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# Boundedness in a two-dimensional chemotaxis system with signal-dependent motility and logistic source

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**Abstract**

In this paper, we study the following chemotaxis system with a signal-dependent motility and logistic source:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1-u^\alpha), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u^r, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^2$, where the motility function $\gamma(v)$ satisfies $\gamma(v) \in C^3([0,\infty))$ with $\gamma(v) > 0$, and $\frac{|\gamma'(v)|^2}{\gamma(v)}$ is bounded for all $v > 0$. The purpose of this paper is to prove that the model possesses globally bounded solutions. In addition, we show that all solutions $(u, v)$ of the model will exponentially converge to the unique constant steady state $(1, 1)$ as $t \to +\infty$ when $\mu \geq \frac{K}{4^{1+r}}$ with $K = \max_{0<v\leq\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$.

**Mathematics Subject Classification:** 35K55; 35Q92; 35Q35; 92C17

**Keywords:** Global boundedness; Logistic source; Chemotaxis; Signal-dependent motility; Exponential decay

## 1 Introduction

Experimental observations show that colonies of bacteria and simple eukaryotes can generate complex shapes and patterns. In order to understand the mechanism of pattern formations, extensive mathematical models were derived, including the Keller–Segel system in [10] modeling the pattern formations driven by chemotactic bacteria

$$\begin{cases} u_t = \nabla\big(\mu(v)\nabla u - u\chi(v)\nabla v\big), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

where the cell diffusion rate $\mu$ and the chemotactic sensitivity $\chi$ are assumed to depend only on the signal concentration. In (1.1), the cell diffusion rate $\mu$ and chemotactic sensitivity $\chi$ are linked via

$$\chi(v) = (\sigma - 1)\mu'(v).$$

Here the parameter $\sigma$ denotes the ratio of effective body length to step size, and $\gamma'(v) < 0$ (resp. $> 0$) if the diffusive motility decreases (resp. increases) with respect to the chemical concentration.

One notices that $\sigma = 0$ implies that $\chi = -\mu'$, the motion of cells is biased by the local concentration chemotactic signal and is prescribed by the motility $\gamma(v)$ of cells, so that the system (1.1) can be written as

$$\begin{cases} u_t = \Delta\big(\gamma(v)u\big), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0 \end{cases} \tag{1.2}$$

with homogeneous Neumann boundary conditions. Recently, the mathematical research on the system (1.2) has attracted a lot of interest. Tao and Winkler [22] considered the chemotaxis model with $\tau = 1$ under the condition that the motility function $\gamma(v)$ satisfies $\gamma \in C^3([0,\infty))$, $C_1 < \gamma(v) < C_2$, and $|\gamma'(v)| \leq C_3$ for all $v > 0$, and they showed that the chemotaxis model has globally bounded solutions. Yoon and Kim [23] proved that there exists a globally bounded solution for the system (1.2) with $\tau = 1$ and $\gamma(v) = \frac{C_0}{v^k}$ $(k > 0)$ for small constant $C_0 > 0$ in any dimension. Lately, the smallness assumption of $C_0 > 0$ was removed for the parabolic–elliptic case of (1.2) with $\tau = 0$ when $n \leq 2$, $k > 0$ or $n \geq 3$ with $k < \frac{2}{n-2}$; Ahn and Yoon in [1] proved that the model (1.2) possesses a globally bounded and classical solution.

In [7], Jiang and Laurençot proved the existence of a global and bounded classical solution to the model (1.2) with $\tau = 0$ when $\gamma \in C^3((0,\infty))$, $\gamma > 0$, $K_v \triangleq \sup_{s\in[v,\infty]}\{\gamma(s)\} < \infty$, $v > 0$ and if $n \geq 2$, $u_0$ satisfying

$$u_0(x) \in C^0(\overline{\Omega}), \qquad u_0(x) \geq 0, \quad u_0(x) \not\equiv 0, x \in \overline{\Omega}. \tag{1.3}$$

In addition, these solutions were shown to be uniformly bounded with respect to time when $\gamma \in C^3((0,\infty))$, $\gamma' \leq 0$, and there are $k \geq l > 0$ such that $\lim_{v\to\infty}\inf v^k\gamma(v) > 0$ and $\lim_{v\to\infty}\sup v^l\gamma(v) < \infty$ for any $k < \frac{n}{n-2}$ and $k - l < \frac{2}{n-2}$ if $n \geq 3$ and $u_0$ satisfies (1.3).

Considering the presence of cell generation and death in a biological realistic setting, this can be expressed through logical sources. Researchers usually considered the following density-suppressed motility chemotaxis model with logical sources:

$$\begin{cases} u_t = \Delta\big(\gamma(v)u\big) + \mu u(1-u), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, t > 0. \end{cases} \tag{1.4}$$

Essentially, (1.4) with $\mu > 0$ has been used in [4] to justify that the bacterial motion with density-suppressed motility (i.e., $\gamma'(v) < 0$) can produce the stripe pattern formation observed in the experiment of [11]. Fujie and Jiang [5] proved the global existence of a classical solution for (1.4) in the two-dimensional setting when $u_0$ was assumed to be as in (1.3) and $\gamma(v)$ satisfied $\gamma \in C^3([0,\infty))$, $\gamma > 0$, $K_v \triangleq \sup_{s\in[v,\infty]}\{\gamma(s)\} < \infty$, $v > 0$. Moreover, if $\mu = 0$

and $\gamma(v)$ also satisfies $\lim_{v\to+\infty} v^k\gamma(v) < +\infty$, for $k > 0$, the global solution to (1.4) in the two-dimensional case is proven to be bounded uniformly in time. Recently, Lyu and Wang [12] explored how strong the logistic damping can warrant the global boundedness of solutions to (1.4) under the minimal conditions for the density-suppressed motility function $\gamma(v)$, and further established the asymptotic behavior of solutions under some conditions.

In order to address the dependence of dynamical behaviors of solutions on the interactions between nonlinear cross-diffusion, generalized logistic source, and nonlinear signal production, Tao and Fang [17] considered the density-suppressed motility mode

$$\begin{cases} u_t = \Delta\big(\gamma(v)u\big) + ru - \mu u^\alpha, & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u^\beta, & x \in \Omega, t > 0. \end{cases}$$

They proved that if $\beta < \frac{2}{N+2}\alpha$, then the system has a globally bounded classical solution, and further established the asymptotic behavior of solutions for sufficiently large $\mu$.

Inspired by the above works, in this paper we consider the initial–Neumann boundary value problem of the following parabolic–elliptic system:

$$\begin{cases} u_t = \Delta\big(\gamma(v)u\big) + \mu u\big(1 - u^\alpha\big), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u^r, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.5}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. Here $\mu, \alpha, r > 0$ are any given constants. We denote the cell density by $u$ and the chemical concentration by $v$. In order to study the large-time behavior of the system (1.5), we assume that the motility function $\gamma(v)$ satisfies:

$$\text{(H)} \quad \gamma(v) \in C^3\big([0,\infty)\big) \quad \text{with } \gamma(v) > 0 \text{ and } \frac{|\gamma'(v)|^2}{\gamma(v)} \text{ is bounded for all } v \geq 0.$$

In this paper we shall develop some results on the global boundedness and large-time behavior of solutions to the system (1.5) with general motility $\gamma(v)$. The main result of this paper reads as follows.

**Theorem 1.1** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Suppose that parameters $\mu, \alpha, r > 0$, and the motility function $\gamma(v)$ satisfies* (H). *If $\frac{1+2\alpha}{2(1+\alpha)} \leq r \leq 1 \leq \alpha$, then for any initial data $u_0$ satisfying the conditions* (1.3), *the system* (1.5) *possesses a globally bounded solution $(u, v)$ which is bounded in $\Omega \times (0,\infty)$ in the sense that there exists $C > 0$ satisfying*

$$\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} + \big\|v(\cdot,t)\big\|_{W^{1,\infty}(\Omega)} \leq C \quad \textit{for all } t > 0.$$

**Theorem 1.2** *Under the same assumptions as in Theorem* 1.1, *if the constant $\mu$ satisfies $\mu \geq \frac{K}{4^{1+r}}$ with $K = \max_{0<v\leq\infty} \frac{|\gamma'(v)|^2}{\gamma(v)}$, then the globally bounded solution of* (1.5) *satisfies*

$$\big\|u(\cdot,t) - 1\big\|_{L^\infty(\Omega)} + \big\|v(\cdot,t) - 1\big\|_{L^\infty(\Omega)} \to 0, \quad \textit{as } t \to \infty.$$

*Moreover, the convergence rate is exponential in the sense that there exist constants* $\lambda$, $C$ *such that*

$$\left\|u(\cdot,t)-1\right\|_{L^\infty(\Omega)}+\left\|v(\cdot,t)-1\right\|_{L^\infty(\Omega)}\le Ce^{-\lambda t}\quad \text{for all } t>0.$$

## 2 Preliminaries

We first recall the local existence of classical solutions to equations (1.5). The proof is based on an appropriate fixed point theorem and the maximum principle, refer to [9, Lemma 2.1] for more details.

