# Scattering threshold for a focusing inhomogeneous non-linear Schrödinger equation with inverse square potential 

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## Abstract

This work studies the three space dimensional focusing inhomogeneous Schrödinger equation with inverse square potential

$$
i \partial_{t} u-\left(-\Delta+\frac{\lambda}{|x|^{2}}\right) u+|x|^{-2 \tau}|u|^{2(q-1)} u=0, \quad u(t, x): \mathbb{R} \times \mathbb{R}^{3} \rightarrow \mathbb{C}
$$

The purpose is to investigate the energy scattering of global inter-critical solutions below the ground state threshold. The scattering is obtained by using the new approach of Dodson-Murphy, based on Tao's scattering criteria and Morawetz estimates. This work naturally extends the recent paper by J. An et al. (Discrete Contin. Dyn. Syst., Ser. B 28(2): 1046-1067 2023). The threshold is expressed in terms the non-conserved potential energy. As a consequence, it can be given with a classical way with the conserved mass and energy. The inhomogeneous term $|x|^{-2 \tau}$ for $\tau>0$ guarantees the existence of ground states for $\lambda \geq 0$, contrarily to the homogeneous case $\tau=0$. Moreover, the decay of the inhomogeneous term enables to avoid any radial assumption on the datum. Since there is no dispersive estimate of $L^{1} \rightarrow L^{\infty}$ for the free Schrödinger equation with inverse square potential for $\lambda<0$, one restricts this work to the case $\lambda \geq 0$.

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Keywords: Nonlinear equations; Inhomogeneous Schrödinger equation; Inverse square potential; Scattering

## 1 Introduction

This paper is concerned with the Cauchy problem for a focusing inhomogeneous Schrödinger equation with inverse square potential

$$
\left\{\begin{array}{l}
i \partial_{t} u-\mathcal{K}_{\lambda} u+|x|^{-2 \tau}|u|^{2(q-1)} u=0  \tag{1.1}\\
u_{\mid t=0}=u_{0}
\end{array}\right.
$$

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Here and hereafter, the space dimension is equal to three and the wave function is denoted by $u:=u(t, x): \mathbb{R} \times \mathbb{R}^{3} \rightarrow \mathbb{C}$. The linear Schrödinger operator is denoted by $\mathcal{K}_{\lambda}:=-\Delta+\frac{\lambda}{|x|^{2}}$, where $\Delta:=\sum_{k=1}^{3} \frac{\partial^{2}}{\partial x_{k}^{2}}$ is the classical Laplacian operator. The inhomogeneous singular decaying term is $|\cdot|^{-2 \tau}$ for some $\tau>0$. Motivated by the next sharp Hardy inequality [5],

$$
\begin{equation*}
\frac{1}{4} \int_{\mathbb{R}^{3}} \frac{|f(x)|^{2}}{|x|^{2}} d x \leq \int_{\mathbb{R}^{3}}|\nabla f(x)|^{2} d x \tag{1.2}
\end{equation*}
$$

one assumes that $\lambda>-\frac{1}{4}$, which guarantees that extension of $-\Delta+\frac{\lambda}{|x|^{2}}$, denoted by $\mathcal{K}_{\lambda}$, is a self-adjoint positive operator. In the range $-\frac{1}{4}<\lambda<\frac{3}{4}$, the extension is not unique [18, 27]. In such a case, one picks the Friedrichs extension [18, 24].

Note that by [20, Theorem 1.2], the assumption $\lambda>-\frac{1}{4}$ implies that

$$
\begin{equation*}
\left\|\sqrt{\mathcal{K}_{\lambda}} \cdot\right\|=\left(\|\nabla \cdot\|^{2}+\lambda\left\|\frac{\cdot}{|x|}\right\|^{2}\right)^{\frac{1}{2}} \simeq\|\cdot\|_{\dot{H}^{1}} \tag{1.3}
\end{equation*}
$$

The problem (1.1) models many physical phenomena. Indeed, they are used in nonlinear optical systems with spatially dependent interactions [6]. In particular, when $\lambda=0$, they can be considered as modeling inhomogeneities in the medium in which the wave propagates $[1,2,19]$. When $\tau=0$, they model a quantum field equations or black hole solutions of the Einstein's equations [12, 18]. See also [3, 22]. In statistical mechanics, the inverse square potential represents the borderline case for phase transition for the long-range 1D Ising model [17]. The quantum mechanics of the inverse square potential is relevant to phenomena as diverse as the Efimov effect for short range interacting bosons [21], the interaction between an electron and a polar neutral molecule [8] and the near-horizon problem for certain black holes [9].
Let us recall some literature dealing with (1.1). Using the energy method, [25] investigated the local well-posedness in the energy space. Moreover, the local solution extends globally in time, either in the defocusing case or in the focusing, mass-subcritical case. Later on, [10] revisits the same problem, where the authors studied the local wellposedness and small data global well-posedness in the energy-sub-critical case by using the standard Strichartz estimates combined with the fixed point argument. See also $[4,11]$ for the ground state threshold of global existence versus blow-up dichotomy in the intercritical regime. Furthermore, [10] showed a scattering criterion and constructed a wave operator for the inter-critical case. The well-posedness and blow-up in the energy critical regime were investigated in [16].
The purpose of this paper is to investigate the scattering of energy global solutions of the Schrödinger problem (1.1) in the inter-critical regime and below the ground state threshold. This naturally extends the recent paper [4], where the global existence versus finite time blow-up below the ground state threshold was proved, but the scattering was not treated. The scattering is obtained by using the new approach of Dodson-Murphy [13]. This method is based on Tao's scattering criteria [26] and Morawetz estimates. The inhomogeneous term $|x|^{-2 \tau}$ for $\tau>0$ guarantees the existence of ground states for $\lambda \geq 0$, contrarily to the homogeneous case $\tau=0$. Moreover, the decay of the inhomogeneous term enables to avoid any radial assumption on the datum. Because for $\lambda<0$ there is no
dispersive estimate of $L^{1} \rightarrow L^{\infty}$ for the free Schrödinger equation with inverse square potential [23], one restricts this work to the case $\lambda \geq 0$.

The rest of this paper is organized as follows. The next section contains the main result and some useful estimates. Section 3 proves the main result. A non-linear estimate is proved in the appendix.

## 2 Background and main result

This section contains the main results and some useful estimates.

### 2.1 Preliminary

Here and hereafter, one denotes for simplicity some standard Lebesgue and Sobolev spaces $L^{r}:=L^{r}\left(\mathbb{R}^{3}\right), W^{1, r}:=W^{1, r}\left(\mathbb{R}^{3}\right)$ and $H^{1}:=W^{1,2}$ and the norms $\|\cdot\|_{r}:=\|\cdot\|_{L^{r}},\|\cdot\|:=\|\cdot\|_{2}$. Similarly, one defines Sobolev spaces in terms of the operator $\mathcal{K}_{\lambda}$, as the completion of $C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$ with respect to the norms

$$
\|\cdot\|_{\dot{W}_{\lambda}^{1, r}}:=\left\|\sqrt{\mathcal{K}_{\lambda}} \cdot\right\|_{r}, \quad\|\cdot\|_{W_{\lambda}^{1, r}}:=\left\|\left\langle\sqrt{\mathcal{K}_{\lambda}}\right\rangle \cdot\right\|_{r},
$$

where $\langle\cdot\rangle:=\left(1+|\cdot|^{2}\right)^{\frac{1}{2}}$. Take also for short $\dot{H}_{\lambda}^{1}:=\dot{W}_{\lambda}^{1,2}$ and $H_{\lambda}^{1}:=W_{\lambda}^{1,2}$. Note that by the definition of the operator $\mathcal{K}_{\lambda}$ and Hardy estimate (1.2), one has

$$
\|\cdot\|_{\dot{H}_{\lambda}^{1}}:=\left\|\sqrt{\mathcal{K}_{\lambda}} \cdot\right\|=\left(\|\nabla \cdot\|^{2}+\lambda\left\|\frac{\cdot}{|x|}\right\|^{2}\right)^{\frac{1}{2}} \simeq\|\cdot\|_{\dot{H}^{1}}
$$

