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The existence, uniqueness, and stability results for a nonlinear coupled system using ψ-Caputo fractional derivatives

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Abstract

In this article, we use coupled boundary conditions on a nonlinear system with ψ-Caputo fractional derivatives to derive new conclusions on the solution's existence, uniqueness, and stability. We use the well-known tools of fixed-point theory to establish the proposed results. We give an example to verify the theoretical findings. The proposed existence, uniqueness, and stability analyses considering the ψ-Caputo fractional derivative are the novelty of this article.

Keywords: ψ-Caputo derivative; Fractional boundary value problem; Existence and uniqueness; Hyers–Ulam stability

1 Introduction

Differential and integral operators are highly useful operators to deal with physical problems [1–3]. Nowadays, fractional calculus [4–7] is one of the highly emerging fields of applied mathematics. Several applications of fractional-order operators (integrals and derivatives) have been recorded in various branches of science and engineering [8–12]. The literature on fractional calculus is increasing day-by-day with theoretical and application-oriented simulations.

Boundary value problems with fractional-order operators are also of interest to several researchers. Fractional derivatives and integrals incorporate memory in the system, which makes a physical phenomenon more realistic than the classical ones. There are several studies proposed to prove the existence, uniqueness, and stability of the solutions of various types of fractional boundary value problems (FBVPs) in terms of different fractional derivatives.

In [13], the authors derive existence results for nonlinear impulsive hybrid FBVPs. In [14], *Erturk et al.* proposed the existence and stability analysis for FBVPs. In [15], the authors derived existence, uniqueness, and stability analyses of the generalized Caputo-type FBVPs. In [16], *Zhang and Liu* provided the existence and Hyers–Ulam stability for a class of FBVPs on a star graph. In [17], an iterative two-point FBVP was considered to prove the existence, uniqueness, and Hyers–Ulam stability. The authors in [18] analyzed an Ulam–Hyers–Rassias-stable multiorder FBVP of the generalized Caputo type with mixed integroderivative conditions at four points. In [19], the authors proved the stability of a coupled system of nonlinear implicit antiperiodic FBVP. In [20], the authors derived the Ulam–Hyers–Rassias stability of a Caputo-type FBVP. In [21], the authors used a nonlinear fractional differential equation for a coupled system of boundary value problems. In [22], the analyses of the existence and stability of a coupled system of implicit-type impulsive FBVPs were proposed. In [23], *Khan et al.* proposed Ulam-type stability for a coupled system of nonlinear FBVPs. In [24], the authors analyzed the FBVPs with a ψ-Caputo fractional derivative.

In [25], the authors derived novel existence and uniqueness results for the following Caputo-type nonlinear coupled system with a new kind of coupled boundary conditions:

$$\begin{cases} {}^{c}D^{\rho}l(x) = \eta(x, l(x), m(x)), & x \in J := [0, \varpi], \\ {}^{c}D^{\tau}m(x) = \zeta(x, l(x), m(x)), & x \in J := [0, \varpi], \\ (l+m)(0) = -(l+m)(\varpi), \qquad \int_{\varrho}^{\varsigma}(l-m)(g)\,dg = B, & 0 < \varrho < \varsigma < \varpi. \end{cases} \tag{1}$$

In this case, operator ${}^{C}D^{\Delta}$ is the $\Delta \in \{\rho, \tau\}$-order Caputo fractional derivative, where $\rho, \tau \in (0, 1]$. B is a nonnegative constant and $\eta, \zeta : [0, \varpi] \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions.