**Lemma 2.1** (Local existence) *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded domain with a smooth boundary and assume that the motility function* $\gamma$ *satisfies condition* (H). *Assume that the initial data* $u_0$ *satisfies the conditions* (1.3). *Then there exists a unique local-in-time nonnegative classical solution*

$$(u,v)\in C^0\big(\overline{\Omega}\times[0,T_{\max})\big)\cap C^{2,1}\big(\overline{\Omega}\times(0,T_{\max})\big)$$

*to* (1.5). *Here,* $T_{\max}\in(0,\infty]$ *denotes the maximal existence time. Moreover, if* $T_{\max}<\infty$, *then*

$$\left\|u(\cdot,t)\right\|_{L^\infty(\Omega)}+\left\|v(\cdot,t)\right\|_{W^{1,\infty}(\Omega)}\to\infty\quad \text{as } t\nearrow T_{\max}. \tag{2.1}$$

At the end of this section, we state some a priori estimates on $u$, $v$, which shall be used in the sequel.

**Lemma 2.2** *Let* $(u,v)$ *be the solution of the system* (1.5). *Then it holds that*

$$\int_\Omega u(\cdot,t)\,dx\le\max\big\{\|u_0\|_{L^1(\Omega)},|\Omega|\big\}:=m_*,\quad \text{for all } t\in(0,T_{\max}), \tag{2.2}$$

*and*

$$\int_\Omega v(\cdot,t)\,dx\le C,\quad \text{for all } t\in(0,T_{\max}). \tag{2.3}$$

*Proof* Integrating the first equation of (1.5) over $\Omega$, we obtain

$$\frac{d}{dt}\int_\Omega u\,dx=\mu\int_\Omega u\,dx-\mu\int_\Omega u^{1+\alpha}\,dx,\quad \text{for all } t\in(0,T_{\max}). \tag{2.4}$$

Due to the Hölder inequality, we conclude that $\int_\Omega u^{1+\alpha}\,dx\ge\frac{1}{|\Omega|^\alpha}(\int_\Omega u\,dx)^{1+\alpha}$, which implies that

$$\frac{d}{dt}\int_\Omega u\,dx\le\mu\int_\Omega u\,dx-\mu|\Omega|^{-\alpha}\left(\int_\Omega u\,dx\right)^{1+\alpha},\quad \text{for all } t\in(0,T_{\max}), \tag{2.5}$$

and hence (2.2). Now (2.3) results from a time integration of to the second equation of (1.5). As the parameters satisfy $\frac{1+2\alpha}{2(1+\alpha)}\le r\le1\le\alpha$, one has

$$\int_\Omega v\,dx=\int_\Omega u^r\,dx\le\int_\Omega u\,dx+C_1\le C_2,\quad \text{for all } t\in(0,T_{\max}) \tag{2.6}$$

and hence (2.3) follows. □

## 3 Boundedness of solutions

In this section, we shall prove Theorem 1.1. First, we show the global existence of uniformly-in-time bounded solutions.

**Lemma 3.1** *Let the same assumptions as in Theorem 1.1 hold. Then there exists a constant $C > 0$ independent of $t$ such that the solution of (1.5) satisfies*

$$\|u\ln u\|_{L^1(\Omega)} \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

*Proof* Multiplying the first equation of (1.5) by $\ln u$, and integrating the result by parts, one has

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u\ln u\,dx - \int_\Omega u\,dx\right) + \int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx \\ &\quad = -\int_\Omega \gamma'(v)\nabla u\cdot\nabla v\,dx + \mu\int_\Omega u\ln u\,dx - \mu\int_\Omega u^{1+\alpha}\ln u\,dx \end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$. From the assumptions in (H), we can find a constant $K > 0$ such that

$$\frac{|\gamma'(v)|^2}{\gamma(v)} \le K, \quad \text{for all } v > 0. \tag{3.3}$$

We now estimate the first integrals on the right of this inequality (3.2). Using the Hölder and Young inequalities, we have

$$\begin{aligned} -\int_\Omega \gamma'(v)\nabla u\cdot\nabla v\,dx &\le \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx + \frac{1}{2}\int_\Omega u|\nabla v|^2\frac{|\gamma'(v)|^2}{\gamma(v)}\,dx \\ &\le \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx + \frac{K}{2}\int_\Omega u|\nabla v|^2\,dx \\ &\le \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx + \frac{K}{2}\|\nabla v\|^2_{L^{\frac{2(1+\alpha)}{\alpha}}(\Omega)}\|u\|_{L^{1+\alpha}(\Omega)} \end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$, and substituting into (3.2) gives

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u\ln u\,dx - \int_\Omega u\,dx\right) + \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx \\ &\quad \le \frac{K}{2}\|\nabla v\|^2_{L^{\frac{2(1+\alpha)}{\alpha}}(\Omega)}\|u\|_{L^{1+\alpha}(\Omega)} + \mu\int_\Omega u\ln u\,dx - \mu\int_\Omega u^{1+\alpha}\ln u\,dx. \end{aligned} \tag{3.5}$$

Applying the Agmon–Douglis–Nirenberg $L^p$ estimates (cf. [1, 2]) to the second equation of (3.5) with homogeneous Neumann boundary conditions, we know that for all $p > 1$, there exists a constant $C_1 > 0$ such that

$$\left\|v(\cdot,t)\right\|_{W^{2,p}(\Omega)} \le C_1\left\|u^r(\cdot,t)\right\|_{L^p(\Omega)}. \tag{3.6}$$

The Sobolev embedding theorem yields $\|\nabla v\|_{L^{\frac{2(1+\alpha)}{\alpha}}(\Omega)} \le C_2\|v\|_{W^{2,\frac{2(1+\alpha)}{1+2\alpha}}(\Omega)}$ in two dimensions (i.e., $n = 2$) which, together with (3.6), implies

$$\|\nabla v\|^2_{L^{\frac{2(1+\alpha)}{\alpha}}(\Omega)} \le C_2^2\|v\|^2_{W^{2,\frac{2(1+\alpha)}{1+2\alpha}}(\Omega)} \le C_3\left\|u^r\right\|^2_{L^{\frac{2(1+\alpha)}{1+2\alpha}}(\Omega)}. \tag{3.7}$$

On the other hand, using the $L^p$-interpolation inequality and the fact $\|u(\cdot,t)\|_{L^1(\Omega)} \le m_*$ (see Lemma 2.2), with positive parameters satisfying $\frac{1+2\alpha}{2(1+\alpha)} \le r \le 1 \le \alpha$, we have

$$\left\|u^r\right\|^2_{L^{\frac{2(1+\alpha)}{1+2\alpha}}(\Omega)} = \|u\|^{2r}_{L^{\frac{2r(1+\alpha)}{1+2\alpha}}(\Omega)} \le \|u\|^{\frac{(1+\alpha)2r-1-2\alpha}{\alpha}}_{L^{1+\alpha}(\Omega)} \|u\|^{\frac{2\alpha-2r+1}{\alpha}}_{L^1(\Omega)} \le C_4\|u\|^{\alpha}_{L^{1+\alpha}(\Omega)} + C_5, \tag{3.8}$$

the last inequality holding due to the Young inequality. We substitute (3.7) and (3.8) into (3.5) to obtain

