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Barycentric Lagrange interpolation method for solving Love's integral equations

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Abstract

In this paper, we present a new simple method for solving two integral equations of Love's type that have many applications, especially in electrostatic systems. The approach of the solution is based on an innovative technique using matrix algebra for the barycentric Lagrange interpolation. The unknown function is expressed through the product of four matrices. The kernel is interpolated twice, so we get it in the product of five matrices. Additionally, we derive an equivalent linear algebraic system to the solution by substituting the matrix-vector barycentric interpolated unknown function together with the double interpolated kernel into both sides of the integral equation. Thus, there was no need to employ the collocation method. The obtained results converge strongly with the approximate analytical solutions, in addition to being uniformly approximated, continuous, and even, which proves the validity of the solution by the presented method.

Mathematics Subject Classification: 00A69

Keywords: Electrostatics; Polymer structures; Aerodynamics; Fracture mechanics hydrodynamics; Elasticity

1 Introduction

For many applied scientific fields, especially in the study of electrostatic systems, it is commonly necessary to solve Love's integral equations. Many techniques and methods have been published for the solution of this kind of equations [1–4]. Love's integral equation was numerically solved for a relatively small parameter by Barerra et al. [1]. They took into account a new unknown function based on the unknown function of the integral equation and the exact solution; they employed the product integration approach based on discrete spline quadratic quasi-interpolation. Fu-Rong Lin et al. [2] solved Love's integral equation for a minimal parameter. They applied a composite Gauss–Legendre quadrature to an alternating integral equation. The coefficient matrix of a corresponding linear system is a nonsymmetric block matrix with Toeplitz blocks. Thus, they transformed the nonsymmetric linear system into a symmetric linear system and introduced a preconditioner, which is a block matrix with circulant blocks. For Boubaker and related polynomials, Gradimir et al. [3] proposed three-term recurrence relations as well as various features, such as zero distribution. These polynomials are used to generate an approximative analytical solution to Love's integral problem. Pastore [4] described a method for solving a

certain Love's integral equation numerically. Using a stable and convergent technique, he reduced the integral problem to an equivalent set of Fredholm integral equations. Most of the methods mentioned above are tedious, make you feel bored in their calculations, and praise the theories of functional analysis, although the solution to these equations has been proven to be convergent, continuous, even, and stable. There are also many different innovative methods and techniques [5–16] for solving integro-differential equations, Volterra and Fredholm weakly singular integral equations. These methods are suitable for solving Love's integral equations for any parameter, but they are expensive because they contain singularities in the kernels, and sometimes the unknown functions are also singular near the endpoints of the domain of integration. Jean-Paul Berrut et al. [17] presented the tradition of the barycentric Lagrange interpolation. Nicholas J. Higham [18] proved the stability of the tradition of the barycentric Lagrange interpolation. For the first time, Shoukralla et al. [19–21] developed matrix-vector formulas of the barycentric Lagrange interpolation versions and successfully used these versions of interpolation to solve the second kind Volterra integral problem. In this paper, we use these new versions of interpolation to solve Love's integral equations, in which the parameter is equal to one. The first step in the solution begins with expressing the unknown function through four matrices. The first matrix is a row matrix that expresses the monomial basis functions, the second is the coefficient matrix of the barycentric functions, the third is a diagonal matrix whose elements represent interpolation weights, and the fourth matrix is a column matrix for the unknown coefficients of the unknown function. For the kernel, it is interpolated twice: the first interpolation for the first variable and the second interpolation for the second variable, but in the order of matrices opposite to the order of the first interpolation. Thus, the kernel is represented by five matrices. One of these matrices is a square matrix whose elements express the functional values of the kernel at the interpolation nodes. This is the most important matrix in the solution procedure. By substituting the unknown interpolation function, and with it the interpolation kernel into both sides of the integral equation, we get an equivalent linear algebraic system to the solution of the integral equation. Thus, we get the unknown function in the form of continuous, uniformly interpolated, and even function. There are two solved cases with equal and opposite potential. The results were extremely accurate. This demonstrates the method's originality and effectiveness.