The existence and stability of solutions in terms of a ψ-Caputo-type derivative is still not studied for the FBVP (1). Therefore, we deal with the existence and Hyers–Ulam stability of the solution for the following ψ-Caputo-type FBVP:

$$\begin{cases} {}^{c}D^{(\rho,\psi(x))}l(x) = \eta(x, l(x), m(x)), & x \in J := [0, \varpi], \\ {}^{c}D^{(\tau,\psi(x))}m(x) = \zeta(x, l(x), m(x)), & x \in J := [0, \varpi], \\ (l+m)(0) = -(l+m)(\varpi), \qquad \int_{\varrho}^{\varsigma}(l-m)(g)\,dg = B, & 0 < \varrho < \varsigma < \varpi. \end{cases} \tag{2}$$

In this case, the operator ${}^{C}D^{\Delta,\psi}$ is the $\Delta \in \{\rho, \tau\}$-order ψ-Caputo fractional derivative and $\psi(x)$ is the unknown function, where $\rho, \tau \in (0, 1]$. B is a nonnegative constant and $\eta, \zeta : [0, \varpi] \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions.

The scientific contribution of this manuscript is to extend the dynamics of the proposed Caputo-type BVP into ψ-Caputo fractional-order sense because fractional derivatives contain memory in the system that defines real-life dynamics in a better way. The rest of this article is organized as follows: in Sect. 2, the necessary definitions and lemmas are provided. In Sect. 3, we derive the existence and uniqueness analysis on the proposed ψ-Caputo-type FBVP (2). In Sect. 4, the Hyers–Ulam stability of the proposed system is derived. In Sect. 5, we verify the proposed theoretical results by implementing them on a problem. Finally, we conclude our observations in Sect. 6.

2 Preliminaries

Here, we recall the following preliminaries:

Definition 1 [26] For, $\rho > 0$, we have a function u, which is integrable on $[\nu_1, \nu_2]$ along with an increasing differentiable function $\psi \in C^n[\nu_1, \nu_2]$ such that $\psi'(x) \neq 0$ for every $x \in [\nu_1, \nu_2]$. The left-sided ψ-Riemann–Liouville integral of order ρ of the function u is given by,

$$I_{\nu_1^+}^{\rho,\psi} u(x) = \frac{1}{\Gamma(\rho)} \int_{\nu_1}^{x} \psi'(g)\big(\psi(x) - \psi(g)\big)^{\rho-1} u(g)\,dg,$$

with $\Gamma(.)$ being the gamma function.

Definition 2 [26] Assume that, $n-1 < \rho < n$, and $u:[\nu_1,\nu_2] \to \mathbb{R}$ is a function that can be integrated and let ψ be defined according to Definition 1. The left-sided fractional derivative of order ρ of the function h in the ψ-Riemann–Liouville sense is defined by:

$$D_{\nu_1^+}^{\rho,\psi} u(x) = \left[\frac{1}{\psi'(x)} \frac{d}{dx}\right]^n I_{\nu_1^+}^{n-\rho,\psi} u(x),$$

with $n = [\rho + 1]$, along with $[\rho]$ denoting the integer component of a real number.

Definition 3 [26] Consider $n-1 < \rho < n$, the function $u \in C^{n-1}[\nu_1,\nu_2]$ with ψ from the preceding definition 1, then the left-side Caputo fractional derivative of ρth order of the function u is given by:

$${}^{c}D^{\rho,\psi} u(x) = D_{\nu_1^+}^{\rho,\psi} \left\{ u(x) - \sum_{\delta=0}^{n-1} \frac{u_\psi^{[\delta]}(\nu_1)}{\delta!} \big(\psi(x) - \psi(\nu_1)\big)^{\delta} \right\},$$

with $u_\psi^{[\delta]}(x) = [\frac{1}{\psi'(x)} \frac{d}{dx}]^{\delta} u(x)$, $n = \rho$ for $\rho \in \mathbb{N}$, and $n = [\rho] + 1$ for $\rho \notin \mathbb{N}$, respectively. In addition, if $u \in C^n[\nu_1,\nu_2]$ and $\rho \notin \mathbb{N}$, then

$$\begin{aligned} D_{\nu_1^+}^{\rho,\psi} u(x) &= I_{\nu_1^+}^{n-\rho,\psi} \left[\frac{1}{\psi'(x)} \frac{d}{dx}\right]^n u(x) \\ &= \frac{1}{\Gamma(n-\rho)} \int_{\nu_1}^{x} \psi'(g)\big(\psi(x) - \psi(g)\big)^{n-\rho-1} u_\psi^{[n]}(g)\,dg. \end{aligned}$$

