{c}=\inf_{\hat{\gamma}\in\hat{\Gamma}}\max_{t\in[0,1]}I\big(\hat{\gamma}(t)\big),
$$

where

$$
\hat{\Gamma}=\big\{\hat{\gamma}\in\mathcal{C}\big([0,1],\hat{E}\big):\hat{\gamma}(0)=(0,0),I\big(\hat{\gamma}(1)\big)<0\big\}.
$$

Note that $\Gamma=\{g\circ\hat{\gamma}:\hat{\gamma}\in\hat{\Gamma}\}$, one has $c=\hat{c}$. For $n\in\mathbb{N}$, from the definition of $c$, one can choose $\gamma_n\in\Gamma$ satisfying

$$
\max_{t\in[0,1]}\psi\big(0,\gamma_n(t)\big)=\max_{[0,1]}I\big(\gamma_n(t)\big)\le c+\frac{1}{n^2}.
$$

By Lemma 2.1, set $G=[0,1]$, $G_0=\{0,1\}$ and $\hat{E}$, $\hat{\Gamma}$ in place of $X$, $\Gamma$. Set $\hat{\gamma}_n(t)=(0,\gamma_n(t))$, $\varepsilon_n=\frac{1}{n^2}$, $\xi_n=\frac{1}{n}$. From (2.12), one has $\varepsilon_n=\frac{1}{n^2}\in(0,\frac{c}{2})$ for $n\in\mathbb{N}$ large. And then, Lemma 2.1 implies that that there exists $(s_n,v_n)\in\hat{E}$ such that, as $n\to\infty$,

$$
\psi(s_n,v_n)\to c,
\tag{2.15}
$$

$$
\big\|\psi'(s_n,v_n)\big\|_{\hat{E}^*}\big(1+\big\|(s_n,v_n)\big\|_{\hat{E}}\big)\to 0,
\tag{2.16}
$$

$$
\operatorname{dist}\big((s_n,v_n),\{0\}\times\gamma_n\big([0,1]\big)\big)\to 0.
\tag{2.17}
$$

It follows from (2.17) that

$$
s_n\to 0.
\tag{2.18}
$$

Since

$$
\big\langle\psi'(s_n,v_n),(w,z)\big\rangle=\big\langle I'\big(g(s_n,v_n)\big),g(s_n,v_n)\big\rangle+J\big(g(s_n,v_n)\big)w,\quad\forall(w,z)\in\hat{E}.
\tag{2.19}
$$

Combining (2.13) with (2.14), let $w = 1$ and $z = 0$ in (2.19), one has

$$J\big(g(s_n, v_n)\big) \to 0, \quad \text{as } n \to \infty. \tag{2.20}$$

Let $u_n := g(s_n, v_n)$, by (2.15) and (2.20), we have

$$I(u_n) \to c, \qquad J(u_n) \to 0 \quad \text{as } n \to \infty.$$

For $v \in E$, define $w_n = e^{-2s_n} v(e^{-s_n} x) \in E$, it follows from (2.16) and (2.19) that

$$\big(1 + \|u_n\|_E\big)\big|I'(u_n)v\big| = \big(1 + \|u_n\|_E\big)\big|I'(u_n)g(s_n, w_n)\big| = o(1)\|w_n\|_E, \quad \text{as } n \to \infty.$$

On the other hand, we deduce from (2.18) that

$$\begin{aligned}
\|w_n\|_E^2 &= \|w_n\|^2 + \|w_n\|_*^2 \\
&= e^{-4s_n}\|\nabla v\|_2^2 + e^{-2s_n}\int_{\mathbb{R}^2} V\big(e^{s_n}x\big)v^2\,\mathrm{d}x + e^{-2s_n}\int_{\mathbb{R}^2}\big[\log\big(1 + e^{s_n}|x|\big)\big]v^2\,\mathrm{d}x \\
&= \big[1 + o(1)\big]\|\nabla v\|_2^2 + \big[1 + o(1)\big]\int_{\mathbb{R}^2} V(x)v^2\,\mathrm{d}x \\
&\quad + \big[1 + o(1)\big]\int_{\mathbb{R}^2}\big[\log\big(1 + |x|\big)\big]v^2\,\mathrm{d}x \\
&= \big[1 + o(1)\big]\|v\|_E^2, \quad \text{as } n \to \infty,
\end{aligned}$$

where $o(1) \to 0$ uniformly in $v \in E$. Consequently,

$$\big(1 + \|u_n\|_E\big)\big\|I'(u_n)\big\|_{E^*} \to 0, \quad \text{as } n \to \infty.$$

The proof is now finished. □

Now, we illustrate the boundedness of the Cerami sequence.

**Lemma 2.3** *Suppose that* $(V)$, $(V_1)$ *and* $(F_1)$–$(F_4)$ *hold. Let* $\{u_n\} \subset E$ *satisfying* (2.8). *Then there exists* $\mu^* > 0$ *such that* $\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^2)$ *with* $\mu \le \mu^*$.

