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Affine periodic solutions for some stochastic differential equations

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Abstract

In this paper, we are study the problem of affine periodicity of solutions in distribution for some nonlinear stochastic differential equation with exponentially stable. We prove the existence and uniqueness of stochastic affine periodic solutions in distribution via the Banach fixed-point theorem.

Keywords: Stochastic differential equations; Exponential stable; Affine periodic solutions in distribution

1 Introduction

In this paper, we consider the following stochastic differential equation

$$dX(t) = A(t)X(t)\,dt + f\big(t, X(t)\big)\,dt + g\big(t, X(t)\big)\,dW(t), \tag{1.1}$$

where $t \in \mathbb{R}$, $A(t)$ is a linear operator, $A(t+T) = QA(t)Q^{-1}$, whose corresponding semigroup has exponential stability. The drift coefficient $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and diffusion coefficient $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are continuous with the following (Q, T)-affine periodicity

$$f(t+T, x) = Qf\big(t, Q^{-1}x\big),$$
$$g(t+T, x) = Qg\big(t, Q^{-1}x\big),$$

for some invertible matrix $Q \in GL(n)$, and positive constant $T > 0$, $\{W(t)\}$ is a two-sided standard m-dimensional Brownian motion.

The existence of periodic solutions for differential equations has been investigated by many mathematicians [1, 6, 11, 12]. The theory of stochastic differential equations has been well developed. Recently, Kolmogorov [8] studied the definition of recurrence for stochastic processes. Liu et al. [2, 9, 10] studied the existence of almost periodic solutions and almost automorphic solutions in distribution for stochastic differential equations. Chen et al. [3, 7] obtained the existence of periodic solutions in the sense of distribution for stochastic differential equations. Jiang et al. [7] obtained smooth Wong–Zakai approximations and periodic solutions in distribution of dissipative stochastic differential equations. However, some natural phenomena such as spiral waves, rotation motions in the body from mechanics, and spiral lines in geometry often exhibit symmetry besides time periodicity. Li et al. [4, 13, 14] introduced another special kind of recurrence, affine periodicity, which contains several special cases, such as periodicity, antiperiodicity, rotation periodicity, and quasiperiodicity.

Motivated by these works, in this paper, we obtain the existence and uniqueness of affine periodic solutions for equation (1.1) in the sense of distribution via Banach's fixed-point theorem, exponential stability, and stochastic analysis techniques.

2 Preliminary

Throughout this paper, we assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, the space $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$ stands for the space of all $\mathbb{R}^d$-valued random variables X such that

$$\mathbf{E}\|X\|^2 = \int_\Omega \|X\|^2 \, d\mathbf{P} < \infty.$$

Then, $\mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$ is a Hilbert space equipped with the norm

$$\|X\|_2 = \left(\int_\Omega |X|^2 \, d\mathbf{P} \right)^{\frac{1}{2}}.$$

Let us recall the definitions of affine periodic functions and affine periodic solution in distribution to be studied in this paper, see [7].

Definition 2.1 A continuous function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is called (Q, T)- affine periodic if for some invertible matrix $Q \in GL(n)$ and periodic $T > 0$,

$$f(t + T, x) = Qf\big(t, Q^{-1}x\big).$$

Definition 2.2 The solution $X(t)$ of the system (1.1) is said to be a (Q, T)-affine periodic solution in distribution if the following conditions hold:

(i) Stochastic process $X(t)$ is (Q, T)-affine periodic in distribution, namely,

$$X(t + T) = QX(t).$$

(ii) There exists a stochastic process W_1, which has the same distribution as W, such that $Q^{-1}X(t + T)$ is a solution of the stochastic differential equation

$$dY(t) = f\big(t, Y(t)\big) \, dt + g\big(t, Y(t)\big) \, dW_1(t). \tag{2.1}$$

We recall the definition of exponential stability for stochastic differential equations, see [5].