Let us also define the real numbers

$$
\gamma:=3 q-3+2 \tau, \quad \rho:=2 q-\gamma .
$$

If $u \in H_{\lambda}^{1}$, one defines the quantities related to energy solutions of (1.1),

$$
\begin{align*}
Q[u] & :=\int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q} d x,  \tag{2.1}\\
\mathcal{I}[u] & :=\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}-\frac{\gamma}{2 q} \mathcal{Q}[u], \\
\mathcal{M}[u] & :=\int_{\mathbb{R}^{3}}|u(x)|^{2} d x,  \tag{2.2}\\
\mathcal{E}[u] & :=\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}-\frac{1}{q} \mathcal{Q}[u] . \tag{2.3}
\end{align*}
$$

The equation (1.1) enjoys the scaling invariance

$$
\begin{equation*}
u_{\kappa}:=\kappa^{\frac{1-\tau}{q-1}} u\left(\kappa^{2} \cdot, \kappa \cdot\right), \quad \kappa>0 . \tag{2.4}
\end{equation*}
$$

The critical exponent $s_{c}$ keeps invariant the following homogeneous Sobolev norm

$$
\left\|u_{\kappa}(t)\right\|_{\dot{H}^{s}}=\kappa^{s-\left(\frac{3}{2}-\frac{1-\tau}{q-1}\right)}\left\|u\left(\kappa^{2} t\right)\right\|_{\dot{H}^{s}}:=\kappa^{s-s_{c}}\left\|u\left(\kappa^{2} t\right)\right\|_{\dot{H}^{s}} .
$$

Two cases are of particular interest in the physical context. The first one $s_{c}=0$ corresponds to the mass-critical case, which is equivalent to $q=q_{c}:=1+\frac{2(1-\tau)}{3}$. This case is related to
the conservation of the mass (2.2). The second one is the energy-critical case $s_{c}=1$, which corresponds to $q=q^{c}:=1+2(1-\tau)$. This case is related to the conservation of the energy (2.3). A particular periodic global solution of (1.1), called standing wave, takes the form $e^{i t} \varphi$, where $\varphi$ satisfies

$$
\begin{equation*}
\mathcal{K}_{\lambda} \varphi+\varphi=|x|^{-2 \tau}|\varphi|^{2(q-1)} \varphi, \quad 0 \neq \varphi \in H_{\lambda}^{1} . \tag{2.5}
\end{equation*}
$$

The standing waves play an important role in the Schrödinger context. Indeed, they give global solutions which don't scatter. The existence of ground states is related to the next Gagliardo-Nirenberg type inequalities [10].

Proposition 2.1 Let $\lambda>-\frac{1}{4}, 0<\tau<1$ and $1<q<q^{c}$. Thus,

1. There exists a sharp constant $C_{q, \tau, \lambda}>0$, such that for all $u \in H_{\lambda}^{1}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q} d x \leq C_{q, \tau, \lambda}\|u\|^{\rho}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{\gamma} \tag{2.6}
\end{equation*}
$$

2. Moreover, there exists $\varphi$, a solution to (2.5), satisfying

$$
\begin{equation*}
C_{q, \tau, \lambda}=\frac{2 q}{\rho}\left(\frac{\rho}{\gamma}\right)^{\frac{\gamma}{2}}\|\varphi\|^{-2(q-1)}, \tag{2.7}
\end{equation*}
$$

3. Furthermore, one has the following Pohozaev identities

$$
\begin{equation*}
\mathcal{Q}[\varphi]=\frac{2 q}{\rho} \mathcal{M}[\varphi]=\frac{2 q}{\gamma}\left\|\sqrt{\mathcal{K}_{\lambda}} \varphi\right\|^{2} \tag{2.8}
\end{equation*}
$$

Here and hereafter, we focus on the inter-critical regime $0<s_{c}<1$. So, we denote the positive real number $\frac{1}{s_{c}}-1:=\alpha_{c} \in(0,1), \varphi$ is a ground state of (2.5), and the scale invariant quantities are

$$
\begin{aligned}
& \mathcal{M E}[u]:=\left(\frac{\mathcal{M}\left[u_{0}\right]}{\mathcal{M}[\varphi]}\right)^{\alpha_{c}}\left(\frac{\mathcal{E}\left[u_{0}\right]}{\mathcal{E}[\varphi]}\right), \\
& \mathcal{M G}[u]:=\left(\frac{\left\|u_{0}\right\|}{\|\varphi\|}\right)^{\alpha_{c}}\left(\frac{\left\|\sqrt{\mathcal{K}_{\lambda}} u_{0}\right\|}{\left\|\sqrt{\mathcal{K}_{\lambda}} \varphi\right\|}\right), \\
& \mathcal{M Q}[u]:=\left(\frac{\mathcal{M}\left[u_{0}\right]}{\mathcal{M}[\varphi]}\right)^{\alpha_{c}}\left(\frac{\mathcal{Q}[u]}{\mathcal{Q}[\varphi]}\right) .
\end{aligned}
$$

Let $e^{-i \cdot \mathcal{K}_{\lambda}}$ be the operator associated to the free Schrödinger equation $\left(i \partial_{t}-\mathcal{K}_{\lambda}\right)=0$. Then, by Duhamel integral formula, energy solutions of the problem (1.1) are fixed points of the function

$$
\begin{equation*}
f(u):=e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}+i \int_{0} e^{-i(\cdot-s) \mathcal{K}_{\lambda}}\left[|x|^{-2 \tau}|u(s)|^{2(q-1)} u(s)\right] d s \tag{2.9}
\end{equation*}
$$

In the next sub-section, we list the contribution of this note.

### 2.2 Main result

The main result of this note is the following energy scattering threshold.

Theorem 2.2 Let $\lambda>0$ and $0<\tau<1$. Take $q_{c}<q<q^{c}$ and $u \in C_{T^{*}}\left(H_{\lambda}^{1}\right)$ be a maximal solution to (1.1). Then, $u$ is global and scatters if one of the next assumptions holds

$$
\begin{align*}
& \sup _{t \in\left[0, T^{*}\right)} \mathcal{M Q}[u(t)]<1  \tag{2.10}\\
& \max \left\{\mathcal{M E}\left[u_{0}\right], \mathcal{M G}\left[u_{0}\right]\right\}<1 \tag{2.11}
\end{align*}
$$

## Remarks 2.3

1. The first point expresses the threshold in terms of the non-conserved potential energy $\mathcal{Q}$,
2. The scattering under the assumption (2.11) is a consequence of the scattering under the condition (2.10),
3. The assumption $\lambda \geq 0$ is needed only in the proof of Proposition 3.3,
4. The proof follows the method of Dodson-Murphy [13] based on Tao's scattering criteria [26] and Morawetz estimates,
5. The scattering means that the global solution of (1.1) is close to the solution of the associated free equation. This means that the source term has a negligible affect for large time. Precisely, the energy scattering reads: there exists $u_{ \pm} \in H^{1}$ such that $\lim _{t \rightarrow \pm \infty}\left\|u(t)-e^{-i t \mathcal{K}_{\lambda}} u_{ \pm}\right\|_{H^{1}}=0$,
6. This theorem extends the recent paper [4], where the global existence versus finite time blow-up below the ground state threshold was proved, but the scattering was not treated.