*Proof* We choose $\mu^* > 0$ small enough to satisfy

$$\frac{a}{4}\|\nabla u_n\|_2^2 - C\mu\|u_n\|_2^3\|\nabla u_n\|_2 \ge 0, \quad \text{for } \mu \le \mu^*. \tag{2.21}$$

By applying $(F_4)$, $(V_1)$, (2.8), (2.21), the Gagliardo-Nirenberg inequality [13, Theorem 1.3.7] and the Hardy-Littlewood-Sobolev inequality [8], one has

$$\begin{aligned}
c + o(1) &= I(u_n) - \frac{1}{8}J(u_n) \\
&= \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{3}{8}\int_{\mathbb{R}^2} V(x)u_n^2\,\mathrm{d}x + \frac{1}{16}\int_{\mathbb{R}^2}\big(\nabla V(x), x\big)u_n^2\,\mathrm{d}x + \frac{\mu}{8}\int_{\mathbb{R}^2}\phi_{u_n}u_n^2\,\mathrm{d}x \\
&\quad + \frac{\mu}{64\pi}\|u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2}\big[f(u_n)u_n - 5F(u_n)\big]\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{a}{4}\|\nabla u_n\|_2^2 + \frac{3}{8}\int_{\mathbb{R}^2} V(x)u_n^2\,\mathrm{d}x + \frac{1}{16}\int_{\mathbb{R}^2}\big(\nabla V(x),x\big)u_n^2\,\mathrm{d}x - C\mu\|u_n\|_{\frac{8}{3}}^4 \\
&\quad + \frac{\mu}{16\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\big(1+|x-y|\big)u_n^2(x)u_n^2(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{\mu}{64\pi}\|u_n\|_2^4 \qquad (2.22)\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^2}\big[f(u_n)u_n - 5F(u_n)\big]\,\mathrm{d}x \\
&\geq \frac{a}{4}\|\nabla u_n\|_2^2 - C\mu\|u_n\|_2^3\|\nabla u_n\|_2 + \frac{3}{8}\int_{\mathbb{R}^2} V(x)u_n^2\,\mathrm{d}x + \frac{1}{16}\int_{\mathbb{R}^2}\big(\nabla V(x),x\big)u_n^2\,\mathrm{d}x \\
&\quad + \frac{\mu}{64\pi}\|u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2}\big[f(u_n)u_n - 5F(u_n)\big]\,\mathrm{d}x \qquad (2.23)\\
&\geq \frac{\mu}{64\pi}\|u_n\|_2^4,
\end{aligned}
$$

which means

$$
\|u_n\|_2 \leq C, \qquad \int_{\mathbb{R}^2}\big[f(u_n)u_n - 5F(u_n)\big]\,\mathrm{d}x \leq C. \tag{2.24}
$$

Next, we show the boundedness of $\{\|u_n\|\}$. With reduction to absurdity, we suppose $\|u_n\| \to \infty$. Set $v_n := u_n/\|u_n\|$, from (2.24), we have $\|v_n\| = 1$, $\|v_n\|_2 \to 0$. Let $q' = \frac{q}{q-1}$, it follows from the Gagliardo-Nirenberg inequality that

$$
\|v_n\|_{2q'}^{2q'} \leq C\|v_n\|_2^2\|\nabla v_n\|_2^{2q'-2} = o(1). \tag{2.25}
$$

Define

$$
G_n := \left\{x \in \mathbb{R}^2 : \left|\frac{f(u_n)}{u_n}\right| \leq \alpha\right\}.
$$

Then,

$$
\int_{G_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2\,\mathrm{d}x \leq \alpha\|v_n\|_2^2 = o(1). \tag{2.26}
$$

Furthermore, from $(F_4)$, (2.24), (2.25) and the Hölder inequality, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^2\setminus G_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2\,\mathrm{d}x \\
&\quad \leq \left(\int_{\mathbb{R}^2\setminus G_n}\left|\frac{f(u_n)}{u_n}\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^2\setminus G_n}|v_n|^{2q'}\,\mathrm{d}x\right)^{\frac{1}{q'}} \\
&\quad \leq C^{\frac{1}{q}}\left(\int_{\mathbb{R}^2\setminus G_n}\big[f(u_n)u_n - 5F(u_n)\big]\,\mathrm{d}x\right)^{\frac{1}{q}}\|v_n\|_{2q'}^2 = o(1).
\end{aligned} \tag{2.27}
$$

It follows from (2.2), (2.24) and the Gagliardo-Nirenberg inequality that

$$
I_2(u_n) \leq C\|u_n\|_{8/3}^4 \leq C\|u_n\|_2^3\|\nabla u_n\|_2 \leq C\|\nabla u_n\|_2. \tag{2.28}
$$

Then, from (2.5), (2.8), (2.26), (2.27) and (2.28) follows that

$$
\min\{a,1\} + o(1) = \frac{\min\{a,1\}\|u_n\|^2 - \langle I'(u_n), u_n\rangle}{\|u_n\|^2}
$$

$$
\begin{aligned}
&\leq \frac{-\mu I_1(u_n) + \mu I_2(u_n) + \int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x}{\|u_n\|^2} \\
&\leq \frac{C\mu}{\|u_n\|} + \int_{G_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \,\mathrm{d}x + \int_{\mathbb{R}^2\setminus G_n} \left|\frac{f(u_n)}{u_n}\right| v_n^2 \,\mathrm{d}x \\
&= o(1),
\end{aligned}
$$

which is a contradiction. Consequently, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. □

To obtain the nontrivial solutions, we also need the following important lemma.

