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Minimax optimal control problems for an extensible beam equation with uncertain initial velocity

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Abstract

This paper is devoted to the problem of minimax optimal control problems of an extensible beam equation with distributed controls and initial velocity disturbances (or noises). The existence of optimal solutions for distributed control with fixed disturbance, namely the (P_v) problem, and the existence of minimax optimal solutions to (P) problem without restricting the initial disturbance are proved. We derive the necessary optimality conditions for optimal solutions of the (P_v) and (P) problems in the form of Pontryagin's maximum principle.

Keywords: Extensible beam equation; Existence of optimal controls; Minimax optimal solutions; Necessary optimality conditions; Pontryagin's maximum principle

1 Introduction

In this paper, we are concerned with the problem of minimax optimal control for the following nonlinear beam equation using distribution control in the presence of uncertain velocity:

$$y'' + \Delta^2 y - \left(1 + \int_\Omega |\nabla y|^2\,dx\right)\Delta y + g(y) = f + u \quad \text{in } Q := \Omega \times (0,T), \tag{1.1}$$

where $' = \frac{\partial}{\partial t}$, Ω is a bounded domain in $\mathbb{R}^N$, $N \in \{1,2,3\}$ (mainly $N = 3$) with smooth boundary $\partial\Omega$, Δ and Δ^2 are the Laplacian and bi-Laplacian operators defined as

$$\Delta y = \sum_{i=1}^N \frac{\partial^2 y}{\partial x_i^2} \quad \text{and} \quad \Delta^2 y = \sum_{i,j=1}^N \frac{\partial^4 y}{\partial x_i^2 x_j^2},$$

respectively, $g(y)$ is a nonlinear term, which we will explain later, f is a forcing function, and u is a control function. We consider either hinged boundary condition

$$y = \Delta y = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T) \tag{1.2}$$

or clamped boundary condition

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T), \tag{1.3}$$

where ν is the unit outward normal vector tailing on $\partial\Omega$. We also consider the initial condition

$$y(x,0) = y_0(x), \qquad y'(x,0) = y_1(x) + v(x) \quad \text{in } \Omega. \tag{1.4}$$

Here we assume that the initial velocity $y'(x,0)$ is not completely known and that v belongs to an admissible set $\mathcal{V}_{\mathrm{ad}}$. For this reason, we consider an *uncertain system* through incomplete velocity data. Many physical systems can be described by equations that contain uncertainties such as noise or disturbances. We can find recent studies on the problem of optimal control or identification of systems with uncertain or missing data. See, for instance, [3, 7, 10, 13, 14] and references therein. We make a very natural assumption, as is the case with many physical systems (cf. [3]), that uncertainty appears through the initial velocity value.

Since the one-dimensional version of Eq. (1.1) was proposed by Woinowsky-Krieger [28], there have been many researches focused on the properties of the solutions such as stability or energy decay, including studies on attractors. To name just a few, we can cite Ball [5], Bernstein [6], Dickey [12], and references therein. For studies of more generalized equations, we can cite several researches: Brito [8], Medeiros [21], Oliveria and Lima [23], Yang [30], etc.

As a contribution to control theory to the nonlinear beam equation, in [15], we studied the optimal control problems in the framework of Lions [19] using distributed forcing control variables. Quite recently, we studied in [16] that the nonlinear solution map of Eq. (1.1) is Fréchet differentiable and applied the result to the bilinear robust control problems using the minimax optimal control strategy (see [17, 29]), which is known to be a useful strategy for dealing with competing control problems. To summarize the results in [16], we showed the existence of an optimal pair and studied the necessary optimal condition of the optimal pair given by the following saddle points:

$$J\big(u^*,v\big) \le J\big(u^*,v^*\big) \le J\big(u,v^*\big) \quad \forall (u,v) \in \mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}, \tag{1.5}$$

where J is a quadratic cost, u and v are the distributed control and disturbance, respectively, and $\mathcal{U}_{\mathrm{ad}}$ and $\mathcal{V}_{\mathrm{ad}}$ are admissible sets.

Inspired by the research of Arada, Bergounioux, and Raymond [4], we set up our control strategy as follows: In finding the optimal control for all admissible *initial velocity disturbances*, we pursue a control strategy that tolerates the worst disturbance, i.e., we look for a safe optimal control value. This control strategy is different from our previous study [16] and from Ahmed and Xiang [2]. For our study, we introduce the cost functional

$$\mathcal{J}(y,u,v) = \int_Q \big(F(x,t,y) + H(x,t,u)\big)\,dx\,dt + \int_\Omega \big(K\big(x,y(T)\big) + L(x,v)\big)\,dx, \tag{1.6}$$

where y is the solution of Eq. (1.1), $(u,v) \in \mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}$, and the properties of the integrands are explicitly given later.

First, we consider the following problem:

$$(P_v) \quad \inf\{\mathcal{J}(y,u,v) \mid u \in \mathcal{U}_{\mathrm{ad}}\}, \tag{1.7}$$

where y is the solution of Eq. (1.1) with (1.2) or (1.3) and (1.4) corresponding to u for given v, and $\mathcal{U}_{\mathrm{ad}}$ is a given admissible control set. We denote by Argmin(P_v) the set of controls to (P_v) and denote $\mathcal{J}(y,u,v)$ by $J(u,v)$. We set up our main control problem (P) as follows: Find $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ such that $\bar{u} \in$ Argmin$(P_{\bar{v}})$ for some $\bar{v} \in \mathcal{V}_{\mathrm{ad}}$ and

$$J(u_v,v) \le J(\bar{u},\bar{v}) \quad \text{for all } v \in \mathcal{V}_{\mathrm{ad}} \text{ and } u_v \in \operatorname{Argmin}(P_v). \tag{1.8}$$

As is indicated in [4], problem (P) can be expressed in the following equivalent form: Find $\bar{v} \in \mathcal{V}_{\mathrm{ad}}$ and $u_{\bar{v}} \in$ Argmin$(P_{\bar{v}})$ such that

$$J(u_v,v) \le J(u_{\bar{v}},\bar{v}) \quad \text{for all } v \in \mathcal{V}_{\mathrm{ad}} \text{ and } u_v \in \operatorname{Argmin}(P_v) \tag{1.9}$$

or, shortly

$$(P) \quad \max_{v\in\mathcal{V}_{\mathrm{ad}}} \min_{u\in\mathcal{U}_{\mathrm{ad}}} J(u,v). \tag{1.10}$$

In this paper, we study the existence of optimal solutions for (P_v) and (P) and the necessary optimal conditions they must satisfy in the form of Pontryagin's maximal principle. First, in the proof of the existence, appropriate assumptions are given to the integrands of (1.6), unlike the assumption given in [16], and the weak lower and upper semicontinuity of convex and concave functionals is used.

Next, to derive necessary conditions for optimal solutions for (P_v) and (P) in the form of Pontryagin's maximum or minimum principle, we need to construct appropriate adjoint equations for (P_v) and (P). For this, the differentiability of the nonlinear solution map is needed, as we did in [16]. As is well known from other related studies ([18, 22, 24], etc.), since the control domain is just a metric space, the perturbation of the control has to be of the *spike (or needle-like)* type. Overcoming the nonlinearity in Eq. (1.1), we successfully derive a sort of *Taylor expansion of first order* for the state variable with respect to the diffuse perturbations of the control and disturbance. This is the main novelty of this paper.

The content of the paper is organized as follows. In Sect. 2, we present notations and preliminary results for Eq. (1.1) with (1.2) or (1.3) and (1.4). In Sect. 3, we state the main results of this paper, including assumptions for the cost function. In Sect. 4, we prove the existence of optimal solutions to (P_v) and (P). In Sect. 5, we show the Taylor expansion of state and disturbance variables for the diffuse perturbation of the reference control variable and the disturbance variable. In Sect. 6, we give the proof of the main results.

2 Preliminaries

Let X be a Banach space. We denote its topological dual by X' and the duality pairing between X' and X by $\langle\cdot,\cdot\rangle_{X',X}$. We also introduce the following abbreviations:

$$\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}, \qquad H = L^2(\Omega), \tag{2.1}$$

where $p \geq 1$. The scalar product and norm on H are denoted by $(\cdot,\cdot)_2$ and $\|\cdot\|_2$, respectively. The scalar product and norm on $[H]^N$ ($N \leq 3$) are also denoted by $(\cdot,\cdot)_2$ and $\|\cdot\|_2$, respectively. As is well known, $H^k(\Omega)$ is the Sobolev space of order $k \geq 1$ on Ω, and $H_0^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $H^k(\Omega)$ norm.

We denote

$$V_2 = \begin{cases} H^2(\Omega) \cap H_0^1(\Omega) & \text{for condition (1.2)}, \\ H_0^2(\Omega) & \text{for condition (1.3)}. \end{cases} \tag{2.2}$$

We define the operator $A : V_2 \to V_2'$ by

$$\langle A\phi, \psi \rangle_{V_2', V_2} = (\Delta\phi, \Delta\psi)_2 \quad \forall \phi, \psi \in V_2, \tag{2.3}$$

$$D(A) = \{\phi \in V_2 \mid A\phi \in H\}. \tag{2.4}$$

Since A is self-adjoint from V_2 into V_2', strictly positive on V_2 due to (2.3), and the injection of V_2 in H is compact, we can use the spectral theory of self-adjoint compact operators in a Hilbert space (see [25, Theorem 7.7]). We infer that there exists a complete orthonormal basis $\{e_k\}_{k=1}^\infty$ of H, which consists of eigenvectors of A,

$$\begin{cases} Ae_k = \lambda_k e_k \quad \forall k, \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lambda_k \to \infty \quad \text{as } k \to \infty. \end{cases}$$

This allows us to define the powers A^s of A for any $s \in \mathbb{R}$ such that for $s \geq 0$,

$$D(A^s) = \left\{ \phi \in H \,\middle|\, \sum_{k=1}^\infty \lambda_k^{2s} (\phi, e_k)_2^2 < \infty \right\},$$

and for $s < 0$, $D(A^s)$ is the completion of H in the norm

$$\left\{ \sum_{k=1}^\infty \lambda_k^{2s} (\phi, e_k)_2^2 \right\}^{\frac{1}{2}}.$$

For $s \in \mathbb{R}$, the scalar product and norm in $D(A^s)$ can be written alternatively as

$$(\phi, \psi)_{D(A^s)} = \left(A^s\phi, A^s\psi\right)_2 = \sum_{k=1}^\infty \lambda_k^{2s} (\phi, e_k)_2 (\psi, e_k)_2, \tag{2.5}$$

$$\|\phi\|_{D(A^s)} = \left\{(\phi,\phi)_{D(A^s)}\right\}^{\frac{1}{2}} = \left\{ \sum_{k=1}^\infty \lambda_k^{2s} (\phi, e_k)_2^2 \right\}^{\frac{1}{2}}. \tag{2.6}$$

Hereafter, we denote $V_s := D(A^{\frac{s}{4}})$. Then for $s \in \mathbb{R}$, V_s are Hilbert spaces with the scalar products and norms

$$((\phi, \psi))_s := (\phi, \psi)_{V_s} = \left(A^{\frac{s}{4}}\phi, A^{\frac{s}{4}}\psi\right)_2, \qquad \|\phi\|_{V_s} = \left\|A^{\frac{s}{4}}\phi\right\|_2 \tag{2.7}$$

for $\phi, \psi \in V_s$. Thus

$$\|\phi\|_{V_2} = \left\|A^{\frac{1}{2}}\phi\right\|_2 = \|\Delta\phi\|_2, \qquad \|\psi\|_{V_1} = \left\|A^{\frac{1}{4}}\psi\right\|_2 = \|\nabla\psi\|_2 \tag{2.8}$$

for all $(\phi, \psi) \in V_2 \times V_1$.

