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Dynamic of the nonclassical diffusion equation with memory

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Abstract

In this paper, we consider the nonclassical diffusion equation with memory and the nonlinearity of the polynomial growth condition of arbitrary order in the time-dependent space. First, the well-posedness of the solution for the equation is obtained in the time-dependent space $\mathscr{U}_t$. Then, we establish the existence and regularity of the time-dependent global attractor. Finally, we also conclude that the fractal dimension of the time-dependent attractor is finite.

Mathematics Subject Classification: 35B25; 35B40; 35B41; 45K05; 35K57

Keywords: Nonclassical diffusion equation; Time-dependent global attractor; Polynomial growth of arbitrary order; Regularity; Finite fractal dimension

1 Introduction

In recent years, exploring the dynamic behavior of dissipative partial differential equations with time-dependent coefficients in the field of infinite-dimensional dynamical systems has attracted much attention. For this kind of problem, Plinio et al. [36] first put forward the concept of the time-dependent global attractor in time-dependent space. Subsequently, improved theoretical results and some applications have emerged widely, see [12, 14, 16, 24, 25, 27–34, 37, 43, 50]. The main characteristic of this type of problem is that the norm of space depends on time explicitly, which will lead to the fact that the considered problem is still nonautonomous even when the forcing term is independent of time t.

In this paper, we are concerned with the following nonclassical diffusion equation with memory

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u - \int_0^\infty \kappa(s)\triangle u(t-s)\,ds + f(u) = g(x), \quad x \in \Omega, t \geq \tau, \\ u|_{\partial\Omega} = 0, \quad t \geq \tau, \\ u(x,t) = u_\tau(x), \quad x \in \Omega, t \leq \tau, \tau \in \mathbb{R} \end{cases} \tag{1.1}$$

in time-dependent space, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $u = u(x,t) : \Omega \times [\tau,\infty) \to \mathbb{R}$ is an unknown function, $u_\tau(x,r) : \Omega \times (-\infty,\tau]$ is the initial value function that characterizes the past time and $g = g(x) \in H^{-1}(\Omega)$ is the external term. In addition, the

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time-dependent coefficient $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0 \tag{1.2}$$

and there is a constant $L > 0$ such that

$$\sup_{t\in\mathbb{R}}\left(\left|\varepsilon(t)\right| + \left|\varepsilon'(t)\right|\right) \le L. \tag{1.3}$$

The nonlinearity $f(s) \in C^1(\mathbb{R})$ with $f(0) = 0$ satisfies the polynomial growth condition of arbitrary order

$$\gamma_1|s|^p - \beta_1 \le f(s)s \le \gamma_2|s|^p + \beta_2, \quad p \ge 2, \tag{1.4}$$

and the dissipation condition

$$f'(s) \ge -l, \tag{1.5}$$

where γ_i, β_i $(i = 1,2)$ and l are positive constants. Setting $F(s) = \int_0^s f(y)\,dy$, it follows from (1.4) that there exist $\tilde{\gamma}_i$, $\tilde{\beta}_i$ $(i = 1,2)$ such that

$$\tilde{\gamma}_1|s|^p - \tilde{\beta}_1 \le F(s) \le \tilde{\gamma}_2|s|^p + \tilde{\beta}_2. \tag{1.6}$$

The memory kernel κ is a nonnegative summable function satisfying $\int_0^\infty \kappa(s)\,ds = 1$ and having the following form

$$\kappa(s) = \int_s^\infty \mu(r)\,dr, \tag{1.7}$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing piecewise absolutely continuous function and is allowed to have infinitely many discontinuity points. We assume

$$\kappa(s) \le \Theta\mu(s), \quad \forall s \in \mathbb{R}^+, \Theta > 0. \tag{1.8}$$

From [17], the above inequality (1.8) is equivalent to the following

$$\mu(r + s) \le Me^{-\delta r}\mu(s), \tag{1.9}$$

where $M \ge 1$, $\delta > 0$, $r \ge 0$.

The nonclassical diffusion equation arising from several physical phenomena, a pseudoparabolic equation, was first proposed by Aifantis in [1]. Then, Jäckle [19] came up with the diffusion equation with memory in the study of heat conduction and relaxation of high-viscosity liquids. Gradually, the study of nonclassical diffusion equations with memory emerged in the time-dependent spaces.

The researches were first focused on the nonclassical diffusion equation with constant coefficient. The kind of problem where the function $\varepsilon(t)$ is a positive constant and the case $\kappa(s) = 0$ or $\kappa(s) \neq 0$ in equation (1.1) has been studied extensively, see, e.g., [2–7, 9–

11, 20–22, 26, 39, 41, 42, 44–48] and the references therein. For the nonclassical diffusion equation with memory (i.e., $\kappa(s) \neq 0$ in (1.1)), the authors [46] first obtained the existence and regularity of a uniform attractor in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ ($\Omega \in \mathbb{R}^N$, $N \geq 3$), when the nonlinearity satisfies the critical exponential growth condition and the memory kernel satisfies

$$\mu'(s) + \delta\mu(s) \leq 0, \quad \delta, s \geq 0, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+). \tag{1.10}$$

Since then, the condition (1.10) has been used for the nonclassical diffusion equation with memory and constant coefficient. In 2014, Conti et al. [11] applied the memory kernel condition (1.8) and proved the existence of global attractor in $H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$ ($\Omega \in \mathbb{R}^3$). Then, they also obtained the existence of the exponential attractor in [10]. In [2], the authors obtained the existence of a global attractor for the nonclassical diffusion equation with memory and a new class of nonlinearity. It is worth noting that (1.9) is weaker than (1.10), which shows (1.8) is more general, see [10, 11] for details.

When the perturbed coefficient $\varepsilon(t)$ is a decreasing function and satisfies (1.2) and (1.3), there are some results for other equations, see [12, 24, 25, 27, 28, 30–32, 40]. For the nonclassical diffusion equation, the case $\kappa(s) = 0$ in equation (1.1) has been investigated by some authors. When the forcing term $g \in L^2(\Omega)$ ($\Omega \subset \mathbb{R}^3$) and the nonlinearity $f(u)$ satisfies $|f'(u)| \leq C(1 + |u|)$, the authors [16] obtained the existence of a time-dependent global attractor by using the decomposition technique. Using the same method, Ma et al. proved the existence, regularity, and asymptotic structure of the time-dependent global attractor in [29], when the forcing term $g \in H^{-1}(\Omega)$ ($\Omega \subset \mathbb{R}^N$, $N \geq 3$) and the nonlinear term $f(u)$ satisfies the critical exponential growth condition. In [43, 50], in order to overcome the difficulty caused by the nonlinearity with a polynomial growth condition of arbitrary order, the authors applied the contractive function method and obtained the existence of the time-dependent global attractor. However, we also proved the regularity and asymptotic structure of the time-dependent global attractor in [43]. However, we find that all the articles mentioned above studied the nonclassical diffusion equations without memory. Recently, we submitted a new manuscript on the existence of the time-dependent global attractor for the nonclassical diffusion equation with memory and the nonlinearity of critical exponential growth. In summary, the researches of the nonclassical diffusion equations are not abundant on time-dependent spaces.

Therefore, these discussions above motivate us to consider the dynamic behavior of problem (1.1) in the time-dependent space in this paper. In contrast to the existing papers, the nonlinear term satisfies the polynomial growth condition of arbitrary order and the memory kernel condition is weaker in this paper, which is a highlight worth mentioning.

In order to obtain the corresponding results for problem (1.1), we follow the ideas from [11, 43]. However, we also encountered some difficulties. First, a weaker memory kernel condition (1.8) makes the directly obtained energy functional unavailable. Secondly, due to the influence of the term $-\varepsilon(t)\triangle u_t$, the solution for the problem (1.1) does not have the higher regularity. Finally, we can not apply the decomposition technique from [29, 36], because of the particularity of the nonlinearity for a polynomial growth condition of arbitrary order. For these reasons, introducing a new function related to the memory kernel, we construct a new energy functional and obtain the existence of the time-dependent absorbing sets. Then, we apply the contractive function method rather than the decomposition method to verify the asymptotic compactness. In addition, we apply a decomposition method from [43] and obtain the regularity of the time-dependent attractor, and we also conclude that the fractal dimension of the time-dependent attractor is finite.

The paper is organized as follows. In Sect. 2, we introduce notations, function spaces involved, some abstract results for the time-dependent global attractor, and some standard conclusions. In Sect. 3, we will prove the well-posedness of the solution. Then, based on the existence of the solution, we obtain the process generated by a weak solution. In Sect. 4, we investigate the existence of the time-dependent global attractor in $\mathscr{U}_t$. In Sect. 5, the regularity of the time-dependent attractor is obtained in $\mathscr{U}_t^1$. Finally, based on the previous results in this paper, we find that the fractal dimension of the time-dependent attractor is finite in $\mathscr{U}_t$.