Also, if $\rho = n \in \mathbb{N}$, one has

$${}^{c}D_{\nu_1^+}^{\rho,\psi} u(x) = u_\psi^{[n]}(x).$$

Lemma 1 [26] *Assume that* $u:[\nu_1,\nu_2] \to \mathbb{R}$ *and* $\rho > 0$. *Then, the following assertions hold*:

(i) *If* $u \in C[\nu_1,\nu_2]$, *subsequently* ${}^{c}D_{\nu_1^+}^{\rho,\psi} I_{\nu_1^+}^{\rho,\psi} u(x) = u(x)$;

(ii) *If* $u \in C^{n-1}[\nu_1,\nu_2]$, *subsequently* $I_{\nu_1^+}^{\rho,\psi}\, {}^{c}D_{\nu_1^+}^{\rho,\psi} u(x) = u(x) - \sum_{\delta=0}^{n-1} C_\delta [\psi(x) - \psi(\nu_1)]^{\delta}$, *where* $C_\delta = \frac{u_\psi^{[\delta]}(a)}{\delta!}$.

Lemma 2 [26] *Let* $u,\psi \in C[\nu_1,\nu_2]$ *and* $\rho > 0$. *Then*,

(i) $I_{\nu_1^+}^{\rho,\psi}(.)$ *is bounded linearly from* $C[\nu_1,\nu_2]$ *to* $C[\nu_1,\nu_2]$;

(ii) $I_{\nu_1^+}^{\rho,\psi} u(\nu_1) = \lim_{x \to \nu_1^+} u(x) = 0$.

Lemma 3 [26] *Let us consider the mapping* $u:[\nu_1,\nu_2] \to \mathbb{R}$ *and* $\rho,\tau > 0$. *Then*,

(i) $I_{\nu_1^+}^{\rho,\psi} [\psi(x) - \psi(\nu_1)]^{\tau-1} = \frac{\Gamma(\tau)}{\Gamma(\rho+\tau)} [\psi(x) - \psi(\nu_1)]^{\rho+\tau-1}$;

(ii) ${}^{c}D_{\nu_1^+}^{\rho,\tau} [\psi(x) - \psi(\nu_1)]^{\tau-1} = \frac{\Gamma(\tau)}{\Gamma(\tau-\rho)} [\psi(x) - \psi(\nu_1)]^{\tau-\rho-1}$;

(iii) ${}^{c}D^{\rho,\tau}_{\nu_1^+}[\psi(x)-\psi(\nu_1)]^{\delta}=0$, $\forall\delta\in\{0,1,\ldots,n-1\}$, *where* $n\in\mathbb{N}$;

(iv) $I^{\rho,\psi}_{\nu_1^+}I^{\tau,\psi}_{\nu_1^+}u(x)=I^{\rho+\tau}_{\nu_1^+}u(x)$.

Lemma 4 *Let* $H,Z\in C[0,\varpi]$ *and* $l,m\in BC(J)$. *Then, this coupled linear system*

$$\begin{cases} {}^{c}D^{\rho,\psi(x)}l(x)=H(x), & x\in J:=[0,\varpi],\\ {}^{c}D^{\tau,\psi(x)}m(x)=Z(x), & x\in J:=[0,\varpi],\\ (l+m)(0)=-(l+m)(\varpi), \qquad \int_{\varrho}^{\varsigma}(l-m)(g)\,dg=B, & \end{cases} \tag{3}$$

has the following solution:

$$\begin{aligned} l(x)={}&\frac{1}{2}\biggl\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\biggl(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg\\ &+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\biggr)\\ &-\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\biggl(\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}Z(\mu)\,d\mu\\ &-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}H(\mu)\,d\mu\biggr)\,dg\biggr\}\\ &-\int_0^{x}\frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg, \end{aligned} \tag{4}$$

$$\begin{aligned} m(x)={}&\frac{1}{2}\biggl\{-\frac{B}{\varsigma-\varrho}+\frac{1}{2}\biggl(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg\\ &+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\biggr)\\ &+\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\biggl(\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}Z(\mu)\,d\mu\\ &-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}H(\mu)\,d\mu\biggr)\,dg\biggr\}\\ &-\int_0^{x}\frac{\psi'(g)(\psi(x)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg. \end{aligned} \tag{5}$$

Proof To solve the FDE in (3), we apply the operators I^{ρ} and I^{τ} to both sides of the equations, respectively, and using Lemma 1, we obtain

$$l(x)=c_0-\frac{1}{\Gamma(\rho)}\int_0^{x}\psi'(g)\bigl(\psi(x)-\psi(g)\bigr)^{\rho-1}H(g)\,dg, \tag{6}$$

$$m(x)=c_1-\frac{1}{\Gamma(\tau)}\int_0^{x}\psi'(g)\bigl(\psi(x)-\psi(g)\bigr)^{\tau-1}Z(g)\,dg. \tag{7}$$

Using the boundary conditions of the problem (3) in (6) and (7), respectively, we obtain the values of $c_0,c_1\in\mathbb{R}$.

$$c_0+c_1=\frac{1}{2}\biggl(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg \tag{8}$$

$$+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\Bigg),$$

$$c_0-c_1=\frac{1}{\varsigma-\varrho}\Bigg(B-\int_{\varrho}^{\varsigma}\Bigg\{\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}Z(\mu)\,dg$$
$$-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}H(\mu)\,d\mu\Bigg\}\Bigg). \tag{9}$$

Solving (8) and (9) together for c_0 and c_1, it is found that

$$c_0=\frac{1}{2}\Bigg\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\Bigg(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg$$
$$+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\Bigg)$$
$$-\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\Bigg(\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}Z(\mu)\,d\mu$$
$$-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}H(\mu)\,d\mu\Bigg)\,dg\Bigg\},$$

$$c_1=\frac{1}{2}\Bigg\{-\frac{B}{\varsigma-\varrho}+\frac{1}{2}\Bigg(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg$$
$$+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\Bigg)$$
$$+\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\Bigg(\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}Z(\mu)\,d\mu$$
$$-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}H(\mu)\,d\mu\Bigg)\,dg\Bigg\}.$$

In (6) and (7), substituting the values of c_0 and c_1 yields the solutions of (3). The converse of this lemma can be computed directly. The proof is concluded. □

3 Existence and uniqueness analysis

$\xi = C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ represents the Banach space endowed with the norm $\|(l,m)\| = \|l\| + \|m\| = \sup_{x\in[0,1]}|l(x)| + \sup_{x\in[0,1]}|m(x)|$, for $(l,m)\in\xi$. In light of Lemma 4, we introduce the following operator $\Pi:\xi\to\xi$ for the problem (2) using Eqs. (4) and (5):

$$\Pi(l,m)(x) := \big(\Pi_1(l,m)(x),\Pi_2(l,m)(x)\big), \tag{10}$$

$$\Pi_1(l,m)(x)=\frac{1}{2}\Bigg\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\Bigg(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\big(g,l(g),m(g)\big)\,dg \tag{11}$$
$$+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\big(g,l(g),m(g)\big)\,dg\Bigg)$$
$$-\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\Bigg(\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\zeta\big(\mu,l(\mu),m(\mu)\big)\,d\mu$$
$$-\int_0^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\eta\big(\mu,l(\mu),m(\mu)\big)\,d\mu\Bigg)\,dg\Bigg\}$$