It becomes apparent that each natural topological imbedding

$$V_2 \hookrightarrow V_1 \hookrightarrow H \hookrightarrow V_1' \hookrightarrow V_2' \tag{2.9}$$

is continuous and compact. According to Adams [1], when $N \leq 3$, the imbedding

$$V_2 \hookrightarrow C_0(\Omega) \tag{2.10}$$

is compact.

We give the following assumptions on $g(y)$ in Eq. (1.1): g in Eq. (1.1) is a C^1 function such that $g(0) = 0$ (without loss of generality);

(H1) The function G defined by $G(s) = \int_0^s g(r)\,dr$ satisfies the condition

$$\liminf_{|s|\to\infty} \frac{G(s)}{s^2} \geq 0; \tag{2.11}$$

(H2) There exists a constant $c_1 > 0$ such that

$$\left|g'(s)\right| \leq c_1\left(1 + |s|^\gamma\right) \quad (0 \leq \gamma < \infty). \tag{2.12}$$

We infer from (2.11) that for every $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$G(s) + \eta s^2 > -C_\eta \quad \forall s \in \mathbb{R}. \tag{2.13}$$

Now we can rewrite Eq. (1.1) with the boundary condition (1.2) or (1.3), $u = 0$, and the initial condition (1.4) with $v = 0$ as the following abstract Cauchy problem:

$$\begin{cases} y'' + Ay + (1 + \|y\|_{V_1}^2)A^{\frac{1}{2}}y + g(y) = f \quad \text{in } (0,T), \\ y(0) = y_0, \qquad y'(0) = y_1. \end{cases} \tag{2.14}$$

Remark 2.1 By **(H1)**, **(H2)**, and (2.10) we can infer or rewrite (2.12) and (2.13) suitably for the nonlinear operator g in Eq. (2.14):

(i) It follows from (2.12) and (2.10) that the nonlinear operator g in Eq. (2.14) is a C^1 bounded operator from V_2 into H, Fréchet differentiable with differential g', and Lipschitzian from the bounded sets of V_2 into H. Indeed, for every $R > 0$, there exists $c(R)$ such that

$$\left\|g(\phi_1) - g(\phi_2)\right\|_2 \leq c(R)\|\phi_1 - \phi_2\|_2 \quad \forall \phi_i \in V_2 \ (i = 1,2), \tag{2.15}$$

where $\|\phi_i\|_{V_2} \leq R$ $(i = 1,2)$.

(ii) As a consequence of (H1) and (2.13), we deduce that there exists $G \in C^1(V_2, \mathbb{R})$, $G(0) = 0$, such that $g(\phi) = G'(\phi)$ for all $\phi \in V_2$ and that for every $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$G(\phi) + \eta\|\phi\|_2^2 > -C_\eta \quad \forall \phi \in V_2. \tag{2.16}$$

The Hilbert space of the weak solutions of Eq. (2.14) is defined as

$$\mathcal{W}(0,T) = \{\phi \mid \phi \in L^2(0,T;V_2), \phi' \in L^2(0,T;H), \phi'' \in L^2(0,T;V_2')\}$$

equipped with the norm

$$\|\phi\|_{\mathcal{W}(0,T)} = \left(\|\phi\|^2_{L^2(0,T;V_2)} + \|\phi'\|^2_{L^2(0,T;H)} + \|\phi''\|^2_{L^2(0,T;V_2')}\right)^{\frac{1}{2}},$$

where ϕ' and ϕ'' denote the first- and second-order derivatives of ϕ in the sense of distribution.

Hereafter, we use C as a generic constant and omit the integral variables in any definite integrals without confusion.

Definition 2.1 A function y is said to be a weak solution of Eq. (2.14) if $y \in \mathcal{W}(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot), \phi\rangle_{V_2',V_2} + ((y(\cdot),\phi))_2 + (1 + \|y(\cdot)\|^2_{V_1})((y(\cdot),\phi))_1 + (g(y(\cdot)),\phi)_2 \\ \quad = (f(\cdot),\phi)_2 \\ \quad \text{for all } \phi \in V_2 \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0, \qquad y'(0) = y_1. \end{cases} \tag{2.17}$$

By referring to Yang [30] in terms of dealing with the nonlinear terms in Eq. (2.14) we can state the following existence theorem (see [27, pp. 212–219] to deal with the term $g(y)$ in Eq. (2.14)).

Theorem 2.1 *Assume that* (***H*1**) *and* (***H*2**) *are fulfilled and that* $y_0 \in V_2$, $y_1 \in H$, *and* $f \in L^2(0,T;H)$. *Then there exists a weak solution* y *of Eq.* (2.14) *satisfying*

$$y \in \mathcal{W}(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;H). \tag{2.18}$$

Let X be a Banach space. Set

$$C_w([0,T];X) = \{\phi \in L^\infty(0,T;X) \mid \langle\phi(\cdot),\xi\rangle_{X,X'} \in C([0,T]) \ \forall \xi \in X'\}.$$

Then, as noted in [16], we can use (2.18) together with the results in Lions and Magenes [20, Lemma 8.2] to prove the following improved regularity for the weak solution y of Eq. (2.14).

Corollary 2.1 *Let* y *be a weak solution of Eq.* (2.14). *Then, we can assert* (*after possibly a modification on a set of measure zero*) *that*

$$y \in C_w([0,T];V_2), \qquad y' \in C_w([0,T];H). \tag{2.19}$$

Proof See [16]. □

The following lemma is not only used to verify the improved regularity of the weak solution of Eq. (2.14), but also plays an important role in obtaining various estimates in this paper.

Lemma 2.1 *Let y be a weak solution of Eq.* (2.14). *Then, for each $t \in [0,T]$, we have the following energy equality*:

$$\begin{aligned} &\|y'(t)\|_2^2 + \|y(t)\|_{V_2}^2 + \frac{1}{2}\big(1 + \|y(t)\|_{V_1}^2\big)^2 + 2G\big(y(t)\big) \\ &\quad = 2\int_0^t \big(f,y'\big)_2\,ds + \|y_1\|_2^2 + \|y_0\|_{V_2}^2 \\ &\qquad + \frac{1}{2}\big(1 + \|y_0\|_{V_1}^2\big)^2 + 2G(y_0). \end{aligned} \tag{2.20}$$

Proof As noted in [16], by the double regularization procedure in Lions and Magenes [20, pp. 276–279], we deduce that the weak solution y of Eq. (2.14) satisfies

$$\begin{aligned} &\|y'(t)\|_2^2 + \|y(t)\|_{V_2}^2 \\ &\quad = \|y_1\|_2^2 + \|y_0\|_{V_2}^2 + 2\int_0^t \big(f,y'\big)_2\,ds - 2\int_0^t \big(1 + \|y\|_{V_1}^2\big)\big(A^{\frac{1}{2}}y,y'\big)_2\,ds \\ &\qquad - 2\int_0^t \big(g(y),y'\big)_2\,ds. \end{aligned} \tag{2.21}$$

Since

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big(1 + \|y(t)\|_{V_1}^2\big)^2 &= \big(1 + \|y(t)\|_{V_1}^2\big)\frac{d}{dt}\|y(t)\|_{V_1}^2 \\ &= 2\big(1 + \|y(t)\|_{V_1}^2\big)\big(A^{\frac{1}{2}}y(t),y'(t)\big)_2 \end{aligned} \tag{2.22}$$

and

$$\frac{d}{dt}G\big(y(t)\big) = \big(g\big(y(t)\big),y'(t)\big)_2, \tag{2.23}$$

we can combine (2.21) with (2.22) and (2.23) to obtain (2.20).

This completes the proof. □

Here we can state the following theorem.

Theorem 2.2 *Let y be the weak solution of Eq.* (2.14). *Then* (*after a possible modification on a set of measure zero*), $y \in C([0,T];V_2) \cap C^1([0,T];H)$. *Moreover, the solution mapping $p = (y_0,y_1,f) \to y(p)$ of $\mathcal{P} \equiv V_2 \times H \times L^2(0,T;H)$ into $\mathcal{S}(0,T) \equiv \mathcal{W}(0,T) \cap C([0,T];V_2) \cap C^1([0,T];H)$ is locally Lipschitz continuous. Letting $p_1 = (y_0^1,y_1^1,f_1) \in \mathcal{P}$ and $p_2 = (y_0^2,y_1^2,f_2) \in \mathcal{P}$, the following is satisfied*:

$$\begin{aligned} &\|y(p_1) - y(p_2)\|_{\mathcal{S}(0,T)} \\ &\quad \le C\big(\|y_0^1 - y_0^2\|_{V_2}^2 + \|y_1^1 - y_1^2\|_2^2 + \|f_1 - f_2\|_{L^2(0,T;H)}^2\big)^{\frac{1}{2}} \end{aligned}$$

$$\equiv C\|p_1 - p_2\|_{\mathcal{P}}, \tag{2.24}$$

where $C > 0$ is a constant depending on the data.