2 Preliminaries

As in [15], we introduce a variable that shows the past history of equation (1.1), that is

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)\,dr, \quad s \geq 0 \tag{2.1}$$

and

$$\eta_t^t(x,s) = u(x,t) - \eta_s^t(x,s), \quad s \geq 0, \tag{2.2}$$

where $\eta_t = \frac{\partial \eta}{\partial t}$, $\eta_s = \frac{\partial \eta}{\partial s}$.

Therefore, according to (1.7), (2.1), and (2.2), the problem (1.1) can be transformed into the following system

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u - \int_0^\infty \mu(s)\triangle\eta^t(s)\,ds + f(u) = g(x), \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{2.3}$$

with the corresponding initial conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, t \geq \tau, \\ \eta^t(x,s) = 0, \quad (x,s) \in \partial\Omega \times \mathbb{R}^+, t \geq \tau, \\ u(x,\tau) = u_\tau(x), \quad x \in \Omega, \tau \in \mathbb{R}, \\ \eta(x,\tau,s) = \eta^\tau(x,s) = \int_0^s u_\tau(x,\tau-r)\,dr, \quad (x,s) \in \Omega \times \mathbb{R}^+, \tau \in \mathbb{R}, \\ \eta^t(x,0) = \eta(x,t,0) = 0. \end{cases} \tag{2.4}$$

First, we give some spaces and the corresponding norms used in the remainder of the paper. Usually, we set $\|\cdot\|_{L^p(\Omega)}$ as the norm of $L^p(\Omega)$ ($p \geq 1$). In particular, set $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ as the scalar product and norm of $\mathrm{H} = L^2(\Omega)$, respectively. The Laplacian $A = -\triangle$ with Dirichlet boundary conditions is a positive operator on H with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Then, we consider the family of Hilbert spaces $\mathrm{H}_s = \mathrm{D}(A^{s/2})$, $\forall s \in \mathbb{R}$, with the standard inner products and norms, respectively,

$$\langle\cdot,\cdot\rangle_s = \langle\cdot,\cdot\rangle_{D(A^{s/2})} = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle, \|\cdot\|_s = \|A^{s/2}\cdot\|.$$

In particular, $\mathrm{H}_{-1} = H^{-1}(\Omega)$, $\mathrm{H}_0 = \mathrm{H}$, $\mathrm{H}_1 = H_0^1(\Omega)$, $\mathrm{H}_2 = H^2(\Omega) \cap H_0^1(\Omega)$.

Hence, for any $t \in \mathbb{R}$, $-1 \leq s \leq 1$, we define the time-dependent spaces $\mathcal{H}_t^s$ with norms

$$\|u\|_{\mathcal{H}_t^s}^2 = \|u\|_s^2 + \varepsilon(t)\|u\|_{s+1}^2$$

and

$$\mathcal{H}_t^{s_1} \hookrightarrow\hookrightarrow \mathcal{H}_t^{s_2}, \quad \forall s_2 < s_1,$$

where the embedding constant is independent of $t \in \mathbb{R}$ and the symbol s is always omitted whenever zero. In particular,

$$\|u\|_{\mathcal{H}_t}^2 = \|u\|^2 + \varepsilon(t)\|u\|_1^2, \qquad \|u\|_{\mathcal{H}_t^1}^2 = \|u\|_1^2 + \varepsilon(t)\|u\|_2^2.$$

According to the definition of the memory kernel, for any $-1 \leq s \leq 1$, we introduce the Hilbert (history) spaces

$$\mathcal{M}^s = L_\mu^2\left(\mathbb{R}^+; \mathrm{H}_s\right) = \left\{\eta^t : \mathbb{R}^+ \to \mathrm{H}_s : \int_0^\infty \mu(s)\left\|\eta^t(s)\right\|_s ds < \infty\right\},$$

with the corresponding inner products and norms

$$\begin{aligned} \left\langle \eta^t, \xi^t\right\rangle_{\mu,s} &= \left\langle \eta^t, \xi^t\right\rangle_{\mathcal{M}^s} = \int_0^\infty \mu(s)\left\langle \eta^t(s), \xi^t(s)\right\rangle_s ds, \\ \left\|\eta^t\right\|_{\mu,s}^2 &= \left\|\eta^t\right\|_{\mathcal{M}^s}^2 = \int_0^\infty \mu(s)\left\|\eta^t(s)\right\|_s^2 ds. \end{aligned}$$

We know that $\mathcal{M}^{s_1} \hookrightarrow \mathcal{M}^{s_2}$, $\forall s_1 > s_2$. Due to the lack of tightness in the memory space, a new weighted space needs to be constructed. According to the literatures [13], let

$$\mathcal{L}^s = \left\{\eta^t \in \mathcal{M}^s : \eta_s^t \in \mathcal{M}, \sup_{y \geq 1} y T_{\eta^t}(y) < \infty\right\}$$

and

$$\left\|\eta^t\right\|_{\mathcal{L}^s}^2 = \left\|\eta^t\right\|_{\mathcal{M}^s}^2 + \left\|\eta_s^t\right\|_{\mathcal{M}}^2 + \sup_{y \geq 1} y T_{\eta^t}(y),$$

where T_{η^t} is the tail function of η^t with the following form

$$T_{\eta^t}(y) = \int_0^{1/y} \mu(s)\left\|A^{1/2}\eta^t(s)\right\|^2 ds + \int_{1/y}^\infty \mu(s)\left\|A^{1/2}\eta^t(s)\right\|^2 ds, \quad y \geq 1.$$

Now, combining the above spaces, we give the following time-dependent space families

$$\begin{aligned} \mathscr{U}_t^s &= \mathcal{H}_t^s \times \mathcal{M}^{s+1}, \\ \mathscr{Z}_t^s &= \mathcal{H}_t^s \times \mathcal{L}^{s+1}, \end{aligned}$$

endowed with the inner products and norms, respectively,

$$\|z\|^2_{\mathscr{U}^s_t} = \left\|\left(u,\eta^t\right)\right\|^2_{\mathscr{U}^s_t} = \|u\|^2_s + \varepsilon(t)\|u\|^2_{s+1} + \left\|\eta^t\right\|^2_{\mu,s+1},$$
$$\|z\|^2_{\mathscr{Z}^s_t} = \left\|\left(u,\eta^t\right)\right\|^2_{\mathscr{Z}^s_t} = \|u\|^2_s + \varepsilon(t)\|u\|^2_{s+1} + \left\|\eta^t\right\|^2_{\mathcal{L}^{s+1}}.$$

In particular,

$$\mathscr{U}_t = \mathcal{H}_t \times \mathcal{M}^1, \qquad \|z\|^2_{\mathscr{U}_t} = \|u\|^2 + \varepsilon(t)\|u\|^2_1 + \left\|\eta^t\right\|^2_{\mu,1},$$
$$\mathscr{U}^1_t = \mathcal{H}^1_t \times \mathcal{M}^2, \qquad \|z\|^2_{\mathscr{U}^1_t} = \|u\|^2_1 + \varepsilon(t)\|u\|^2_2 + \left\|\eta^t\right\|^2_{\mu,2},$$
$$\mathscr{Z}^1_t = \mathcal{H}^1_t \times \mathcal{L}^2, \qquad \|z\|^2_{\mathscr{Z}^1_t} = \|u\|^2_1 + \varepsilon(t)\|u\|^2_2 + \left\|\eta^t\right\|^2_{\mathcal{L}^2}.$$

Note that the dual space of X is denoted as X^*. As a convenience, we choose C as the positive constant depending on the subscript that may be different from line to line or in the same line throughout the paper.

Secondly, we recall some notations, concepts, and abstract results, see, e.g., [14, 33, 36] for more details. For every $t \in \mathbb{R}$, let X_t be a family of normed spaces, we introduce the R-ball of X_t

$$\mathbb{B}_{X_t}(R) = \left\{z \in X_t : \|z\|^2_{X_t} \le R\right\}.$$

In addition, we denote the Hausdorff semidistance of two nonempty sets $B, C \subset X_t$ by

$$\operatorname{dist}_{X_t}(B,C) = \sup_{x\in B}\inf_{y\in C}\|x-y\|_{X_t}.$$

Definition 2.1 Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two-parameter family of mappings $\{U(t,\tau) : X_\tau \to X_t, t \ge \tau \in \mathbb{R}\}$ with properties

(i) $U(\tau,\tau) = \mathrm{Id}$ is the identity on X_τ, $\tau \in \mathbb{R}$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall t \ge s \ge \tau$.