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u\ln u\,dx - \int_\Omega u\,dx\right) + \frac{1}{2}\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u}\,dx + \int_\Omega u\ln u\,dx - \int_\Omega u\,dx \\ &\quad\le \frac{KC_3}{2}\|u\|_{L^{1+\alpha}(\Omega)}\left(C_4\|u\|^{\alpha}_{L^{1+\alpha}(\Omega)} + C_5\right) + (\mu+1)\int_\Omega u\ln u\,dx \\ &\qquad - \mu\int_\Omega u^{1+\alpha}\ln u\,dx - \int_\Omega u\,dx \\ &\quad\le C_6\|u\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + C_7 + (\mu+1)\int_\Omega u\ln u\,dx - \mu\int_\Omega u^{1+\alpha}\ln u\,dx \\ &\quad\le C_8, \end{aligned} \tag{3.9}$$

where we have used the following fact (see [21, Lemma 3.1]): Let $\mu > 0$, $\alpha \ge 1$, and $b \ge 0$, then there exists a constant $l := l(\mu, b, \alpha) > 0$ such that

$$(1+\mu)z\ln z + bz^{1+\alpha} - \mu z^{1+\alpha}\ln z \le l, \quad \text{for all } z > 0.$$

Hence from (3.9), we obtain

$$\frac{d}{dt}\left(\int_\Omega u\ln u\,dx - \int_\Omega u\,dx\right) + \int_\Omega u\ln u\,dx - \int_\Omega u\,dx \le C_8,$$

which gives $\int_\Omega u\ln u\,dx - \int_\Omega u\,dx \le C_9$ and then

$$\int_\Omega u\ln u\,dx \le \int_\Omega u\,dx + C_9 \le C_{10}.$$

Since $u\ln u \ge -\frac{1}{e}$, we derive

$$\int_\Omega |u\ln u|\,dx \le \int_\Omega u\ln u\,dx + \frac{2|\Omega|}{e} \le C_{11},$$

which yields (3.1). □

Next, we will show that there exists some $p > 1$ close to 1 such that $\int_\Omega u^p\,dx$ is uniformly bounded in time. The following basic statement can be found in [6, Chap. 4, Sect. 21] or [14, p. 43].

**Lemma 3.2** *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary, and let $G$ denote Green's function of $-\Delta + 1$ in $\Omega$ subject to Neumann boundary conditions. Then we have*

$$\left|G(x,y)\right| \le L\ln\frac{A}{|x-y|}, \quad \textit{for all } x, y \in \Omega \textit{ with } x \ne y. \tag{3.10}$$

In order to estimate the product $uv$ in terms of $u\ln u$ and $e^{\beta v}$ below, we shall refer to a variant of the Young inequality.

**Lemma 3.3** *Let $\beta > 0$. Then*

$$xy \le \frac{1}{\beta} x \ln x + \frac{1}{e\beta} e^{\beta y}, \quad \textit{for all } x > 0 \textit{ and } y > 0. \tag{3.11}$$

*Proof* The assertion easily follows by simply maximizing $x \in (0,\infty) \mapsto xy - \frac{1}{\beta} x \ln x$ for fixed $\beta > 0$ and $y > 0$. □

We next provide a specific estimate for the solution of a linear elliptic equation with a source term in $L^1$.

**Lemma 3.4** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. Then for all $M > 0$ there exist $\beta > 0$ and $C > 0$ such that*

$$\|u^r\|_{L^1(\Omega)} \le M \tag{3.12}$$

*and the solution $v$ of*

$$\begin{cases} 0 = \Delta v - v + u^r, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.13}$$

*satisfies*

$$\int_\Omega e^{\beta v}\, dx \le C. \tag{3.14}$$

*Proof* Since (3.14) is trivial in the special case $u \equiv 0$, we may assume that $u \not\equiv 0$. Then with $G$ denoting Green's function of $-\Delta + 1$ in $\Omega$ under homogeneous Neumann boundary conditions, $v$ can be represented as

$$v(x) = \int_\Omega G(x,y) u^r(y)\, dy, \quad \text{a.e. } x \in \Omega,$$

see [14, p. 43]. Now Lemma 3.2 provides $A > 0$ and $L > 0$ such that

$$v(x) \le L \int_\Omega \ln \frac{A}{|x-y|} \cdot \left|u^r(y)\right| dy, \quad \text{a.e. } x \in \Omega.$$

Therefore, invoking Jensen inequality, (3.12), and Fubini theorem, we find that for each $\beta > 0$,

$$\begin{aligned} \int_\Omega e^{\beta v(x)}\, dx &\le \int_\Omega e^{\beta L \|u^r\|_{L^1(\Omega)} \cdot \int_\Omega \ln \frac{A}{|x-y|} \cdot \frac{|u^r(y)|}{\|u^r\|_{L^1(\Omega)}}\, dy}\, dx \\ &\le \int_\Omega \left\{ \int_\Omega e^{\beta L \|u^r\|_{L^1(\Omega)} \cdot \ln \frac{A}{|x-y|}} \cdot \frac{|u^r(y)|}{\|u^r\|_{L^1(\Omega)}}\, dy \right\} dx \end{aligned}$$

$$
\begin{aligned}
&= A^{\beta L\|u^r\|_{L^1(\Omega)}} \cdot \int_\Omega \int_\Omega |x-y|^{-\beta L\|u^r\|_{L^1(\Omega)}} \cdot \frac{|u^r(y)|}{\|u^r\|_{L^1(\Omega)}}\,dy\,dx \\
&\le A^{\beta LM} \int_\Omega \int_\Omega |x-y|^{-\beta LM} \cdot \frac{|u^r(y)|}{\|u^r\|_{L^1(\Omega)}}\,dy\,dx \\
&= A^{\beta LM} \int_\Omega \left\{ \int_\Omega |x-y|^{-\beta LM}\,dx \right\} \cdot \frac{|u^r(y)|}{\|u^r\|_{L^1(\Omega)}}\,dy.
\end{aligned} \tag{3.15}
$$

If we now fix $\beta > 0$ to be small enough fulfilling $\beta LM < 2$, then $C_1 := \sup_{y\in\Omega} \int_\Omega |x-y|^{-\beta LM}\,dx$ is finite due to the boundedness of $\Omega$, and therefore

$$
\int_\Omega e^{\beta v}\,dx \le C_1 A^{\beta LM} \cdot \int_\Omega \frac{|u^r(y)|}{\|u\|_{L^r(\Omega)}}\,dy = C_2 A^{\beta LM}
$$

holds for any such $\beta > 0$. □

**Lemma 3.5** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. There exists $C > 0$ such that the solution of* (3.13) *satisfies*

$$
\int_\Omega |\nabla v|^2\,dx \le C. \tag{3.16}
$$

*Proof* Now in order to prove (3.16), we use $v$ as a test function in (3.13) to obtain

$$
\int_\Omega |\nabla v|^2\,dx + \int_\Omega v^2\,dx = \int_\Omega u^r v\,dx. \tag{3.17}
$$

We find $\beta > 0$ and $C_1, C_2 > 0$ such that $\int_\Omega e^{\beta v}\,dx \le C_1$, $\int_\Omega u\ln u\,dx \le C_2$. We employ Lemma 3.3 to estimate

$$
\begin{aligned}
\int_\Omega u^r v\,dx &\le \int_\Omega uv\,dx + C_3 \int_\Omega v\,dx \\
&\le \frac{1}{\beta}\int_\Omega u\ln u\,dx + \frac{1}{e\beta}\int_\Omega e^{\beta v}\,dx + C_3\int_\Omega v\,dx \le C_4,
\end{aligned}
$$

which, together with (3.17), establishes (3.16). □

Next, we will show that there exists some $p > 1$ close to 1 such that $\int_\Omega u^p\,dx$ is uniformly bounded in time.