### 2.3 Useful estimates

In this sub-section, some standard tools needed in the sequel are given.

Definition 2.4 A couple of real numbers ( $q, r$ ) is said to be $\mu$ admissible (admissible if $\mu=0$ ) if

$$
\begin{equation*}
3\left(\frac{1}{2}-\frac{1}{r}\right)=\frac{2}{q}+\mu, \quad \frac{6}{3-2 \mu}<r<6 . \tag{2.12}
\end{equation*}
$$

For simplicity, we denote by $\Gamma^{\mu}$ the set of $\mu$ admissible pairs and $\Gamma:=\Gamma^{0}$. Let also for any real interval $I$,

$$
\begin{aligned}
& \Lambda_{\mu}(I):=\bigcap_{(q, r) \in \Gamma^{\mu}} L^{q}\left(I, L^{r}\right), \quad\|\cdot\|_{\Lambda_{\mu}(I)}:=\sup _{(q, r) \in \Gamma^{\mu}}\|\cdot\|_{L^{q}\left(I, L^{r}\right)}, \\
& \|\cdot\|_{\Lambda^{\prime}-\mu(I)}:=\inf _{(q, r) \in \Gamma^{-\mu}}\|\cdot\|_{L^{q^{\prime}}\left(I, L^{\prime}\right)} .
\end{aligned}
$$

Take also the particular cases

$$
\Lambda(I):=\Lambda_{0}(I), \quad \Lambda^{\prime}(I):=\Lambda_{0}^{\prime}(I), \quad \Lambda_{\mu}:=\Lambda_{\mu}((0, \infty)), \quad \Lambda_{-\mu}^{\prime}:=\Lambda_{-\mu}^{\prime}((0, \infty))
$$

An essential tool used in this note is Strichartz estimate [7, 15, 28].

Proposition 2.5 Let $\lambda>-\frac{1}{4}, \mu \in \mathbb{R}$ and $0 \in I$ be a real interval. Then, there exists $C>0$ such that

1. $\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u\right\|_{\Lambda_{\mu}(I)} \leq C\|u\|_{\dot{H}^{\mu}}$,
2. $\left\|\int_{0} e^{-i(-\tau) \mathcal{K}_{\lambda}} f(\tau) d \tau\right\|_{\Lambda(I)} \leq C\|f\|_{\Lambda^{\prime}(I)}$,
3. if $\lambda \geq 0$, so $\left\|\int_{0}^{0} e^{-i(-\tau) \mathcal{K}_{\lambda}} f(\tau) d \tau\right\|_{\Lambda_{\mu}(I)} \leq C\|f\|_{\Lambda_{-\mu}^{\prime}(I)}$.

The above Strichartz estimates are consequence of the next dispersive estimates [14, 23].

## Proposition 2.6 There exists $C>0$ such that

1. $\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u\right\|_{r^{\prime}} \leq C \frac{\|u\|_{r}}{|t|^{3\left(\frac{1}{r}-\frac{1}{2}\right)}}$, whenever $\frac{1}{2} \leq \frac{1}{r}<\min \left\{1,1-\frac{\kappa}{3}\right\}$,
2. $\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u\right\|_{r^{\prime}} \leq C \frac{\|u\|_{r}}{|t|^{3\left(\frac{1}{r}-\frac{1}{2}\right)}}$, whenever $r \in[2, \infty]$ and $\lambda \geq 0$.

From now on, we hide the time variable $t$ for simplicity, spreading it out only when necessary. Moreover, we denote the centered ball of $\mathbb{R}^{3}$ with radius $R>0$ and its complementary, respectively $B(R)$ and $B^{c}(R)$. Furthermore $C\left(R, R^{\prime}\right)$ is the centered annulus of $\mathbb{R}^{3}$ with small radius $R$ and large radius $R^{\prime}$. Finally, the critical Sobolev embedding $H^{1} \hookrightarrow L^{2^{*}}$ gives the index $2^{*}:=6$. In what follows, one proves the main result of this note.

## 3 Proof of Theorem 2.2

### 3.1 Global existence

The global existence of energy solutions of (1.1) follows from the conservation laws via the next coercivity result.

Lemma 3.1 Let $u \in H_{\lambda}^{1}$ and $0<\nu<1$ satisfying

$$
\begin{equation*}
\mathcal{M Q}[u]<v . \tag{3.1}
\end{equation*}
$$

Then, there is $c(v, \varphi)>0$ such that

$$
\begin{align*}
& \left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}<c(\nu, \varphi) \mathcal{E}[u],  \tag{3.2}\\
& \mathcal{I}[u]>c(\nu, \varphi)\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2} . \tag{3.3}
\end{align*}
$$

Proof A direct computation gives the useful identities

$$
\begin{align*}
& 2(q-1) s_{c}=\gamma-2,  \tag{3.4}\\
& \alpha_{c}(\gamma-2)=\rho . \tag{3.5}
\end{align*}
$$

Using the Gagliardo-Nirenberg inequality (2.6) via Pohozaev identities (2.8), the explicit expression (2.7) and the equalities (3.4)-(3.5), are written

$$
\begin{aligned}
{[\mathcal{Q}[u]]^{\frac{\gamma}{2}} } & \leq C_{q, \tau, \lambda}\left(\|u\|^{2 \alpha_{c}} \mathcal{Q}[u]\right)^{\frac{\gamma}{2}-1}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{\gamma} \\
& \leq \frac{2 q}{\rho}\left(\frac{\rho}{\gamma}\right)^{\frac{\gamma}{2}}\|\varphi\|^{-2(q-1)}\left(\mathcal{M}[u]^{\alpha_{c}} \mathcal{Q}[u]\right)^{\frac{\gamma}{2}-1}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{\gamma} \\
& \leq \frac{2 q}{\rho}\left(\frac{\rho}{\gamma}\right)^{\frac{\gamma}{2}} \mathcal{M}[\varphi]^{\frac{\rho-2(q-1)}{2}}[\mathcal{Q}[\varphi]]^{\frac{\gamma}{2}-1}(\mathcal{M} \mathcal{Q}[u])^{\frac{\gamma}{2}-1}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{\gamma}
\end{aligned}
$$

$$
\begin{align*}
& \leq\left(\frac{\rho}{\gamma} \frac{\mathcal{Q}[\varphi]}{\mathcal{M}[\varphi]}\right)^{\frac{\gamma}{2}}(\mathcal{M} \mathcal{Q}[u])^{\frac{\gamma}{2}-1}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{\gamma} \\
& \leq(\mathcal{M} \mathcal{Q}[u])^{\frac{\gamma}{2}-1}\left(\frac{2 q}{\gamma}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}\right)^{\frac{\gamma}{2}} \tag{3.6}
\end{align*}
$$

Thus, taking (3.6) to the exponent $\frac{2}{\gamma}$, we get

$$
\begin{equation*}
\mathcal{Q}[u] \leq \frac{2 q}{\gamma}(\mathcal{M} \mathcal{Q}[u])^{\frac{\gamma-2}{\gamma}}\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2} \tag{3.7}
\end{equation*}
$$

This implies that

$$
\begin{aligned}
\mathcal{E}[u] & =\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}-\frac{1}{q} \mathcal{Q}[u] \\
& \geq\left(1-\frac{2}{\gamma}(\mathcal{M Q}[u])^{\frac{\gamma-2}{\gamma}}\right)\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2} .
\end{aligned}
$$

The proof of (3.2) follows from (3.1) via the assumption $s_{c}>0$ which gives $\gamma>2$. Moreover, by (3.7) and (3.1), we have

$$
\begin{aligned}
\mathcal{I}[u] & =\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}-\frac{\gamma}{2 q} \mathcal{Q}[u] \\
& \geq\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2}\left(1-(\mathcal{M Q}[u])^{\frac{\gamma-2}{\gamma}}\right) \\
& \gtrsim\left\|\sqrt{\mathcal{K}_{\lambda}} u\right\|^{2} .
\end{aligned}
$$

This proves (3.3).