Proof As explained in [16], we can make use of (2.21) to show that

$$y \in C\big([0,T]; V_2\big) \cap C^1\big([0,T]; H\big). \tag{2.25}$$

Thus we can verify that every weak solution y of Eq. (2.14) with the data $(y_0, y_1, f) \in V_2 \times H \times L^2(0,T;H)$ exists in $\mathcal{S}(0,T)$. Based on this result, we can verify (2.24): for all $p_1 = (y_0^1, y_1^1, f_1) \in \mathcal{P}$ and $p_2 = (y_0^2, y_1^2, f_2) \in \mathcal{P}$, we denote $y_1 - y_2 \equiv y(p_1) - y(p_2)$ by ψ. Then from Eq. (2.14) we know that ψ satisfies the following in weak sense:

$$\begin{cases} \psi'' + A\psi + (1 + \|y_1\|_{V_1}^2)A^{\frac{1}{2}}\psi + g(y_1) - g(y_2) = \epsilon(\psi) + f_1 - f_2 & \text{in } (0,T), \\ \psi(0) = y_0^1 - y_0^2, \qquad \psi'(0) = y_1^1 - y_1^2, \end{cases} \tag{2.26}$$

where

$$\epsilon(\psi) = -\big(\|y_1\|_{V_1}^2 - \|y_2\|_{V_1}^2\big)A^{\frac{1}{2}}y_2 = -((\psi, y_1 + y_2))_1 A^{\frac{1}{2}}y_2. \tag{2.27}$$

Applying the energy equality (2.21) to Eq. (2.26), we obtain

$$\begin{aligned} &\big\|\psi'(t)\big\|_2^2 + \big\|\psi(t)\big\|_{V_2}^2 \\ &\quad = -2\int_0^t \big(g(y_1) - g(y_2), \psi'\big)_2\,ds + 2\int_0^t \big(\epsilon(\psi), \psi'\big)_2\,ds \\ &\qquad - 2\int_0^t \big(1 + \|y_1\|_{V_1}^2\big)\big(A^{\frac{1}{2}}\psi, \psi'\big)_2\,ds \\ &\qquad + 2\int_0^t \big(f_1 - f_2, \psi'\big)_2\,ds + \big\|\psi'(0)\big\|_2^2 + \big\|\psi(0)\big\|_{V_2}^2. \end{aligned} \tag{2.28}$$

Here we just estimate the first term on the right side of (2.28). Then we can refer to [16] to get estimations of the other terms in (2.28). From (2.15) and the fact that $\mathcal{S}(0,T) \hookrightarrow C(\bar{Q})$ we have

$$\big\|g(y_1) - g(y_2)\big\|_2^2 \le C\|\psi\|_2^2, \tag{2.29}$$

where C depends only on the data of the weak solutions y_1 and y_2. Thanks to (2.29), we have

$$\begin{aligned} \left|2\int_0^t \big(g(y_1) - g(y_2), \psi'\big)_2\,ds\right| &\le 2\int_0^t \big\|g(y_1) - g(y_2)\big\|_2 \big\|\psi'\big\|_2\,ds \\ &\le C\int_0^t \|\psi\|_2 \big\|\psi'\big\|_2\,ds \\ &\le C\int_0^t \|\psi\|_{V_2} \big\|\psi'\big\|_2\,ds \\ &\le C\int_0^t \big(\|\psi\|_{V_2}^2 + \big\|\psi'\big\|_2^2\big)\,ds. \end{aligned} \tag{2.30}$$

Then by analogy with [16] in terms of estimating the other terms on the right hand side of (2.28), we arrive at

$$\|\psi\|_{C([0,T];V_2)\cap C^1([0,T];H)} \leq C\|p_1 - p_2\|_{\mathcal{P}}. \tag{2.31}$$

Since $A : V_2 \to V_2'$ is an isomorphism, we infer from (2.26) and (2.31) that

$$\left\|\psi''\right\|_{L^2(0,T;V_2')} \leq C\|p_1 - p_2\|_{\mathcal{P}}. \tag{2.32}$$

Hence we have proved (2.24).

This completes the proof. □

3 Basic assumptions and statement of the main theorem

We consider the following uncertain control system:

$$\begin{cases} y'' + Ay + (1 + \|y\|_{V_1}^2)A^{\frac{1}{2}}y + g(y) = f + u & \text{in } (0,T), \\ y(0) = y_0, \qquad y'(0) = y_1 + v, \end{cases} \tag{3.1}$$

where $(y_0, y_1, f) \in V_2 \times H \times L^2(0,T;H)$, u is a control function belonging to the admissible control set $\mathcal{U}_{\mathrm{ad}}$, and v denotes the noise or disturbance variable described by the *uncertainty* that belongs to the admissible set $\mathcal{V}_{\mathrm{ad}}$.

We give the following assumptions for problem (P).

(A1) The control set is defined as

$$\mathcal{U}_{\mathrm{ad}} = \left\{u \in L^2(0,T;H) \mid u(x,t) \in K_u(x,t) \text{ for almost all } (x,t) \in Q\right\},$$

where $K_u(\cdot,\cdot)$ is a measurable multivalued mapping with nonempty, convex, and closed values in $\mathcal{P}(\mathbb{R})$.

(A2) The set of v (noises) is given by

$$\mathcal{V}_{\mathrm{ad}} = \left\{v \in L^\infty(\Omega) \mid v(x) \in K_v(x) \subset G \text{ for almost all } x \in \Omega\right\},$$

where $K_v(\cdot)$ is a measurable multivalued mapping with nonempty, convex, and closed values in $\mathcal{P}(\mathbb{R})$, and G is a bounded subset of $\mathbb{R}$.

(A3) F and H are Carathéodory functions from $Q \times \mathbb{R}$ to $\mathbb{R}$, and $F(\cdot,\cdot,0)$ and $H(\cdot,\cdot,0)$ belong to $L^1(Q)$. For a.e. $(x,t) \in Q$, $F(x,t,\cdot)$ is of class C^1 and satisfies

$$\left|F_y(x,t,y)\right| \leq c\left(1 + |y|^{\sigma_1}\right) \tag{3.2}$$

for some constants $c > 0$ and $0 < \sigma_1 < 1$. $H(x,t,\cdot)$ is *convex* such that

$$c_1|u|^2 \leq H(x,t,u) \leq h(x,t) + c_2|u|^2, \tag{3.3}$$

where $c_1, c_2 > 0$ and $h \in L^1(Q)$.

(A4) K and L are Carathéodory functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$, and $K(\cdot, 0)$ belongs to $L^1(\Omega)$. For a.e. $x \in \Omega$, $K(x, \cdot)$ is of class C^1 and satisfies

$$\left|K_y(x,y)\right| \leq c\left(1 + |y|^{\sigma_2}\right) \tag{3.4}$$

for some constants $c > 0$ and $0 < \sigma_2 < 1$. $L(x, \cdot)$ is *concave* and satisfies

$$\left|L(x,v)\right| \leq L_1(x)\eta\left(|v|\right), \quad \text{a.e. } x \in \Omega, v \in \mathcal{V}_{\mathrm{ad}}, \tag{3.5}$$

where $L_1 \in L^1(\Omega)$, and η is a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Remark 3.1 From **(A1)** we deduce that $\mathcal{U}_{\mathrm{ad}}$ is a closed and convex subset of $L^2(0, T; H)$. Also, we deduce from **(A2)** that $\mathcal{V}_{\mathrm{ad}}$ is convex, bounded, and closed in $L^p(\Omega)$ for any $p \geq 1$.

Thanks to Theorem 2.2, we have the well-defined solution mapping $(u, v)(\in \mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}) \longmapsto y(\in \mathcal{S}(0, T))$ given by the solution of Eq. (3.1). Thus, throughout the paper, we employ the notation $y(u, v)$ to denote the solution of Eq. (3.1) corresponding to $(u, v) \in \mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}$.

To present our main result, we define the Hamiltonian functions for the control problems (P_v) and (P) as follows:

$$\begin{aligned}
&\mathcal{H}_1(x,t,u,p) := H(x,t,u) + pu \quad \forall (x,t,u,p) \in Q \times \mathbb{R} \times \mathbb{R}, \\
&\mathcal{H}_2(x,v,p) := L(x,v) + pv \quad \forall (x,v,p) \in \Omega \times \mathbb{R} \times \mathbb{R}.
\end{aligned}$$

We present the following necessary optimality conditions given as a *Pontryagin's principle* for optimal solutions to (P_v), where $v \in \mathcal{V}_{\mathrm{ad}}$ is fixed.

Theorem 3.1 *Assume that conditions* **(A1)–(A4)** *are fulfilled. For any fixed $v \in \mathcal{V}_{\mathrm{ad}}$, let u_v be an optimal solution of (P_v). Then there exists $p \in \mathcal{S}(0, T)$ such that*

$$\begin{cases}
p'' + Ap + \mathcal{G}(y(u_v, v), p) + g'(y(u_v, v))p = F_y(y(u_v, v)) \quad \textit{in } (0, T), \\
p(T) = 0, \qquad p'(T) = -K_y(y(u_v, v; T)),
\end{cases} \tag{3.6}$$

where

$$\mathcal{G}\left(y(u_v, v), p\right) = \left(1 + \left\|y(u_v, v)\right\|_{V_1}^2\right)A^{\frac{1}{2}}p + 2\left(\left(y(u_v, v), p\right)\right)_1 A^{\frac{1}{2}}y(u_v, v), \tag{3.7}$$

and

$$\mathcal{H}_1\left(x, t, u_v(x,t), p(x,t)\right) = \min_{u \in K_u(x,t)} \mathcal{H}_1\left(x, t, u(x,t), p(x,t)\right) \tag{3.8}$$

for a.e. $(x, t) \in Q$.

Next, we give necessary optimality conditions, also given as a *Pontryagin's principle* for optimal solutions to (P).

Theorem 3.2 *Assume that conditions* **(A1)–(A4)** *are fulfilled. Then* (P) *admits an optimal solution* $\bar{v}$. *Furthermore, there exists an optimal solution* $\bar{u} \in \operatorname{Argmin}(P_{\bar{v}})$ *and* $p \in \mathcal{S}(0,T)$ *such that*

$$\begin{cases} p'' + Ap + \mathcal{G}(y(\bar{u},\bar{v}),p) + g'(y(\bar{u},\bar{v}))p = F_y(y(\bar{u},\bar{v})) \quad \text{in } (0,T), \\ p(T) = 0, \qquad p'(T) = -K_y(y(\bar{u},\bar{v};T)), \end{cases} \tag{3.9}$$

where

$$\mathcal{G}\big(y(\bar{u},\bar{v}),p\big) = \big(1 + \|y(\bar{u},\bar{v})\|_{V_1}^2\big)A^{\frac{1}{2}}p + 2\big(\big(y(\bar{u},\bar{v}),p\big)\big)_1 A^{\frac{1}{2}}y(\bar{u},\bar{v}), \tag{3.10}$$

and

$$\mathcal{H}_1\big(x,t,\bar{u}(x,t),p(x,t)\big) = \min_{u \in K_u(x,t)} \mathcal{H}_1\big(x,t,u(x,t),p(x,t)\big) \tag{3.11}$$

for a.e. $(x,t) \in Q$,

$$\mathcal{H}_2\big(x,\bar{v}(x),p(x,0)\big) = \max_{v \in K_v(x)} \mathcal{H}_2\big(x,v(x),p(x,0)\big) \tag{3.12}$$

for a.e. $x \in \Omega$.

Remark 3.2 Considering (3.7) and (3.10), we can infer for any $\phi \in \mathcal{S}(0,T)$ that $\mathcal{G}(\phi,\cdot) \in \mathcal{L}(V_2,H)$. Thus, together with **(A3)–(A4)**, we can refer to the linear theory of Dautray and Lions [11, pp. 570–589] to ensure that Eqs. (3.6) and (3.9), after reversing the direction of time $t \to T-t$, admit a unique weak solution $p \in \mathcal{S}(0,T)$.