Definition 2.2 A family $\mathfrak{D} = \{D_t\}_{t\in\mathbb{R}}$ of bounded sets $D_t \subset X_t$ is called uniformly bounded if there exist a constant $R > 0$ such that $D_t \subset \mathbb{B}_{X_t}(R)$, $\forall t \in \mathbb{R}$.

Definition 2.3 A time-dependent absorbing set for the process $\{U(t,\tau)\}_{t\ge\tau}$ is a uniformly bounded family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R > 0$ there exists a t_0 such that

$$U(t,\tau)\mathbb{B}_{X_\tau}(R) \subset B_t, \quad \text{for all } \tau \le t - t_0.$$

Definition 2.4 We say that a process $\{U(t,\tau)\}_{t\ge\tau}$ in a family of normed spaces $\{X_t\}_{t\in\mathbb{R}}$ is pullback asymptotically compact if and only if for any fixed $t \in \mathbb{R}$, bounded sequence $\{x_n\}_{n=1}^\infty \subset X_{\tau_n}$ and any $\{\tau_n\}_{n=1}^\infty \subset \mathbb{R}^{-t}$ with $\tau_n \to -\infty$ as $n \to \infty$, sequence $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ has a convergent subsequence, where $\mathbb{R}^{-t} = \{\tau : \tau \in \mathbb{R}, \tau \le t\}$.

Definition 2.5 The time-dependent global attractor for $\{U(t,\tau)\}_{t\ge\tau}$ is the smallest family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ such that

(i) each A_t is compact in X_t;
(ii) $\mathfrak{A}$ is pullback attracting, i.e., it is uniformly bounded and the limit

$$\lim_{\tau\to-\infty} \mathrm{dist}_{X_t}\big(U(t,\tau)D_\tau, A_t\big) = 0$$

holds for every uniformly bounded family $\mathfrak{D} = \{D_t\}_{t\in\mathbb{R}}$ and every fixed $t \in \mathbb{R}$.

Definition 2.6 We say $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant if

$$U(t,\tau)A_\tau = A_t, \quad \forall t \geq \tau.$$

Definition 2.7 Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\mathfrak{C} = \{C_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subsets of $\{X_t\}_{t\in\mathbb{R}}$. We call a function $\psi_\tau^t(\cdot,\cdot)$, defined on $X_t \times X_t$, a contractive function on $C_\tau \times C_\tau$ if for any fixed $t \in \mathbb{R}$ and any sequence $\{x_n\}_{n=1}^\infty \subset C_\tau$, there is a subsequence $\{x_{n_k}\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \psi_\tau^t(x_{n_k}, x_{n_l}) = 0.$$

Theorem 2.8 *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process $\{X_t\}_{t\in\mathbb{R}}$ and has a pullback absorbing family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$. Moreover, assume that for any $\epsilon > 0$ there exists $T = T(\epsilon) \leq t$, $\psi_T^t \in \mathfrak{C}(B_T)$ such that*

$$\big\|U(t,T)x - U(t,T)y\big\|_{X_t} \leq \epsilon + \psi_T^t(x,y), \quad \forall x,y \in B_T,$$

for any fixed $t \in \mathbb{R}$. Then $\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact.

Theorem 2.9 *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process in a family of Banach spaces $\{X_t\}_{t\in\mathbb{R}}$. Then, $U(\cdot,\cdot)$ has a time-dependent global attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ satisfying $A_t = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} U(t,\tau)B_\tau}$ if and only if*
(i) *$\{U(t,\tau)\}_{t\geq\tau}$ has a pullback absorbing family $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}}$;*
(ii) *$\{U(t,\tau)\}_{t\geq\tau}$ is pullback asymptotically compact.*

For the sake of estimation, we next recall some standard conclusions, see [18, 23, 35].

Lemma 2.10 *Assume that the memory function κ satisfies (1.7) and (1.8), then for any $T > \tau$, $\eta^t \in C([\tau,T], L_\mu^2(\mathbb{R}^+;\mathrm{H}_1))$ such that*

$$\begin{aligned}
-\langle\eta_s^t,\eta^t\rangle_{\mu,1} &= -\frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\big\|\nabla\eta^t(s)\big\|^2\,ds \\
&= \left[-\frac{1}{2}\mu(s)\big\|\nabla\eta^t(s)\big\|^2\right]_0^\infty + \frac{1}{2}\int_0^\infty \mu'(s)\big\|\nabla\eta^t(s)\big\|^2\,ds \\
&\leq 0.
\end{aligned}$$

Lemma 2.11 *If $\mathcal{K} \subset \mathcal{M}$ satisfies*
(i) $\sup_{\eta^t\in\mathcal{K}} \|\eta^t\|_{\mathcal{M}^1} < \infty$;
(ii) $\sup_{\eta^t\in\mathcal{K}} \|\eta_s^t\|_{\mathcal{M}} < \infty$;

(iii) $\lim_{y\to\infty}[\sup_{\eta^t\in\mathcal{K}} T_{\eta^t}(y)] = 0$,
then $\mathcal{K}$ is relatively compact in $\mathcal{M}$.

Lemma 2.12 (*Aubin–Lions Lemma*). *Assume that X, B, and Y are three Banach spaces with $X \hookrightarrow\hookrightarrow B$ and $B \hookrightarrow Y$. Let f_n be bounded in $L^p([0,T],B)$ $(1 \le p < \infty)$. If f_n satisfies*
(i) *f_n is bounded in $L^p([0,T],X)$;*
(ii) *$\frac{\partial f_n}{\partial t}$ is bounded in $L^p([0,T],Y)$,*
then f_n is relatively compact in $L^p([0,T],B)$.

3 Well-posedness

Now, we give the definition of a weak solution and prove the well-posedness of the weak solution for the problem (2.3) and (2.4) by using the Faedo–Galerkin method from [8, 23, 38].

Definition 3.1 The function $z = (u,\eta^t) = (u(x,t),\eta^t(x,s))$ defined in $\Omega\times[\tau,T]$ is said to be a weak solution for the problem (2.3) and (2.4) with the initial data $z_\tau \in \mathbb{B}_{\mathscr{U}_\tau}(R_0) \subset \mathscr{U}_\tau$, $-\infty < \tau < T < +\infty$ if z satisfies
(i) $z \in C([\tau,T],\mathscr{U}_t)$, $(x,t)\in\Omega\times[\tau,T]$;
(ii) for any $\theta = (v,\xi^t)\in\mathscr{U}_t$, the equality

$$\langle u_t,v\rangle + \varepsilon(t)\langle\nabla u_t,\nabla v\rangle + \langle\nabla u,\nabla v\rangle + \langle\eta^t,v\rangle_{\mu,1} + \langle f(u),v\rangle = \langle g,v\rangle$$

and

$$\langle\eta^t_t,\xi^t\rangle_{\mu,1} = -\langle\eta^t_s,\xi^t\rangle_{\mu,1} + \langle u,\xi^t\rangle_{\mu,1}$$

hold for a.e. $[\tau,T]$.

Theorem 3.2 *Assume that (1.2)–(1.8) hold and $g\in H^{-1}(\Omega)$, then for any initial data $z_\tau = (u_\tau,\eta^\tau)\in\mathbb{B}_{\mathscr{U}_\tau}(R_0)\subset\mathscr{U}_\tau$ and any $\tau\in\mathbb{R}$, there exists a unique solution z for the problem (2.3) and (2.4) such that $z = (u,\eta^t)\in C([\tau,T],\mathscr{U}_t)$ for any fixed $T>\tau$. Furthermore, the solution depends on the initial data continuously in $\mathscr{U}_t$.*

Proof Assume that ω_k is the eigenfunction of $A = -\triangle$ with a Dirichlet boundary value in H_1, then $\{\omega_k\}_{k=1}^\infty$ is a standard orthogonal basis of H and is also an orthogonal basis in H_1. The corresponding eigenvalues are denoted by $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\le\cdots$, $\lambda_j\to\infty$ with $A\omega_k = \lambda_k\omega_k$, $\forall k\in\mathbb{N}$. Next, we will complete our proof.