### 3.2 Scattering criteria

Here and hereafter, we denote a smooth function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$ such that $\psi=1$ on $B(1)$ and $\psi=0$ on $B^{c}(2)$. Take also $\psi_{R}:=\psi(\dot{\bar{R}})$. In this sub-section, we prove the next scattering criteria.

Proposition 3.2 Take the assumptions of Theorem 2.2. Let $u \in C\left(\mathbb{R}, H_{\lambda}^{1}\right)$ be a global solution to (1.1). Assume that

$$
\begin{equation*}
0<\sup _{t \in \mathbb{R}}\|u(t)\|_{H_{\lambda}^{1}}:=E<\infty \tag{3.8}
\end{equation*}
$$

There exist $R, \varepsilon>0$ depending on $E, d, q, \tau$ such that $u$ scatters if

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{B(R)}|u(t, x)|^{2} d x<\varepsilon^{2} . \tag{3.9}
\end{equation*}
$$

Proof Using an interpolation via the bound in $L^{\infty}\left(H_{\lambda}^{1}\right)$, it is sufficient to prove that

$$
\begin{equation*}
u \in L^{4}\left(L^{2^{*}}\right) . \tag{3.10}
\end{equation*}
$$

Moreover, by Sobolev embeddings and Hölder estimate, we write

$$
\|u\|_{L_{T}^{4}\left(L^{2^{*}}\right)} \leq T^{\frac{1}{4}}\|u\|_{L^{\infty}\left(H_{\lambda}^{1}\right)} .
$$

So, it is sufficient to prove that there is $T>0$ such that

$$
\begin{equation*}
u \in L^{4}\left((T, \infty), L^{2^{*}}\right) \tag{3.11}
\end{equation*}
$$

By continuity argument, Strichartz estimate and Sobolev embedding, the key of the proof of the scattering criterion is the next result.

Proposition 3.3 Take the assumptions of Proposition 3.2. Then, for any $\varepsilon>0$, there exist T, $\mu>0$ satisfying

$$
\begin{equation*}
\left\|e^{i(\cdot-T) \mathcal{K}_{\lambda}} u(T)\right\|_{L^{4}\left((T, \infty), L^{2^{*}}\right)} \lesssim \varepsilon^{\mu} . \tag{3.12}
\end{equation*}
$$

Proof By the integral formula

$$
\begin{align*}
e^{-i(t-T) \mathcal{K}_{\lambda}} u(T) & =e^{-i t \mathcal{K}_{\lambda}} u_{0}+i \int_{0}^{T} e^{-i(t-\tau) \mathcal{K}_{\lambda}}\left[|x|^{-2 \tau}|u|^{2(q-1)} u\right] d \tau \\
& =e^{-i t \mathcal{K}_{\lambda}} u_{0}+i\left(\int_{0}^{T-\varepsilon^{-\beta}}+\int_{T-\varepsilon^{-\beta}}^{T}\right) e^{-i(t-\tau) \mathcal{K}_{\lambda}}\left[|x|^{-2 \tau}|u|^{2(q-1)} u\right] d \tau \\
& :=e^{-i t \mathcal{K}_{\lambda}} u_{0}+i\left(\int_{J_{1}}+\int_{J_{2}}\right) e^{-i(t-\tau) \mathcal{K}_{\lambda}}\left[|x|^{-2 \tau}|u|^{2(q-1)} u\right] d \tau \\
& :=e^{-i t \mathcal{K}_{\lambda}} u_{0}+F_{1}+F_{2} . \tag{3.13}
\end{align*}
$$

Now, we estimate the three different parts in (3.13).

- The linear term. By Hölder and Strichartz estimates via Sobolev injections, we have

$$
\begin{align*}
\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{4}\left((T, \infty), L^{2^{*}}\right)} & \leq\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{\infty}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}}\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{2}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}} \\
& \leq c\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{\infty}\left((T, \infty), H_{\lambda}^{1}\right)}^{\frac{1}{2}}\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{2}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}} \\
& \leq c\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{2}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}} \tag{3.14}
\end{align*}
$$

Thus, by the Dominated convergence Theorem via Strichartz estimates and the fact that $\left(2,2^{*}\right) \in \Gamma$, one may choose $T_{0}>\varepsilon^{-\beta}>0$, where $\beta>0$ is to choose later, such that

$$
\begin{equation*}
\left\|e^{-i \cdot \mathcal{K}_{\lambda}} u_{0}\right\|_{L^{4}\left(\left(T_{0}, \infty\right), L^{2^{*}}\right)} \leq \varepsilon^{2} \tag{3.15}
\end{equation*}
$$

- The term $F_{1}$. First, the integral formula (2.9) gives

$$
\begin{equation*}
F_{1}=e^{-i t \mathcal{K}_{\lambda}}\left(e^{-i\left(-T+\varepsilon^{-\beta}\right) \mathcal{K}_{\lambda}} u\left(T-\varepsilon^{-\beta}\right)-u_{0}\right) . \tag{3.16}
\end{equation*}
$$

So, using Strichartz estimate via (3.16), the fact that $\left(2,2^{*}\right) \in \Gamma$ and an interpolation, we write

$$
\begin{align*}
\left\|F_{1}\right\|_{L^{4}\left((T, \infty), L^{*}\right)} & \leq\left\|F_{1}\right\|_{L^{\infty}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}}\left\|F_{1}\right\|_{L^{2}\left((T, \infty), L^{2^{*}}\right)}^{\frac{1}{2}} \\
& \leq c\left\|F_{1}\right\|_{L^{\infty}\left((T, \infty), L^{*}\right)}^{\frac{1}{2}} . \tag{3.17}
\end{align*}
$$

Now, an interpolation via (3.16), (A.1) and Proposition 2.6, implies that

$$
\begin{align*}
\left\|F_{1}(t)\right\|_{2^{*}} & \leq\left\|F_{1}(t)\right\|^{\frac{1}{3}}\left\|F_{1}(t)\right\|_{\infty}^{\frac{2}{3}} \\
& \leq c\left\|F_{1}(t)\right\|_{\infty}^{\frac{2}{3}} \\
& \leq c\left(\int_{0}^{T-\varepsilon^{-\beta}}|t-s|^{-\frac{3}{2}}\left\||x|^{-2 \tau}|u|^{2(q-1)} u\right\|_{1} d s\right)^{\frac{2}{3}} \\
& \leq c\left(\left(t-T+\varepsilon^{-\beta}\right)^{1-\frac{3}{2}}\|u\|_{H_{\lambda}^{1}}^{2 q-1}\right)^{\frac{2}{3}} \\
& \leq c \varepsilon^{\frac{\beta}{3}} . \tag{3.18}
\end{align*}
$$

So, it follows that

$$
\begin{equation*}
\left\|F_{1}\right\|_{L^{\infty}\left((T, \infty), L^{2^{*}}\right)} \leq c \varepsilon^{v}, \quad v>0 . \tag{3.19}
\end{equation*}
$$

Finally, with an interpolation via (3.19), we get

$$
\begin{align*}
\left\|F_{1}\right\|_{L^{4}\left((T, \infty), L^{2^{*}}\right)}^{2} & \leq\left\|F_{1}\right\|_{L^{\infty}\left((T, \infty), L^{2^{*}}\right)}\left\|F_{1}\right\|_{L^{2}\left((T, \infty), L^{2^{*}}\right)} \\
& \leq \varepsilon^{v}, \quad v>0 . \tag{3.20}
\end{align*}
$$