Definition 2.3 A semigroup of operators $\{U(t)\}_{t\geq 0}$ is said to be exponentially stable, if there are positive numbers $K > 0$, $\omega > 0$ such that

$$\big\|U(t)\big\| \leq Ke^{-\omega t}, \tag{2.2}$$

for all $t \geq 0$.

For later use, we recall the definition of a mild solution, see [5]. We set $\mathcal{F}_t = \sigma\{W(u) : u \leq t\}$.

Definition 2.4 An $\mathcal{F}_t$-adapted stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is said to be a mild solution of (1.1) if it satisfies the stochastic integral equation

$$X(t) = U(t-a)X(a) + \int_a^t U(t-s)f\bigl(s,X(s)\bigr)\,ds + \int_a^t U(t-s)g\bigl(s,X(s)\bigr)\,dW(s),$$

for all $t \geq a$, $a \in \mathbb{R}$.

3 Main results and proof

Now, we can state our main result, which is a result of the existence and uniqueness of (Q,T)-affine periodic solutions in distribution for the stochastic differential equation (1.1).

Theorem 3.1 *Assume that $A(t), f(t,x)$, and $g(t,x)$ are (Q,T)-affine periodic functions satisfying the following assumptions*:

(H1) *The semigroup $\{U(t)\}_{t\geq 0}$ generated by $A(t)$ is exponentially stable.*

(H2) *The drift coefficient f and diffusion coefficient g satisfy the Lipschitz conditions in X, that is, for all $X \in \mathcal{L}^2(\mathbf{P},\mathbb{R}^d)$ and $t \in \mathbb{R}$,*

$$\mathbf{E}\bigl\|f(t,X_1) - f(t,X_2)\bigr\|^2 \vee \mathbf{E}\bigl\|g(t,X_1) - g(t,X_2)\bigr\|^2 \leq L\mathbf{E}\|X_1 - X_2\|^2, \tag{3.1}$$

where $L > 0$ is a constant such that

$$\frac{2K^2L}{w^2} + \frac{K^2L}{w} < 1.$$

Then, there exists the unique $\mathcal{L}^2$-bounded (Q,T)-affine periodic solution in distribution of (1.1).

Proof Since the semigroup $\{U(t)\}_{t\leq 0}$ is exponentially stable, if $X(t)$ is $\mathcal{L}^2$-bounded, the $X(t)$ is a mild solution of (1.1) if and only if it satisfies the integral equation

$$X(t) = U(t-r)X(r) + \int_r^t U(t-s)f\bigl(s,X(s)\bigr)\,ds + \int_r^t U(t-s)g\bigl(s,X(s)\bigr)\,dW(s).$$

We set $r \to \infty$ in the above integral equation, by the exponentially stability of $U(t)$, we obtain that $X(t)$ satisfies the stochastic integral equation

$$X(t) = \int_{-\infty}^t U(t-s)f\bigl(s,X(s)\bigr)\,ds + \int_{-\infty}^t U(t-s)g\bigl(s,X(s)\bigr)\,dW(s).$$

Let $s = \sigma + T$ and $\widetilde{W}(\sigma) := W(s) - W(T)$. $\widetilde{W}(\sigma)$ coincides with the law of $W(s)$. Thus,

$$\begin{aligned} &X(t+T) \\ &\quad= \int_{-\infty}^{t+T} U(t+T-s)f\bigl(s,X(s)\bigr)\,ds + \int_{-\infty}^{t+T} U(t+T-s)g\bigl(s,X(s)\bigr)\,dW(s) \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{t} U(t-\sigma) f\big(\sigma+T, X(\sigma+T)\big)\, d\sigma + \int_{-\infty}^{t} U(t-\sigma) g\big(\sigma+T, X(\sigma+T)\big)\, d\widetilde{W}(\sigma) \\
&= \int_{-\infty}^{t} U(t-\sigma) Q f\big(\sigma, Q^{-1} Q X(\sigma)\big)\, d\sigma + \int_{-\infty}^{t} U(t-\sigma) Q g\big(\sigma, Q^{-1} Q X(\sigma)\big)\, d\widetilde{W}(\sigma) \\
&= Q \int_{-\infty}^{t} U(t-\sigma) f\big(\sigma, X(\sigma)\big)\, d\sigma + Q \int_{-\infty}^{t} U(t-\sigma) g\big(\sigma, X(\sigma)\big)\, d\widetilde{W}(\sigma) \\
&= Q X(t).
\end{aligned}
$$