4 Existence of minimax optimal solutions

In this section, we study the existence of minimax optimal solutions to control problems (P_v) and (P). The following results play an important role in proving the existence.

Proposition 4.1 *The solution mapping* $(u,v) \longmapsto y(u,v)$ *from* $\mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}$, *endowed with weak-*$L^2(0,T;H)\times$ *weakly-star-*$L^\infty(\Omega)$ *topology, into* $C([0,T];V_1) \cap C(\bar{Q})$ *is sequentially continuous.*

Before we prove Proposition 4.1, we need the following well-known compactness results.

Lemma 4.1 *Let* X, B, *and* Y *be Banach spaces with* $X \hookrightarrow B \hookrightarrow Y$, *where* X *is compactly embedded in* B. *Let*

$$W = \big\{\phi \in L^\infty(0,T;X) \mid \phi' \in L^r(0,T;Y)\big\} \quad \text{with } r > 1.$$

Then W *is compactly embedded in* $C([0,T];B)$.

Proof See [26]. □

Proof of Proposition 4.1 Let $(u^*,v^*) \in \mathcal{U}_{\text{ad}} \times \mathcal{V}_{\text{ad}}$, and let $(u_n,v_n) \in \mathcal{U}_{\text{ad}} \times \mathcal{V}_{\text{ad}}$ be a sequence such that

$$(u_n,v_n) \rightharpoonup (u^*,v^*) \quad \text{weakly and weakly-star in } L^2(0,T;H) \text{ and } L^\infty(\Omega), \tag{4.1}$$

respectively, as $n \to \infty$. From now on, we denote by y_n the solution $y(u_n,v_n)$ of Eq. (3.1) in which u and v are replaced by u_n and v_n, respectively. From Theorem 2.2 we have

$$\|y_n\|_{\mathcal{S}(0,T)} \le C\big\|(y_0,y_1+v_n,f+u_n)\big\|_{\mathcal{P}}, \tag{4.2}$$

which implies that y_n and y_n' remain in bounded sets of $L^\infty(0,T;V_2) \cap \mathcal{W}(0,T)$ and $L^\infty(0,T;H)$, respectively. Therefore by using Rellich's extraction theorem we can find a subsequence of $\{y_n\}$, say again $\{y_n\}$, and $y^* \in \mathcal{W}(0,T)$ such that

$$y_n \rightharpoonup y^* \quad \text{weakly in } \mathcal{W}(0,T) \text{ as } n \to \infty, \tag{4.3}$$

$$y_n \rightharpoonup y^* \quad \text{weakly-star in } L^\infty(0,T;V_2) \text{ as } n \to \infty, \tag{4.4}$$

$$y_n' \rightharpoonup y^{*\prime} \quad \text{weakly-star in } L^\infty(0,T;H) \text{ as } n \to \infty. \tag{4.5}$$

Since $V_2 \hookrightarrow V_1$ is compact, in view of (2.10), we can apply Lemma 4.1 to (4.4) and (4.5) with $X = V_2$, $Y = V_1$ (or $C(\bar{\Omega})$), and $Z = H$ to verify that

$$\{y_n\} \text{ is precompact in } C\big([0,T];V_1\big) \cap C(\bar{Q}). \tag{4.6}$$

Hence we can find a subsequence $\{y_{n_k}\} \subset \{y_n\}$, if necessary, such that

$$y_{n_k}(t) \to y^*(t) \quad \text{in } V_1 \text{ for all } t \in [0,T] \text{ as } k \to \infty. \tag{4.7}$$

Therefore (2.12), (4.3), (4.6), and (4.7) imply

$$\|y_{n_k}\|_{V_1}^2 A y_{n_k} \rightharpoonup \|y^*\|_{V_1}^2 A y^* \quad \text{weakly in } L^2(0,T;H) \text{ as } k \to \infty, \tag{4.8}$$

$$g(y_{n_k}) \to g(y^*) \quad \text{strongly in } L^2(0,T;H) \text{ as } k \to \infty. \tag{4.9}$$

Then by the standard arguments in Dautray and Lions [11, pp. 561–565] we conclude that the limit y^* is a weak solution of Eq. (3.1) with u and v replaced by u^* and v^*, respectively. Moreover, from the uniqueness of weak solutions we conclude that $y^* = y(u^*,v^*)$ in $\mathcal{S}(0,T)$, which implies that $y(u_n,v_n) \to y(u^*,v^*)$ in $C([0,T];V_1) \cap C(\bar{Q})$.

This completes the proof. □

Now, we study the existence of an optimal solution to problem (P_v).

Theorem 4.1 *Let $v \in \mathcal{V}_{\text{ad}}$. Under* **(A1)–(A4)**, *problem (P_v) admits at least one optimal solution.*

Proof Let (u,v) be given in $\mathcal{U}_{\mathrm{ad}} \times \mathcal{V}_{\mathrm{ad}}$, and let $y(u,v)$ be the associated solution of Eq. (3.1). By **(A3)–(A4)**, Theorem 2.2, and Young's inequality it follows that

$$\begin{aligned}
J(u,v) &\geq c_1 \|u\|^2_{L^2(0,T;H)} - \int_Q \bigl|F\bigl(x,t,y(u,v)\bigr)\bigr|\,dx\,dt \\
&\quad - \int_\Omega \bigl(\bigl|K\bigl(x,y(u,v)(T)\bigr)\bigr| + \bigl|L(x,v)\bigr|\bigr)\,dx \\
&\geq c_1 \|u\|^2_{L^2(0,T;H)} - c\int_Q \bigl(1 + \bigl|y(u,v)\bigr|^{\sigma_1}\bigr)\bigl|y(u,v)\bigr|\,dx\,dt \\
&\quad - \int_Q \bigl|F(x,t,0)\bigr|\,dx\,dt - c\int_\Omega \bigl(1 + \bigl|y(u,v;T)\bigr|^{\sigma_2}\bigr)\bigl|y(u,v;T)\bigr|\,dx\,dt \\
&\quad - \int_\Omega \bigl|K(x,0)\bigr|\,dx - \|L_1\|_{L^1(\Omega)}\bigl\|\eta\bigl(|v|\bigr)\bigr\|_{L^\infty(\Omega)} \\
&\geq c_1 \|u\|^2_{L^2(0,T;H)} - C\bigl(\bigl\|y(u,v)\bigr\|_{\mathcal{S}(0,T)} + \bigl\|y(u,v)\bigr\|^{1+\sigma_1}_{\mathcal{S}(0,T)} \\
&\quad + \bigl\|y(u,v)\bigr\|^{1+\sigma_2}_{\mathcal{S}(0,T)} + 1\bigr) - \|L_1\|_{L^1(\Omega)}\bigl\|\eta\bigl(|v|\bigr)\bigr\|_{L^\infty(\Omega)} \\
&\geq c_1 \|u\|^2_{L^2(0,T;H)} - C\bigl(c(y_0,y_1+v,f,\sigma_1,\sigma_2) + \|u\|_{L^2(0,T;H)} \\
&\quad + \|u\|^{1+\sigma_1}_{L^2(0,T;H)} + \|u\|^{1+\sigma_2}_{L^2(0,T;H)} + 1\bigr) - \|L_1\|_{L^1(\Omega)}\bigl\|\eta\bigl(|v|\bigr)\bigr\|_{L^\infty(\Omega)} \\
&\geq \frac{c_1}{2}\|u\|^2_{L^2(0,T;H)} - C\bigl(c(y_0,y_1+v,f,\sigma_1,\sigma_2) + 1\bigr) \\
&\quad - \|L_1\|_{L^1(\Omega)}\bigl\|\eta\bigl(|v|\bigr)\bigr\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.10}$$

Let $\{u_n\} \subset \mathcal{U}_{\mathrm{ad}}$ be a minimizing sequence for problem (P_v). Thanks to (4.10), $\{u_n\}$ is bounded in $L^2(0,T;H)$. Since $\mathcal{U}_{\mathrm{ad}}$ is weakly closed, there exist $u^* \in \mathcal{U}_{\mathrm{ad}}$ and a subsequence of $\{u_n\}$, indexed again by n, such that

$$u_n \rightharpoonup u^* \quad \text{weakly in } L^2(0,T;H) \text{ as } n \to \infty. \tag{4.11}$$

For simplicity, we denote the associated solutions $y(u_n,v)$ and $y(u^*,v)$ of Eq. (3.1) by y_n and y^*, respectively. We note that

$$\begin{aligned}
J\bigl(u^*,v\bigr) - J(u_n,v) &= \int_Q F_n(x,t)\bigl(y^* - y_n\bigr)\,dx\,dt + \int_\Omega K_n(x)\bigl(y^*(T) - y_n(T)\bigr)\,dx \\
&\quad + \int_Q \bigl(H\bigl(x,t,u^*\bigr) - H(x,t,u_n)\bigr)\,dx\,dt,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
F_n(x,t) &= \int_0^1 F_y\bigl(x,t,\theta y^* + (1-\theta)y_n\bigr)\,d\theta, \\
K_n(x) &= \int_0^1 K_y\bigl(x,\theta y^*(T) + (1-\theta)y_n(T)\bigr)\,d\theta.
\end{aligned}$$

From **(A3)–(A4)** and Proposition 4.1 we easily see that

$$\lim_{n\to\infty} \int_Q F_n(x,t)\bigl(y^* - y_n\bigr)\,dx\,dt = \lim_{n\to\infty} \int_\Omega K_n(x)\bigl(y^*(T) - y_n(T)\bigr)\,dx = 0. \tag{4.13}$$

Due to **(A3)** together with the well-known classical Mazur theorem, we deduce that

$$\int_Q H(x,t,u^*)\,dx\,dt \le \liminf_{n\to\infty} \int_Q H(x,t,u_n)\,dx\,dt. \tag{4.14}$$

Applying (4.13) and (4.14) to (4.12), we obtain that

$$J(u^*,v) \le \liminf_{n\to\infty} J(u_n,v) \le \inf(P_v). \tag{4.15}$$

This completes the proof. □

To study the existence of a solution to problem (P), we need the following weak convergence result.

Proposition 4.2 *Let* **(A1)–(A4)** *be fulfilled, and let* $\{v_n\} \subset \mathcal{V}_{\mathrm{ad}}$ *be a sequence converging to some* $v \in \mathcal{V}_{\mathrm{ad}}$ *for the weak-star topology of* $L^\infty(\Omega)$. *Then the corresponding sequence* $\{u_{v_n}\}$ (*where* $u_{v_n} \in \mathrm{Argmin}(P_{v_n})$) *also converges to some* u_v ($\in \mathrm{Argmin}(P_v)$) *for the weak topology of* $L^2(0,T;H)$.