- The term $F_{2}$. By the assumption (3.9), one has for $T>\varepsilon^{-\beta}$ large enough,

$$
\int_{\mathbb{R}^{3}} \psi_{R}(x)|u(T, x)|^{2} d x<\varepsilon^{2}
$$

Moreover, a computation with use of (1.1) and Hölder estimate gives

$$
\begin{align*}
\left.\left.\left|\frac{d}{d t} \int_{\mathbb{R}^{3}} \psi_{R}\right| u\right|^{2} d x \right\rvert\, & =\left|-2 \Im \int_{\mathbb{R}^{3}} \psi_{R} \bar{u} \Delta u d x\right| \\
& =\left|2 \Im \int_{\mathbb{R}^{3}} \bar{u} \nabla \psi_{R} \cdot \nabla u d x\right| \\
& \lesssim \frac{1}{R} . \tag{3.21}
\end{align*}
$$

Take the real function $g_{R}(t):=\int_{\mathbb{R}^{3}} \psi_{R}(x)|u(t, x)|^{2} d x$. By (3.21), we write

$$
\begin{align*}
\left\|\psi_{R} u(t)\right\|^{2} & \leq g_{R}(t) \\
& \leq g_{R}(T)+\int_{t}^{T}\left|g_{R}^{\prime}(s)\right| d s \\
& \leq \int_{\mathbb{R}^{3}} \psi_{R}(x)|u(T, x)|^{2} d x+C \frac{T-t}{R} . \tag{3.22}
\end{align*}
$$

Then, for any $T-\varepsilon^{-\beta} \leq t \leq T$ and $R>\varepsilon^{-(2+\beta)}$, yields by (3.22),

$$
\left\|\psi_{R} u(t)\right\| \leq\left(\int_{\mathbb{R}^{3}} \psi_{R}(x)|u(T, x)|^{2} d x+C \frac{T-t}{R}\right)^{\frac{1}{2}} \leq C \varepsilon
$$

This gives

$$
\begin{equation*}
\left\|\psi_{R} u\right\|_{L^{\infty}\left(\left[T-\varepsilon^{-\beta}, T\right], L^{2}\right)} \leq C \varepsilon . \tag{3.23}
\end{equation*}
$$

Using Strichartz estimate in Proposition 2.5, we write

$$
\begin{align*}
\left\|F_{2}\right\|_{L^{4}\left(L^{2^{*}}\right)} & \leq\left\|F_{2}\right\|_{\Lambda_{\frac{1}{2}}} \\
& \leq\left\||x|^{-2 \tau}|u|^{2(q-1)} u\right\|_{\Lambda_{-\frac{1}{2}}^{\prime}\left(J_{2}\right)} \\
& \leq\left\||x|^{-2 \tau}|u|^{2(q-1)} u\right\|_{L^{4}\left(J_{2}, L^{5}\right)} \\
& \leq\left\|\psi_{R}|x|^{-2 \tau}|u|^{2(q-1)} u\right\|_{L^{4}\left(J_{2}, L^{5}\right)}+\left\|\left(1-\psi_{R}\right)|x|^{-2 \tau}|u|^{2(q-1)} u\right\|_{L^{4}\left(J_{2}, L^{\left.\frac{6}{5}\right)}\right.} \\
& :=\|(I)\|_{\left.L^{4} J_{2}\right)}+\|(I I)\|_{\left.L^{4} J_{2}\right)} . \tag{3.24}
\end{align*}
$$

Now, by Hölder estimate via (3.24) and (3.23), we write for certain $0<\theta \leq 1$,

$$
\begin{align*}
(I) & \leq\left\|\psi_{R} u\right\|_{b}\left\||x|^{-2 \tau}\right\|_{L^{a}(|x|<R)}\|u\|_{b}^{2(q-1)} \\
& \leq c\left\|\psi_{R} u\right\|^{\theta}\|u\|_{H_{\lambda}^{1}}^{2(q-1)+1-\theta} \\
& \leq c \varepsilon^{\theta} . \tag{3.25}
\end{align*}
$$

Here,

$$
\left\{\begin{array}{l}
\frac{5}{6}=\frac{1}{a}+\frac{2 q-1}{b},  \tag{3.26}\\
\frac{3}{a}>2 \tau \\
\frac{1}{6}<\frac{1}{b} \leq \frac{1}{2} .
\end{array}\right.
$$

This reads

$$
\left\{\begin{array}{l}
\frac{5}{2}-\frac{3(2 q-1)}{b}=\frac{3}{a}>2 \tau,  \tag{3.27}\\
\frac{1}{6}<\frac{1}{b} \leq \frac{1}{2} .
\end{array}\right.
$$

So, we get

$$
\begin{equation*}
q-\frac{1}{2}<\frac{3(2 q-1)}{b}<\frac{5}{2}-2 \tau \tag{3.28}
\end{equation*}
$$

This is possible because $q<q^{c}$ and $0<\tau<1$.
Moreover, by Hölder estimate via (3.24) and the properties of $\psi$, we write

$$
\begin{align*}
(I I) & \leq c\left\||x|^{-2 \tau}\right\|_{L^{g}(|x|>R)}\|u\|_{e}^{2 q-1} \\
& \leq c R^{-(2 g \tau-3)}\|u\|_{H_{\lambda}^{1}}^{2 q-1} \\
& \leq c R^{-(2 g \tau-3)} . \tag{3.29}
\end{align*}
$$

Here,

$$
\left\{\begin{array}{l}
\frac{4}{6}=\frac{1}{g}+\frac{2 q-1}{e},  \tag{3.30}\\
\frac{3}{g}<2 \tau \\
\frac{1}{6} \leq \frac{1}{e} \leq \frac{1}{2}
\end{array}\right.
$$

This reads

$$
\left\{\begin{array}{l}
\frac{4}{2}-\frac{3(2 q-1)}{e}=\frac{3}{g}<2 \tau,  \tag{3.31}\\
\frac{1}{6} \leq \frac{1}{e} \leq \frac{1}{2} .
\end{array}\right.
$$

So,

$$
\left\{\begin{array}{l}
\frac{1}{e}>\frac{2(1-\tau)}{3(2 q-1)},  \tag{3.32}\\
\frac{1}{6} \leq \frac{1}{e} \leq \frac{1}{2}
\end{array}\right.
$$

This is possible because $q>q_{c}$ gives

$$
\begin{equation*}
q-1>\frac{1-4 \tau}{6} \tag{3.33}
\end{equation*}
$$

Now, by (3.24), (3.25) and (3.29), we get for $0<\beta<\theta$ and $R^{-(2 g \tau-3)}<\varepsilon^{\beta}$,

$$
\begin{align*}
\left\|F_{2}\right\|_{L^{4}\left(L^{2^{*}}\right)} & \leq\|(I)\|_{L^{2}\left(J_{2}\right)}+\|(I I)\|_{L^{2}\left(J_{2}\right)} \\
& \leq c\left|J_{2}\right|^{\frac{1}{2}}\left(R^{-(2 g \tau-3)}+\varepsilon^{\theta}\right) \\
& \leq c \varepsilon^{-\frac{\beta}{2}}\left(R^{-(2 g \tau-3)}+\varepsilon^{\theta}\right) \\
& \leq c \varepsilon^{\nu} . \tag{3.34}
\end{align*}
$$

The proof is closed via (3.15), (3.19) and (3.34).