2 The barycentric matrix-vector interpolate solutions to Love's integral equations

Consider the following Love's integral equation:

$$u(x) = 1 + \int_{-1}^{1} \frac{d}{\pi(d^2 + (x-t)^2)} u(t)\,dt, \quad -1 < x < 1, \tag{1}$$

which describes the electrostatic potential in space, generated by a condenser consisting of two parallel equal circular plates of some radius and separated by a distance. In the case when the potentials of the plates are equal in magnitude and sign, the corresponding integral equation (1) becomes

$$\tilde{u}(x) = 1 - \int_{-1}^{1} \frac{d}{\pi(d^2 + (x-t)^2)} \tilde{u}(t)\,dt, \quad -1 < x < 1, \tag{2}$$

which occurs in the problem of determining the capacity of circular plate condenser, where $d = 1$ is the equation's parameter, $\tilde{u}(x)$ is the unknown function to be determined. It is known that the approximate analytic solution of Love's integral equation found by Love (1912–2001) is continuous, real, and even. The kernel $k(x,t) = k(x-t) = \frac{d}{\pi(d^2+(x-t)^2)}$ is called the difference kernel. The presented procedure focuses on finding the matrix-vector barycentric interpolated solution u_n of degree n to equation (1), similarly we find for $\tilde{u}_n$ to equation (2).

We assume that the unknown function $u(x)$ is given in the form of a tabulated function $u(x_i) = u_i$, where the $(n+1)$ equidistant distinct node distributions are selected with regards to the variable x with a step-size $h = \frac{b-a}{n}$ to be $x_i = a + ih$, $i = \overline{0,n}$. Let $u_n(x)$ be the matrix-vector barycentric Lagrange interpolating polynomial of degree n that interpolates $u(x)$. Then $u_n(x)$ takes the form [12]

$$u_n(x) = \mathrm{X}(x)\mathrm{C}^T\mathrm{DU}, \tag{3}$$

where $\mathrm{U} = [u_i]_{i=0}^n$ is the $(n+1)\times 1$ unknown coefficient column matrix such that the entries u_i satisfy the interpolation conditions $u_n(x_i) = u_i$ for $i = \overline{0:n}$, $\mathrm{D} = \mathrm{diag}\{\gamma_i\}_{i=0}^n$ is a square diagonal matrix whose entries are the weights of the barycentric functions that is defined by $\gamma_i = (-1)^i\binom{n}{i}$, $\mathrm{X}(x) = [x^i]_{i=0}^n$ is the $1\times(n+1)$ monomial basis row matrix, and $\mathrm{C} = [c_i]_{i=0}^n$ is the $(n+1)\times(n+1)$ known barycentric Lagrange coefficients matrix with each row containing the coefficients of the barycentric Lagrange function corresponding to the node x_i for $i = \overline{0:n}$, that is, the coefficients of the polynomials $c_i = \frac{C_i(x)}{\vartheta(x)}$, where

$$C_i(x) = \frac{1}{x-x_i}; \qquad \vartheta(x) = \sum_{i=0}^{n}\gamma_i C_i(x). \tag{4}$$

The kernel $k(x,t)$ is now interpolated twice with regard to the two variables x and t to get the double matrix vector barycentric interpolated formula $k_{n,n}(x_i,t_j)$ as follows [12]:

$$k_{n,n}(x_i,t_j) = \mathrm{X}(x)\mathrm{C}^T\mathrm{KCX}^T(t), \quad t_j = a + jh; j = \overline{0,n}. \tag{5}$$

Here, $\mathrm{K} = [\gamma_{ij}\delta_{ij}]_{i,j=0}^n$ is a square known matrix whose entries $\gamma_{ij}\delta_{ij}$ are defined by

$$\delta_{ij} = k(x_i,t_j); \qquad \gamma_{ij} = \gamma_i\times\gamma_j, \quad x_i = a + ih, t_j = a + jh; i,j = \overline{0,n}. \tag{6}$$