**Lemma 3.6** *Suppose the conditions in Theorem 1.1 hold. Then there exists $p > 1$ close to 1 such that*

$$
\|u(\cdot,t)\|_{L^p(\Omega)} \le C, \quad \textit{for all } t \in (0, T_{\max}), \tag{3.18}
$$

*where $C > 0$ is a constant independent of $t$.*

*Proof* We multiply the first equation of (1.5) by $u^{p-1}$ to obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + (p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2\,dx \\
&\quad = -(p-1)\int_\Omega \gamma'(v)u^{p-1}\nabla u\cdot\nabla v\,dx + \mu\int_\Omega u^p\,dx - \mu\int_\Omega u^{p+\alpha}\,dx.
\end{aligned} \tag{3.19}
$$

The Cauchy–Schwarz inequality and (3.3) allow us to have

$$
\begin{aligned}
&-(p-1)\int_\Omega u^{p-1}\gamma'(v)\nabla u\cdot\nabla v\,dx\\
&\quad\le \frac{(p-1)}{2}\int_\Omega u^{p-2}\gamma(v)|\nabla u|^2\,dx+\frac{(p-1)}{2}\int_\Omega u^p\frac{|\gamma'(v)|^2}{\gamma(v)}|\nabla v|^2\,dx\\
&\quad\le \frac{(p-1)}{2}\int_\Omega u^{p-2}\gamma(v)|\nabla u|^2\,dx+\frac{(p-1)K}{2}\int_\Omega u^p|\nabla v|^2\,dx.
\end{aligned}
\tag{3.20}
$$

Using the Hölder and Gagliardo–Nirenberg inequalities, as well as (3.16) and (3.6), one has

$$
\begin{aligned}
\int_\Omega u^p|\nabla v|^2\,dx &\le \|u\|^p_{L^{p+r}(\Omega)}\|\nabla v\|^2_{L^{\frac{2(p+r)}{r}}(\Omega)}\\
&\le C_1\|u\|^p_{L^{p+r}(\Omega)}\|v\|_{W^{2,\frac{p+r}{r}}(\Omega)}\|\nabla v\|_{L^2(\Omega)}\\
&\le C_2\|u\|^p_{L^{p+r}(\Omega)}\|v\|_{W^{2,\frac{p+r}{r}}(\Omega)}\\
&= C_3\|u\|^p_{L^{p+r}(\Omega)}\left\|u^r\right\|_{L^{\frac{p+r}{r}}(\Omega)}=C_3\|u\|^{p+r}_{L^{p+r}(\Omega)}.
\end{aligned}
\tag{3.21}
$$

Then we can substitute (3.20) and (3.21) into (3.19) to obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx+\frac{(p-1)}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2\,dx\\
&\quad\le \frac{KC_3(p-1)}{2}\int_\Omega u^{p+r}\,dx+\mu\int_\Omega u^p\,dx-\mu\int_\Omega u^{p+\alpha}\,dx,
\end{aligned}
\tag{3.22}
$$

Using the Young and Hölder inequalities, we can prove that

$$
(\mu+1)\int_\Omega u^p\,dx\le(\mu+1)|\Omega|^{\frac{\alpha}{p+\alpha}}\left(\int_\Omega u^{p+\alpha}\,dx\right)^{\frac{p}{p+\alpha}}\le\frac{\mu}{2}\int_\Omega u^{p+\alpha}\,dx+C_4.
\tag{3.23}
$$

Moreover, we can choose $p>1$ satisfying $\frac{KC_3(p-1)}{2}<\frac{\mu}{2}$ to derive that

$$
\frac{KC_3(p-1)}{2}\int_\Omega u^{p+r}\,dx\le\frac{\mu}{2}\int_\Omega u^{p+\alpha}\,dx.
\tag{3.24}
$$

Then the combination of (3.22), (3.23), and (3.24) gives

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx+\int_\Omega u^p\,dx\le C_4.
\tag{3.25}
$$

Applying the ODE comparison principle to (3.25), we have (3.18) for some $p>1$ close to 1. □

Next, we will show $\|v(\cdot,t)\|_{L^\infty(\Omega)}$ is uniformly bounded in time, which rules out the possibility of degeneracy.

**Lemma 3.7** *Suppose the conditions in Theorem 1.1 hold. Then there exist constants $C, \gamma_1, \gamma_2 > 0$ such that*

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \text{for all } t \in (0, T_{\max}). \tag{3.26}$$

*and*

$$0 < \gamma_1 \le \gamma(v) \le \gamma_2. \tag{3.27}$$

*Proof* From Lemma 3.6, we can find a constant $C_1 > 0$ such that $\|u(\cdot,t)\|_{L^p(\Omega)} \le C_1$ for some $p > 1$. Then applying the elliptic regularity estimate to the second equation of (1.5), one has

$$\|v(\cdot,t)\|_{W^{2,\frac{p}{r}}(\Omega)} \le C_2 \|u^r(\cdot,t)\|_{L^{\frac{p}{r}}(\Omega)} \le C_2 \|u(\cdot,t)\|^r_{L^p(\Omega)} \le C_1^r C_2, \tag{3.28}$$

which, along with the Sobolev inequality, gives (3.26). Then since $0 < \gamma(v) \in C^3([0,\infty))$, we can find two positive constants $\gamma_1$ and $\gamma_2$ such that (3.27) holds. $\square$

**Lemma 3.8** *Suppose the conditions in Theorem 1.1 hold. Then there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{3.29}$$

*Proof* Multiplying the first equation of (1.5) by $u$ and integrating the result by parts, using the Young inequality and (3.3), we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega \gamma(v)|\nabla u|^2\,dx + \mu\int_\Omega u^{2+\alpha}\,dx \\
&\quad = -\int_\Omega \gamma'(v)u\nabla u\cdot\nabla v\,dx + \mu\int_\Omega u^2\,dx \\
&\quad \le \frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2\,dx + \frac{1}{2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^2|\nabla v|^2\,dx + \mu\int_\Omega u^2\,dx \\
&\quad \le \frac{1}{2}\int_\Omega \gamma(v)|\nabla u|^2\,dx + \frac{K}{2}\int_\Omega u^2|\nabla v|^2\,dx + \mu\int_\Omega u^2\,dx,
\end{aligned} \tag{3.30}$$

which, combined with (3.27), gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^2\,dx + \gamma_1\int_\Omega |\nabla u|^2\,dx + 2\mu\int_\Omega u^{2+\alpha}\,dx \\
&\quad \le K\int_\Omega u^2|\nabla v|^2\,dx + 2\mu\int_\Omega u^2\,dx.
\end{aligned} \tag{3.31}$$

We differentiate the second equation of the system (1.5) and multiply the result by $2\nabla v$ to obtain

$$\begin{aligned}
0 &= 2\nabla v\cdot\nabla\Delta v + 2\nabla v\cdot\nabla u^r - 2|\nabla v|^2 \\
&= \Delta|\nabla v|^2 - 2\left|D^2 v\right|^2 + 2\nabla v\cdot\nabla u^r - 2|\nabla v|^2,
\end{aligned} \tag{3.32}$$

where we have used the identity $\Delta|\nabla v|^2 = 2\nabla v \cdot \nabla\Delta v + 2|D^2 v|^2$. Then multiplying (3.32) by $|\nabla v|^2$ and integrating the results, we have

$$\begin{aligned}
&\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + 2\int_\Omega |\nabla v|^2\left|D^2 v\right|^2 dx + 2\int_\Omega |\nabla v|^4\,dx \\
&\quad= \int_{\partial\Omega} |\nabla v|^2 \frac{\partial|\nabla v|^2}{\partial\nu}\,dS + 2\int_\Omega |\nabla v|^2 \nabla v\cdot\nabla u^r\,dx \\
&\quad= \int_{\partial\Omega} |\nabla v|^2 \frac{\partial|\nabla v|^2}{\partial\nu}\,dS - 2\int_\Omega u^r\Delta v|\nabla v|^2\,dx - 2\int_\Omega u^r\nabla\left(|\nabla v|^2\right)\cdot\nabla v\,dx \\
&\quad\le \int_{\partial\Omega} |\nabla v|^2 \frac{\partial|\nabla v|^2}{\partial\nu}\,dS + 2\int_\Omega u^r\left(|\Delta v||\nabla v|^2 + \left|\nabla|\nabla v|^2\right|\cdot|\nabla v|\right)dx.
\end{aligned} \tag{3.33}$$

With the inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le 2\lambda|\nabla v|^2$ on $\partial\Omega$ (see [13, Lemma 4.2]) and the following trace inequality (see [16, Remark 52.9]) for any $\varepsilon > 0$:

$$\|\varphi\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\varphi\|_{L^2(\Omega)} + C_\varepsilon\|\varphi\|_{L^2(\Omega)},$$

we have

$$\begin{aligned}
\int_{\partial\Omega} |\nabla v|^2 \frac{\partial|\nabla v|^2}{\partial\nu}\,dS &\le 2\lambda\left\||\nabla v|^2\right\|^2_{L^2(\partial\Omega)} \\
&\le \frac{1}{4}\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + C_1\left\||\nabla v|^2\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.34}$$