**Lemma 2.4** ([5, Lemma 2.1]) *Let $\{u_n\}$ be the sequence in $L^2(\mathbb{R}^2)$ such that $u_n \to u \in L^2(\mathbb{R}^2)\backslash\{0\}$ a.e., on $\mathbb{R}^2$. If $\{v_n\}$ is a bounded sequence in $L^2(\mathbb{R}^2)$ such that $\sup_{n\in\mathbb{N}} \mathcal{A}_1(u_n^2, v_n^2) < \infty$, then $\{\|v_n\|_*\}$ is bounded. If, moreover, $\mathcal{A}_1(u_n^2, v_n^2) \to 0$, $\|v_n\|_2 \to 0$ as $n \to \infty$, then $\|v_n\|_* \to 0$ as $n \to \infty$.*

## 3 Lowest energy solutions

*Proof of Theorem 1.1* In terms of Lemma 2.2 and Lemma 2.3, there is a sequence $\{u_n\} \subset E$ such that $\|u_n\|^2 \leq K_1$ for some constant $K_1 > 0$ and (2.8) hold. If

$$
\delta := \lim_{n\to\infty} \sup \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |u_n|^2 \,\mathrm{d}x = 0,
$$

by applying Lions' concentration compactness principle [21, Lemma 1.21], we have $u_n \to 0$ as $n \to \infty$ in $L^s(\mathbb{R}^2)$, $s \in (2,\infty)$. And then, by (2.2), we deduce that $I_2(u_n) \to 0$ as $n \to \infty$. From (2.10), let $\varepsilon = \frac{c}{3K_1}$, there is a constant $C(\varepsilon) > 0$ satisfying

$$
\int_{\mathbb{R}^2} \left|\frac{1}{2} f(u_n)u_n - F(u_n)\right| \mathrm{d}x \leq \frac{3}{2}\varepsilon \|u_n\|_2^2 + C(\varepsilon)\|u_n\|_p^p \leq \frac{c}{2} + o(1). \tag{3.1}
$$

Then, by (2.5), (2.6), (2.8) and (3.1), one has

$$
\begin{aligned}
c + o(1) &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\
&= -\frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\mu}{4} I_1(u_n) + \frac{\mu}{4} I_2(u_n) + \int_{\mathbb{R}^2} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] \mathrm{d}x \\
&\leq \frac{c}{2} + o(1),
\end{aligned}
$$

which is absurd, thus, $\delta > 0$.

Up to a subsequence if necessary, we suppose that there is $y_n \in \mathbb{R}^2$ such that

$$
\int_{B_1(y_n)} |u_n|^2 \,\mathrm{d}x > \frac{\delta}{2}.
$$

Set $\hat{u}_n(x) = u_n(x + y_n)$, then

$$
\int_{B_1(0)} |\hat{u}_n|^2 \,\mathrm{d}x > \frac{\delta}{2}. \tag{3.2}
$$

Since

$$\|\hat{u}_n\|_*^2 = \int_{\mathbb{R}^2} \log\bigl(1+|x-y_n|\bigr)u_n^2\,\mathrm{d}x \le \|u_n\|_*^2 + \log\bigl(1+|y_n|\bigr)\|u_n\|_2^2.$$

Then, for any $n \in \mathbb{N}$, $\hat{u}_n \in E$. Note that $\|\hat{u}_n\| = \|u_n\|$, $I_i(\hat{u}_n) = I_i(u_n)$, here $i = 0,1,2$, by (2.8), we have

$$I(\hat{u}_n) \to c > 0, \qquad \bigl\langle I'(\hat{u}_n), \hat{u}_n\bigr\rangle \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Going if necessary to a subsequence, one has $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^2)$, $\hat{u}_n \to \hat{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s \ge 2$, $\hat{u}_n \to \hat{u}$ a.e. on $\mathbb{R}^2$ as $n \to \infty$. Then, from (3.2), we have $\hat{u} \ne 0$. It follows from (2.2), (2.6), (2.10), (3.3) and Sobolev embedding inequality that

$$\begin{aligned} a\|\nabla\hat{u}_n\|_2^2 + b\|\nabla\hat{u}_n\|_2^4 + \int_{\mathbb{R}^2} V(x)\hat{u}_n^2\,\mathrm{d}x + \mu I_1(\hat{u}_n) + o(1) &= \mu I_2(\hat{u}_n) + \int_{\mathbb{R}^2} f(\hat{u}_n)\hat{u}_n\,\mathrm{d}x \\ &\le C\|\hat{u}_n\|_{\frac{8}{3}}^4 + \|\hat{u}_n\|_2^2 + C\|\hat{u}_n\|_p^p \\ &\le C\|\hat{u}_n\|^4 + C\|\hat{u}_n\|^2 + C\|\hat{u}_n\|^p. \end{aligned}$$

Note that $\{\|\hat{u}_n\|\}$ is bounded, one can conclude that $\sup_{n\in\mathbb{N}} I_1(\hat{u}_n) = \sup_{n\in\mathbb{N}} \mathcal{A}_1(\hat{u}_n^2, \hat{u}_n^2) < \infty$. By Lemma 2.4, we obtain the boundedness of $\{\|\hat{u}_n\|_*\}$. Thus, $\{\hat{u}_n\}$ is bounded in $E$. Passing to a subsequence if necessary, we have

$$\begin{aligned} &\hat{u}_n \rightharpoonup \hat{u} \quad \text{in } E, \qquad \hat{u}_n \to \hat{u} \quad \text{in } L^s\bigl(\mathbb{R}^2\bigr) \text{ for } s \ge 2, \\ &\hat{u}_n \to \hat{u} \quad \text{a.e. on } \mathbb{R}^2 \text{ as } n \to \infty. \end{aligned} \tag{3.4}$$

Next, we show that $I'(\hat{u}) = 0$. We claim

$$\bigl\langle I'(\hat{u}), w\bigr\rangle = \lim_{n\to\infty}\bigl\langle I'(\hat{u}_n), w\bigr\rangle = \lim_{n\to\infty}\bigl\langle I'(u_n), w(x-y_n)\bigr\rangle = 0, \quad \forall w \in E. \tag{3.5}$$