Moreover, we get from (3) and (5)

$$k_{n,n}(x,t)u_n(t) = \mathrm{X}(x)\mathrm{C}^T\mathrm{KC}\tilde{\mathrm{X}}(t)\mathrm{C}^T\mathrm{DU}, \qquad \tilde{\mathrm{X}}(t) = \mathrm{X}^T(t)\mathrm{X}(t). \tag{7}$$

Hence, the matrix-vector interpolated unknown function $u_n(t)$ is replaced by $u(x)$ of (1) to get

$$u_n(x) = 1 + \frac{1}{\pi}\int_{-1}^{1}\mathrm{X}(x)\mathrm{C}^T\mathrm{KC}\tilde{\mathrm{X}}(t)\mathrm{C}^T\mathrm{DU}\,dt = 1 + \frac{1}{\pi}\mathrm{X}(x)\mathrm{C}^T\mathrm{KC}\tilde{\tilde{\mathrm{X}}}(t)\mathrm{C}^T\mathrm{DU}, \tag{8}$$

where $\tilde{\tilde{\mathrm{X}}}(t) = \int_{-1}^{1}\tilde{\mathrm{X}}(t)\,dt$. Furthermore, by substituting $u_n(x)$ and $u_n(t)$ in both sides of equation (1) and replacing $k_{n,n}(x,t)$ from (5) with $k(x,t)$ of (1), and after matrix abbreviations,

we get

$$\mathrm{X}(x)\mathrm{C}^T\mathrm{KC}\tilde{\tilde{\mathrm{X}}}(t)\mathrm{C}^T\mathrm{DU} - \frac{1}{\pi}\mathrm{X}(x)\mathrm{C}^T\mathrm{KC}\Psi\mathrm{C}^T\mathrm{DU} = \mathrm{X}(x)\mathrm{C}^T\mathrm{KCM}, \tag{9}$$

where

$$\mathrm{M} = \int_{-1}^{1} \mathrm{X}^T(t)\,dt, \qquad \Psi = \int_{-1}^{1} \tilde{\mathrm{X}}(t)\mathrm{C}^T\mathrm{KC}\tilde{\tilde{\mathrm{X}}}(t)\,dt. \tag{10}$$

Simplifying (9) gives

$$\tilde{\tilde{\mathrm{X}}}(t)\mathrm{C}^T\mathrm{DU} - \frac{1}{\pi}\Psi\mathrm{C}^T\mathrm{DU} = \mathrm{M}. \tag{11}$$

Consequently, this idea of double substitution allows us to get the following algebraic linear system without applying the collocation method:

$$\left(\tilde{\tilde{\mathrm{X}}}(t)\mathrm{C}^T\mathrm{D} - \tilde{\Psi}\mathrm{C}^T\mathrm{D}\right)\mathrm{U} = \mathrm{M}, \qquad \tilde{\Psi} = \frac{1}{\pi}\Psi. \tag{12}$$

Hence, the values to the unknown coefficient column matrix U can be obtained in the form

$$\mathrm{U} = \left(\tilde{\tilde{\mathrm{X}}}(t)\mathrm{C}^T\mathrm{D} - \tilde{\Psi}\mathrm{C}^T\mathrm{D}\right)^{-1}\mathrm{M} = \mathrm{D}^{-1}\left(\mathrm{C}^T\right)^{-1}\left(\tilde{\tilde{\mathrm{X}}}(t) - \tilde{\Psi}\right)^{-1}\mathrm{M}. \tag{13}$$

Substituting U from (13) into (3) yields the barycentric matrix-vector interpolated solution

$$u_n(x) = \mathrm{X}(x)\left(\tilde{\tilde{\mathrm{X}}}(t) - \tilde{\Psi}\right)^{-1}\mathrm{M}. \tag{14}$$

The solution of equation (2) is similar to the solution of equation with some changes. The barycentric matrix-vector interpolated solution $\tilde{u}_n(x)$ can be obtained by

$$\tilde{u}_n(x) = \mathrm{X}(x)\left(\tilde{\tilde{\mathrm{X}}}(t) + \tilde{\Psi}\right)^{-1}\mathrm{M}. \tag{15}$$