♡ *Faedo–Galerkin scheme.* If we give an integer m, we denote by P_m the projection on the subspace $\operatorname{span}\{\omega_1,\dots,\omega_m\}$ in $H_0^1(\Omega)$, and Q_m the projection on the subspace $\operatorname{span}\{e_1,\dots,e_m\}\subset L^2_\mu(\mathbb{R}^+,\mathrm{H}_1)$ in $L^2_\mu(\mathbb{R}^+,\mathrm{H}_1)$. For every fixed m, we look for the function $u^m(t) = P_m u = \sum_{k=1}^m a_m^k(t)\omega_k$ and $\eta^{t,m}(s) = Q_m\eta^t = \sum_{k=1}^m b_m^k(t)e_k(s)$, where $a_m^k(t)$ and $b_m^k(t)$ satisfy

$$\begin{cases}\langle u_t^m,\omega_k\rangle + \langle\varepsilon(t)Au_t^m,\omega_k\rangle + \langle Au^m,\omega_k\rangle + \langle\eta^{t,m},\omega_k\rangle_{\mu,1} = \langle g,\omega_k\rangle - \langle f(u^m),\omega_k\rangle,\\ \langle\eta_t^{t,m},e_k\rangle_{\mu,1} = -\langle\eta_s^{t,m},e_k\rangle_{\mu,1} + \langle u^m,e_k\rangle_{\mu,1},\\ a_m^k(\tau) = \langle u_\tau,\omega_k\rangle, b_m^k(\tau) = \langle\eta^\tau,e_k\rangle_{\mu,1}.\end{cases} \tag{3.1}$$

On account of the standard existence theory for ordinary differential equations, there exists a continuous solution of the problem (2.3) and (2.4) on an interval $[\tau, T]$. Then, we will prove the convergence of $z^m(t) = (u^m, \eta^{t,m})$.

♡ *Energy estimates*. Multiplying the first and the second equation of (3.1) by a_m^k and b_m^k, respectively, and summing from 1 to m about k, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2 + \|\eta^{t,m}\|_{\mu,1}^2\right) + \left(2 - \varepsilon'(t)\right)\|u^m\|_1^2 \\ &\quad = -2\langle \eta^{t,m}, \eta_s^{t,m}\rangle_{\mu,1} - 2\langle f(u^m), u^m\rangle + 2\langle g, u^m\rangle. \end{aligned} \tag{3.2}$$

By (1.4), Hölder's inequality, and Young's inequality, we obtain

$$\langle f(u^m), u^m\rangle \ge \gamma_1 \int_\Omega |u^m|^p\, dx - \beta_1|\Omega|, \tag{3.3}$$

$$\langle g, u^m\rangle \le \|g\|_{-1}^2 + \frac{1}{4}\|u^m\|_1^2. \tag{3.4}$$

According to Lemma 2.10 and (3.2)–(3.4), we have

$$\begin{aligned} &\frac{d}{dt}\left(\|u^m\|^2 + \varepsilon(t)\|u^m\|_1^2 + \|\eta^t\|_{\mu,1}^2\right) + \left(1 - \varepsilon'(t)\right)\|u^m\|_1^2 \\ &\qquad + \frac{1}{2}\|u^m\|_1^2 + 2\gamma_1 \int_\Omega |u^m|^p\, dx \\ &\quad \le 2\|g\|_{-1}^2 + 2\beta_1|\Omega|. \end{aligned} \tag{3.5}$$

Applying the decreasing property of $\varepsilon(t)$ and integrating from τ to t at the sides of (3.5) we obtain

$$\|z^m\|_{\mathcal{U}_t}^2 + \frac{1}{2}\int_\tau^t \|u^m(s)\|_1^2\, ds + 2\gamma_1 \int_\tau^t \int_\Omega |u^m(s)|^p\, dx\, ds \le R, \tag{3.6}$$

where

$$R = \|z_\tau^m\|_{\mathcal{U}_\tau}^2 + (t-\tau)\left(2\|g\|_{-1}^2 + 2\beta_1|\Omega|\right).$$

Thereby, we infer from (3.6) that

$$\{u^m\}_m^\infty \quad \text{is bounded in } L^\infty\left([\tau, T], \mathcal{H}_t\right) \cap L^2\left([\tau, T], \mathrm{H}_1\right) \cap L^2\left([\tau, T], L^p(\Omega)\right), \tag{3.7}$$

$$\{\eta^{t,m}\}_m^\infty \quad \text{is bounded in } L^\infty\left([\tau, T], L_\mu^2(\mathbb{R}^+, \mathrm{H}_1)\right), \tag{3.8}$$

for any fixed $T > t$. It follows from (1.4) that

$$\int_\tau^t \int_\Omega |f(u^m)|^q\, dx\, dt \le C_{q,\gamma_2} \int_\tau^t \|u^m(s)\|_{L^p(\Omega)}^p\, ds + C_{q,\beta_2,|\Omega|,t-\tau}, \tag{3.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence, we infer from (3.9) that

$$\{f(u^m)\}_{m=1}^\infty \quad \text{is bounded in } L^q\left([\tau, T], L^q(\Omega)\right). \tag{3.10}$$

Next, we verify the uniform estimate for u_t^m. Multiplying the first equation of (3.1) by $\partial_t a_m^k$ and summing from 1 to m yields

$$\frac{d}{dt}E(t) + \langle \eta^{t,m}, u_t^m \rangle_{\mu,1} + \| u_t^m \|^2 + \varepsilon(t) \| u_t^m \|_1^2 = 0, \tag{3.11}$$

where

$$E(t) = \frac{1}{2}\| u^m \|_1^2 + \langle F(u^m), 1 \rangle - \langle g, u^m \rangle.$$

Applying (1.6) arrives at

$$\langle F(u^m), 1) \geq \tilde{\gamma}_1 \int_\Omega |u^m|^p \, dx - \tilde{\beta}_1 |\Omega|, \tag{3.12}$$

$$\langle F(u^m), 1) \leq \tilde{\gamma}_2 \int_\Omega |u^m|^p \, dx + \tilde{\beta}_2 |\Omega|. \tag{3.13}$$

Hence, combining with Poincaré's inequality, (3.4), (3.12), and (3.13), we obtain

$$E(t) \geq \frac{1}{4}\| u^m \|_1^2 + \tilde{\gamma}_1 \int_\Omega |u^m|^p \, dx - \tilde{\beta}_1 |\Omega| - \|g\|_{-1}^2, \tag{3.14}$$

$$E(t) \leq C_{\tilde{\gamma}_2} \left(\| u^m \|_1^2 + \| u^m \|_{L^p(\Omega)}^p \right) + \tilde{\beta}_2 |\Omega| + \|g\|_{-1}^2. \tag{3.15}$$

In addition,

$$\left| \langle \eta^{t,m}, u_t^m \rangle_{\mu,1} \right| \leq \frac{\kappa(0)}{2\varepsilon(t)} \| \eta^{t,m} \|_{\mu,1}^2 + \frac{\varepsilon(t)}{2} \| u_t^m \|_1^2. \tag{3.16}$$

Then, it follows from (3.6), (3.11), and (3.16) that

$$\frac{d}{dt}E(t) + \frac{1}{2}\| u_t^m \|^2 + \frac{\varepsilon(t)}{2} \| u_t^m \|_1^2 \leq \frac{R\kappa(0)}{2\varepsilon(t)} \leq \frac{R\kappa(0)}{2\varepsilon(T)}, \tag{3.17}$$

for $t \in [\tau, T]$. Integrating from s to t at the sides of (3.17), for any $s \in (\tau, T]$, we obtain

$$E(t) \leq E(s) + \frac{R\kappa(0)}{2\varepsilon(T)}(t-s). \tag{3.18}$$

Then, integrating from τ to T about variable s for (3.18), we have

$$E(t) \leq \frac{1}{T-\tau}\int_\tau^T E(r)\, dr + \frac{R\kappa(0)}{2\varepsilon(T)}(T-\tau). \tag{3.19}$$

From (3.6), (3.14), (3.15), and (3.19), then there exists $\rho_1 > 0$ such that

$$\| u^m \|_1^2 + \| u^m \|_{L^p(\Omega)}^p \leq \rho_1, \tag{3.20}$$

where

$$\rho_1 = C_{\tilde{\gamma}_1}\left(C_{\tilde{\gamma}_2, \gamma_1, (T-\tau)} R + (\tilde{\beta}_1 + \tilde{\beta}_2)|\Omega| + 2\|g\|_{-1}^2 + \frac{R\kappa(0)}{2\varepsilon(T)}(T-\tau) \right).$$

Combining with (3.14), (3.15), and (3.20) as well as integrating from τ to t at the side of (3.17), we obtain

$$\int_\tau^t \left(\left\|u_t^m(r)\right\|^2 + \varepsilon(r)\left\|u_t^m(r)\right\|_1^2\right) dr \leq 2E(\tau) + 2\tilde{\beta}_1|\Omega| + 2\|g\|_{-1}^2 + \frac{R\kappa(0)}{\varepsilon(T)} \leq \rho_2, \tag{3.21}$$

where

$$\rho_2 = C_{\tilde{\gamma}_2}\rho_1 + 2(\tilde{\beta}_1 + \tilde{\beta}_2)|\Omega| + 4\|g\|_{-1}^2 + \frac{R\kappa(0)}{\varepsilon(T)}.$$