Indeed, we have

$$\begin{aligned} \bigl\|w(x-y_n)\bigr\|_E^2 &= \|w\|^2 + \int_{\mathbb{R}^2} \log\bigl(1+|x+y_n|\bigr)w^2\,\mathrm{d}x \\ &\le \|w\|_E^2 + \log\bigl(1+|y_n|\bigr)\|w\|_2^2, \quad \forall w \in E. \end{aligned} \tag{3.6}$$

And, from (3.2), one has

$$\begin{aligned} \|u_n\|_*^2 &= \int_{\mathbb{R}^2} \log\bigl(1+|x+y_n|\bigr)\hat{u}_n^2\,\mathrm{d}x \\ &\ge \int_{B_1(0)} \log\bigl(1+|x+y_n|\bigr)\hat{u}_n^2\,\mathrm{d}x \\ &\ge \frac{\delta}{2}\log|y_n| \\ &\ge \frac{\delta}{4}\log\bigl(1+|y_n|\bigr), \quad \forall |y_n| \ge 2. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$\left\|w(x-y_n)\right\|_E^2 \le \|w\|_E^2 + \left(\frac{4}{\delta}\|u_n\|_*^2 + \log 3\right)\|w\|_2^2, \quad \forall w \in E. \tag{3.8}$$

Hence, by (2.6), (2.8) and (3.8), we deduce

$$\begin{aligned} |\langle I'(\hat{u}_n), w\rangle| &= \left|\langle I'(u_n), w(x-y_n)\rangle\right| \\ &\le \left\|I'(u_n)\right\|_{E^*}\left[\|w\|_E^2 + \left(\frac{4}{\delta}\|u_n\|_*^2 + \log 3\right)\|w\|_2^2\right]^{\frac{1}{2}} \\ &= o(1), \quad \forall w \in E. \end{aligned} \tag{3.9}$$

Then,

$$\langle I'(\hat{u}_n), \hat{u}\rangle = o(1). \tag{3.10}$$

Moreover, from (2.2) and (3.4), we have

$$\left|\mathcal{A}_2\left(\hat{u}_n^2, \hat{u}_n(\hat{u}_n - \hat{u})\right)\right| \le C\|\hat{u}_n\|_{8/3}^3\|\hat{u}_n - \hat{u}\|_{8/3} = o(1). \tag{3.11}$$

By applying $(F_1)$, $(F_2)$, (3.4) and the Lebesgue's dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^2} f(\hat{u}_n)(\hat{u}_n - \hat{u})\,\mathrm{d}x = o(1). \tag{3.12}$$

Similar to [5, Lemma 2.6], one has

$$\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})w\right) = o(1), \quad \forall w \in E. \tag{3.13}$$

Let $w = \hat{u}_n - \hat{u}$, one has

$$\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})^2\right) = o(1). \tag{3.14}$$

Then, by (3.3), (3.4), (3.10)–(3.12) and (3.14), we have

$$\begin{aligned} o(1) &= \langle I'(\hat{u}_n), \hat{u}_n - \hat{u}\rangle \\ &= a\|\nabla\hat{u}_n\|_2^2 - a\|\nabla\hat{u}\|_2^2 + b\|\nabla\hat{u}_n\|_2^4 - b\|\nabla\hat{u}\|_2^4 + \int_{\mathbb{R}^2} V(x)\hat{u}_n^2\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} V(x)\hat{u}^2\,\mathrm{d}x + \mu\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})^2\right) + \mu\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})\hat{u}\right) \\ &\quad - \mu\mathcal{A}_2\left(\hat{u}_n^2, \hat{u}_n(\hat{u}_n - \hat{u})\right) - \int_{\mathbb{R}^2} f(\hat{u}_n)(\hat{u}_n - \hat{u})\,\mathrm{d}x \\ &= a\|\nabla\hat{u}_n\|_2^2 - a\|\nabla\hat{u}\|_2^2 + b\|\nabla\hat{u}_n\|_2^4 - b\|\nabla\hat{u}\|_2^4 \\ &\quad + \mu\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})^2\right) + o(1), \end{aligned}$$

which, together with $\hat{u}_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^2)$, implies

$$\|\hat{u}_n - \hat{u}\| \to 0.$$

Using Lemma 2.4, one has $\|\hat{u}_n - \hat{u}\|_* \to 0$. Thus, $\|\hat{u}_n - \hat{u}\|_E \to 0$. By (2.3), one has

$$\begin{aligned} \left|\mathcal{A}_1\left(\hat{u}_n^2 - \hat{u}^2, \hat{u}w\right)\right| &\le \|\hat{u}_n - \hat{u}\|_* \|\hat{u}_n + \hat{u}\|_* \|\hat{u}\|_2 \|w\|_2 + \|\hat{u}_n - \hat{u}\|_2 \|\hat{u}_n + \hat{u}\|_2 \|\hat{u}\|_* \|w\|_* \\ &= o(1), \quad \forall w \in E. \end{aligned} \tag{3.15}$$

Analogously to (3.11) and (3.12), we deduce

$$\mathcal{A}_2\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})w\right) = o(1), \qquad \mathcal{A}_2\left(\hat{u}_n^2 - \hat{u}^2, \hat{u}w\right) = o(1) \tag{3.16}$$

and

$$\int_{\mathbb{R}^2} \left[f(\hat{u}_n) - f(\hat{u})\right] w \,\mathrm{d}x = o(1), \quad \forall w \in E. \tag{3.17}$$