3 Computational results

We designed a code by applying MATLAB2019a for solving the Love's integral equation (1). An approximate analytic solution [3] is given by

$$u(x) = 1.919200 - 0.311717x^2 + 0.015676x^4 + 0.019682x^6 - 0.000373x^8. \tag{16}$$

Table 1 shows the approximate analytic solution and the matrix-vector barycentric interpolated polynomials $u_n(x_i)$ for $n = 2, 3, 5, 15, 20$ at the set of points $x_i = -1 : 0.2 : 1$. The CPU total time was 8.147 sec., 9.905 sec., 10.927 sec., 29.200 sec., and 46.723 sec., respectively. Table 2 shows the absolute errors $R_n(x_i) = |u(x_i) - u_n(x_i)|$. It turns out that the matrix-vector barycentric polynomials and the absolute errors are uniformly distributed and symmetric. Figure 1 shows the graphs of the approximate analytic solution and the obtained interpolated solution for $n = 2$. Table 3 shows the matrix-vector barycentric interpolated polynomial solutions $\tilde{u}_n(x_i)$ for $n = 2, 3, 5, 15, 20$ at $x_i = -1 : 0.2 : 1$. Figure 2 shows the graphs of the matrix-vector barycentric interpolated polynomial solutions $\tilde{u}_n(x_i)$ for $n = 2$.

Table 1 A comparison of the approximate analytic solution with the matrix-vector barycentric polynomials $u_n(x_i)$ for $n = 2, 3, 5, 15, 20$ at $x_i = -1 : 0.2 : 1$

x_i	$u(x_i)$	$u_2(x_i)$	$u_3(x_i)$	$u_5(x_i)$	$u_{15}(x_i)$	$u_{20}(x_i)$
−1	1.6425	1.6388	1.6345	1.6403	1.6397	1.6397
−0.8	1.7312	1.7731	1.7423	1.7344	1.7307	1.7307
−0.6	1.8099	1.8776	1.8262	1.8119	1.8097	1.8097
−0.4	1.8698	1.9522	1.8861	1.8696	1.8696	1.8696
−0.2	1.9068	1.997	1.922	1.9052	1.9066	1.9066
0	1.9192	2.0119	1.934	1.9172	1.919	1.919
0.2	1.9068	1.997	1.922	1.9052	1.9066	1.9066
0.4	1.8698	1.9522	1.8861	1.8696	1.8696	1.8696
0.6	1.8099	1.8776	1.8262	1.8119	1.8097	1.8097
0.8	1.7312	1.7731	1.7423	1.7344	1.7307	1.7307
1	1.642	1.6388	1.6345	1.6403	1.6397	1.6397

Table 2 The absolute errors $R_n(x_i)$ of the interpolated matrix-vector barycentric polynomials $u_n(x_i)$ for $n = 2, 3, 5, 15, 20$ at $x_i = -1 : 0.2 : 1$

x_i	$R_2(x_i)$	$R_3(x_i)$	$R_5(x_i)$	$R_{15}(x_i)$	$R_{20}(x_i)$
−1	0.0037	0.008	0.0022	0.0028	0.0028
−0.8	0.0419	0.0111	0.0032	0.0005	0.0005
−0.6	0.0677	0.0163	0.002	0.0002	0.0002
−0.4	0.0824	0.0163	0.0002	0.0002	0.0002
−0.2	0.0902	0.0152	0.0016	0.0002	0.0002
0	0.0927	0.0148	0.002	0.0002	0.0002
0.2	0.0902	0.0152	0.0016	0.0002	0.0002
0.4	0.0824	0.0163	0.0002	0.0002	0.0002
0.6	0.0677	0.0163	0.002	0.0002	0.0002
0.8	0.0419	0.0111	0.0032	0.0005	0.0005
1	0.0032	0.0075	0.0017	0.0023	0.0023