$$
\begin{aligned}
&- \int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)} \zeta\big(g, l(g), m(g)\big)\, dg, \\
\Pi_2(l,m)(x) = {}& \frac{1}{2}\bigg\{ -\frac{B}{\varsigma-\varrho} + \frac{1}{2}\bigg(\int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)} \eta\big(g, l(g), m(g)\big)\, dg \\
&+ \int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)} \zeta\big(g, l(g), m(g)\big)\, dg \bigg) \\
&+ \frac{1}{\varsigma-\varrho} \int_{\varrho}^{\varsigma} \bigg(\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)} \zeta\big(\mu, l(\mu), m(\mu)\big)\, d\mu \\
&- \int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)} \eta\big(\mu, l(\mu), m(\mu)\big)\, d\mu \bigg)\, dg \bigg\} \\
&- \int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\tau-1}}{\Gamma(\tau)} \zeta\big(g, l(g), m(g)\big)\, dg.
\end{aligned}
\tag{12}
$$

Next, we present the assumptions required to establish the paper's primary findings.

(M1) Existence of nonnegative functions $\varkappa_i, \kappa_i \in C([0,1], \mathbb{R}^+)$, $i = 1,2,3$, that are continuous such that

$$
\begin{aligned}
&\big|\eta(x,l,m)\big| \le \varkappa_1(x) + \varkappa_2(x)|l| + \varkappa_3(x)|m|, \quad \forall (x,l,m) \in J \times \mathbb{R}^2, \\
&\big|\zeta(x,l,m)\big| \le \kappa_1(x) + \kappa_2(x)|l| + \kappa_3(x)|m|, \quad \forall (x,l,m) \in J \times \mathbb{R}^2.
\end{aligned}
$$

(M2) There exist positive constants U_i, V_i, $i = 1,2$, such that

$$
\begin{aligned}
&\big|\eta(x,l,h_1) - \eta(x,m,h_2)\big| \le U_1|l-m| + U_2|h_1-h_2|, \quad \forall x \in J, U_i, V_i \in \mathbb{R}, i = 1,2, \\
&\big|\zeta(x,l,h_1) - \zeta(x,m,h_2)\big| \le V_1|l-m| + V_2|h_1-h_2|, \quad \forall x \in J, U_i, V_i \in \mathbb{R}, i = 1,2.
\end{aligned}
$$

We propose the following notation for computational convenience.

$$
\sigma_1 = \frac{(\psi(\varpi)-\psi(0))^{\rho}}{4\Gamma(\rho+1)} + \frac{(\psi(\varsigma)-\psi(0))^{(\rho+1)}\psi'(\varrho) - \psi'(\varsigma)(\psi(\varrho)-\psi(0))^{(\rho+1)}}{2(\varsigma-\varrho)(\rho+2)\psi'(\varrho)\psi'(\varsigma)}, \tag{13}
$$

$$
\sigma_2 = \frac{(\psi(\varpi)-\psi(0))^{\tau}}{4\Gamma(\tau+1)} + \frac{(\psi(\varsigma)-\psi(0))^{\tau+1}\psi'(\varrho) - \psi'(\varsigma)(\psi(\varrho)-\psi(0))^{(\tau+1)}}{2(\varsigma-\varrho)(\tau+2)\psi'(\varrho)\psi'(\varsigma)} \tag{14}
$$

and

$$
\begin{aligned}
\omega = \min\bigg\{ & 1 - \bigg(\|\varkappa_2\| \bigg[2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)} \bigg] + \|\kappa_2\| \bigg[2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)} \bigg] \bigg), \\
& 1 - \bigg(\|\varkappa_3\| \bigg[2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)} \bigg] + \|\kappa_3\| \bigg[2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)} \bigg] \bigg) \bigg\}.
\end{aligned}
$$

To derive the first existence proof for the problem (2), we use the following fixed-point theorem [27].