Thus, by (2.6), (3.4), (3.13), (3.15), (3.16) and (3.17), one has

$$\begin{aligned} &\left\langle I'(\hat{u}_n) - I'(\hat{u}), w\right\rangle \qquad (3.18) \\ &= a(\nabla\hat{u}_n - \nabla\hat{u}, \nabla w) + b\|\nabla\hat{u}_n\|_2^2(\nabla\hat{u}_n, \nabla w) - b\|\nabla\hat{u}\|_2^2(\nabla\hat{u}, \nabla w) \\ &\quad + \int_{\mathbb{R}^2} V(x)\hat{u}_n w \,\mathrm{d}x - \int_{\mathbb{R}^2} V(x)\hat{u}w \,\mathrm{d}x + \mu\mathcal{A}_1\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})w\right) \\ &\quad + \mu\mathcal{A}_1\left(\hat{u}_n^2 - \hat{u}^2, \hat{u}w\right) - \mu\mathcal{A}_2\left(\hat{u}_n^2, (\hat{u}_n - \hat{u})w\right) - \mu\mathcal{A}_2\left(\hat{u}_n^2 - \hat{u}^2, \hat{u}w\right) \\ &\quad - \int_{\mathbb{R}^2} \left[f(\hat{u}_n) - f(\hat{u})\right] w \,\mathrm{d}x \\ &= o(1), \quad \forall w \in E. \qquad (3.19) \end{aligned}$$

Hence, it follows from (3.9) and (3.19) that (3.5) holds. Therefore, $\hat{u} \in E$ is a nontrivial solution of (1.1), and $I(\hat{u}) = c > 0$.

Define

$$\mathcal{N} := \left\{u \in E\backslash\{0\} : I'(u) = 0\right\}.$$

Note that $\hat{u} \in \mathcal{N}$, one has $\mathcal{N} \neq \emptyset$. Using $(F_1)$ and $(F_2)$, we have

$$\left|f(u)u\right| \le \frac{1}{2}\min\{a, 1\}u^2 + C|u|^p, \quad \forall u \in \mathbb{R}. \tag{3.20}$$

Due to $\langle I'(u), u\rangle = 0$ for $u \in \mathcal{N}$, by (2.6), (3.20) and Sobolev embedding inequality, one has

$$\begin{aligned} \min\{a, 1\}\|u\|^2 &\le a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 + \int_{\mathbb{R}^2} V(x)u^2 \,\mathrm{d}x + \mu I_1(u) \\ &= \mu I_2(u) + \int_{\mathbb{R}^2} f(u)u \,\mathrm{d}x \end{aligned}$$

$$\leq \hat{C}\|u\|^4 + \frac{1}{2}\min\{a,1\}\|u\|^2 + \bar{C}\|u\|^p, \quad \forall u \in \mathcal{N}, \tag{3.21}$$

which yields

$$\|u\| \geq \delta_0 := \min\left\{1, \left(\frac{1}{2}\min\{a,1\}\right)^{\frac{1}{2}} (\hat{C}+\bar{C})^{-\frac{1}{2}}\right\} > 0, \quad \forall u \in \mathcal{N}. \tag{3.22}$$

One can easily deduce that $\inf_{\mathcal{N}} I > -\infty$. Now, we choose $\{u_n\} \subset \mathcal{N}$ satisfying $I(u_n) \to \inf_{\mathcal{N}} I$. It is easy to see that $\{u_n\}$ satisfies (2.8). Using Lemma 2.3, we obtain the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^2)$. Next, we claim that $\{u_n\}$ does not vanish. In fact, if not, applying Lions' concentration compactness principle [12], one has $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2,\infty)$. Then, by (2.2) and (2.10), we have

$$I_2(u_n) = o(1), \qquad \int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x = o(1),$$

which, together with (3.21) and (3.22), we obtain a contradiction. Therefore, using the same argument as above, there is $u_0 \in \mathcal{N}$ such that $I(u_0) = \inf_{\mathcal{N}} I > -\infty$. Then, $u_0 \in E$ is a lowest energy solution of problem (1.1). □

## 4 Existence of ground state solutions

Now, we consider the existence of ground state solutions for problem (1.1). Here we give some key lemmas.

**Lemma 4.1** *Assume that* $(F_1)$, $(F_2)$ *and* $(F_5)$ *hold. Then*

$$h(s,u) := \frac{1-s^4}{2} f(u)u + \frac{s^4-3}{2}F(u) + \frac{1}{s^2}F\left(s^2u\right) \geq 0, \quad \forall s > 0, u \in \mathbb{R}. \tag{4.1}$$

*Proof* It follows from $(F_1)$ and $(F_2)$ that, for $u = 0$, (4.1) holds. In case $u \neq 0$, from $(F_5)$, one has

$$\frac{\mathrm{d}(h(s,u))}{\mathrm{d}s} = 2s^3|u|^3\left[\frac{f(s^2u)s^2u - F(s^2u)}{s^6|u|^3} - \frac{f(u)u - F(u)}{|u|^3}\right]$$
$$\begin{cases} \geq 0, & s \geq 1, \\ \leq 0, & 0 < s < 1, \end{cases}$$

then, $h(s,u) \geq h(1,u) = 0$ for $s > 0$. □

**Lemma 4.2** *Assume that* $(V)$, $(V_2)$, $(F_1)$, $(F_2)$ *and* $(F_5)$ *hold. Then there is* $\mu^{**} > 0$ *such that, for* $\mu \geq \mu^{**}$,

$$I(u) \geq I\left(s^2u_s\right) + \frac{1-s^4}{4}J(u), \quad \forall u \in E, s > 0, \tag{4.2}$$

$$I(u) \geq \frac{1}{4}J(u) + \frac{\mu}{64\pi}\|u\|_2^4, \quad \forall u \in E. \tag{4.3}$$