Proof Since $\{v_n\}$ is bounded in $L^\infty(\Omega)$, we have

$$J(u_{v_n},v_n) \le J(u_0,v_n) < \infty, \tag{4.16}$$

where u_0 is any fixed element of $\mathcal{U}_{\mathrm{ad}}$. From (4.10) we know that the sequence $\{u_{v_n}\}$ is bounded in $L^2(0,T;H)$. Then there exist a subsequence of $\{u_{v_n}\}$, still denoted by itself, and $u^* \in L^2(0,T;H)$ such that

$$u_{v_n} \rightharpoonup u^* \quad \text{weakly in } L^2(0,T;H) \text{ as } n \to \infty. \tag{4.17}$$

Since $\mathcal{U}_{\mathrm{ad}}$ is weakly closed by **(A1)**, we see that $u^* \in \mathcal{U}_{\mathrm{ad}}$. By (4.17) and Proposition 4.1 we deduce that

$$y(u_{v_n},v_n) \to y(u^*,v) \quad \text{strongly in } C([0,T];V_1) \cap C(\bar{Q}) \tag{4.18}$$

as $n \to \infty$. By the definition of u_{v_n} ($\in \mathrm{Argmin}(P_{v_n})$), we have

$$J(u_{v_n},v_n) \le J(u,v_n) \quad \forall u \in \mathcal{U}_{\mathrm{ad}}. \tag{4.19}$$

This can be written again as

$$\begin{aligned} &\int_Q \big(F(x,t,y(u_{v_n},v_n)) + H(x,t,u_{v_n})\big)\,dx\,dt + \int_\Omega K(x,y(u_{v_n},v_n;T))\,dx \\ &\quad\le \int_Q \big(F(x,t,y(u,v_n)) + H(x,t,u)\big)\,dx\,dt + \int_\Omega K(x,y(u,v_n;T))\,dx \end{aligned} \tag{4.20}$$

for all $u \in \mathcal{U}_{\mathrm{ad}}$. Let L_n and R_n be the left and right sides of (4.20), respectively. Due to (4.17) and (4.18), we have

$$\begin{aligned}\liminf_{n\to\infty} L_n &\geq \int_Q \bigl(F\bigl(x,t,y\bigl(u^*,v\bigr)\bigr) + H\bigl(x,t,u^*\bigr)\bigr)\,dx\,dt \\ &\quad + \int_\Omega K\bigl(x,y\bigl(u^*,v;T\bigr)\bigr)\,dx, \qquad (4.21)\\ \lim_{n\to\infty} R_n &= \int_Q \bigl(F\bigl(x,t,y(u,v)\bigr) + H(x,t,u)\bigr)\,dx\,dt \\ &\quad + \int_\Omega K\bigl(x,y(u,v;T)\bigr)\,dx. \qquad (4.22)\end{aligned}$$

Adding $\int_\Omega L(x,v)\,dx$ to the right sides of (4.21) and (4.22), we obtain by (4.20)–(4.22) that

$$J\bigl(u^*,v\bigr) \leq J(u,v) \quad \forall u \in \mathcal{U}_{\mathrm{ad}}. \qquad (4.23)$$

This implies that $u^* \in \operatorname{Argmin}(P_v)$. Therefore we can set $u^* = u_v$.

This completes the proof. □

Theorem 4.2 *Assume* **(A1)–(A4)** *are fulfilled. Then problem* (P) *admits at least one optimal solution* v^*.

Proof Problem (P) is to find

$$\sup\bigl\{J(u_v,v) \mid v \in \mathcal{V}_{\mathrm{ad}}, u_v \in \operatorname{Argmin}(P_v)\bigr\}. \qquad (4.24)$$

Let $\{v_n\}$ $(\subset \mathcal{V}_{\mathrm{ad}})$ be a maximizing sequence for problem (P). Since $\mathcal{V}_{\mathrm{ad}}$ is bounded in $L^\infty(\Omega)$, we can find a subsequence of $\{v_n\}$, still dented by itself, and $v^* \in L^\infty(\Omega)$ such that

$$v_n \rightharpoonup v^* \quad \text{weak-star in } L^\infty(\Omega) \text{ as } n \to \infty. \qquad (4.25)$$

As is noted in Remark 3.1, **(A2)** ensures that $\mathcal{V}_{\mathrm{ad}}$ is weakly closed in $L^p(\Omega)$ $(p \geq 1)$. So we know that the limit v^* in (4.25) belongs to $\mathcal{V}_{\mathrm{ad}}$. With a symmetric reasoning of (4.14), we infer from (4.25) that assumption **(A4)** ensures that

$$\limsup_{n\to\infty} \int_\Omega L(x,v_n)\,dx \leq \int_\Omega L\bigl(x,v^*\bigr)\,dx. \qquad (4.26)$$

Meanwhile, by Proposition 4.2 we know that there exists a subsequence of $\{u_{v_n}\}$ (where $u_{v_n} \in \operatorname{Argmin}(P_{v_n})$), denoted by itself, and $u_{v^*} \in \operatorname{Argmin}(P_{v^*})$ such that

$$u_{v_n} \rightharpoonup u_{v^*} \quad \text{weakly in } L^2(0,T;H) \text{ as } n \to \infty. \qquad (4.27)$$

We can also combine Proposition 4.1 with (4.25) and (4.27) to obtain that

$$y(u_{v_n},v_n) \to y\bigl(u_{v^*},v^*\bigr) \quad \text{strongly in } C\bigl([0,T];V_1\bigr) \cap C(\bar{Q}) \qquad (4.28)$$

as $n \to \infty$, where $y(u_{v_n}, v_n)$ and $y(u_{v^*}, v^*)$ are associated solutions of Eq. (3.1). Thus we have

$$\begin{aligned} & J(u_{v_n}, v_n) - J(u_{v^*}, v^*) \\ &\quad = \bigl(J(u_{v_n}, v_n) - J(u_{v^*}, v_n)\bigr) + \bigl(J(u_{v^*}, v_n) - J(u_{v^*}, v^*)\bigr) \\ &\quad \le J(u_{v^*}, v_n) - J(u_{v^*}, v^*) \\ &\quad = \int_Q \bigl(F(x,t,y(u_{v^*}, v_n)) - F\bigl(x,t,y(u_{v^*}, v^*)\bigr)\bigr)\,dx\,dt \\ &\qquad + \int_\Omega \bigl(K\bigl(x, y(u_{v^*}, v_n; T)\bigr) - K\bigl(x, y(u_{v^*}, v^*; T)\bigr)\bigr)\,dx \\ &\qquad + \int_\Omega \bigl(L(x, v_n) - L(x, v^*)\bigr)\,dx. \end{aligned} \tag{4.29}$$

From **(A3)–(A4)**, (4.26), (4.28), and (4.29) we deduce

$$\limsup_{n\to\infty} J(u_{v_n}, v_n) \le J(u_{v^*}, v^*). \tag{4.30}$$

Thus it readily follows that

$$\begin{aligned} \sup\bigl\{J(u_v, v) \mid v \in \mathcal{V}_{\mathrm{ad}}, u_v \in \operatorname{Argmin}(P_v)\bigr\} &= \limsup_{n\to\infty} J(u_{v_n}, v_n) \\ &\le J(u_{v^*}, v^*). \end{aligned} \tag{4.31}$$

Therefore v^* ($\in \mathcal{V}_{\mathrm{ad}}$) (with $u_{v^*} \in \operatorname{Argmin}(P_{v^*})$) is an optimal solution for problem (P).

This completes the proof. □

5 Taylor expansions

As pointed out before, to derive necessary conditions for the optimal solution, it is necessary to study the state perturbation of the cost function corresponding to admissible control variables and disturbance variables. This amounts to finding a sort of *Taylor expansion of first order* for the state and the cost with respect to the perturbations of the control and disturbance. As is well known in other related studies ([4, 18, 22, 24], etc.), since the control domain is just a metric space, the perturbation of the control has to be of the *spike (or needle-like)* type. Here we follow the construction developed in [4] and [22].

Given a reference control $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$, an admissible control $u \in \mathcal{U}_{\mathrm{ad}}$, and a number $\rho \in (0,1)$, a diffuse perturbation of $\bar{u}$ is defined by

$$u_\rho(x,t) := \begin{cases} \bar{u}(x,t) & \text{in } Q \setminus Q_\rho, \\ u(x,t) & \text{in } Q_\rho, \end{cases} \tag{5.1}$$

where Q_ρ is a measurable subset of Q explicitly described below.

We present the following increment formula due to Taylor expansion for the solution of Eq. (3.1) with respect to diffuse perturbations of the reference control.

Theorem 5.1 *For any given $\bar{u}, u \in \mathcal{U}_{\mathrm{ad}}$ and $\rho \in (0,1)$, we consider the diffuse perturbation in (5.1) and the solutions $\bar{y}$ ($\equiv y(\bar{u}, v)$) and y_ρ ($\equiv y(u_\rho, v)$) of Eq. (3.1) corresponding to $(\bar{u}, v)$*

and (u_ρ, v), respectively. Then there is a measurable subset $Q_\rho \subset Q$ such that

$$\tau^{N+1}(Q_\rho) = \rho\tau^{N+1}(Q), \tag{5.2}$$

$$\int_{Q_\rho} \big(H(x,t,u) - H(x,t,\bar{u})\big)\,dx\,dt = \rho \int_Q \big(H(x,t,u) - H(x,t,\bar{u})\big)\,dx\,dt, \tag{5.3}$$

$$y_\rho = \bar{y} + \rho z + \rho\delta_\rho \quad \textit{with} \lim_{\rho\to 0^+} \|\delta_\rho\|_{C([0,T];V_1)\cap C(\bar{Q})} = 0, \tag{5.4}$$

$$J(u_\rho, v) = J(\bar{u}, v) + \rho\Delta J + o(\rho), \tag{5.5}$$

where τ^{N+1} is the $(N+1)$-dimensional Lebesgue measure,

$$\Delta J := J'_y(\bar{u}, v)z + \int_Q \big(H(x,t,u) - H(x,t,\bar{u})\big)\,dx\,dt,$$

and z is the weak solution to

$$\begin{cases} z'' + Az + \mathcal{G}(\bar{y}, z) + g'(\bar{y})z = u - \bar{u} & \textit{in } (0,T), \\ z(0) = 0, \qquad z'(0) = 0, \end{cases} \tag{5.6}$$

where $\mathcal{G}(\bar{y}, z) = (1 + \|\bar{y}\|^2_{V_1})A^{\frac{1}{2}}z + 2((\bar{y}, z))_1 A^{\frac{1}{2}}\bar{y}$.

To prove Theorem 5.1, we need the following technical lemma, which was used in [9, 22, 24], etc.