Hence, we infer from (3.21) that

$$\left\{u_t^m\right\}_{m=1}^\infty \quad \text{is bounded in } L^2\left([\tau, T], \mathcal{H}_t\right). \tag{3.22}$$

♡ *Existence of solution*. It follows from (3.7), (3.8), (3.10), and (3.22) that there exist $u \in L^\infty([\tau, T], \mathcal{H}_t) \cap L^2([\tau, T], \mathrm{H}_1) \cap L^2([\tau, T], L^p(\Omega))$, $\eta^t \in L^\infty([\tau, T], L^2_\mu(\mathbb{R}^+, \mathrm{H}_1))$, $\chi \in L^q([\tau, T], L^q(\Omega))$, $u_t \in L^2([\tau, T], \mathcal{H}_t)$ and a subsequence of $\{u^m\}_{m=1}^\infty$ (still denoted as $\{u^m\}_{m=1}^\infty$) such that

$$u^m \to u \quad \text{weak-star in } L^\infty\left([\tau, T], \mathcal{H}_t\right)), \tag{3.23}$$

$$u^m \to u \quad \text{weakly in } L^2\left([\tau, T], \mathrm{H}_1\right), \tag{3.24}$$

$$u^m \to u \quad \text{weakly in } L^p\left([\tau, T], L^p(\Omega)\right), \tag{3.25}$$

$$\eta^{t,m} \to \eta^t \quad \text{weak-star in } L^\infty\left([\tau, T], \mathcal{M}^1\right), \tag{3.26}$$

$$f\left(u^m\right) \to \chi \quad \text{weakly in } L^q\left([\tau, t], L^q(\Omega)\right), \tag{3.27}$$

$$u_t^m \to u_t \quad \text{weakly in } L^2\left([\tau, T], \mathcal{H}_t\right). \tag{3.28}$$

We find from (3.6), (3.21), and Lemma 2.12 that there is a subsequence of $\{u^m\}_{m=1}^\infty$ (still denoted as $\{u^m\}_{m=1}^\infty$) such that

$$u^m \to u \quad \text{in } L^2\left([\tau, T], L^2(\Omega)\right),$$

which shows that

$$u^m \to u, \quad \text{a.e. in } \Omega \times [\tau, T]. \tag{3.29}$$

Due to (3.29) and the continuity of f, we have

$$f\left(u^m\right) \to f(u), \quad \text{a.e. in } \Omega \times [\tau, T],$$

which combines with the term-by-term of the integral theorem of Lebesgue and the uniqueness of limit, we verify $\chi = f(u)$.

Moreover, we can obtain that $z \in C([\tau, T], \mathscr{U}_t)$, the conclusion (ii) in Definition 3.1 and $z(\tau) = z_\tau$ hold. The proof of these conclusions is similar to Theorem 3.2 from [43] and also trivial, hence, we will omit it.

♡ *Uniqueness and continuity of solution*. Assume that $(u^i, \eta^{t,i})$ $(i = 1, 2)$ are two solutions of the problem (2.3) and (2.4) with the initial data $(u^i_\tau, \eta^{\tau,i})$, respectively. For convenience, define $\bar{u} = u^1 - u^2$, $\bar{\eta}^t = \eta^{t,1} - \eta^{t,2}$, then $(\bar{u}, \bar{\eta}^t)$ satisfies the following problem

$$\begin{cases} \bar{u}_t - \varepsilon(t)\Delta\bar{u}_t - \Delta\bar{u} - \int_0^\infty \mu(s)\Delta\bar{\eta}^t(s)\,ds + f(u^1) - f(u^2) = 0, \\ \bar{\eta}^t_t = -\bar{\eta}^t_s + \bar{u}, \\ \bar{u}(x,\tau) = \bar{u}_\tau, \bar{\eta}^\tau(x,s) = \bar{\eta}^\tau. \end{cases} \tag{3.30}$$

Multiplying the first equation (3.30) by $\bar{u}$ and integrating on Ω, we obtain

$$\frac{d}{dt}\big(\|\bar{u}\|^2 + \varepsilon(t)\|\bar{u}\|_1^2 + \|\bar{\eta}^t\|^2_{\mu,1}\big) + \big(2 - \varepsilon'(t)\big)\|\bar{u}\|_1^2 = -2\langle\bar{\eta}^t, \bar{\eta}^t_s\rangle - 2\langle f(u^1) - f(u^2), \bar{u}\rangle.$$

In view of (1.2), (1.3), (1.5), and Lemma 2.10, then

$$\frac{d}{dt}\big(\|\bar{u}\|^2 + \varepsilon(t)\|\bar{u}\|_1^2 + \|\bar{\eta}^t\|^2_{\mu,1}\big) \le 2l\|\bar{u}\|^2 \le 2l\big(\|\bar{u}\|^2 + \varepsilon(t)\|\bar{u}\|_1^2 + \|\bar{\eta}^t\|^2_{\mu,1}\big).$$

Then, by the Gronwall lemma, we obtain

$$\big\|(\bar{u}, \bar{\eta}^t)\big\|^2_{\mathscr{U}_t} \le e^{2l(t-\tau)}\big\|(\bar{u}_\tau, \bar{\eta}^\tau)\big\|^2_{\mathscr{U}_\tau},$$

which shows the uniqueness and continuous dependence of the solution on the initial value. □

According to Theorem 3.2, we can define a continuous process $\{U(t,\tau)\}_{t\ge\tau}$ generated by the solution of the problem (2.3) and (2.4), where the mapping

$$U(t,\tau): \mathscr{U}_\tau \to \mathscr{U}_t, \quad t \ge \tau \in \mathbb{R}$$

and $U(t,\tau)z_\tau = z(t)$, $z_\tau \in \mathscr{U}_\tau$.

4 The time-dependent global attractor

In this subsection, we first consider a time-dependent absorbing family for the solution process to prove the existence of the time-dependent global attractor.

Theorem 4.1 *Assume that* (1.2)–(1.4), (1.6)–(1.8), $g \in H^{-1}(\Omega)$ *hold and* $z_\tau = (u_\tau, \eta^\tau) \in \mathbb{B}_{\mathscr{U}_\tau}(R_0) \subset \mathscr{U}_\tau$, *then there exists* $R_1 > 0$ *such that* $\mathfrak{B} = \{B_t\}_{t\in\mathbb{R}} = \{\mathbb{B}_{\mathscr{U}_t}(R_1)\}_{t\in\mathbb{R}}$ *is a time-dependent absorbing set in* $\mathscr{U}_t$ *for the process* $\{U(t,\tau)\}_{t\ge\tau}$ *corresponding to the problem* (2.3) *and* (2.4).

Proof Multiplying (2.3) by z and repeating the estimates of Theorem 3.2, we can obtain

$$\frac{d}{dt}E_1(t) + \big(1 - \varepsilon'(t)\big)\|u\|_1^2 + 2\gamma_1\int_\Omega |u|^p\,dx + \frac{1}{2}\|u\|_1^2 \le 2\|g\|^2_{-1} + 2\beta_1|\Omega|, \tag{4.1}$$

where

$$E_1(t) = \|u\|^2 + \varepsilon(t)\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}.$$

According to (1.2), (1.3), (4.1), and Poincaré's inequality, we have

$$\frac{d}{dt}E_1(t)+\frac{\varepsilon(t)}{L}\|u\|_1^2+\frac{1}{4}\|u\|_1^2+\frac{\lambda_1}{4}\|u\|^2\leq 2\|g\|_{-1}^2+2\beta_1|\Omega|. \tag{4.2}$$

To reconstruct $E_1(t)$, we assume a new function

$$\Psi_1(t)=\int_0^\infty \kappa(s)\left\|\eta^t(s)\right\|_1^2\,ds. \tag{4.3}$$

We find from (1.8) that

$$\Psi_1(t)\leq \Theta\left\|\eta^t\right\|_{\mu,1}^2\leq \Theta E_1(t). \tag{4.4}$$

In addition, taking the derivative with respect to t at the side of (4.3) and combining with (1.7) and (1.8), we obtain

$$\begin{aligned}\frac{d}{dt}\Psi_1(t)&=-\left\|\eta^t\right\|_{\mu,1}^2+2\int_0^\infty \kappa(s)\left\langle\nabla\eta^t(s),\nabla u(s)\right\rangle ds\\&\leq -\frac{1}{2}\left\|\eta^t\right\|_{\mu,1}^2+2\Theta^2\kappa(0)\|u\|_1^2.\end{aligned} \tag{4.5}$$