By the Gagliardo–Nirenberg inequality and the fact $\||\nabla v|^2\|_{L^1(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)} \le C_2$ (see Lemma 3.5), we have

$$\begin{aligned}
C_1\left\||\nabla v|^2\right\|^2_{L^2(\Omega)} &\le C_3\left\|\nabla|\nabla v|^2\right\|_{L^2(\Omega)}\left\||\nabla v|^2\right\|_{L^1(\Omega)} + C_3\left\||\nabla v|^2\right\|^2_{L^1(\Omega)} \\
&\le \frac{1}{4}\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + C_4.
\end{aligned} \tag{3.35}$$

Then a combination of (3.34) and (3.35) gives

$$\int_{\partial\Omega} |\nabla v|^2 \frac{\partial|\nabla v|^2}{\partial\nu}\,dS \le \frac{1}{2}\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + C_4. \tag{3.36}$$

Next, we will estimate the last term on the right of (3.33). To this end, we use the Young inequality and the facts $|\Delta v| \le \sqrt{2}|D^2 v|$ and $\nabla|\nabla v|^2 = 2D^2 v\cdot\nabla v$ to derive

$$\begin{aligned}
&2\int_\Omega u^r\left(|\Delta v||\nabla v|^2 + \left|\nabla|\nabla v|^2\right|\cdot|\nabla v|\right)dx \\
&\quad\le 2\sqrt{2}\int_\Omega u^r|\nabla v|^2\left|D^2 v\right|dx + 4\int_\Omega u^r|\nabla v|^2\left|D^2 v\right|dx \\
&\quad\le 2(\sqrt{2}+2)\int_\Omega u^r|\nabla v|^2\left|D^2 v\right|dx \\
&\quad\le 2\int_\Omega |\nabla v|^2\left|D^2 v\right|^2 dx + \frac{(\sqrt{2}+2)^2}{2}\int_\Omega u^{2r}|\nabla v|^2\,dx.
\end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.33), one has

$$\begin{aligned}\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + 4\int_\Omega |\nabla v|^4\,dx &\le (2+\sqrt{2})^2\int_\Omega u^{2r}|\nabla v|^2\,dx + 2C_4\\ &\le (2+\sqrt{2})^2\int_\Omega u^2|\nabla v|^2\,dx + C_5.\end{aligned}\tag{3.38}$$

Combining (3.31) and (3.38) and using the Young inequality, we can find some $\varsigma > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2\,dx + \gamma_1\int_\Omega |\nabla u|^2\,dx + 2\mu\int_\Omega u^{2+\alpha}\,dx\\ &\quad+\int_\Omega \left|\nabla|\nabla v|^2\right|^2 dx + 4\int_\Omega |\nabla v|^4\,dx\\ &\le \left[K + (2+\sqrt{2})^2\right]\int_\Omega u^2|\nabla v|^2\,dx + 2\mu\int_\Omega u^2\,dx + C_5\\ &\le \left[K + (2+\sqrt{2})^2\right]\|u\|^2_{L^3(\Omega)}\|\nabla v\|^2_{L^6(\Omega)} + 2\mu|\Omega|^{\frac{\alpha}{2+\alpha}}\|u\|^2_{L^{2+\alpha}(\Omega)} + C_5\\ &\le C_6\|u\|^3_{L^3(\Omega)} + \varsigma\|\nabla v\|^6_{L^6(\Omega)} + \mu\|u\|^{2+\alpha}_{L^{2+\alpha}(\Omega)} + C_7.\end{aligned}\tag{3.39}$$

With the boundedness of $\|u\|_{L^1(\Omega)}$ and $\|u\ln u\|_{L^1(\Omega)}$ and the inequality in [15, Lemma 3.5], we can choose $\varepsilon$ small enough to obtain

$$\begin{aligned}\|u\|^3_{L^3(\Omega)} &\le \varepsilon\|\nabla u\|^2_{L^2(\Omega)}\|u\ln u\|_{L^1(\Omega)} + C_\varepsilon\left(\|u\ln u\|^3_{L^1(\Omega)} + \|u\|_{L^1(\Omega)}\right)\\ &\le \frac{\gamma_1}{C_6}\|\nabla u\|^2_{L^2(\Omega)} + C_8.\end{aligned}\tag{3.40}$$

On the other hand, using the Gagliardo–Nirenberg inequality, we can derive that

$$\begin{aligned}\|\nabla v\|^6_{L^6(\Omega)} &= \left\||\nabla v|^2\right\|^3_{L^3(\Omega)}\\ &\le C_9\left(\left\|\nabla|\nabla v|^2\right\|^2_{L^2(\Omega)}\left\||\nabla v|^2\right\|_{L^1(\Omega)} + \left\||\nabla v|^2\right\|^3_{L^1(\Omega)}\right)\\ &\le C_9C_2\left\|\nabla|\nabla v|^2\right\|^2_{L^2(\Omega)} + C_9C_2^3.\end{aligned}\tag{3.41}$$

Substituting (3.40) and (3.41) into (3.39), and choosing $\varsigma = \frac{1}{C_2C_9}$, we end up with $\frac{d}{dt}\int_\Omega u^2\,dx + \mu\int_\Omega u^{2+\alpha}\,dx \le C_{10}$ which, along with the Young inequality and $\int_\Omega u^2\,dx \le \mu\int_\Omega u^{2+\alpha}\,dx + C_{11}$ yields

$$\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega u^2\,dx \le C_{10} + C_{11}.$$

This gives (3.29) with the help of the ODE comparison principle. □

Next, we shall show the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$. To this end, we first improve the regularity of $v$. More precisely, we have the following results.

**Lemma 3.9** *Suppose the conditions in Theorem 1.5 hold. Then we have*

$$\left\|\nabla v(\cdot,t)\right\|_{L^\infty(\Omega)} \le C,\tag{3.42}$$

*where $C > 0$ is a constant independent of $t$.*

*Proof* Using (3.6) and (3.29), we can derive that $\|v(\cdot,t)\|_{W^{2,2}(\Omega)} \leq C_1\|u^r(\cdot,t)\|_{L^2(\Omega)} \leq C_1\|u(\cdot,t)\|_{L^2(\Omega)} + C_2 \leq C_3$, which, by the Gagliardo–Nirenberg inequality, gives

$$\|\nabla v\|_{L^4(\Omega)} \leq C_4\|v\|^{\frac{1}{2}}_{W^{2,2}(\Omega)}\|\nabla v\|^{\frac{1}{2}}_{L^2(\Omega)} + C_4\|\nabla v\|_{L^2(\Omega)} \leq C_5. \tag{3.43}$$

Then multiplying the first equation of (1.5) by $u^2$ and integrating over $\Omega$ by parts, one obtains

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\,dx + 2\int_\Omega \gamma(v)u|\nabla u|^2\,dx + \mu\int_\Omega u^{3+\alpha}\,dx \\
&\quad = -2\int_\Omega \gamma'(v)u^2\nabla u\cdot\nabla v\,dx + \mu\int_\Omega u^3\,dx \\
&\quad \leq \int_\Omega \gamma(v)u|\nabla u|^2\,dx + \int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}u^3|\nabla v|^2\,dx + \frac{\mu}{2}\int_\Omega u^{3+\alpha}\,dx + C_6,
\end{aligned}$$

which, subject to the facts (3.3) and (3.43), gives rise to

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\,dx + \frac{4\gamma_1}{9}\int_\Omega \left|\nabla u^{\frac{3}{2}}\right|^2 dx + \frac{\mu}{2}\int_\Omega u^{3+\alpha}\,dx \\
&\quad \leq K\int_\Omega u^3|\nabla v|^2\,dx + C_6 \\
&\quad \leq K\|u\|^3_{L^6(\Omega)}\|\nabla v\|^2_{L^4(\Omega)} + C_6 \\
&\quad \leq C_5^2K\|u\|^3_{L^6(\Omega)} + C_6.
\end{aligned} \tag{3.44}$$