### 3.3 Virial/morawetz estimate

Let $\zeta: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a convex smooth function. Define the variance potential

$$
\begin{equation*}
V_{\zeta}:=\int_{\mathbb{R}^{3}} \zeta(x)|u(\cdot, x)|^{2} d x, \tag{3.35}
\end{equation*}
$$

and the Morawetz action

$$
\begin{equation*}
M_{\zeta}=2 \mathfrak{\Im} \int_{\mathbb{R}^{3}} \bar{u}(\nabla \zeta \cdot \nabla u) d x:=2 \mathfrak{\Im} \int_{\mathbb{R}^{3}} \bar{u}\left(\zeta_{j} u_{j}\right) d x \tag{3.36}
\end{equation*}
$$

where repeated indices are summed here and subsequently. Let us give a Morawetz-type estimate for the Schrödinger equation with inverse square potential [4].

Proposition 3.4 Take $u \in C\left([0, T], H_{\lambda}^{1}\right)$ to be a local solution of (1.1) and $\zeta: \mathbb{R}^{3} \rightarrow \mathbb{R}$ be a smooth function. Then, the following equality holds on $[0, T]$,

$$
\begin{aligned}
V_{\zeta}^{\prime \prime}[u]=M_{\zeta}^{\prime}[u]= & 4 \int_{\mathbb{R}^{3}} \partial_{l} \partial_{k} \zeta \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x-\int_{\mathbb{R}^{3}} \Delta^{2} \zeta|u|^{2} d x+4 \lambda \int_{\mathbb{R}^{3}} \nabla \zeta \cdot x \frac{|u|^{2}}{|x|^{4}} d x \\
& -2\left(1-\frac{1}{q}\right) \int_{\mathbb{R}^{3}} \Delta \zeta|x|^{-2 \tau}|u|^{2 q} d x+\frac{2}{q} \int_{\mathbb{R}^{3}} \nabla \zeta \cdot \nabla\left(|x|^{-2 \tau}\right)|u|^{2 q} d x .
\end{aligned}
$$

The next radial identities will be useful in the sequel.

$$
\begin{align*}
& \nabla=\frac{x}{r} \partial_{r}  \tag{3.37}\\
& \frac{\partial^{2}}{\partial x_{l} \partial x_{k}}:=\partial_{l} \partial_{k}=\left(\frac{\delta_{l k}}{r}-\frac{x_{l} x_{k}}{r^{3}}\right) \partial_{r}+\frac{x_{l} x_{k}}{r^{2}} \partial_{r}^{2},  \tag{3.38}\\
& \Delta=\partial_{r}^{2}+\frac{2}{r} \partial_{r} . \tag{3.39}
\end{align*}
$$

In the rest of this note, we take a smooth radial function $\zeta(x):=\zeta(|x|)$ such that

$$
\zeta: r \rightarrow \begin{cases}r^{2}, & \text { if } 0 \leq r \leq \frac{1}{2} \\ r, & \text { if } r>1\end{cases}
$$

Now, for $R>0$, take via (3.36),

$$
\zeta_{R}:=R^{2} \zeta\left(\frac{|\cdot|}{R}\right) \quad \text { and } \quad M_{R}:=M_{\zeta_{R}}
$$

Moreover, we assume that in the centered annulus $C\left(0, \frac{R}{2}, R\right)$,

$$
\begin{equation*}
\partial_{r} \zeta>0, \quad \partial_{r}^{2} \zeta \geq 0 \quad \text { and } \quad\left|\partial^{\alpha} \zeta\right| \leq C_{\alpha}|\cdot|^{1-\alpha}, \quad \forall|\alpha| \geq 1 . \tag{3.40}
\end{equation*}
$$

Note that on the centered ball of radius $\frac{R}{2}$, we have

$$
\begin{equation*}
\partial_{j k} \zeta_{R}=2 \delta_{j k}, \quad \Delta \zeta_{R}=6 \quad \text { and } \quad \Delta^{2} \zeta_{R}=0 \tag{3.41}
\end{equation*}
$$

Moreover, for $|x|>R$,

$$
\begin{equation*}
\partial_{j k} \zeta_{R}=\frac{R}{|x|}\left(\delta_{j k}-\frac{x_{j} x_{k}}{|x|^{2}}\right), \quad \Delta \zeta_{R}=\frac{2 R}{|x|} \quad \text { and } \quad \Delta^{2} \zeta_{R}=0 . \tag{3.42}
\end{equation*}
$$

Now, one states a Morawetz-type estimate.

Proposition 3.5 There is $t_{n}, R_{n} \rightarrow \infty$ such that

$$
\begin{align*}
& \int_{0}^{T} \int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u(t, x)|^{2 q} d x d t \lesssim T^{\frac{1}{1+2 \tau}}  \tag{3.43}\\
& \lim _{n} \int_{B\left(R_{n}\right)}|x|^{-2 \tau}\left|u\left(t_{n}, x\right)\right|^{2 q} d x=0 . \tag{3.44}
\end{align*}
$$

Proof Taking account of Proposition 3.4, we write

$$
\begin{align*}
V_{R}^{\prime \prime}[u]:= & V_{\zeta_{R}}^{\prime \prime}[u] \\
= & 4 \int_{\mathbb{R}^{3}} \partial_{l} \partial_{k} \zeta_{R} \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x-\int_{\mathbb{R}^{3}} \Delta^{2} \zeta_{R}|u|^{2} d x+4 \lambda \int_{\mathbb{R}^{3}} \nabla \zeta_{R} \cdot x \frac{|u|^{2}}{|x|^{4}} d x \\
& -2\left(1-\frac{1}{q}\right) \int_{\mathbb{R}^{3}} \Delta \zeta_{R}|x|^{-2 \tau}|u|^{2 q} d x+\frac{2}{q} \int_{\mathbb{R}^{3}} \nabla \zeta_{R} \cdot \nabla\left(|x|^{-2 \tau}\right)|u|^{2 q} d x \\
:= & (I)+(I)+(I I I), \tag{3.45}
\end{align*}
$$

where one decomposes the above integrals as $\left(\int_{B\left(\frac{R}{2}\right)}+\int_{C\left(\frac{R}{2}, R\right)}+\int_{B^{c}(R)}\right)$. Let us denote for short the source term $\mathcal{N}[u]:=|x|^{-2 \tau}|u|^{2(q-1)} u$. For the first term, we have

$$
\begin{align*}
(I):= & 4 \int_{B\left(\frac{R}{2}\right)} \partial_{l} \partial_{k} \zeta_{R} \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x-\int_{B\left(\frac{R}{2}\right)} \Delta^{2} \zeta_{R}|u|^{2} d x+4 \lambda \int_{B\left(\frac{R}{2}\right)} \nabla \zeta_{R} \cdot x \frac{|u|^{2}}{|x|^{4}} d x \\
& +2\left(\frac{1}{q}-1\right) \int_{B\left(\frac{R}{2}\right)} \Delta \zeta_{R}|x|^{-2 \tau}|u|^{2 q} d x+\frac{2}{q} \int_{B\left(\frac{R}{2}\right)} \nabla \zeta_{R} \cdot \nabla\left(|x|^{-2 \tau}\right)|u|^{2 q} d x \\
= & 8 \int_{B\left(\frac{R}{2}\right)}|\nabla u|^{2} d x+8 \lambda \int_{B\left(\frac{R}{2}\right)} \frac{|u|^{2}}{|x|^{2}} d x \\
& +12\left(\frac{1}{q}-1\right) \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x-\frac{8 \tau}{q} \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x \\
= & 8\left(\int_{B\left(\frac{R}{2}\right)}|\nabla u|^{2} d x-\frac{\gamma}{2 q} \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x+\lambda \int_{B\left(\frac{R}{2}\right)} \frac{|u|^{2}}{|x|^{2}} d x\right) . \tag{3.46}
\end{align*}
$$