Therefore, for fixed $\nu>0$, we definite the function

$$\Phi_1(t)=E_1(t)+\frac{\nu}{8\Theta^2\kappa(0)}\Psi_1(t). \tag{4.6}$$

It follows from (4.2), (4.5), and (4.6) that

$$\frac{d}{dt}\Phi_1(t)+\frac{\varepsilon(t)}{L}\|u\|_1^2+\frac{\lambda_1}{4}\|u\|^2+\frac{\nu}{16\Theta^2\kappa(0)}\left\|\eta^t\right\|_{\mu,1}^2+\frac{1}{4}(1-\nu)\|u\|_1^2\leq 2\|g\|_{-1}^2+2\beta_1|\Omega|.$$

Let $\sigma_1=\min\{\frac{1}{2L},\frac{\lambda_1}{8},\frac{\nu}{32\Theta^2\kappa(0)}\}>0$, then

$$\frac{d}{dt}\Phi_1(t)+2\sigma_1E_1(t)\leq 2\|g\|_{-1}^2+2\beta_1|\Omega|, \tag{4.7}$$

for small enough ν. By (4.6), we also yield

$$E_1(t)\leq\Phi_1(t)\leq 2E_1(t). \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\frac{d}{dt}\Phi_1(t)+\sigma_1\Phi_1(t)\leq 2\|g\|_{-1}^2+2\beta_1|\Omega|. \tag{4.9}$$

By the Gronwall lemma, we obtain

$$\Phi_1(t)\leq e^{-\sigma_1(t-\tau)}\Phi_1(\tau)+\frac{2}{\sigma_1}\left(\|g\|_{-1}^2+\beta_1|\Omega|\right). \tag{4.10}$$

Hence, from (4.8) and (4.10), we conclude that

$$E_1(t) \le 2e^{-\sigma_1(t-\tau)}E_1(\tau) + \frac{2}{\sigma_1}\left(\|g\|_{-1}^2 + \beta_1|\Omega|\right),$$

that is,

$$\|u\|^2 + \varepsilon(t)\|u\|_1^2 + \left\|\eta^t\right\|_{\mu,1}^2 \le R_1$$

for any $t \ge t^* = \tau + \frac{1}{\sigma_1}\ln\frac{4E_1(\tau)}{R_1}$, where $R_1 = \frac{4}{\sigma_1}(\|g\|_{-1}^2 + \beta_1|\Omega|)$.

Therefore, $B_t = \{z = (u,\eta^t) \in \mathscr{U}_t : \|z(t)\|_{\mathscr{U}_t}^2 \le R_1\}$ is a time-dependent absorbing set in $\mathscr{U}_t$ for the solution process $\{U(t,\tau)\}_{t\ge\tau}$. □

We next verify the pullback asymptotically compact for the process $\{U(t,\tau)\}_{t\ge\tau}$ corresponding to the problem (2.3) and (2.4).

Theorem 4.2 *Assume that* (1.2), (1.3), (1.4), *and* (1.8) *hold, then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *of the problem* (2.3) *and* (2.4) *is pullback asymptotically compact in* $\mathscr{U}_t$.

Proof Assume that $z^n = (u^n,\eta^{t,n})$, $z^m = (u^m,\eta^{t,m})$ are two solutions of the problem (2.3) and (2.4) with initial data $z_\tau^n, z_\tau^m \in \mathbb{B}_{\mathscr{U}_\tau}(R_0)$, respectively. Without loss of generality, we assume $\tau \le T_1 < t$ for every fixed T_1. As a convenience, let $w(t) = u^n(t) - u^m(t)$, $\zeta^t = \eta^{t,n} - \eta^{t,m}$, then $(w(t),\zeta^t)$ satisfies the following system

$$\begin{cases} w_t - \varepsilon(t)\triangle w_t - \triangle w - \int_0^\infty \mu(s)\triangle\zeta^t(s)\,ds + f(u^n) - f(u^m) = 0, \\ \zeta_t^t = -\zeta_s^t + w, \quad t \ge \tau, \\ w(x,T_1) = w_{T_1} = u_{T_1}^n - u_{T_1}^m, \qquad \zeta^{T_1} = \eta^{T_1,n} - \eta^{T_1,m}. \end{cases} \tag{4.11}$$

Taking w as a test function for the first equation of (4.11), we can obtain

$$\frac{d}{dt}E_2(t) + \left(1 - \varepsilon'(t)\right)\|w\|_1^2 + \|w\|_1^2 \le 2l\|w\|^2, \tag{4.12}$$

where

$$E_2(t) = \|w\|^2 + \varepsilon(t)\|w\|_1^2 + \left\|\zeta^t\right\|_{\mu,1}^2.$$

Combining with (1.2), (1.3), and Poincaré's inequality, we have

$$\frac{d}{dt}E_2(t) + \frac{\varepsilon(t)}{L}\|w\|_1^2 + \frac{\lambda_1}{2}\|w\|^2 + \frac{1}{2}\|w\|_1^2 \le 2l\|w\|^2. \tag{4.13}$$

Let

$$\Psi_2(t) = \int_0^\infty \kappa(s)\left\|\zeta^t(s)\right\|_1^2\,ds,$$

$$\Phi_2(t) = E_2(t) + \frac{\nu}{4\Theta^2\kappa(0)}\Psi_2(t),$$

then applying the similar arguments used in the proof of Theorem 4.1, we find that

$$\frac{d}{dt}\Phi_2(t) + 2\sigma_2 E_2(t) \le 2l\|w\|^2, \tag{4.14}$$

$$E_2(t) \le \Phi_2(t) \le 2E_2(t), \tag{4.15}$$

where $0 < \sigma_2 = \min\{\frac{1}{2L}, \frac{\lambda_1}{4}, \frac{\nu}{16\Theta^2\kappa(0)}\}$, $\nu > 0$ is small enough. It follows from (4.14) and (4.15) that

$$\frac{d}{dt}\Phi_2(t) + \sigma_2\Phi_2(t) \le 2l\|w\|^2. \tag{4.16}$$

By the Gronwall lemma, we obtain

$$\Phi_2(t) \le e^{-(t-T_1)}\Phi_2(T_1) + 2l\int_{T_1}^t \|w(r)\|^2\,dr. \tag{4.17}$$

Hence, for any $\epsilon > 0$ and some given t, set $t > T_1 \ge \tau$ such that $t - T_1$ is enough large, we can conclude from (4.15) and (4.17) that

$$\begin{aligned} E_2(t) &\le 2e^{-(t-T_1)}E_2(T_1) + \psi_{T_1}^t\big(u_{T_1}^n, u_{T_1}^m\big) \\ &\le \epsilon + \psi_{T_1}^t\big(u_{T_1}^n, u_{T_1}^m\big), \end{aligned} \tag{4.18}$$

where

$$\psi_{T_1}^t\big(u_{T_1}^n, u_{T_1}^m\big) = 2l\int_{T_1}^t \big\|u^n(r) - u^m(r)\big\|^2\,dr.$$

Now, assume that $z^k = (u^k, \eta^{t,k})$ is a solution of the problem (2.3) and (2.4) with initial data $z_\tau^k \in \mathbb{B}_{\mathscr{U}_\tau}(R_0)$, then we find that $u_t^k \in L^2([T_1,t], \mathcal{H}_t)$ and $u^k \in L^2([T_1,t], H_0^1(\Omega))$ for some given t by applying the same arguments of Theorem 3.2. Thereby, we infer from Lemma 2.12 that there exists a convergent subsequence of u^k (denoted as u^{k_i}) such that

$$\lim_{i\to\infty}\lim_{j\to\infty}\psi_{T_1}^t\big(u_{T_1}^n, u_{T_1}^m\big) = 2l\lim_{i\to\infty}\lim_{j\to\infty}\int_{T_1}^t \big\|u^{k_i}(r) - u^{k_j}(r)\big\|^2\,dr = 0, \tag{4.19}$$

which implies that $\psi_{T_1}^t \in \mathfrak{C}(B_{T_1})$. We conclude from (4.18), (4.19), and Theorem 2.8 that

$$\big\|U(t,T_1)u_{T_1}^n - U(t,T_1)u_{T_1}^m\big\| \le \epsilon + \psi_{T_1}^t\big(u_{T_1}^m, u_{T_1}^m\big).$$

This shows that the process $\{U(t,\tau)\}_{t\ge\tau}$ is pullback asymptotic compact in $\mathscr{U}_t$. □