*Proof* By direct calculation, we have

$$1 - s^4 + 4s^4\log s > 0, \quad \forall s > 0. \tag{4.4}$$

So we can choose $\mu^{**} > 0$ sufficiently large to satisfy

$$\frac{\mu(1-s^4)}{32\pi}\|u\|_2^4 + \frac{\mu s^4 \log s}{8\pi}\|u\|_2^4 - \frac{(1-s^4)^2}{4}b\|\nabla u\|_2^4 \geq 0, \tag{4.5}$$

for $\mu \geq \mu^{**}$. Then, it follows from $(V_2)$, (2.7), (2.9), (4.1) and (4.5) that

$$\begin{aligned}
I(u) - I(s^2 u_s) &= \frac{a}{2}\int_{\mathbb{R}^2}(1-s^4)|\nabla u|^2\,\mathrm{d}x + \frac{b\|\nabla u\|_2^2}{4}\int_{\mathbb{R}^2}(1-s^8)|\nabla u|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[V(x) - s^2 V(s^{-1}x)\right]u^2\,\mathrm{d}x \\
&\quad + \frac{\mu}{4}(1-s^4)I_0(u) + \frac{\mu s^4\log s}{8\pi}\|u\|_2^4 \\
&\quad + \int_{\mathbb{R}^2}\left[\frac{1}{s^2}F(s^2u) - F(u)\right]\mathrm{d}x \\
&= \frac{1-s^4}{4}J(u) - \frac{(1-s^4)^2}{4}b\|\nabla u\|_2^4 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x - \frac{1}{2}s^2\int_{\mathbb{R}^2}V(s^{-1}x)u^2\,\mathrm{d}x \\
&\quad - \frac{1-s^4}{4}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x + \frac{1-s^4}{8}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\,\mathrm{d}x + \frac{\mu s^4\log s}{8\pi}\|u\|_2^4 \\
&\quad + \frac{\mu(1-s^4)}{32\pi}\|u\|_2^4 + \int_{\mathbb{R}^2}\left[\frac{1}{s^2}F(s^2u) + \frac{1-s^4}{2}f(u)u + \frac{s^4-3}{2}F(u)\right]\mathrm{d}x \\
&\geq \frac{1-s^4}{4}J(u), \quad \forall u \in E, s > 0,
\end{aligned}$$

which implies that (4.2) holds. Moreover, by $(F_1)$, $(F_2)$ and (4.1), one has

$$\lim_{s\to 0} h(s,u) = \frac{1}{2}f(u)u - \frac{3}{2}F(u) \geq 0, \quad u \in \mathbb{R}. \tag{4.6}$$

Then, from $(V_2)$, (2.5), (2.7) and (4.6), we know

$$\begin{aligned}
I(u) - \frac{1}{4}J(u) &= -\frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2}V(x)u^2\,\mathrm{d}x + \frac{1}{8}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\,\mathrm{d}x \\
&\quad + \frac{\mu}{32\pi}\|u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^2}\left[f(u)u - 3F(u)\right]\mathrm{d}x \\
&\geq \frac{\mu}{64\pi}\|u\|_2^4, \quad \forall u \in E.
\end{aligned}$$

Then, (4.3) holds. □

Using Lemma 4.2, we obtain the corollary as follows.

**Corollary 4.1** *Assume that $(V)$, $(V_2)$, $(F_1)$, $(F_2)$ and $(F_5)$ hold. Then there is $\mu^{**} > 0$ such that, for $\mu \geq \mu^{**}$,*

$$I(u) = \max_{s>0} I(s^2 u_s), \quad \forall u \in \mathcal{M}. \tag{4.7}$$

**Lemma 4.3** *Assume that* $(V)$, $(F_1)$–$(F_3)$ *and* $(F_5)$ *hold. Then, for* $u \in E\backslash\{0\}$, *there is a constant* $s(u) > 0$ *such that* $[s(u)]^2 u_{s(u)} \in \mathcal{M}$.

*Proof* Fix $u \in E\backslash\{0\}$, now we define the function $\eta(s) := I(s^2 u_s)$ for $s \in (0,\infty)$. Then,

$$\begin{aligned}
\eta'(s) = 0 \quad \Leftrightarrow \quad & 2as^3\|\nabla u\|_2^2 + 2bs^7\|\nabla u\|_2^4 + s\int_{\mathbb{R}^2} V(s^{-1}x)u^2\,\mathrm{d}x \\
& - \frac{1}{2}\int_{\mathbb{R}^2}(\nabla V(s^{-1}x),x)u^2\,\mathrm{d}x \\
& + s^3\mu I_0(u) - \frac{4s^3\log s + s^3}{8\pi}\mu\|u\|_2^4 + \frac{2}{s^3}\int_{\mathbb{R}^2}F(s^2u)\,\mathrm{d}x \\
& - \frac{2}{s}\int_{\mathbb{R}^2} f(s^2u)u\,\mathrm{d}x = 0 \\
\Leftrightarrow \quad & J(s^2u_s) = 0 \\
\Leftrightarrow \quad & s^2u_s \in \mathcal{M}, \quad \forall s > 0.
\end{aligned}$$

From $(F_1)$–$(F_3)$, one can clearly know that $\lim_{s\to 0}\eta(s) = 0$, $\eta(s) > 0$ for $s$ small and $\eta(s) < 0$ for $s$ large. Then, there is $s(u) > 0$ such that $\eta(s(u)) = \max_{s>0}\eta(s)$. Thus, $\eta'(s(u)) = 0$, $s(u)^2u_{s(u)} \in \mathcal{M}$. □

Using Corollary 4.1 and Lemma 4.3, we get the following lemma immediately.