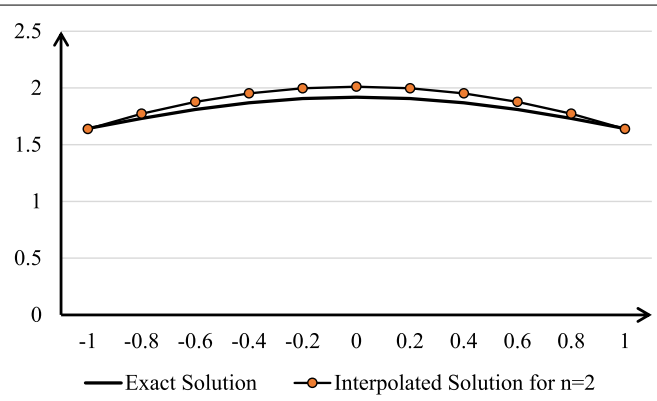

Figure 1 A comparison of the exact solution with the interpolated solution for $n = 2$

4 Conclusions

The interpolated solutions of two Love's integral equations were found. The solutions were continuous, uniformity interpolated, and even functions. The method of the solutions is based on vector-matrix versions of the barycentric Lagrange interpolation. The unknown function was expressed in four matrices, and the kernel in five matrices. From these five matrices, we only need to compute the square matrix whose elements represent the functional values of the kernel at the interpolation nodes. The fact that there are only a few steps of the solution's procedure and that only one matrix must be calculated to get the

Table 3 The matrix-vector barycentric interpolated polynomial solutions $\tilde{u}_n(x_i)$ for $n = 2, 3, 5, 15, 20$ at $x_i = -1 : 0.2 : 1$

x_i	$u_2(x_i)$	$u_3(x_i)$	$u_5(x_i)$	$u_{15}(x_i)$	$u_{20}(x_i)$
−1	0.76732	0.76044	0.7557	0.75572	0.75572
−0.8	0.72189	0.72214	0.72101	0.72249	0.72249
−0.6	0.68656	0.69234	0.69374	0.69448	0.69448
−0.4	0.66132	0.67106	0.67412	0.67389	0.67389
−0.2	0.64617	0.6583	0.66229	0.66152	0.66152
0	0.64113	0.65404	0.65833	0.65741	0.65741
0.2	0.64617	0.6583	0.66229	0.66152	0.66152
0.4	0.66132	0.67106	0.67412	0.67389	0.67389
0.6	0.68656	0.69234	0.69374	0.69448	0.69448
0.8	0.72189	0.72214	0.72101	0.72249	0.72249
1	0.76732	0.76044	0.7557	0.75572	0.75572

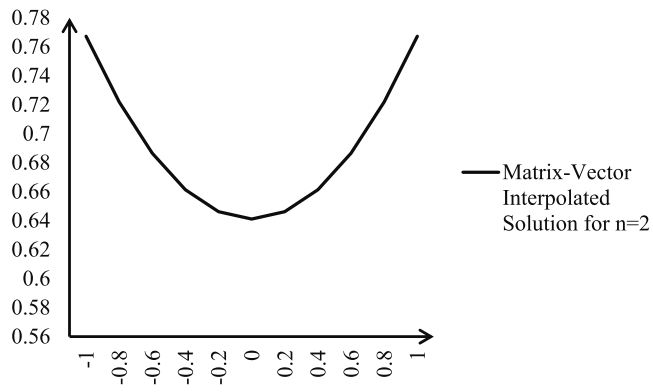

Figure 2 The matrix-vector interpolated solution for $n = 2$

solution is an advantage of the method. The most important advantage of the presented method is that we get a linear algebraic system without using the collocation method. This occurs as an inevitable consequence of substituting the interpolated unknown function into both sides of the integral equation. The obtained results are highly accurate.

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Declarations

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Competing interests
The authors declare no competing interests.

Author contributions
ESS put the idea of research article, presented the method, wrote the research, and revised it. BMA designed the MATLAB code. All authors read and approved the final manuscript.

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