Theorem 4.3 *The process $\{U(t,\tau)\}_{t\ge\tau}$ generated by the problem (2.3) and (2.4) has an invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathscr{U}_t$.*

Proof Combining with Theorem 4.1 and Theorem 4.2, we obtain easily the existence of the invariant time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ for the problem (2.3) and (2.4). □

5 Regularity of the attractor

In this subsection, based on the ideas from [49], we obtain the uniform boundedness (i.e., regularity) of the attractor. Now, we resolve the solution $U(t,\tau)z_\tau = z(t) = (u(t),\eta^t)$ with $z_\tau \in A_\tau$ into the sum

$$U(t,\tau)z_\tau = U_0(t,\tau)z_\tau + U_1(t,\tau)z_\tau,$$

where $U_0(t,\tau)z_\tau = (v(t),\xi^t)$, $U_1(t,\tau)z_\tau = (y(t),\zeta^t)$ solve the following system, respectively,

$$\begin{cases} v_t - \varepsilon(t)\triangle v_t - \triangle v - \int_0^\infty \mu(s)\triangle\xi^t(s)\,ds = g - g^0, \quad x \in \Omega, \\ \xi_t^t = -\xi_s^t + v \\ v(x,t)|_{\partial\Omega} = 0, v(x,\tau) = v_\tau(x), \quad t \geq \tau, \tau \in \mathbb{R}, \\ \xi^t(x,s)|_{\partial\Omega} = 0, \xi^\tau(x,s) = \int_0^s u(x,\tau - r)\,dr, \quad s \in \mathbb{R}^+ \end{cases} \tag{5.1}$$

and

$$\begin{cases} y_t - \varepsilon(t)\triangle y_t - \triangle y - \int_0^\infty \mu(s)\triangle\zeta^t(s)\,ds + f(u) = g^0, \quad x \in \Omega, \\ \zeta_t^t = -\zeta_s^t + y \\ y(x,t)|_{\partial\Omega} = 0, \qquad y(x,\tau) = 0, \quad t \geq \tau, \tau \in \mathbb{R} \\ \zeta^t(x,s)|_{\partial\Omega} = 0, \qquad \zeta^\tau(x,s) = 0, \quad s \in \mathbb{R}^+. \end{cases} \tag{5.2}$$

Note that for every $g \in H^{-1}(\Omega)$ and any $\vartheta > 0$, there exists a $g^0 \in L^2(\Omega)$ such that

$$\|g - g^0\|_{-1} < \vartheta, \tag{5.3}$$

as $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is dense.

Lemma 5.1 *Assume that* (1.2)–(1.8) *hold and* $g \in H^{-1}(\Omega)$, *then*

$$\|U_0(t,\tau)z_\tau\|_{\mathscr{U}_t}^2 \leq 2e^{-\sigma_1(t-\tau)}\|z_\tau\|_{\mathscr{U}_t}^2 + \frac{2\vartheta^2}{\sigma_1}, \tag{5.4}$$

where σ_1 *is given in Theorem* 4.1.

Proof Similar to the proof of Theorem 4.1, we find easily that (5.4) holds. Therefore, the proof is omitted. □

Lemma 5.2 *Assume that* (1.2)–(1.8) *hold and* $g^0 \in L^2(\Omega)$, *then there exists* $R_2 > 0$ *such that*

$$\sup_{t\geq\tau}\|U_1(t,\tau)z_\tau\|_{\mathscr{U}_t^1}^2 \leq R_2.$$

Proof Taking the inner product of $-\triangle y$ with the first equation of (5.2) in $L^2(\Omega)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|y\|_1^2 + \varepsilon(t)\|y\|_2^2 + \|\zeta^t\|_{\mu,2}^2\right) + \left(2 - \varepsilon'(t)\right)\|y\|_2^2 \\ &\quad = -2\langle\zeta^t,\zeta_s^t\rangle_{\mu,2} + 2\langle f(u),\triangle y\rangle + 2\langle g^0,-\triangle y\rangle. \end{aligned} \tag{5.5}$$

From (1.5) and Young's inequality, then

$$\langle f(u), \triangle y\rangle = \langle f(u) - f(0), \triangle y\rangle \le 2l^2\|u\|^2 + \frac{1}{8}\|y\|_2^2, \tag{5.6}$$

$$\left|\langle g^0, -\triangle y\rangle\right| \le 2\|g^0\|^2 + \frac{1}{8}\|y\|_2^2. \tag{5.7}$$

Similar to Lemma 2.10, we also obtain

$$-\langle \zeta^t, \zeta_s^t\rangle_{\mu,2} \le 0. \tag{5.8}$$

By (1.2), (1.3), and (5.5)–(5.8), we have

$$\frac{d}{dt}\left(\|y\|_1^2 + \varepsilon(t)\|y\|_2^2 + \|\zeta^t\|_{\mu,2}^2\right) + \frac{\varepsilon(t)}{L}\|y\|_2^2 + \frac{\lambda_1}{4}\|y\|_1^2 + \frac{1}{4}\|y\|_2^2 \le 4l^2\|u\|^2 + 4\|g^0\|^2.$$

Set

$$\Psi_1(t) = \int_0^\infty \kappa(s)\|\zeta^t(s)\|_{\mu,2}^2\,ds,$$

then it is easy to obtain that

$$\|y\|_1^2 + \varepsilon(t)\|y\|_2^2 + \|\zeta^t\|_{\mu,2}^2 \le \frac{4}{\sigma_1}\left(l^2R_1 + \|g^0\|^2\right),$$

by using the same discussion of Theorem 4.1. Further, we have

$$\sup_{t\ge\tau}\|U_1(t,\tau)z_\tau\|_{\mathscr{U}_t^1}^2 \le R_2,$$

where σ_1 is given in Theorem 4.1,

$$R_2 = \frac{4}{\sigma_1}\left(l^2R_1 + \|g^0\|^2\right). \qquad \square$$

Theorem 5.3 *Assume that* (1.2)–(1.8) *hold and* $g \in H^{-1}(\Omega)$, *then* $\{A_t\}_{t\in\mathbb{R}}$ *is bounded in* $\mathscr{U}_t^1$.

Proof Thanks to Lemma 5.1 and Lemma 5.2, then for any $t \in \mathbb{R}$, we have

$$\operatorname{dist}_{\mathscr{U}_t}\left(A_t, \mathbb{B}_{\mathscr{U}_t^1}(R_2)\right) = \operatorname{dist}_{\mathscr{U}_t}\left(U(t,\tau)A_\tau, \mathbb{B}_{\mathscr{U}_t^1}(R_2)\right) \le Ce^{-\sigma_3(t-\tau)} \to 0, \quad \tau \to -\infty,$$

where $\sigma_3 > 0$,

$$\mathbb{B}_{\mathscr{U}_t^1}(R_2) = \left\{z(t) \in \mathscr{U}_t^1 : \|z(t)\|_{\mathscr{U}_t^1}^2 \le R_2\right\}.$$

Thereby, the above conclusion holds. $\square$

6 Fractal dimension of the attractor

In what follows, in order to study the attractor further, we discuss the fractal dimension of the time-dependent global attractor by Lemma 6.1 of [36].

We decompose the solution $U(t,\tau)z_\tau = z(t) = (u,\eta^t)$ with initial value $z_\tau = (u_\tau,\eta^\tau) \in \mathscr{U}_\tau$ into the sum

$$U(t,\tau)z_\tau = D(t,\tau)z_\tau + K(t,\tau)z_\tau,$$

then the corresponding solutions are $U(t,\tau)z_\tau^i = z^i(t)$ for any initial value $z_\tau^i \in \mathscr{U}_\tau$ $(i = 1,2)$, respectively. Now, we split the solution $z^1 - z^2$ with initial value $z_\tau^1 - z_\tau^2$ into the following form

$$U(t,\tau)z_\tau^1 - U(t,\tau)z_\tau^2 = D(t,\tau)z_\tau^1 - D(t,\tau)z_\tau^2 + K(t,\tau)z_\tau^1 - K(t,\tau)z_\tau^2,$$

where $(\tilde{v},\tilde{\zeta}^t) = D(t,\tau)z_\tau^1 - D(t,\tau)z_\tau^2$ and $(\tilde{w},\tilde{\xi}^t) = K(t,\tau)z_\tau^1 - K(t,\tau)z_\tau^2$ are the solutions of the following problem, respectively,

$$\begin{cases} \tilde{v}_t - \varepsilon(t)\triangle\tilde{v}_t - \triangle\tilde{v} - \int_0^\infty \triangle\mu(s)\tilde{\zeta}^t(s)\,ds = 0, \\ \zeta_t^t = -\zeta_s^t + \tilde{v}, \\ (\tilde{v}_\tau,\tilde{\zeta}^\tau) = z_\tau^1 - z_\tau^2, \qquad \tilde{v}|_{\partial\Omega} = 0, \qquad \tilde{\zeta}|_{\partial\Omega\times\mathbb{R}^+} = 0 \end{cases} \tag{6.1}$$

and

$$\begin{cases} \tilde{w}_t - \varepsilon(t)\triangle\tilde{w}_t - \triangle\tilde{w} - \int_0^\infty \triangle\mu(s)\tilde{\xi}^t(s)\,ds + f(u^1) - f(u^2) = 0, \\ \tilde{\xi}_t^t = -\tilde{\xi}_s^t + \tilde{w}, \\ (\tilde{w}_\tau,\tilde{\xi}^\tau) = 0, \qquad \tilde{w}|_{\partial\Omega} = 0, \qquad \tilde{\xi}|_{\partial\Omega\times\mathbb{R}^+} = 0. \end{cases} \tag{6.2}$$