Lemma 5.1 *Let $u, \bar{u} \in \mathcal{U}_{\mathrm{ad}}$. For every $\rho \in (0,1)$, there is a sequence of measurable subsets Q^n_ρ in Q such that*

$$\tau^{N+1}\big(Q^n_\rho\big) = \rho\tau^{N+1}(Q), \tag{5.7}$$

$$\int_{Q^n_\rho} \big(H(x,t,\bar{u}) - H(x,t,u)\big)\,dx\,dt = \rho \int_Q \big(H(x,t,\bar{u}) - H(x,t,u)\big)\,dx\,dt, \tag{5.8}$$

$$\frac{1}{\rho}\chi_{Q^n_\rho} \rightharpoonup 1 \quad \textit{weak-star in } L^\infty(Q) \textit{ as } n \to \infty, \tag{5.9}$$

where $\chi_{Q^n_\rho}$ means the characteristic function of Q^n_ρ.

Proof See [24]. □

We also need the following lemma.

Lemma 5.2 *Let y_1 and y_2 be the weak solutions of Eq. (3.1) corresponding to (u_1, v) and (u_2, v) with other data fixed, respectively. Then we have*

$$\|y_1 - y_2\|_{\mathcal{S}(0,T)} \le C\|u_1 - u_2\|_{L^2(0,T;H)}. \tag{5.10}$$

Proof The proof is an immediate consequence of Theorem 2.2. □

Proof of Theorem 5.1 The existence of the subset Q_ρ satisfying (5.2) and (5.3) immediately follows from Lemma 5.1. We easily infer that the increment formula (5.5) follows from (5.3) and (5.4). Thus we focus on justifying (5.4), that is, the Taylor expansion for the weak solution y of Eq. (3.1) corresponding to the diffuse control perturbations.

For any $\rho \in (0,1)$, we take the sets Q_ρ^n as in Lemma 5.1 and consider the following diffuse control perturbations:

$$u_\rho^n(x,t) := \begin{cases} \bar{u}(x,t) & \text{in } Q \setminus Q_\rho^n, \\ u(x,t) & \text{in } Q_\rho^n. \end{cases} \tag{5.11}$$

Let y_ρ^n ($\equiv y(u_\rho^n, v)$) and $\bar{y}$ ($\equiv y(\bar{u}, v)$) be the weak solutions of Eq. (3.1) corresponding to (u_ρ^n, v) and $(\bar{u}, v)$ with other data fixed, respectively. Let z be the weak solution of Eq. (5.6). Now we construct an equation that admits $\delta_\rho^n := \frac{y_\rho^n - \bar{y}}{\rho} - z$ as a unique weak solution: We observe that

$$\begin{aligned}
&\frac{1}{\rho}\Big(\big(1 + \|y_\rho^n\|_{V_1}^2\big)A^{\frac12} y_\rho^n - \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12}\bar{y}\Big) \\
&\qquad - \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12} z - 2((\bar{y}, z))_1 A^{\frac12}\bar{y} \\
&\quad = \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12}\delta_\rho^n + \frac{1}{\rho}\big(\|y_\rho^n\|_{V_1}^2 - \|\bar{y}\|_{V_1}^2\big)A^{\frac12} y_\rho^n - 2((\bar{y}, z))_1 A^{\frac12}\bar{y} \\
&\quad = \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12}\delta_\rho^n + \frac{1}{\rho}\big(\big(y_\rho^n - \bar{y}, y_\rho^n + \bar{y}\big)\big)_1 A^{\frac12} y_\rho^n - 2((\bar{y}, z))_1 A^{\frac12}\bar{y} \\
&\quad = \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12}\delta_\rho^n + \big(\big(\delta_\rho^n, y_\rho^n + \bar{y}\big)\big)_1 A^{\frac12} y_\rho^n \\
&\qquad + \big(\big(z, y_\rho^n + \bar{y}\big)\big)_1 A^{\frac12} y_\rho^n - 2((\bar{y}, z))_1 A^{\frac12}\bar{y} \\
&\quad = \big(1 + \|\bar{y}\|_{V_1}^2\big)A^{\frac12}\delta_\rho^n + \big(\big(\delta_\rho^n, y_\rho^n + \bar{y}\big)\big)_1 A^{\frac12} y_\rho^n \\
&\qquad + \big(\big(z, y_\rho^n - \bar{y}\big)\big)_1 A^{\frac12} y_\rho^n + 2((\bar{y}, z))_1 A^{\frac12}\big(y_\rho^n - \bar{y}\big).
\end{aligned} \tag{5.12}$$

Thus we infer that δ_ρ^n is a unique weak solution of

$$\begin{cases} \delta_\rho^{n\prime\prime} + A\delta_\rho^n + (1 + \|\bar{y}\|_{V_1}^2)A^{\frac12}\delta_\rho^n + ((\delta_\rho^n, y_\rho^n + \bar{y}))_1 A^{\frac12} y_\rho^n + B(y_\rho^n, \bar{y})\delta_\rho^n \\ \quad = \mathcal{F}(y_\rho^n, \bar{y}) + (\frac{1}{\rho}\chi_{Q_\rho^n} - 1)(u - \bar{u}) \quad \text{in } (0,T), \\ \delta_\rho^n(0) = 0, \qquad \delta_\rho^{n\prime}(0) = 0, \end{cases} \tag{5.13}$$

where

$$B\big(y_\rho^n, \bar{y}\big) = \int_0^1 g'\big(\theta\bar{y} + (1-\theta)y_\rho^n\big)\, d\theta, \tag{5.14}$$

$$\begin{aligned}
\mathcal{F}\big(y_\rho^n, \bar{y}\big) &= 2((\bar{y}, z))_1 A^{\frac12}\big(\bar{y} - y_\rho^n\big) + \big(\big(z, \bar{y} - y_\rho^n\big)\big)_1 A^{\frac12} y_\rho^n \\
&\quad + \left(g'(\bar{y}) - \int_0^1 g'\big(\theta\bar{y} + (1-\theta)y_\rho^n\big)\, d\theta\right) z.
\end{aligned} \tag{5.15}$$

From (5.9) we deduce that

$$\left(\frac{1}{\rho}\chi_{Q_\rho^n} - 1\right)(u - \bar{u}) \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;H) \text{ as } n \to \infty. \tag{5.16}$$

Hence for given $\rho \in (0,1)$, by the uniform boundedness principle we can find $n(\rho) \in \mathbb{N}$ ($n(\rho) \to \infty$ as $\rho \to 0^+$) and a positive constant M such that

$$\left\| \left(\frac{1}{\rho} \chi_{Q_\rho^{n(\rho)}} - 1 \right)(u - \bar{u}) \right\|_{L^2(0,T;H)} \leq M. \tag{5.17}$$

Since

$$u_\rho^{n(\rho)} \to \bar{u} \quad \text{strongly in } L^2(0,T;H) \text{ as } \rho \to 0^+, \tag{5.18}$$

we infer from Lemma 5.2 that

$$y_\rho^{n(\rho)} \to \bar{y} \quad \text{strongly in } \mathcal{S}(0,T) \text{ as } \rho \to 0^+. \tag{5.19}$$

Moreover, since V_2 is compactly embedded in $C_0(\Omega)$ when $N \in \{1,2,3\}$, we can use Lemma 4.1 by taking $X = V_2$, $B = C_0(\Omega)$ (or V_1), and $Y = H$ in Lemma 4.1 to obtain that $\mathcal{S}(0,T)$ is compactly embedded in $C([0,T];V_1) \cap C(\bar{Q})$. Hence from **(H1)**, **(H2)**, and (5.19), together with the fact that $\mathcal{S}(0,T)$ is compactly embedded in $C([0,T];V_1) \cap C(\bar{Q})$, we get that there is a subsequence of $\{y_\rho^{n(\rho)}\}$, still denoted by itself, such that

$$\mathcal{F}\big(y_\rho^{n(\rho)}, \bar{y}\big) \to 0 \quad \text{strongly in } L^2(0,T;H) \text{ as } \rho \to 0^+. \tag{5.20}$$

To estimate δ_ρ^n in Eq. (5.13), we apply the energy equality (2.21) to Eq. (5.13). Then we have

$$\begin{aligned}
&\big\|\delta_\rho^{n\prime}(t)\big\|_2^2 + \big\|\delta_\rho^n(t)\big\|_{V_2}^2 \\
&\quad = -2\int_0^t \big(\big(\delta_\rho^n, y_\rho^n + \bar{y}\big)\big)_1 \big(A^{\frac{1}{2}} y_\rho^n, \delta_\rho^{n\prime}\big)_2 \, ds - 2\int_0^t \big(B\big(y_\rho^n, \bar{y}\big)\delta_\rho^n, \delta_\rho^{n\prime}\big)_2 \, ds \\
&\qquad - 2\int_0^t \big(1 + \|\bar{y}\|_{V_1}^2\big)\big(A^{\frac{1}{2}}\delta_\rho^n, \delta_\rho^{n\prime}\big)_2 \, ds \\
&\qquad + 2\int_0^t \left(\mathcal{F}\big(y_\rho^n, \bar{y}\big) + \left(\frac{1}{\rho}\chi_{Q_\rho^n} - 1 \right)(u - \bar{u}), \delta_\rho^{n\prime} \right)_2 ds.
\end{aligned} \tag{5.21}$$

Estimating (5.21) as in the proof of Theorem 2.2, we arrive at

$$\big\|\delta_\rho^n\big\|_{\mathcal{S}(0,T)} \leq C\left(\big\|\mathcal{F}\big(y_\rho^n, \bar{y}\big)\big\|_{L^2(0,T;H)} + \left\| \left(\frac{1}{\rho}\chi_{Q_\rho^n} - 1 \right)(u - \bar{u}) \right\|_{L^2(0,T;H)} \right), \tag{5.22}$$

where C dose not depend on n and ρ. From (5.17) and (5.20) we deduce that

$$\big\{\delta_\rho^{n(\rho)}\big\} \quad \text{is bounded in } \mathcal{W}(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;H). \tag{5.23}$$

Therefore by Rellich's extraction theorem we can extract a subsequence of $\{\delta_\rho^{n(\rho)}\}$, say again $\{\delta_\rho^{n(\rho)}\}$, and find $\delta \in \mathcal{W}(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;H)$ such that

$$\delta_\rho^{n(\rho)} \rightharpoonup \delta \quad \text{weakly in } \mathcal{W}(0,T) \text{ as } \rho \to 0^+, \tag{5.24}$$

$$\delta_\rho^{n(\rho)} \rightharpoonup \delta \quad \text{weak-star in } L^\infty(0,T;V_2) \text{ as } \rho \to 0^+, \tag{5.25}$$

$$\delta_\rho^{n(\rho)\prime} \rightharpoonup \delta' \quad \text{weak-star in } L^\infty(0,T;H) \text{ as } \rho \to 0^+. \tag{5.26}$$

Considering (5.16), (5.19), (5.20), (5.24), and (5.25), we replace δ_ρ^n by $\delta_\rho^{n(\rho)}$ in the weak form of Eq. (5.13) (in view of Definition 2.1) and let $\rho \to 0^+$. Then by standard arguments as in Dautray and Lions [11, pp. 561–565] we conclude that the limit δ is the weak solution of

$$\begin{cases} \delta'' + A\delta + (1 + \|\bar{y}\|_{V_1}^2)A^{\frac{1}{2}}\delta + 2((\delta, \bar{y}))_1 A^{\frac{1}{2}}\bar{y} + g'(\bar{y})\delta = 0 \quad \text{in } (0,T), \\ \delta(0) = 0, \qquad \delta'(0) = 0. \end{cases} \tag{5.27}$$

By the uniqueness of the weak solution in Theorem 2.2 we know clearly that $\delta = 0$. Finally, we apply again Lemma 4.1 to (5.23) with $X = V_2$, $B = C_0(\Omega)$ (or V_1), and $Y = H$ to verify that $\{\delta_\rho^{n(\rho)}\}$ is precompact in $C([0,T]; V_1) \cap C(\bar{Q})$. Hence we can find a subsequence $\{\delta_\rho^{n_k(\rho)}\} \subset \{\delta_\rho^{n(\rho)}\}$, if necessary, such that

$$\delta_\rho^{n_k(\rho)} \to \delta(=0) \quad \text{strongly in } C\big([0,T]; V_1\big) \cap C(\bar{Q}) \text{ as } \rho \to 0^+. \tag{5.28}$$

This proves (5.4).