Moreover,

$$
\begin{align*}
(I I I):= & 4 \int_{B^{c}(R)} \partial_{l} \partial_{k} \zeta_{R^{\prime}} \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x-\int_{B^{c}(R)} \Delta^{2} \zeta_{R}|u|^{2} d x+4 \lambda \int_{B^{c}(R)} \nabla \zeta_{R} \cdot x \frac{|u|^{2}}{|x|^{4}} d x \\
& +2\left(\frac{1}{q}-1\right) \int_{B^{c}(R)} \Delta \zeta_{R}|x|^{-2 \tau}|u|^{2 q} d x+\frac{2}{q} \int_{B^{c}(R)} \nabla \zeta_{R} \cdot \nabla\left(|x|^{-2 \tau}\right)|u|^{2 q} d x \\
= & 4 \int_{B^{c}(R)} \frac{R}{|x|}\left(\delta_{j k}-\frac{x_{j} x_{k}}{|x|^{2}}\right) \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x+4 \lambda \int_{B^{c}(R)} \frac{R}{|x|} \frac{|u|^{2}}{|x|^{2}} d x \\
& -2\left(1-\frac{1}{q}\right) \int_{B^{c}(R)} \frac{2 R}{|x|}|x|^{-2 \tau}|u|^{2 q} d x-\frac{4 \tau}{q} \int_{B^{c}(R)} \frac{R}{|x|}|x|^{-2 \tau}|u|^{2 q} d x . \tag{3.47}
\end{align*}
$$

Thus, taking $\nabla:=\nabla-\frac{x \cdot \nabla}{|x|^{2}} x$ the angular gradient, (3.47) gives

$$
\begin{align*}
(I I I)= & 4 \int_{B^{c}(R)} \frac{R}{|x|}|\nabla u|^{2} d x+4 \lambda \int_{B^{c}(R)} \frac{R}{|x|} \frac{|u|^{2}}{|x|^{2}} d x \\
& -2\left(1-\frac{1}{q}\right) \int_{B^{c}(R)} \frac{2 R}{|x|}|x|^{-2 \tau}|u|^{2 q} d x-\frac{4 \tau}{q} \int_{B^{c}(R)} \frac{R}{|x|}|x|^{-2 \tau}|u|^{2 q} d x \\
\gtrsim & -R^{-2} \int_{\mathbb{R}^{3}}|u|^{2} d x-R^{-2 \tau} \int_{\mathbb{R}^{3}}|u|^{2 q} d x . \tag{3.48}
\end{align*}
$$

Furthermore, by (3.40), we have

$$
\begin{align*}
(I I):= & 4 \int_{C\left(\frac{R}{2}, R\right)} \partial_{l} \partial_{k} \zeta_{R} \Re\left(\partial_{k} u \partial_{l} \bar{u}\right) d x-\int_{C\left(\frac{R}{2}, R\right)} \Delta^{2} \zeta_{R}|u|^{2} d x+4 \lambda \int_{C\left(\frac{R}{2}, R\right)} \nabla \zeta_{R} \cdot x \frac{|u|^{2}}{|x|^{4}} d x \\
& +2\left(\frac{1}{q}-1\right) \int_{C\left(\frac{R}{2}, R\right)} \Delta \zeta_{R}|x|^{-2 \tau}|u|^{2 q} d x+\frac{2}{q} \int_{C\left(\frac{R}{2}, R\right)} \nabla \zeta_{R} \cdot \nabla\left(|x|^{-2 \tau}\right)|u|^{2 q} d x \\
\gtrsim & -R^{-3} \int_{\mathbb{R}^{3}}|u|^{2} d x-R^{1+2 \tau} \int_{\mathbb{R}^{3}}|u|^{2 q} d x . \tag{3.49}
\end{align*}
$$

Now, by (3.45), (3.46), (3.48), and (3.49) via (3.2), it follows that

$$
\begin{align*}
V_{R}^{\prime \prime} & {[u] } \\
\gtrsim & \int_{B\left(\frac{R}{2}\right)}|\nabla u|^{2} d x-\frac{\gamma}{2 q} \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x \\
& +\lambda \int_{B\left(\frac{R}{2}\right)} \frac{|u|^{2}}{|x|^{2}} d x-R^{-2} \int_{\mathbb{R}^{3}}|u|^{2} d x-R^{-2 \tau} \int_{\mathbb{R}^{3}}|u|^{2 q} d x \\
\gtrsim & \int_{B\left(\frac{R}{2}\right)}|\nabla u|^{2} d x-\frac{\gamma}{2 q} \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x \\
& +\lambda \int_{B\left(\frac{R}{2}\right)} \frac{|u|^{2}}{|x|^{2}} d x-R^{-2} \int_{\mathbb{R}^{3}}|u|^{2} d x-R^{-2 \tau}\|u\|_{H_{\lambda}^{1}}^{2 q} \\
\gtrsim & \int_{B\left(\frac{R}{2}\right)}|\nabla u|^{2} d x-\frac{\gamma}{2 q} \int_{B\left(\frac{R}{2}\right)} \bar{u} \mathcal{N}[u] d x+\lambda \int_{B\left(\frac{R}{2}\right)} \frac{|u|^{2}}{|x|^{2}} d x-c R^{-2}-c R^{-2 \tau} \tag{3.50}
\end{align*}
$$

Indeed, by Sobolev embeddings via $1<q<q^{c}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}|u|^{2 q} d x \lesssim\|u\|_{H^{1}}^{2 q} \tag{3.51}
\end{equation*}
$$

Moreover, by (3.50), (3.3), (3.7) and (3.51), we get

$$
\begin{align*}
V_{R}^{\prime \prime}[u]+c R^{-2}+c R^{-2 \tau} & \gtrsim \mathcal{I}\left(\psi_{R} u\right) \\
& \gtrsim\left\|\sqrt{\mathcal{K}}_{\lambda}\left(\psi_{R} u\right)\right\|^{2} \\
& \gtrsim \int_{\mathbb{R}^{3}}|x|^{-2 \tau}\left|\psi_{R} u\right|^{2 q} d x \\
& \gtrsim \int_{B(R)}|x|^{-2 \tau}|u|^{2 q} d x \\
& \gtrsim \int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q} d x-R^{-2 \tau} . \tag{3.52}
\end{align*}
$$

Integrating in time the estimate (3.52) via the fact that $0<\tau<1$, it follows that

$$
\begin{align*}
\int_{0}^{T} \int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q} d x d s & \lesssim V_{R}^{\prime}[u(T)]-V_{R}^{\prime}\left[u_{0}\right]+c T R^{-2}+c T R^{-2 \tau} \\
& \lesssim R+c T R^{-2 \tau} \tag{3.53}
\end{align*}
$$

So, (3.43) follows by taking $R=T^{\frac{1}{1+2 \tau}}$. Moreover, (3.43) gives

$$
\frac{2}{T} \int_{\frac{T}{2}}^{T} \int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q} d x d s \lesssim T^{-\frac{2 \tau}{1+2 \tau}}
$$

We conclude the proof of (3.44) by using the mean value Theorem.