Then, $X(t)$ is (Q,T)-affine periodic in distribution. Furthermore, $(Q^{-1}X(t+T), \widetilde{W})$ is also a solution of (2.1) with $W_1 = \widetilde{W}$. By Definition 2.2, then $X(t)$ is a (Q,T)-affine periodic solution in distribution of (1.1).

Let $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ be the space of all bounded $\mathcal{L}^2$-continuous affine periodic functions from $\mathbb{R} \to \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d)$ equipped with norm $\|y(t)\|_\infty = \sup_{s\in\mathbb{R}} \|y(t)\|_2$. Define an operator $\mathcal{S}$ on $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ by

$$
(\mathcal{S}Y)(t) \triangleq \int_{-\infty}^{t} U(t-s) f\big(s, Y(s)\big)\, ds + \int_{-\infty}^{t} U(t-s) g\big(s, Y(s)\big)\, dW(s).
$$

Now, we verify that operator $\mathcal{S}$ maps $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ into itself. Let us consider the nonlinear operators $\mathcal{S}_1 Y$ and $\mathcal{S}_2 Y$ on $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ given by

$$
\begin{aligned}
(\mathcal{S}_1 Y)(t) &\triangleq \int_{-\infty}^{t} U(t-s) f\big(s, Y(s)\big)\, ds, \\
(\mathcal{S}_2 Y)(t) &\triangleq \int_{-\infty}^{t} U(t-s) g\big(s, Y(s)\big)\, dW(s),
\end{aligned}
$$

respectively. As $f(t,x(t))$ and $g(t,x(t))$ are (Q,T)-affine periodic, then we know that $S_1 Y$ and $S_2 Y$ are (Q,T)-affine periodic. That is, the operator S maps $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ into itself.

Next, we prove $\mathcal{S}$ is a contraction mapping on $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$.

For $Y_1, Y_2 \in C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ and $t \in \mathbb{R}$, we have

$$
\begin{aligned}
&\mathbf{E}\big\|(\mathcal{S}Y_1)(t) - (\mathcal{S})Y_2(t)\big\|^2 \\
&\quad = \mathbf{E}\bigg\| \int_{-\infty}^{t} U(t-s)\big[f\big(s, Y_1(s)\big) - f\big(s, Y_2(s)\big)\big]\, ds \\
&\qquad + \int_{-\infty}^{t} U(t-s)\big[g\big(s, Y_1(s)\big) - g\big(s, Y_2(s)\big)\big]\, dW(s) \bigg\|^2 \\
&\quad \le 2\mathbf{E}\bigg\| \int_{-\infty}^{t} U(t-s)\big[f\big(s, Y_1(s)\big) - f\big(s, Y_2(s)\big)\big]\, ds \bigg\|^2 \\
&\qquad + 2\mathbf{E}\bigg\| \int_{-\infty}^{t} U(t-s)\big[g\big(s, Y_1(s)\big) - g\big(s, Y_2(s)\big)\big]\, dW(s) \bigg\|^2 \\
&\quad \triangleq 2(D_1 + D_2).
\end{aligned}
$$