Lemma 6.1 *Assume that the process* $\{U(t,\tau)\}_{t\geq\tau}$ *is broken down into*

$$U(t,\tau) = D(t,\tau) + K(t,\tau), \qquad z_\tau \in A_\tau, \tau \leq t,$$

then for every $\tau \in \mathbb{R}$, $z_\tau^1, z_\tau^2 \in A_\tau$ *there exists* $t_\star > 0$ *such that*

$$\left\|D(\tau + t_\star,\tau)z_\tau^1 - D(\tau + t_\star,\tau)z_\tau^2\right\|_{\mathscr{U}_{\tau+t_\star}}^2 \leq \rho\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2, \tag{6.3}$$

$$\left\|K(\tau + t_\star,\tau)z_\tau^1 - K(\tau + t_\star,\tau)z_\tau^2\right\|_{\mathscr{Z}_{\tau+t_\star}^1}^2 \leq Q_{t_\star}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2, \tag{6.4}$$

where $Q_{t_\star} > 0$ *only depends on* $t_\star$ *and* $0 \leq \rho < \frac{1}{4}$.

Proof Taking the inner product of $\tilde{v}$ with the first equation of (6.1) in $L^2(\Omega)$, we arrive at

$$\frac{d}{dt}\left(\|\tilde{v}\|^2 + \varepsilon(t)\|\tilde{v}\|_1^2 + \left\|\tilde{\zeta}^t\right\|_{\mu,1}\right) + \frac{\varepsilon(t)}{L}\|\tilde{v}\|_1^2 + \|\tilde{v}\|_1^2 \leq 0.$$

Similar to the proof of Theorem 4.2, we infer that

$$\|\tilde{v}\|^2 + \varepsilon(t)\|\tilde{v}\|_1^2 + \left\|\tilde{\zeta}^t\right\|_{\mu,1} \leq 2e^{-\sigma_2(t-\tau)}\left(\|\tilde{v}_\tau\|^2 + \varepsilon(\tau)\|\tilde{v}_\tau\|_1^2 + \left\|\tilde{\zeta}^\tau\right\|_{\mu,1}\right),$$

where σ_2 is given in Theorem 4.2. That is,

$$\left\|D(t,\tau)z_\tau^1 - D(t,\tau)z_\tau^2\right\|_{\mathscr{U}_t}^2 \le e^{-\sigma_2(t-\tau)}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2.$$

For any $z_\tau^1, z_\tau^2 \in A_\tau$, choose $t_\star = \frac{3\ln 2}{\sigma_2} > 0$ such that

$$\left\|D(\tau+t_\star,\tau)z_\tau^1 - D(\tau+t_\star,\tau)z_\tau^2\right\|_{\mathscr{U}_{\tau+t_\star}}^2 \le \frac{1}{8}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2.$$

Thereby, (6.3) holds.

Taking the inner product of $-\triangle\tilde{w}$ with the first equation of (6.2) in $L^2(\Omega)$, we have

$$\frac{d}{dt}\left(\|\tilde{w}\|_1^2 + \varepsilon(t)\|\tilde{w}\|_2^2 + \left\|\tilde{\xi}^t\right\|_{\mu,2}^2\right) + \frac{\varepsilon(t)}{L}\|\tilde{w}\|_2^2 + \frac{1}{2}\|\tilde{w}\|_1^2 \le 2l^2\left\|u^1 - u^2\right\|^2. \tag{6.5}$$

Combining with (1.3), (6.5), and the continuous dependence of the solution on the initial value, we obtain

$$\frac{d}{dt}\left(\|\tilde{w}\|_1^2 + \varepsilon(t)\|\tilde{w}\|_2^2 + \left\|\tilde{\xi}^t\right\|_{\mu,2}^2\right) \le 2l^2\left\|z^1 - z^2\right\|_{\mathscr{U}_t}^2 \le 2l^2 e^{2l(t-\tau)}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2. \tag{6.6}$$

Integrating both sides of (6.6) from τ to t, we see that

$$\|\tilde{w}\|_1^2 + \varepsilon(t)\|\tilde{w}\|_2^2 + \left\|\tilde{\xi}^t\right\|_{\mu,2}^2 \le le^{2l(t-\tau)}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2, \tag{6.7}$$

for any $z_\tau^i \in \mathbb{B}_{\mathscr{U}_\tau}(R_0)$. Hence,

$$\left\|K(t,\tau)z_\tau^1 - K(t,\tau)z_\tau^2\right\|_{\mathscr{U}_t^1}^2 \le le^{2l(t-\tau)}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2. \tag{6.8}$$

Thanks to Lemma 3.3 and Lemma 3.4 from [13], then

$$\left\|\tilde{\xi}_s^t\right\|_{\mathcal{M}}^2 + xT_{\tilde{\xi}^t}(x) \le C, \tag{6.9}$$

where C is a positive constant. By (6.8) and (6.9), we have

$$\begin{aligned}
&\left\|K(t,\tau)z_\tau^1 - K(t,\tau)z_\tau^2\right\|_{\mathscr{Z}_t^1}^2 \\
&\quad\le \left\|K(t,\tau)z_\tau^1 - K(t,\tau)z_\tau^2\right\|_{\mathscr{U}_t^1}^2 + \left\|\tilde{\xi}_s^t\right\|_{\mathcal{M}}^2 + xT_{\tilde{\xi}^t}(x) \\
&\quad\le Cle^{2l(t-\tau)}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_\tau}^2.
\end{aligned} \tag{6.10}$$

It follows from (6.10) that for any $z_\tau^1, z_\tau^2 \in A_\tau$ there exists $t_\star > 0$ such that

$$\left\|K(\tau+t_\star,\tau)z_\tau^1 - K(\tau+t_\star,\tau)z_\tau^2\right\|_{\mathscr{Z}_{\tau+t_\star}^1}^2 \le Cle^{2lt_\star}\left\|z_\tau^1 - z_\tau^2\right\|_{\mathscr{U}_t}^2, \tag{6.11}$$

where $Q_\star = Cle^{2lt_\star}$. Hence, (6.11) holds. □

Theorem 6.2 *Assume that (1.2), (1.3), (1.5), (1.7), and (1.8) hold, then the time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ has a finite fractal dimension in $\mathscr{U}_t$, i.e.,*

$$\dim_{\mathscr{U}_t} A_t \le \log_2 \kappa_{(8Q_\star)^{-1}},$$

where

$$\dim_X Y = \limsup_{q\to 0^+} \frac{\log_2 N_q(X,Y)}{\log_2 \frac{1}{q}},$$

$N_q(X,Y)$ is the minimum number of the q-ball of Y that covers K.

Proof Repeating the similar proof of Lemma 5.2 for (6.5), we can obtain that $\|\tilde{\xi}^t\|^2_{\mu,1}$ is bounded. Then, combining with (6.9) and Lemma 2.11, we find that $\mathcal{L}^2 \hookrightarrow\hookrightarrow \mathcal{M}^1$. Hence, $\mathscr{Z}^1_{\tau+t_\star} \hookrightarrow\hookrightarrow \mathscr{U}_{\tau+t_\star}$. Then, we conclude from Lemma 6.1 ([36]) that

$$\dim_{\mathscr{U}_t} A_t \leq \frac{\log_2 \kappa_{\rho Q_\star^{-1}}}{\log_2(4\rho)^{-1}} = \log_2 \kappa_{(8Q_\star)^{-1}},$$

where $\kappa_q = \sup_{t\geq 0} N_q(\mathbb{B}_{\mathscr{Z}^1_t}(1), Y) < \infty$. □

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