Using the Gagliardo–Nirenberg inequality with the fact $\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}(\Omega)} = \|u\|^{\frac{3}{2}}_{L^2(\Omega)} \leq C_7$, we can show that

$$\begin{aligned}
C_5^2K\|u\|^3_{L^6(\Omega)} &= C_5^2K\left\|u^{\frac{3}{2}}\right\|^2_{L^4(\Omega)} \\
&\leq C_8\left(\left\|\nabla u^{\frac{3}{2}}\right\|^{\frac{4}{3}}_{L^2(\Omega)}\left\|u^{\frac{3}{2}}\right\|^{\frac{2}{3}}_{L^{\frac{4}{3}}(\Omega)} + \left\|u^{\frac{3}{2}}\right\|^2_{L^{\frac{4}{3}}(\Omega)}\right) \\
&\leq C_9\left\|\nabla u^{\frac{3}{2}}\right\|^{\frac{4}{3}}_{L^2(\Omega)} + C_9 \\
&\leq \frac{4\gamma_1}{9}\int_\Omega \left|\nabla u^{\frac{3}{2}}\right|^2 dx + C_{10}.
\end{aligned} \tag{3.45}$$

On the other hand, using the Hölder and Young inequalities, one has

$$\int_\Omega u^3\,dx \leq |\Omega|^{\frac{\alpha}{3+\alpha}}\left(\int_\Omega u^{3+\alpha}\,dx\right)^{\frac{3}{3+\alpha}} \leq \frac{\mu}{2}\int_\Omega u^{3+\alpha}\,dx + C_{11}. \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.44) gives

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\,dx + \int_\Omega u^3\,dx \leq C_{12}.$$

By the ODE comparison principle, we have

$$\|u(\cdot,t)\|_{L^3(\Omega)} \le C_{13}. \tag{3.47}$$

Using the elliptic regularity (3.6) and Sobolev embedding theorem again, from (3.47) we derive

$$\|\nabla v\|_{L^\infty(\Omega)} \le C_{14}\|v\|_{W^{2,3}(\Omega)} \le C_{15}\|u^r\|_{L^3(\Omega)} \le C_{16}\|u\|_{L^3(\Omega)} + C_{17} \le C_{18}.$$

This finishes the proof. □

*Proof of Theorem* 1.1 With the aid of Lemma 3.9 and a Moser-type iteration (cf. Lemma 3.6 in [9] or Lemma A.1 in [18]), we obtain that $u$ is bounded in $(0, T_{\max})$. Thus, we can find a positive constant $C$ independent of $t$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}), \tag{3.48}$$

which, together with Lemma 2.1, shows that $T_{\max} = \infty$. Therefore, $(u, v)$ is a global bounded classical solution to the system (1.1) and the proof of Theorem 1.1 is completed. □

## 4 Large time behavior

In this section, we will study the large-time behavior of the solution for the system (1.5). Let

$$K = \max_{0<v\le\infty} \frac{|\gamma'(v)|^2}{\gamma(v)} \tag{4.1}$$

and

$$A(t) := \int_\Omega (u - 1 - \ln u)\,dx. \tag{4.2}$$

Then based on some ideas in [8, 19], we shall show that the constant steady state $(1, 1)$ is globally asymptotically stable by showing $A(t)$ is a Lyapunov functional under the conditions $\mu \ge \frac{K}{4^{1+r}}$.

We next introduce the following lemma, which is a useful tool in this section.

**Lemma 4.1** ([2, Lemma 3.1]) *Let* $f : (1,\infty) \to [0,\infty)$ *be uniformly continuous such that*

$$\int_1^\infty f(t)\,dt < \infty.$$

*Then*

$$f(t) \to 0, \quad \text{as } t \to \infty. \tag{4.3}$$

We construct an appropriate energy function to the system (1.5), which is prepared for the proof of the large-time behavior.

**Lemma 4.2** *Suppose* $(u,v)$ *is the solution of* (1.5) *obtained in Lemma* 2.1. *Let* $K$ *and* $A(t)$ *be defined by* (4.1) *and* (4.2), *respectively. Then we have the following results*:

(1) $A(t) \geq 0$ *for any* $t > 0$.

(2) *If* $\mu \geq \frac{K}{4^{1+r}}$, *then there exists a positive constant* $\delta$ *such that for all* $t > 0$,

$$\frac{d}{dt}A(t) \leq -F(t), \tag{4.4}$$

*where*

$$F(t) := \delta \int_\Omega (u-1)^2\,dx. \tag{4.5}$$

*Proof* First, we will show the nonnegativity of $A(t)$. In fact, letting $\varphi(u) := u - 1 - \ln u$, $u > 0$, noting that $\varphi(1) = \varphi'(1) = 0$, and applying Taylor's formula to $\varphi(u)$ at $u = 1$ gives

$$\varphi(u) := \frac{1}{2}\varphi''(\tilde{u})(u-1)^2 = \frac{1}{2\tilde{u}^2}(u-1)^2 \geq 0,$$

where $\tilde{u}$ is between 1 and $u$, which implies $A(t) \geq 0$.

Taking the time derivative of (4.2), we get

$$\frac{d}{dt}A(t) = -\int_\Omega \frac{\gamma'(v)}{u}\nabla u \cdot \nabla v\,dx - \int_\Omega \frac{\gamma(v)}{u^2}\left|\nabla u^2\right|dx - \mu\int_\Omega (u-1)\left(u^\alpha - 1\right)dx. \tag{4.6}$$

Using the Young inequality and assumptions (4.1), we have the following estimates:

$$\begin{aligned} -\int_\Omega \frac{\gamma'(v)}{u}\nabla u \cdot \nabla v\,dx &\leq \int_\Omega \frac{\gamma(v)}{u^2}|\nabla u|^2\,dx + \frac{1}{4}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}|\nabla v|^2\,dx \\ &\leq \int_\Omega \frac{\gamma(v)}{u^2}|\nabla u|^2\,dx + \frac{K}{4}\int_\Omega |\nabla v|^2\,dx. \end{aligned} \tag{4.7}$$

Substituting into the above formula (4.6), when $t > t_0$, $t_0 > 0$, we have

$$\frac{d}{dt}A(t) \leq \frac{K}{4}\int_\Omega |\nabla v|^2\,dx - \mu\int_\Omega (u-1)\left(u^\alpha - 1\right)dx. \tag{4.8}$$

Because $u > 0$ and $\alpha \geq 1$, through the calculation $(u-1)(u^\alpha - 1) \geq (u-1)^2$, we can get

$$\frac{d}{dt}A(t) \leq \frac{K}{4}\int_\Omega |\nabla v|^2\,dx - \mu\int_\Omega (u-1)^2\,dx. \tag{4.9}$$

By simply treating the second equation of the model (1.5), we get

$$0 = \Delta v - v + u^r = \Delta v - (v-1) + \left(u^r - 1\right). \tag{4.10}$$

Multiplying both sides of the above equation (4.10) by $(v-1)$ and integrating at the same time, using the Young inequality, we obtain

$$\begin{aligned}\int_\Omega |\nabla v|^2\,dx &= -\int_\Omega (v-1)^2\,dx + \int_\Omega \left(u^r-1\right)(v-1)\,dx \\ &\le -\int_\Omega (v-1)^2\,dx + \int_\Omega (v-1)^2\,dx + \frac{1}{4}\int_\Omega \left(u^r-1\right)^2\,dx \\ &= \frac{1}{4}\int_\Omega \left(u^r-1\right)^2\,dx.\end{aligned} \tag{4.11}$$

When $\alpha \ge 1$, one has $\frac{1+2\alpha}{2(1+\alpha)} \le r \le 1$, and then the point $(\tilde{x},\tilde{t}) \in \Omega \times (0,\infty)$ has the property that $u(\tilde{x},\tilde{t}) \le \frac{1}{2}$, so we can easily draw a conclusion

$$\left|u^r-1\right| \le |u-1|^r \le 2^{1-r}|u-1|. \tag{4.12}$$

As a point $(\tilde{x},\tilde{t}) \in \Omega \times (0,\infty)$ satisfies $u(\tilde{x},\tilde{t}) > \frac{1}{2}$, $\tilde{g}(s) := s^r$, $s \in (\frac{1}{4},\infty)$, according to the mean value theorem, there exists $\theta \in (0,1)$ such that

$$\left|u^r-1\right| = \left|\tilde{g}(u)-\tilde{g}(1)\right| \le \tilde{g}'(u-u\theta+\theta)|u-1|. \tag{4.13}$$