Lemma 5 *In the Banach space A, let the operator $H : A \to A$ be completely continuous and assume the set $\Omega = \{\mu \in A | \mu = \varkappa H\mu; 0 < \varkappa < 1\}$ is bounded. Consequently, H holds a fixed point in A.*

Theorem 1 *Let $\eta, \zeta : J \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions, satisfying the assumption $(M1)$. Then, there exists at least one solution on J for problem (2).*

$$\|\varkappa_2\|\left[2\sigma_1 + \frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)}\right] + \|\kappa_2\|\left[2\sigma_2 + \frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)}\right] < 1, \tag{15}$$

$$\|\varkappa_3\|\left[2\sigma_1 + \frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)}\right] + \|\kappa_3\|\left[2\sigma_2 + \frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)}\right] < 1, \tag{16}$$

where σ_i $(i = 1, 2)$ are defined by (13) and (14)

Proof We have an operator $\Pi : \xi \to \xi$ determined by (10) to be completely continuous. Clearly, Π is a continuous mapping between two subsets of ξ that are bounded and compact, respectively. Note that the continuity of functions η and ζ implies that the operator $\Pi : \xi \to \xi$ is continuous. Let $\Omega_{\bar{s}} \subset \xi$ be bounded. Then, there exist positive constants I_η, I_ζ such that

$$\left|\eta\big(x, l(x), m(x)\big)\right| \le I_\eta, \qquad \left|\zeta\big(x, l(x), m(x)\big)\right| \le I_\zeta, \quad \forall (l, m) \in \Omega_{\bar{s}}.$$

Hence, for any $(l, m) \in \Omega_{\bar{s}}$, $x \in J$, we obtain

$$\begin{aligned}
\left|\Pi_1(l,m)(x)\right| &\le I_\eta\left(\frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)} + \sigma_1\right) + I_\zeta\sigma_2 + \frac{B}{\varsigma - \varrho}, \\
\left|\Pi_2(l,m)(x)\right| &\le I_\zeta\left(\frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)} + \sigma_2\right) + \frac{B}{\varsigma - \varrho} + I_\eta\sigma_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|\Pi(l,m)\right\| &= \left\|\Pi_1(l,m)\right\| + \left\|\Pi_2(l,m)\right\| \\
&\le I_\eta\left(\frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)} + 2\sigma_1\right) + I_\zeta\left(\frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)} + 2\sigma_2\right) + \frac{2B}{\varsigma - \varrho}.
\end{aligned}$$

The operator Π is consequently uniformly bounded from the inequality given above. Let $x_1, x_2 \in [0, \varpi]$, $x_1 < x_2$, and $(l, m) \in \Omega_{\bar{s}}$ to show the mapping of Π from bounded sets to equicontinuous sets of ξ. Then,

$$\begin{aligned}
&\left|\Pi_1(l,m)(x_2) - \Pi_1(l,m)(x_1)\right| \\
&\quad\le \left|\frac{\psi'(g)}{\Gamma(\rho)}\left(\int_0^{x_1}\left[\big(\psi(x_2) - \psi(g)\big)^{\rho-1} - \big(\psi(x_1) - \psi(g)\big)^{\rho-1}\right]\right)\right. \\
&\qquad\left.\times\, \eta\big(g, l(g), m(g)\big)\, dg + \int_{x_1}^{x_2}\big(\psi(x_2) - \psi(g)\big)^{\rho-1}\eta\big(g, l(g), m(g)\big)\, dg\right| \\
&\quad\le I_\eta\left(\frac{2(\psi(x_2 - \psi(x_1)))^\rho + (\psi(x_2) - \psi(0))^\rho - (\psi(x_1) - \psi(0))^\rho}{\Gamma(\rho+1)}\right).
\end{aligned}$$

In a similar manner, we can obtain

$$\left|\Pi_2(l,m)(x_2) - \Pi_2(l,m)(x_1)\right|$$

$$\leq I_\zeta\left(\frac{2(\psi(x_2)-\psi(x_1))^\tau+(\psi(x_2)-\psi(0))^\tau-(\psi(x_1)-\psi(0))^\tau}{\Gamma(\tau+1)}\right).$$

In the limit of $x_1 \to x_2$, the right-hand sides of the above inequality tend to zero based on (l,m). As a result, $\Pi : \xi \to \xi$ is completely continuous, as shown by the Arzelá–Ascoli theorem.