By the Cauchy–Schwarz inequality, we have the following estimate

$$\begin{aligned}
D_1 &= \mathbf{E}\left\| \int_{-\infty}^{t} U(t-s)\big[f\big(s, Y_1(s)\big) - f\big(s, Y_2(s)\big)\big]\, ds \right\|^2 \\
&\le K^2 \mathbf{E}\left(\int_{-\infty}^{t} e^{-w(t-s)} \big\| f\big(s, Y_1(s)\big) - f\big(s, Y_2(s)\big) \big\|\, ds \right)^2 \\
&\le K^2 \int_{-\infty}^{t} e^{-w(t-s)}\, ds \int_{-\infty}^{t} e^{-w(t-s)} \mathbf{E}\big\| f\big(s, Y_1(s)\big) - f\big(s, Y_2(s)\big) \big\|^2\, ds \\
&\le \frac{K^2 L}{w^2} \cdot \sup_{s\in\mathbb{R}} \mathbf{E}\big\| Y_1(s) - Y_2(s) \big\|^2.
\end{aligned}$$

By the Itô isometry property, we have the other terms as follows

$$\begin{aligned}
D_2 &= \mathbf{E}\left\| \int_{-\infty}^{t} U(t-s)\big[g\big(s, Y_1(s)\big) - g\big(s, Y_2(s)\big)\big]\, dW(s) \right\|^2 \\
&= \mathbf{E} \int_{-\infty}^{t} \big\| U(t-s)\big[g\big(s, Y_1(s)\big) - g\big(s, Y_2(s)\big)\big] \big\|^2\, ds \\
&\le \int_{-\infty}^{t} K^2 e^{-2w(t-s)} \mathbf{E}\big\| g\big(s, Y_1(s)\big) - g\big(s, Y_2(s)\big) \big\|^2\, ds \\
&\le \frac{K^2 L}{2w} \cdot \sup_{s\in\mathbb{R}} \mathbf{E}\big\| Y_1(s) - Y_2(s) \big\|^2.
\end{aligned}$$

Then, for each $t \in \mathbb{R}$,

$$\mathbf{E}\big\| (\mathcal{S}Y_1)(t) - (\mathcal{S}Y_2)(t) \big\|^2 \le \left(\frac{2K^2 L}{w^2} + \frac{K^2 L}{w} \right) \sup_{s\in\mathbb{R}} \mathbf{E}\big\| Y_1(s) - Y_2(s) \big\|^2,$$

that is,

$$\big\| (\mathcal{S}Y_1)(t) - (\mathcal{S}Y_2)(t) \big\|_2^2 \le \eta \cdot \sup_{s\in\mathbb{R}} \big\| Y_1(s) - Y_2(s) \big\|_2^2, \tag{3.2}$$

with $\eta = \frac{2K^2 L}{w^2} + \frac{K^2 L}{w}$, according to

$$\sup_{s\in\mathbb{R}} \big\| Y_1(s) - Y_2(s) \big\|_2^2 \le \Big(\sup_{s\in\mathbb{R}} \big\| Y_1(s) - Y_2(s) \big\|_2 \Big)^2. \tag{3.3}$$

By (3.2) and (3.3), for each $t \in \mathbb{R}$,

$$\big\| (\mathcal{S}Y_1)(t) - (\mathcal{S}Y_2)(t) \big\|_2 \le \sqrt{\eta} \big\| Y_1(s) - Y_2(s) \big\|_\infty.$$

Thus,

$$\big\| (\mathcal{S}Y_1)(t) - (\mathcal{S}Y_2)(t) \big\|_\infty = \sup_{s\in\mathbb{R}} \big\| Y_1(s) - Y_2(s) \big\|_2 \le \sqrt{\eta} \big\| Y_1(s) - Y_2(s) \big\|_\infty.$$

By the assumption (H2)

$$\frac{2K^2 L}{w^2} + \frac{K^2 L}{w} < 1,$$

it follows that $\mathcal{S}$ is a contraction mapping on $C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$. By the Banach fixed-point theorem, there exists a unique solution $y^* \in C_{RP}(\mathbb{R}, \mathcal{L}^2(\mathbf{P}, \mathbb{R}^d))$ such that $\mathcal{S}y^* = y^*$, which is the unique (Q, T)-affine periodic solution of (1.1). □

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Declarations

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