It can infer (5.5) from (5.4), but we prove it simply as follows: First, we have

$$\begin{aligned} &\left| \frac{J(u_\rho, v) - J(\bar{u}, v)}{\rho} - \Delta J \right| \\ &\quad \le \left| \int_Q \left(\frac{F(x,t,y(u_\rho,v)) - F(x,t,y(\bar{u},v))}{\rho} - F_y\big(x,t,y(\bar{u},v)\big)z \right) dx\,dt \right| \\ &\qquad + \left| \int_\Omega \left(\frac{K(x,y(u_\rho,v;T)) - K(x,y(\bar{u},v;T))}{\rho} - K_y\big(x,y(\bar{u},v;T)\big)z(T) \right) dx \right| \\ &\quad \le \left| \int_Q F_\rho(x,t)\delta_\rho \,dx\,dt \right| + \left| \int_Q \big(F_\rho(x,t) - F_y\big(x,t,y(\bar{u},v)\big)\big) z\,dx\,dt \right| \\ &\qquad + \left| \int_\Omega K_\rho(x)\delta_\rho(T)\,dx \right| + \left| \int_\Omega \big(K_\rho(x) - K_y\big(x,y(\bar{u},v;T)\big)\big) z(T)\,dx \right| \\ &\quad = I_\rho^1 + I_\rho^2 + I_\rho^3 + I_\rho^4, \end{aligned} \tag{5.29}$$

where I_ρ^i $(1 \le i \le 4)$ are given in the order of the last terms of (5.29), and

$$\begin{aligned} &F_\rho(x,t) = F_y\big(x,t,\theta_1 y(u_\rho,v) + (1-\theta_1)y(\bar{u},v)\big), \\ &K_\rho(x) = K_y\big(x,\theta_2 y(u_\rho,v;T) + (1-\theta_2)y(\bar{u},v;T)\big) \end{aligned}$$

for some $\theta_1, \theta_2 \in (0,1)$. From (5.4) we deduce that

$$I_\rho^1, I_\rho^3 \to 0 \quad \text{as } \rho \to 0^+. \tag{5.30}$$

Thanks to **(A3)**, **(A4)**, and (5.19), we obtain that

$$I_\rho^2, I_\rho^4 \to 0 \quad \text{as } \rho \to 0^+. \tag{5.31}$$

This completes the proof. □

We need another Taylor expansion of y with respect to v.

Theorem 5.2 *For any given $\bar{v}, v \in \mathcal{V}_{\mathrm{ad}}$ and $\rho \in (0,1)$, there are a measurable subset $\Omega_\rho \subset \Omega$ and $u_{\bar{v}} \in \operatorname{Argmin}(P_{\bar{v}})$ such that*

$$\tau^N(\Omega_\rho) = \rho \tau^N(\Omega), \tag{5.32}$$

$$\int_{\Omega_\rho} \big(L(x,v) - L(x,\bar{v})\big)\,dx = \rho \int_\Omega \big(L(x,v) - L(x,\bar{v})\big)\,dx, \tag{5.33}$$

$$y(u_{v_\rho}, v_\rho) = y(u_{v_\rho}, \bar{v}) + \rho z + \rho \delta_\rho \quad \text{with } \lim_{\rho \to 0^+} \|\delta_\rho\|_{C([0,T];V_1)\cap C(\bar{Q})} = 0, \tag{5.34}$$

$$J(u_{v_\rho}, v_\rho) = J(u_{v_\rho}, \bar{v}) + \rho \Delta J + o(\rho), \tag{5.35}$$

where

$$\Delta J := J'_y(u_{\bar{v}}, \bar{v})z + \int_\Omega \big(L(x,v) - L(x,\bar{v})\big)\,dx,$$

$$v_\rho(x) = \begin{cases} \bar{v}(x) & \text{in } \Omega \setminus \Omega_\rho, \\ v(x) & \text{in } \Omega_\rho, \end{cases}$$

$$u_{v_\rho} \in \operatorname{Argmin}(P_{v_\rho}),$$

and z is the weak solution to

$$\begin{cases} z'' + Az + \mathcal{G}(y(u_{\bar{v}}, \bar{v}), z) + g'(y(u_{\bar{v}}, \bar{v}))z = 0 & \text{in } (0,T), \\ z(0) = 0, \qquad z'(0) = v - \bar{v}, \end{cases} \tag{5.36}$$

where $\mathcal{G}(y(u_{\bar{v}}, \bar{v}), z) = (1 + \|y(u_{\bar{v}}, \bar{v})\|_{V_1}^2) A^{\frac{1}{2}} z + 2((y(u_{\bar{v}}, \bar{v}), z))_1 A^{\frac{1}{2}} y(u_{\bar{v}}, \bar{v})$.

The proof relies on the following lemmas.

Lemma 5.3 *Let $v, \bar{v} \in L^\infty(\Omega)$. Then for every $\rho \in (0,1)$, there is a sequence of measurable subsets Ω_ρ^n in Ω such that*

$$\tau^N(\Omega_\rho^n) = \rho \tau^N(\Omega), \tag{5.37}$$

$$\int_{\Omega_\rho^n} \big(L(x,\bar{v}) - L(x,v)\big)\,dx = \rho \int_\Omega \big(L(x,\bar{v}) - L(x,v)\big)\,dx, \tag{5.38}$$

$$\frac{1}{\rho} \chi_{\Omega_\rho^n} \rightharpoonup 1 \quad \text{weak-star in } L^\infty(\Omega) \text{ as } n \to \infty, \tag{5.39}$$

where $\chi_{\Omega_\rho^n}$ means the characteristic function of Ω_ρ^n.

Proof See [18]. □

Lemma 5.4 *Let ρ be in $(0,1)$, and let $\{\Omega_\rho^n\}$ be the sets in Lemma 5.3. Set*

$$v_\rho^n(x) := \begin{cases} \bar{v}(x) & \text{in } \Omega \setminus \Omega_\rho^n, \\ v(x) & \text{in } \Omega_\rho^n. \end{cases} \tag{5.40}$$

Then

$$v^n_\rho \rightharpoonup \rho v + (1-\rho)\bar{v} \quad \textit{weak-star in } L^\infty(\Omega) \textit{ as } n \to \infty. \tag{5.41}$$

Proof See [4]. □

Proof of Theorem 5.2 Let $u_{v^n_\rho} \in \operatorname{Argmin}(P_{v^n_\rho})$. By Propositions 4.1 and 4.2 we can show that the sequence $\{u_{v^n_\rho}\}$ (or a subsequence) admits a weak limit $u_\rho \in L^2(0,T;H)$, where $u_\rho \in \operatorname{Argmin}(P_{\rho v+(1-\rho)\bar{v}})$, and

$$y\big(u_{v^n_\rho}, v^n_\rho\big) \to y\big(u_\rho, \rho v + (1-\rho)\bar{v}\big) \quad \text{strongly in } C\big([0,T];V_1\big) \cap C(\bar{Q}) \tag{5.42}$$

as $n \to \infty$. By similar arguments the sequence $\{u_\rho\}$ (or a subsequence) admits a weak limit $u_{\bar{v}} \in L^2(0,T;H)$, where $u_{\bar{v}} \in \operatorname{Argmin}(P_{\bar{v}})$, such that

$$y\big(u_\rho, \rho v + (1-\rho)\bar{v}\big) \to y(u_{\bar{v}}, \bar{v}) \quad \text{strongly in } C\big([0,T];V_1\big) \cap C(\bar{Q}) \tag{5.43}$$

as $\rho \to 0^+$. Thus we can deduce from (5.42) and (5.43) that there exists a sequence $\{n(\rho)\} \subset \mathbb{N}$ ($n(\rho) \to \infty$ as $\rho \to 0^+$) such that

$$y\big(u_{v^{n(\rho)}_\rho}, v^{n(\rho)}_\rho\big) \to y(u_{\bar{v}}, \bar{v}) \quad \text{strongly in } C\big([0,T];V_1\big) \cap C(\bar{Q}) \tag{5.44}$$

as $\rho \to 0^+$. Let z be the weak solution of Eq. (5.36), $y^n_\rho := y(u_{v^{n(\rho)}_\rho}, v^{n(\rho)}_\rho)$, and $\bar{y} := y(u_{\bar{v}}, \bar{v})$. By an argument similar to that in the proof of Theorem 5.1 we can construct the following equation that admits $\delta^n_\rho := \frac{y^n_\rho - \bar{y}}{\rho} - z$ as the solution:

$$\begin{cases} \delta^{n\prime\prime}_\rho + A\delta^n_\rho + (1 + \|\bar{y}\|^2_{V_1})A^{\frac{1}{2}}\delta^n_\rho + ((\delta^n_\rho, y^n_\rho + \bar{y}))_1 A^{\frac{1}{2}} y^n_\rho + B(y^n_\rho, \bar{y})\delta^n_\rho \\ \quad = \mathcal{F}(y^n_\rho, \bar{y}) \quad \text{in } (0,T), \\ \delta^n_\rho(0) = 0, \qquad \delta^{n\prime}_\rho(0) = (\frac{1}{\rho}\chi_{\Omega^{n(\rho)}_\rho} - 1)(v - \bar{v}), \end{cases} \tag{5.45}$$

where

$$B\big(y^n_\rho, \bar{y}\big) = \int_0^1 g'\big(\theta\bar{y} + (1-\theta)y^n_\rho\big)\, d\theta, \tag{5.46}$$

$$\begin{aligned} \mathcal{F}\big(y^n_\rho, \bar{y}\big) &= 2((\bar{y}, z))_1 A^{\frac{1}{2}}\big(\bar{y} - y^n_\rho\big) + \big(\big(z, \bar{y} - y^n_\rho\big)\big)_1 A^{\frac{1}{2}} y^n_\rho \\ &\quad + \left(g'(\bar{y}) - \int_0^1 g'\big(\theta\bar{y} + (1-\theta)y^n_\rho\big)\, d\theta\right) z. \end{aligned} \tag{5.47}$$