Next, we prove the set $\vartheta = \{(l,m) \in \xi \,|\, (l,m) = \lambda\Pi(l,m), 0 < \lambda < 1\}$ is bounded. Let $(l,m) \in \vartheta$, then $(l,m) = \lambda\Pi(l,m)$, $0 < \lambda < 1$. For any $x \in J$, we have

$$l(x) = \lambda\Pi_1(l,m)(x), \qquad m(x) = \lambda\Pi_2(l,m)(x).$$

Using σ_i $(i = 1,2)$ given by (13) and (14), we find that

$$\begin{aligned}
\left|l(x)\right| &= \lambda\left|\Pi_1(l,m)(x)\right| \\
&\leq \left(\|\varkappa_1\| + \|\varkappa_2\|\|l\| + \|\varkappa_3\|\|m\|\right)\left(\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)} + \sigma_1\right) \\
&\quad + \left(\|\kappa_1 + \|\kappa_2\|\,\|l\| + \|\kappa_3\|\|m\|\right)\sigma_2 + \frac{B}{\varrho - \varsigma}, \\
\left|m(x)\right| &= \lambda\left|\Pi_2(l,m)(x)\right| \\
&\leq \left(\|\varkappa_1\| + \|\varkappa_2\|\|l\| + \|\varkappa_3\|\|m\|\right)\sigma_1 \\
&\quad + \left(\|\kappa_1\| + \|\kappa_2\|\|l\| + \|\kappa_3\|\|m\|\right)\left(\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)} + \sigma_2\right) + \frac{B}{\varsigma - \varrho}.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&\|l\| + \|m\| \\
&\leq \|\varkappa_1\|\left(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right) + \|\kappa_1\|\left(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}\right) + \frac{2B}{\varsigma-\varrho} \\
&\quad + \left[\|\varkappa_3\|\left(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right) + \|\kappa_3\|\left(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right)\right]\|m\| \\
&\quad + \left[\|\varkappa_2\|\left(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right) + \|\kappa_2\|\left(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau)}\right)\right]\|l\|.
\end{aligned}$$

Thus, by condition (15) and (16), we obtain

$$\left\|(l,m)\right\| \leq \frac{\|\varkappa_1\|(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}) + \|\kappa_1\|(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}) + \frac{2B}{\varsigma-\varrho}}{\omega},$$

which proves $\|(l,m)\|$ is bounded. Since $x \in J$, we conclude the set ϑ is bounded. From Lemma 5, the conclusion holds, and there is at least one fixed point of the operator Π, which solves the problem (2). □

Suppose $\varkappa_2(x) = \varkappa_3(x) \equiv 0$ and $\kappa_2(x) = \kappa_3(x) \equiv 0$. In this case, we obtain the following specific form of Theorem 1:

Corollary 1 *Consider two continuous functions $\eta, \zeta : J \times \mathbb{R}^2 \to \mathbb{R}$ and assume there exist nonnegative continuous functions $\varkappa_1, \kappa_1 \in C([0,\varpi], \mathbb{R}^+)$, such that*

$$\left|\eta(x,l,m)\right| \le \varkappa_1(x), \qquad \left|\zeta(x,l,m)\right| \le \kappa_1(x), \quad \forall (x,l,m) \in J \times \mathbb{R}^2.$$

Thus, the set J has at least one solution to problem (2).