**Lemma 4.4** *Assume that* $(V)$, $(V_2)$, $(F_1)$–$(F_3)$ *and* $(F_5)$ *hold. Then*

$$\inf_{u\in\mathcal{M}} I(u) := m = \inf_{u\in E\backslash\{0\}}\max_{s>0} I(s^2u_s).$$

**Lemma 4.5** *Assume that* $(V_2)$, $(F_1)$–$(F_3)$ *and* $(F_5)$ *hold. Then*

(i) *there is* $\delta > 0$ *such that* $\|u\| \geq \delta$, $\forall u \in \mathcal{M}$;

(ii) $m = \inf_{u\in\mathcal{M}} I(u) > 0$.

*Proof* (i) From $(F_1)$ and $(F_2)$, we have

$$|f(u)u| + |F(u)| \leq \frac{\min\{2a,1\}}{4}u^2 + C|u|^p, \quad \forall u \in \mathbb{R}. \tag{4.8}$$

Note that $J(u) = 0$ for every $u \in \mathcal{M}$, by $(V_2)$, (2.7), (4.8), Hardy-Littlewood-Sobolev inequality and Sobolev embedding inequality, one has

$$\begin{aligned}
\min\{2a,1\}\|u\|^2 &\leq 2a\|\nabla u\|_2^2 + 2b\|\nabla u\|_2^4 + 6\int_{\mathbb{R}^2} V(x)u^2\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}(\nabla V(x),x)u^2\,\mathrm{d}x + \mu I_1(u) \\
&= \mu I_2(u) + \frac{\mu}{8\pi}\|u\|_2^4 + 2\int_{\mathbb{R}^2}[f(u)u - F(u)]\,\mathrm{d}x \\
&\leq \hat{C}_0\|u\|^4 + \frac{1}{2}\min\{2a,1\}\|u\|^2 + \bar{C}_0\|u\|^p, \quad \forall u \in \mathcal{M},
\end{aligned}$$

which yields

$$\|u\| \geq \delta := \min\left\{1, \left(\frac{1}{2}\min\{2a,1\}\right)^{\frac{1}{2}} (\hat{C}_0 + \bar{C}_0)^{-\frac{1}{2}}\right\}, \quad \forall u \in \mathcal{M}. \tag{4.9}$$

(ii) Choosing $\{u_n\} \subset \mathcal{M}$ such that $I(u_n) \to m$. Now, we distinguish two cases: $\inf_{n\in\mathbb{N}} \|u_n\|_2 > 0$ and $\inf_{n\in\mathbb{N}} \|u_n\|_2 = 0$. If $\inf_{n\in\mathbb{N}} \|u_n\|_2 := \delta_1 > 0$, by (4.3), we have

$$m + o(1) = I(u_n) \geq \frac{\mu}{64\pi}\|u_n\|_2^4 \geq \frac{\mu}{64\pi}\delta_1^4.$$

If $\inf_{n\in\mathbb{N}} \|u_n\|_2 = 0$, in terms of (4.9), up to a subsequence, one has

$$\|u_n\|_2 \to 0, \qquad \|\nabla u_n\|_2 \geq \delta. \tag{4.10}$$

Furthermore, in view of (4.10), one has

$$\frac{|\log(\|\nabla u_n\|_2)|}{\|\nabla u_n\|_2^2} \leq C. \tag{4.11}$$

Fix $t_n = \|\nabla u_n\|_2^{-\frac{1}{2}}$. Due to $J(u_n) = 0$, by (2.2), (2.9), (2.10), (4.7), (4.10), (4.11) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
m + o(1) &= I(u_n) \\
&\geq I\big(t_n^2 (u_n)_{t_n}\big) \\
&= \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 + \frac{b}{4}t_n^8\|\nabla u_n\|_2^4 + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V\big(t_n^{-1}x\big)u_n^2\,\mathrm{d}x + \frac{\mu t_n^4}{4}\big[I_1(u_n) - I_2(u_n)\big] \\
&\quad - \frac{\mu t_n^4 \log t_n}{8\pi}\|u_n\|_2^4 - \frac{1}{t_n^2}\int_{\mathbb{R}^2} F\big(t_n^2 u_n\big)\,\mathrm{d}x \\
&\geq \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 - \frac{\mu t_n^4}{4}I_2(u_n) - \frac{\mu t_n^4 \log t_n}{8\pi}\|u_n\|_2^4 - t_n^2\|u_n\|_2^2 - \frac{C}{t_n^2}\int_{\mathbb{R}^2}\big|t_n^2 u_n\big|^p\,\mathrm{d}x \\
&\geq \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 - Ct_n^4\mu\|u_n\|_2^3\|\nabla u_n\|_2 - \frac{\mu t_n^4 \log t_n}{8\pi}\|u_n\|_2^4 - t_n^2\|u_n\|_2^2 \\
&\quad - Ct_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} \\
&= \frac{a}{2} - C\mu\frac{\|u_n\|_2^3}{\|\nabla u_n\|_2} + \frac{\mu\log(\|\nabla u_n\|_2)}{16\pi\|\nabla u_n\|_2^2}\|u_n\|_2^4 - \frac{\|u_n\|_2^2}{\|\nabla u_n\|_2} - C\frac{\|u_n\|_2^2}{\|\nabla u_n\|_2} \\
&= \frac{a}{2} + o(1).
\end{aligned}$$

Both cases imply that $m = \inf_{u\in\mathcal{M}} I(u) > 0$. □

Similar to [3, Lemma 4.7], now we prove that the Cerami sequence obtained in Lemma 2.2 is a minimizing sequence.