As seen in (5.16), we deduce from (5.39) that

$$\left(\frac{1}{\rho}\chi_{\Omega^n_\rho} - 1\right)(v - \bar{v}) \rightharpoonup 0 \quad \text{weakly in } H \text{ as } n \to \infty. \tag{5.48}$$

Hence for every $\rho \in (0,1)$, by the uniform boundedness principle we can find a subsequence of $\{n(\rho)\} \subset \mathbb{N}$, if necessary, still denoted by itself, and a positive constant M_1 such that

$$\left\| \left(\frac{1}{\rho} \chi_{\Omega_\rho^{n(\rho)}} - 1 \right) (v - \bar{v}) \right\|_H \leq M_1. \tag{5.49}$$

Applying again the energy equality (2.21) to Eq. (5.45) and estimating the equality as shown in (5.22), we arrive at

$$\|\delta_\rho^n\|_{\mathcal{S}(0,T)} \leq C \left(\|\mathcal{F}(y_\rho^n, \bar{y})\|_{L^2(0,T;H)} + \left\| \left(\frac{1}{\rho} \chi_{\Omega_\rho^{n(\rho)}} - 1 \right) (v - \bar{v}) \right\|_H \right), \tag{5.50}$$

where C dose not depend on n and ρ. From (5.49) and (5.50) we know that

$$\{\delta_\rho^n\} \quad \text{is bounded in } \mathcal{W}(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;H). \tag{5.51}$$

Therefore by Rellich's extraction theorem we can extract a subsequence of $\{\delta_\rho^n\}$, still denoted by itself, and find $\delta \in \mathcal{W}(0,T) \cap L^\infty(0,T;V_2) \cap W^{1,\infty}(0,T;H)$ such that

$$\delta_\rho^n \rightharpoonup \delta \quad \text{weakly in } \mathcal{W}(0,T) \text{ as } \rho \to 0^+, \tag{5.52}$$

$$\delta_\rho^n \rightharpoonup \delta \quad \text{weak-star in } L^\infty(0,T;V_2) \text{ as } \rho \to 0^+, \tag{5.53}$$

$$\delta_\rho^{n\prime} \rightharpoonup \delta' \quad \text{weak-star in } L^\infty(0,T;H) \text{ as } \rho \to 0^+. \tag{5.54}$$

Noting (4.3), we can deduce that $\mathcal{F}(y_\rho^n, \bar{y})$ converge to 0 for the weak topology of $L^2(0,T;H)$, and considering (5.44) and (5.52)–(5.54), we conclude by arguments similar to the proof of Theorem 5.1 that the limit δ is the weak solution of Eq. (5.27). By the uniqueness of the weak solution in Theorem 2.2 we know clearly that $\delta = 0$. Thus by following arguments similar to the proof of Theorem 5.1 we can find a subsequence $\{\delta_\rho^{n_k}\} \subset \{\delta_\rho^n\}$, if necessary, such that

$$\delta_\rho^{n_k} \to \delta(=0) \quad \text{strongly in } C\big([0,T];V_1\big) \cap C(\bar{Q}) \text{ as } \rho \to 0^+. \tag{5.55}$$

This proves (5.34).

Thanks to (5.34), by analogy to the proof of (5.5), we can show (5.35).

This completes the proof. □

6 Necessary optimality conditions

In this section, we prove necessary optimality conditions that have to be satisfied by optimal solutions to (P_v) and (P).

6.1 Proof of Theorem 3.1

Let $\rho \in (0,1)$, $v \in \mathcal{V}_{\mathrm{ad}}$, $u_v \in \mathrm{Argmin}(P_v)$, and $u \in \mathcal{U}_{\mathrm{ad}}$. By Theorem 5.1 it is clear that there exists a measurable subset Q_ρ such that

$$\tau^{N+1}(Q_\rho) = \rho \tau^{N+1}(Q), \tag{6.1}$$

$$y\big(u_\rho(v), v\big) = y(u_v, v) + \rho z + \rho \delta_\rho, \quad \text{with } \lim_{\rho \to 0^+} \|\delta_\rho\|_{C([0,T];V_1) \cap C(\bar{Q})} = 0, \tag{6.2}$$

$$J\big(u_\rho(v), v\big) = J(u_v, v) + \rho \Delta J + o(\rho), \tag{6.3}$$

where $u_\rho(v)$ is given by

$$u_\rho(v)(x,t) := \begin{cases} u_v(x,t) & \text{in } Q \setminus Q_\rho, \\ u(x,t) & \text{in } Q_\rho, \end{cases}$$

and

$$\begin{aligned} \Delta J = & \int_Q \big(F_y\big(x,t,y(u_v,v)\big)z(x,t) + H(x,t,u) - H(x,t,u_v)\big)\,dx\,dt \\ & + \int_\Omega K_y\big(x,y(u_v,v;T)\big)z(x,T)\,dx, \end{aligned} \tag{6.4}$$

where z is the solution of

$$\begin{cases} z'' + Az + \mathcal{G}(y(u_v,v),z) + g'(y(u_v,v))z = u - u_v & \text{in } (0,T), \\ z(0) = 0, \qquad z'(0) = 0, \end{cases} \tag{6.5}$$

with $\mathcal{G}(y(u_v,v),z) = (1 + \|y(u_v,v)\|_{V_1}^2)A^{\frac{1}{2}}z + 2((y(u_v,v),z))_1 A^{\frac{1}{2}}y(u_v,v)$. Since $u_v \in \operatorname{Argmin}(P_v)$ and $(u_\rho(v),v)$ is admissible for (P_v), we have with (6.3) that

$$-\Delta J = \lim_{\rho\to 0^+} \frac{J(u_v,v) - J(u_\rho(v),v)}{\rho} \le 0. \tag{6.6}$$

Let p be the solution of (3.6). Multiplying both sides of Eq. (6.5) by p and noting that

$$\int_0^T \big(\mathcal{G}\big(y(u_v,v),z\big),p\big)_2\,dt = \int_0^T \big(z,\mathcal{G}\big(y(u_v,v),p\big)\big)_2\,dt,$$

we obtain

$$\begin{aligned} & \int_0^T (p, u - u_v)_2\,dt \\ & \quad = \int_0^T \big\langle p, z'' + Az + \mathcal{G}\big(y(u_v,v),z\big) + g'\big(y(u_v,v)\big)z\big\rangle_{V_2,V_2'}\,dt \\ & \quad = \int_0^T \big(F_y\big(y(u_v,v)\big),z\big)_2\,dt + \big(K_y\big(y(u_v,v;T)\big),z(T)\big)_2. \end{aligned} \tag{6.7}$$

From (6.4), (6.6), and (6.7) we arrive at the following condition:

$$\int_Q \mathcal{H}_1(x,t,u_v,p)\,dx\,dt \le \int_Q \mathcal{H}_1(x,t,u,p)\,dx\,dt \quad \forall u \in \mathcal{U}_{\text{ad}}. \tag{6.8}$$

The pointwise Pontryagin principle (3.8) can be given by referring to [18, pp. 157–158].

6.2 Proof of Theorem 3.2

Let $\bar{v}$ be the optimal solution to (P) we want to characterize. By Theorem 5.2 it is clear that there exist $\bar{u} \in \operatorname{Argmin}(P_{\bar{v}})$ and a measurable subsets Ω_ρ such that

$$\tau^N(\Omega_\rho) = \rho\tau^N(\Omega), \tag{6.9}$$

$$y(u_{v_\rho}, v_\rho) = y(u_{v_\rho}, \bar{v}) + \rho z + \rho\delta_\rho, \quad \text{with } \lim_{\rho\to 0^+} \|\delta_\rho\|_{C([0,T];V_1)\cap C(\bar{Q})} = 0, \tag{6.10}$$

$$J(u_{v_\rho}, v_\rho) = J(u_{v_\rho}, \bar{v}) + \rho\Delta J + o(\rho), \tag{6.11}$$

where v_ρ is given by

$$v_\rho(v)(x) := \begin{cases} \bar{v}(x) & \text{in } \Omega \setminus \Omega_\rho, \\ v(x) & \text{in } \Omega_\rho, \end{cases}$$

and

$$\Delta J = \int_Q F_y\big(x,t,y(\bar{u},\bar{v})\big)z(x,t)\,dx\,dt + \int_\Omega \big(K_y\big(x,y(\bar{u},\bar{v};T)\big)z(x,T) + L(x,v) - L(x,\bar{v})\big)\,dx, \tag{6.12}$$

where z is the solution of

$$\begin{cases} z'' + Az + \mathcal{G}(y(\bar{u},\bar{v}),z) + g'(y(\bar{u},\bar{v}))z = 0 & \text{in } (0,T), \\ z(0) = 0, \qquad z'(0) = v - \bar{v}. \end{cases} \tag{6.13}$$

Since $\bar{v}$ is an optimal solution to (P) and $\bar{u} \in \mathrm{Argmin}(P_{\bar{v}})$, we deduce that

$$0 \le \frac{J(\bar{u},\bar{v}) - J(u_{v_\rho}, v_\rho)}{\rho} \le \frac{J(u_{v_\rho},\bar{v}) - J(u_{v_\rho}, v_\rho)}{\rho}. \tag{6.14}$$

Thus it follows from (6.11) and (6.14) that

$$-\Delta J = \lim_{\rho\to 0^+} \frac{J(u_{v_\rho},\bar{v}) - J(u_{v_\rho}, v_\rho)}{\rho} \ge 0. \tag{6.15}$$

Let p be the solution of (3.9). Multiplying both sides of Eq. (6.13) by p, we obtain

$$\big(p(0), v - \bar{v}\big)_2 = \int_0^T \big(F_y\big(y(\bar{u},\bar{v})\big), z\big)_2\,dt + \big(K_y\big(y(\bar{u},\bar{v};T)\big), z(T)\big)_2. \tag{6.16}$$

From (6.12), (6.15), and (6.16) we arrive at the following condition:

$$\int_\Omega \mathcal{H}_2\big(x,v,p(x,0)\big)\,dx \le \int_\Omega \mathcal{H}_2\big(x,\bar{v},p(x,0)\big)\,dx \quad \forall v \in \mathcal{V}_{\mathrm{ad}}. \tag{6.17}$$

The pointwise Pontryagin principle (3.12) can also be derived by referring to [18, pp. 157–158]. Finally, since $\bar{u} \in \mathrm{Argmin}(P_{\bar{v}})$, (3.11) follows from Theorem 3.1.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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