### 3.4 Proof of the scattering in Theorem 2.2 under (2.10)

Take $R, \varepsilon>0$ given by Proposition 3.2 and $t_{n}, R_{n} \rightarrow \infty$ given by Proposition 3.5. Letting $n \gg 1$ such that $R_{n}>R$, one gets by Hölder's inequality

$$
\begin{aligned}
\int_{|x| \leq R}\left|u\left(t_{n}, x\right)\right|^{2} d x & =R^{\frac{2 \tau}{q}} \int_{|x| \leq R}|x|^{-\frac{2 \tau}{q}}\left|u\left(t_{n}, x\right)\right|^{2} d x \\
& \leq R^{\frac{2 \tau}{q}}|B(R)|^{\frac{q-1}{q}}\left\||x|^{-\frac{2 \tau}{q}}\left|u\left(t_{n}, x\right)\right|^{2}\right\|_{L^{q}\left(|x| \leq R_{n}\right)} \\
& \leq R^{\frac{2 \tau+3(q-1)}{q}}\left(\int_{|x| \leq R_{n}}|x|^{-2 \tau}\left|u\left(t_{n}, x\right)\right|^{2 q} d x\right)^{\frac{1}{q}} \\
& \lesssim \varepsilon^{2} .
\end{aligned}
$$

Hence, the scattering of energy global solutions of the focusing problem (1.1) follows from Proposition 3.2.

### 3.5 Proof of the scattering in Theorem 2.2 under (2.11)

This part follows from Theorem 2.2 with the next result.

Lemma 3.6 The assumption (2.11) implies (2.10).
Proof Take the real function $g: t \mapsto t^{2}-\frac{C_{q, \tau, \lambda}}{q} t^{\gamma}$ and compute using (3.5),

$$
\begin{align*}
E[u][M[u]]^{\alpha_{c}} & \geq\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|^{2}\|u\|^{2 \alpha_{c}}-\frac{C_{q, \tau, \lambda}}{q}\|u\|^{\rho+2 \alpha_{c}}\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|^{\gamma} \\
& =g\left(\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|\|u\|^{\alpha_{c}}\right) . \tag{3.54}
\end{align*}
$$

Now, with Pohozaev identities (2.8) via (2.11) and the conservation laws, we have for some $0<\varepsilon<1$,

$$
\begin{align*}
g\left(\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|\|u\|^{\alpha_{c}}\right) & \leq E[u][M[u]]^{\alpha_{c}} \\
& <(1-\varepsilon) E[\varphi][M[\varphi]]^{\alpha_{c}} \\
& =(1-\varepsilon) g\left(\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}\right) \tag{3.55}
\end{align*}
$$

Thus, with time continuity, the assumption (2.11) is invariant under the flow of (1.1) and $T^{*}=\infty$. Moreover, by Pohozaev identities (2.8), we write

$$
E[\varphi][M[\varphi]]^{\alpha_{c}}=\frac{\gamma-2}{\gamma}\left(\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}\right)^{2}=\frac{C_{q, \tau, \lambda}(\gamma-2)}{2 q}\left(\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}\right)^{\gamma} .
$$

So, with (3.54) and (3.55), we get

$$
1-\varepsilon \geq \frac{\gamma}{\gamma-2}\left(\frac{\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|\|u\|^{\alpha_{c}}}{\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}}\right)^{2}-\frac{2}{\gamma-2}\left(\frac{\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|\|u\|^{\alpha_{c}}}{\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}}\right)^{\gamma} .
$$

Following the variations of $t \mapsto \frac{\gamma}{\gamma-2} t^{2}-\frac{2}{\gamma-2} t^{\gamma}$ via the assumption (2.11) and a continuity argument, there is a real number denoted also by $0<\varepsilon<1$, such that

$$
\begin{equation*}
\left\|\sqrt{\mathcal{K}}_{\lambda} u(t)\right\|\|u(t)\|^{\alpha_{c}} \leq(1-\varepsilon)\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}} \quad \text { on } \mathbb{R} \tag{3.56}
\end{equation*}
$$

Now, by (3.56) and Pohozaev identities (2.8) via (3.5), it follows that for some real number denoted also by $0<\varepsilon<1$,

$$
\begin{aligned}
\mathcal{Q}[u][M[u]]^{\alpha_{c}} & \leq C_{q, \tau, \lambda}\left\|\sqrt{\mathcal{K}}_{\lambda} u\right\|^{\gamma}\|u\|^{\rho+2 \alpha_{c}} \\
& \leq C_{q, \tau, \lambda}(1-\varepsilon)\left(\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}\right)^{\gamma} \\
& \leq(1-\varepsilon) \frac{2 q}{\gamma}\left(\left\|\sqrt{\mathcal{K}}_{\lambda} \varphi\right\|\|\varphi\|^{\alpha_{c}}\right)^{2} \\
& \leq(1-\varepsilon) \mathcal{Q}[\varphi] M[\varphi]^{\alpha_{c}} .
\end{aligned}
$$

This finishes the proof.

## 4 Conclusion

The key finding of this note is Theorem 2.2 about the energy scattering of inter-critical global solutions of the inhomogeneous focusing Schrödinger problem (1.1). It is established using the new method of Dodson-Murphy [13] based on Tao's scattering criteria [26] and Morawetz estimates. The scattering means that the global solution to (1.1) is close to the solution of the associate free equation. This means that the source term has a negligible affect for large time. Precisely, the energy scattering reads: there exists $u_{ \pm} \in H^{1}$ such that

$$
\lim _{t \rightarrow \pm \infty}\left\|u(t)-e^{-i t \mathcal{K}_{\lambda}} u_{ \pm}\right\|_{H^{1}}=0
$$

This result naturally extends the recent paper [4], where the global existence versus finite time blow-up below the ground state threshold was proved, but the scattering was not treated.

## Appendix

In this scetion, we prove a useful non-linear estimate.
Lemma A. 1 Let $\lambda>-\frac{1}{4}, q_{c}<q<q^{c}$ and $0<\tau<\frac{3}{2}$. Then, for $u \in H_{\lambda}^{1}$, one has

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q-1} d x \lesssim\|u\|_{H_{\lambda}^{1}}^{2 q-1} . \tag{A.1}
\end{equation*}
$$

Proof Using Hölder estimate and Sobolev injections, we have

$$
\int_{\mathbb{R}^{3}}|x|^{-2 \tau}|u|^{2 q-1} d x \leq\left\||x|^{-2 \tau}\right\|_{L^{a_{1}(B)}}\|u\|_{r_{1}}^{2 q-1}+\left\||x|^{-2 \tau}\right\|_{L^{a_{2}(B)}}\|u\|_{r_{2}}^{2 q-1} \lesssim\|u\|_{H_{\lambda}^{1}}^{2 q-1} .
$$

Here,

$$
\left\{\begin{array}{l}
1=\frac{1}{a_{i}}+\frac{2 q-1}{r_{i}},  \tag{A.2}\\
\frac{3}{a_{1}}>2 \tau>\frac{3}{a_{2}}, \\
\frac{1}{6}<\frac{1}{r_{i}} \leq \frac{1}{2} .
\end{array}\right.
$$

Thus,

$$
\left\{\begin{array}{l}
1-\frac{2 q-1}{r_{1}}=\frac{1}{a_{1}}>\frac{2}{3} \tau>\frac{1}{a_{2}}=1-\frac{2 q-1}{r_{2}}  \tag{A.3}\\
\frac{1}{6}<\frac{1}{r_{i}} \leq \frac{1}{2}
\end{array}\right.
$$

This requires

$$
\left\{\begin{array}{l}
\frac{2 q-1}{6}<\frac{2 q-1}{r_{1}}<1-\frac{2}{3} \tau  \tag{A.4}\\
1-\frac{2}{3} \tau<\frac{2 q-1}{r_{2}}<\frac{2 q-1}{2} \\
0<\tau<\frac{3}{2}
\end{array}\right.
$$

This is possible because

$$
\frac{1}{2}-\frac{2}{3} \tau<q_{c}-1<q-1<q^{c}-1<\frac{5}{2}-2 \tau .
$$

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The authors declare no competing interests.

## Author contributions

SB: Corresponding author, Writing review and editing; RG: writing original draft, Methodology, Resources, formal analysis, Conceptualization; TS: Supervision, Methodology, Resources, formal analysis, Conceptualization.

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