**Lemma 4.6** *Assume that* $(F_1)$*–*$(F_3)$ *and* $(F_5)$ *hold. Then there is a sequence* $\{u_n\} \subset E$ *such that*

$$I(u_n) \to c \in (0, m], \qquad \big\|I'(u_n)\big\|_{E^*}\big(1 + \|u_n\|_E\big) \to 0, \qquad J(u_n) \to 0. \tag{4.12}$$

*Proof* In terms of Lemma 4.4 and Lemma 4.5, we may choose $v_n \in \mathcal{M}$ satisfying

$$0 < m \le I(v_n) < m + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{4.13}$$

By Lemma 2.2, for $n \in \mathbb{N}$, there is a sequence $\{u_n\} \subset E$ satisfying (2.8). Next, choosing $T_n > 0$ satisfying $I(T_n^2(v_n)_{T_n}) < 0$. Define $\gamma_n(t) = (tT_n)^2(v_n)_{tT_n}$, $t \in [0,1]$. One can easily get that $\gamma_n \in \Gamma$. In addition, from (2.11), we have

$$c \in \Big[d, \sup_{t>0} I\big(t^2(v_n)_t\big)\Big].$$

Applying Corollary 4.1, we deduce

$$I(v_n) = \sup_{t>0} I\big(t^2(v_n)_t\big).$$

Thus, it follows from (4.13) that

$$c \le \sup_{t>0} I\big(t^2(v_n)_t\big) < m + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Then, let $n \to \infty$ in above inequality, by virtue of Lemma 2.2, we get the desired conclusion. □

*Proof of Theorem* 1.2 From Lemma 4.6, there is a sequence $\{u_n\} \subset E$ satisfying (4.12). By (4.3) and (4.12), we have

$$c + o(1) = I(u_n) - \frac{1}{4}J(u_n) \ge \frac{\mu}{64\pi}\|u_n\|_2^4, \tag{4.14}$$

which implies that $\{\|u_n\|_2\}$ is bounded. Next, we verify the boundedness of $\{\|\nabla u_n\|_2\}$. With reduction to absurdity, we may assume that $\|\nabla u_n\|_2 \to \infty$. Fix $t_n = (\frac{2\sqrt{m}}{\sqrt{a}\|\nabla u_n\|_2})^{\frac{1}{2}}$. Note that $t_n \to 0$, then $t_n^4 \log t_n \to 0$. Hence, by (2.2), (2.9), (2.10), (4.2), (4.12), (4.14) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
m + o(1) \ge c + o(1) &= I(u_n) \\
&\ge I\big(t_n^2(u_n)_{t_n}\big) + \frac{1-t_n^4}{4}J(u_n) \\
&= \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 + \frac{b}{4}t_n^8\|\nabla u_n\|_2^4 + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V\big(t_n^{-1}x\big)u_n^2\,\mathrm{d}x + \frac{\mu t_n^4}{4}\big[I_1(u_n) - I_2(u_n)\big] \\
&\quad - \frac{\mu t_n^4 \log t_n}{8\pi}\|u_n\|_2^4 - \frac{1}{t^2}\int_{\mathbb{R}^2} F\big(t_n^2 u_n\big)\,\mathrm{d}x \\
&\ge \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 - \frac{\mu t_n^4}{4}I_2(u_n) - t_n^2\|u_n\|_2^2 - Ct_n^{2p-2}\|u_n\|_p^p + o(1) \\
&\ge \frac{a}{2}t_n^4\|\nabla u_n\|_2^2 - Ct_n^4\|u_n\|_2^3\|\nabla u_n\|_2 - t_n^2\|u_n\|_2^2 - Ct_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} + o(1) \\
&= 2m - \frac{Cm}{a\|\nabla u_n\|_2}\|u_n\|_2^3 - \frac{2\sqrt{m}}{\sqrt{a}\|\nabla u_n\|_2}\|u_n\|_2^2 - \frac{C(\sqrt{m})^{p-1}}{(\sqrt{a})^{p-1}\|\nabla u_n\|_2}\|u_n\|_2^2 + o(1) \\
&= 2m + o(1),
\end{aligned}$$

which is a contradiction, then, $\{\|\nabla u_n\|_2\}$ is bounded. Thus, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Using the same arguments as the proof of Theorem 1.1, we deduce that there is $\tilde{u} \in E\backslash\{0\}$ satisfying

$$I'(\tilde{u}) = 0, \qquad I(\tilde{u}) = c \in (0, m].$$

Furthermore, note that $\tilde{u} \in \mathcal{M}$, we obtain $I(\tilde{u}) \geq m$. Therefore, $\tilde{u} \in E$ is a ground state solution of (1.1). This completes the proof. □

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**Author contributions**
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**Author details**
[1]College of Science, Heilongjiang Institute of Technology, Harbin, 150050, P.R. China. [2]College of Civil and Architectural Engineering, Heilongjiang Institute of Technology, Harbin, 150050, P.R. China.

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