Calculating the derivatives $\tilde{g}'(s) = rs^{r-1} > 0$, $\tilde{g}''(s) = r(r-1)s^{r-2} < 0$, we see that $\tilde{g}'(s)$ decreases monotonically for $s \in (\frac{1}{4},\infty)$. If $u(\tilde{x},\tilde{t}) > \frac{1}{2}$ and $u-u\theta+\theta > \frac{1}{2}$, then

$$\begin{aligned}\left|u^r-1\right| &\le r(u-u\theta+\theta)^{r-1}|u-1| \\ &\le r2^{1-r}|u-1| \\ &\le 2^{1-r}|u-1|.\end{aligned} \tag{4.14}$$

Collecting (4.12)–(4.14), we have

$$\int_\Omega \left(u^r-1\right)^2\,dx \le 4^{1-r}\int_\Omega (u-1)^2\,dx. \tag{4.15}$$

Substituting (4.11) and (4.15) into (4.9), we have

$$\frac{d}{dt}A(t) \le -\left(\mu - \frac{K}{4^{1+r}}\right)\int_\Omega (u-1)^2\,dx. \tag{4.16}$$

When $\mu$ is appropriately large, we have $\mu \ge \frac{K}{4^{1+r}}$, $\mu - \frac{K}{4^{1+r}} = \delta > 0$, and we can substitute it into the above equation (4.16) to get

$$\frac{d}{dt}A(t) \le -\delta\int_\Omega (u-1)^2\,dx. \tag{4.17}$$

Therefore $F(t) := \delta\int_\Omega (u-1)^2\,dx$, completing the proof. □

**Lemma 4.3** *Suppose that $\mu \ge \frac{K}{4^{1+r}}$ and let $(u,v)$ be the global classical solution of the system (1.5). Then it follows that*

$$\left\|u(\cdot,t)-1\right\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t \to \infty, \tag{4.18}$$

*and*

$$\left\|v(\cdot,t)-1\right\|_{L^\infty(\Omega)} \to 0, \quad as\ t\to\infty. \tag{4.19}$$

*Proof* From Lemma 4.2, for any $\delta$, integrating inequality (4.17) from $t_0 > 0$ to $\infty$, we can get

$$\int_{t_0}^{\infty}\int_\Omega (u-1)^2\,dx \le \frac{A(t_0)}{\delta} < \infty. \tag{4.20}$$

Due to the elliptic regularity [24] and the global boundedness of solutions, we conclude that there exist $\sigma\in(0,1)$ and $L_1 > 0$ such that

$$\left\|u(\cdot,t)\right\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[t,t+1])}, \left\|v(\cdot,t)\right\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[t,t+1])} \le L_1 \tag{4.21}$$

for all $t > 0$. So $\int_\Omega (v-1)^2 + (u-1)^2\,dx$ is uniformly continuous. Since $A(t)\ge 0$, combining (4.20) and Lemma 4.1, we can get $\int_\Omega (u-1)^2\,dx \to 0$, as $t\to\infty$. From the second equation of the chemotaxis model (1.5), using the Young inequality, we can obtain

$$\begin{aligned}\int_\Omega |\nabla v|^2\,dx &= -\int_\Omega (v-1)^2\,dx + \int_\Omega \left(u^r-1\right)(v-1)\,dx\\ &\le -\int_\Omega (v-1)^2\,dx + \frac{1}{2}\int_\Omega (v-1)^2\,dx + \frac{1}{2}\int_\Omega \left(u^r-1\right)^2\,dx.\end{aligned} \tag{4.22}$$

From this, it can be directly concluded that

$$\int_\Omega (v-1)^2\,dx \le \int_\Omega \left(u^r-1\right)^2\,dx. \tag{4.23}$$

From (4.15), the following formula can be obtained:

$$\int_\Omega (v-1)^2\,dx \le 4^{1-r}\int_\Omega (u-1)^2\,dx \to 0, \quad \text{as } t\to\infty. \tag{4.24}$$

The following formula can be obtained from the Gagliardo–Nirenberg inequality and (4.21):

$$\|\varphi\|_{L^\infty(\Omega)} \le C_{CN}\|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{1}{2}}\|\varphi\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{4.25}$$

whenever $\varphi\in W^{1,\infty}(\Omega)\cap L^2(\Omega)$, in particular, if $\varphi$ is taken to be $u-1$ or $v-1$, from (4.23) to (4.25) we can get

$$\|u-1\|_{L^\infty(\Omega)} \to 0, \qquad \|v-1\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t\to\infty,$$

finishing the proof. □

Next, we shall show that the convergence rate is exponential.

**Lemma 4.4** *Assume that* $\mu \geq \frac{K}{4^{1+r}}$ *and* $(u,v)$ *is the global classical solution of the system* (1.5). *Then there exist two positive constants* $\varepsilon > 0$, $C > 0$ *such that for all* $t > T_1 > 0$,

$$\|u-1\|_{L^2(\Omega)} \leq Ce^{-\varepsilon(t-T_1)}, \qquad \|v-1\|_{L^2(\Omega)} \leq Ce^{-\varepsilon(t-T_1)}. \tag{4.26}$$

*Proof* By using L'Hôpital's rule, we obtain

$$\lim_{u\to 1} \frac{u-1-\ln u}{(u-1)^2} = \lim_{u\to 1} \frac{1-\frac{1}{u}}{2(u-1)} = \frac{1}{2},$$

thus,we can pick $T_1 > 0$ such that

$$\frac{1}{4}\int_\Omega (u-1)^2\,dx \leq A(t) \leq \int_\Omega (u-1)^2\,dx. \tag{4.27}$$

According to (4.5) and (4.27), there exists $\varepsilon > 0$ such that

$$\frac{d}{dt}A(t) \leq -F(t) \leq -\varepsilon A(t), \quad t > T_1.$$

By the Grönwall inequality, we readily conclude that

$$A(t) \leq A(T_1)e^{-\varepsilon(t-T_1)}, \quad t > T_1.$$

Therefore

$$\int_\Omega (u-1)^2\,dx \leq 4A(t) \leq C_1e^{-\varepsilon(t-T_1)}, \quad t > T_1. \tag{4.28}$$

Similarly, combining with (4.24), we can obtain

$$\int_\Omega (v-1)^2\,dx \leq C_2e^{-\varepsilon(t-T_1)}, \quad t > T_1, \tag{4.29}$$

which finishes the proof of Lemma 4.4. □

Next, we shall show the boundedness of $\|\nabla u\|_{L^4}$ to obtain the convergence rate with the $L^\infty(\Omega)$-norm. More precisely, we have the following result.

**Lemma 4.5** *There exists a constant* $C > 0$ *independent of* $t$ *such that the solution* $(u,v)$ *of* (1.5) *satisfies*

$$\big\|\nabla u(\cdot,t)\big\|_{L^4} \leq C \tag{4.30}$$

*for all* $t \in (0, T_{\max})$.

*Proof* Using the first equation of (1.5), we obtain

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4\,dx &= \int_\Omega |\nabla u|^2\nabla u\cdot\nabla u_t\,dx\\ &= \int_\Omega |\nabla u|^2\nabla u\cdot\nabla\big(\nabla\cdot\big(\gamma(v)\nabla u\big)\big)\,dx\\ &\quad+\int_\Omega |\nabla u|^2\nabla u\cdot\nabla\big(\nabla\cdot\big(\gamma'(v)u\nabla v\big)\big)\,dx\\ &\quad+\mu\int_\Omega\big(1-u^\alpha-\alpha u^\alpha\big)|\nabla u|^4\,dx\\ &=: J_1+J_2+J_3.\end{aligned}\tag{4.31}$$

Using the identity $\Delta|\nabla u|^2 = 2\nabla u\cdot\nabla\Delta u + 2|D^2u|^2$, we can estimate the term $J_1$ as follows:

$$\begin{aligned}J_1 &= -\int_\Omega |\nabla u|^2\Delta u\nabla\cdot\big(\gamma(v)\nabla u\big)\,dx-\int_\Omega \nabla|\nabla u|^2\cdot\nabla u\nabla\cdot\big(\gamma(v)\nabla u\big)\,dx\\ &= \int_\Omega \gamma(v)|\nabla u|^2\nabla\Delta u\cdot\nabla u\,dx-\int_\Omega\gamma'(v)\nabla|\nabla u|^2\cdot\nabla u\nabla u\cdot\nabla v\,dx\\ &= \frac{1}{2}\int_\Omega\gamma(v)|\nabla u|^2\Delta|\nabla u|^2\,dx-\int_\Omega\gamma(v)|\nabla u|^2\big|D^2u\big|^2\,dx\\ &\quad-\int_\Omega\gamma'(v)\nabla|\nabla u|^2\cdot\nabla u\nabla u\cdot\nabla v\,dx\\ &\le\frac{1}{2}\int_{\partial\Omega}\gamma(v)|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}\,dS-\frac{1}{2}\int_\Omega\gamma(v)\big|\nabla|\nabla u|^2\big|^2\,dx\\ &\quad-\int_\Omega\gamma(v)|\nabla u|^2\big|D^2u\big|^2\,dx\\ &\quad+\frac{3}{2}\int_\Omega\big|\gamma'(v)\big|\big|\nabla|\nabla u|^2\big||\nabla u|^2|\nabla v|\,dx.\end{aligned}\tag{4.32}$$

Using the boundedness of $\|u\|_{L^\infty(\Omega)}$ and $\|v\|_{W^{1,\infty}(\Omega)}$ obtained in Theorem 1.1, assumptions (H), as well as the fact $\Delta v = v-u^r$, we have

$$\begin{aligned}\nabla\cdot\big(\gamma'(v)u\nabla v\big) &= \gamma''(v)u|\nabla v|^2+\gamma'(v)\nabla u\cdot\nabla v+\gamma'(v)u\Delta v\\ &= \gamma''(v)u|\nabla v|^2+\gamma'(v)\nabla u\nabla v+\gamma'(v)uv-\gamma'(v)u^{r+1}\\ &\le C_1\big(1+|\nabla u|\big),\end{aligned}\tag{4.33}$$

which substituted into $J_2$ gives

$$\begin{aligned}J_2 &= -\int_\Omega\nabla|\nabla u|^2\cdot\nabla u\nabla\cdot\big(\gamma'(v)u\nabla v\big)\,dx-\int_\Omega|\nabla u|^2\Delta u\nabla\cdot\big(\gamma'(v)u\nabla v\big)\,dx\\ &\le C_1\int_\Omega|\nabla u|\big|\nabla|\nabla u|^2\big|\big(1+|\nabla u|\big)\,dx+C_1\int_\Omega|\nabla u|^2|\Delta u|\big(1+|\nabla u|\big)\,dx.\end{aligned}\tag{4.34}$$

Moreover, the boundedness of $\|u\|_{L^\infty(\Omega)}$ directly gives

$$J_3\le C_2\int_\Omega|\nabla u|^4\,dx.\tag{4.35}$$

Substituting (4.32)–(4.35) into (4.31), and noting that $0 < \gamma_1 \le \gamma(v)$ and $|\Delta u| \le \sqrt{2}|D^2u|$, we have

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4\,dx + \frac{\gamma_1}{2}\int_\Omega \big|\nabla|\nabla u|^2\big|^2\,dx + \gamma_1\int_\Omega |\nabla u|^2\big|D^2u\big|^2\,dx \\
&\quad\le \frac{1}{2}\int_{\partial\Omega}\gamma(v)|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}\,dS + \frac{3}{2}\int_\Omega \big|\gamma'(v)\big|\big|\nabla|\nabla u|^2\big||\nabla u|^2|\nabla v|\,dx \\
&\qquad + C_1\int_\Omega |\nabla u|\big|\nabla|\nabla u|^2\big|\big(1+|\nabla u|\big)\,dx \\
&\qquad + C_1\int_\Omega |\nabla u|^2|\Delta u|\big(1+|\nabla u|\big)\,dx + C_2\int_\Omega |\nabla u|^4\,dx \\
&\quad\le \frac{\gamma_1}{4}\int_\Omega \big|\nabla|\nabla u|^2\big|^2\,dx + \frac{\gamma_1}{2}\int_\Omega |\nabla u|^2\big|D^2u\big|^2\,dx + C_3\int_\Omega |\nabla u|^4\,dx + C_4,
\end{aligned} \tag{4.36}$$

which leads to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^4\,dx + \gamma_1\int_\Omega \big|\nabla|\nabla u|^2\big|^2\,dx + 2\gamma_1\int_\Omega |\nabla u|^2\big|D^2u\big|^2\,dx \\
&\quad\le 4C_3\int_\Omega |\nabla u|^4\,dx + 4C_4.
\end{aligned} \tag{4.37}$$

On the other hand, using the boundedness of $\|u\|_{L^\infty(\Omega)}$ and the fact $|\Delta u| \le \sqrt{2}|D^2u|$ again, we have

$$\begin{aligned}
&\left(\frac{3}{2}+4C_3\right)\int_\Omega |\nabla u|^4\,dx \\
&\quad= \left(\frac{3}{2}+4C_3\right)\int_\Omega |\nabla u|^2\nabla u\cdot\nabla u\,dx \\
&\quad= -\left(\frac{3}{2}+4C_3\right)\int_\Omega u\nabla|\nabla u|^2\cdot\nabla u\,dx - \left(\frac{3}{2}+4C_3\right)\int_\Omega u|\nabla u|^2\Delta u\,dx \\
&\quad\le \gamma_1\int_\Omega \big|\nabla|\nabla u|^2\big|^2\,dx + 2\gamma_1\int_\Omega |\nabla u|^2\big|D^2u\big|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^4\,dx + C_6,
\end{aligned} \tag{4.38}$$

which substituted into (4.37) gives

$$\frac{d}{dt}\int_\Omega |\nabla u|^4\,dx + \int_\Omega |\nabla u|^4\,dx \le C_7. \tag{4.39}$$

Then applying the ODE comparison principle to (4.39) yields (4.30), and the proof is completed. □

**Lemma 4.6** *Suppose $\mu \ge \frac{K}{4^{1+r}}$. There exist constants $\varepsilon > 0$, $C > 0$ independent of $t$ such that the solution $(u,v)$ of* (1.5) *satisfies*

$$\|u-1\|_{L^\infty(\Omega)} \le Ce^{-\frac{\varepsilon}{6}(t-T_1)} \tag{4.40}$$

*and*

$$\|v-1\|_{L^\infty(\Omega)} \le Ce^{-\frac{\varepsilon}{6}(t-T_1)}. \tag{4.41}$$

*Proof* Using the Gagliardo–Nirenberg inequality, as well as (4.26) and (4.30), we have

$$\begin{aligned} \|u-1\|_{L^\infty(\Omega)} &\le C_1\|\nabla u\|_{L^4(\Omega)}^{\frac{2}{3}}\|u-1\|_{L^2(\Omega)}^{\frac{1}{3}} + C_1\|u-1\|_{L^2(\Omega)} \\ &\le C_2e^{-\frac{\varepsilon}{6}(t-T_1)} + C_2e^{-\varepsilon(t-T_1)} \\ &\le 2C_2e^{-\frac{\varepsilon}{6}(t-T_1)}. \end{aligned} \tag{4.42}$$

On the other hand, from the second equation of (1.5), we infer that $\psi(x,t) := v(x,t) - 1$ satisfies

$$\begin{cases} -\Delta\psi + \psi = u^r - 1, & x \in \Omega, t > 0, \\ \frac{\partial\psi}{\partial\nu} = 0, & x \in \Omega, t > 0. \end{cases} \tag{4.43}$$

Then using the elliptic maximum principle, we obtain from (4.43) and (4.12), (4.14) that

$$\|v-1\|_{L^\infty(\Omega)} = \|\psi\|_{L^\infty(\Omega)} \le \left\|u^r - 1\right\|_{L^\infty(\Omega)} \le C_3\|u-1\|_{L^\infty(\Omega)} \le C_4e^{-\frac{\varepsilon}{6}(t-T_1)}.$$

The proof is completed. □

*Proof of Theorem* 1.2 This is an immediate consequence of Lemmas 4.3 and 4.6. □

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