Corollary 2 *In the Theorem 1 statement, if $\varkappa_i(t) = \lambda_i$, $\kappa_i(t) = \xi_i$, $i = 1,2,3$ (λ_i, ξ_i are examined positive constants), then the functions' conditions η, ζ yield the form:*
(M_1')

$$\left|\eta(x,l,m)\right| \le \lambda_1 + \lambda_2|l| + \lambda_3|m|, \quad \forall (x,l,m) \in J \times \mathbb{R}^2,$$
$$\left|\eta(x,l,m)\right| \le \xi_1 + \xi_2|l| + \xi_3|m|, \quad \forall (x,l,m) \in J \times \mathbb{R}^2$$

and (15) and (16) become

$$\lambda_2\left(2\sigma_1 + \frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)}\right) + \xi_2\left(2\sigma_2 + \frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)}\right) < 1,$$
$$\lambda_3\left(2\sigma_1 + \frac{(\psi(\varpi) - \psi(0))^\rho}{\Gamma(\rho+1)}\right) + \xi_3\left(2\sigma_2 + \frac{(\psi(\varpi) - \psi(0))^\tau}{\Gamma(\tau+1)}\right) < 1.$$

Applying Banach's contraction mapping principle, we express the existence of a unique solution for problem (2) in the next result.

Theorem 2 *Consider two continuous functions $\eta, \zeta : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and the presumption $(M2)$ holds. Subsequently, the problem (2) has a unique solution on J if*

$$\mathcal{U}\Lambda_\eta + \mathcal{V}\Lambda_\zeta < 1, \tag{17}$$

where $\mathcal{U} = \max\{\mathcal{U}_1, \mathcal{U}_2\}$, $\mathcal{V} = \max\{V_1, V_2\}$, $\Lambda_\eta = (\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)} + 2\sigma_1)$ and $\Lambda_\zeta = (2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)})$, and σ_i, $i = 1,2$ are defined by (13) and (14).

Proof Let us consider the operator $\Pi : \xi \to \xi$ defined by (10) and fix

$$r > \frac{Q_1(\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)} + 2\sigma_1) + Q_2(\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)} + 2\sigma_2)}{1 - (\mathcal{U}(\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)} + 2\sigma_1) + \mathcal{V}(\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)} + 2\sigma_2))},$$

where $Q_1 = \sup_{x\in[0,\varpi]} |\eta(x,0,0)|$, and $Q_2 = \sup_{x\in[0,\varpi]} |\zeta(x,0,0)|$. Then, we show that $\Pi A_r \subset A_r$, where $A_r = \{(l,m) \in \xi : \|(l,m)\| \le t\}$. For $(l,m) \in A_r$, we have

$$\begin{aligned}
&\left|\Pi_1(l,m)(x)\right| \\
&\quad \le \frac{1}{2}\left\{\frac{B}{\varsigma - \varrho} + \frac{1}{2}\left(\int_0^\varpi \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\rho-1}}{\Gamma(\rho)}\left[\left|\eta\big(g, l(g), m(g)\big) - \eta(g,0,0)\right| + Q_1\right] dg\right.\right. \\
&\qquad + \left.\int_0^\varpi \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\tau-1}}{\Gamma(\tau)}\left[\left|\zeta\big(g, l(g), m(g)\big) - \zeta(g,0,0)\right| + Q_2\right] dg\right) \\
&\qquad + \frac{1}{\varsigma - \varrho}\int_\varrho^\varsigma
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\left[\left|\zeta(\mu,l(\mu),m(\mu))-\zeta(\mu,0,0)\right|+Q_2\right]d\mu\right.\\
&\left.+\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\left[\left|\eta(\mu,l(\mu),m(\mu))-\eta(\mu,0,0)\right|+Q_1\right]d\mu\right)dg\Bigg\}\\
&+\int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\left[\left|\eta(g,l(g),m(g))-\eta(g,0,0)\right|+Q_1\right]dg\\
\le{}&\left(\mathcal{U}\left(\frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)}+\sigma_1\right)+\mathcal{V}\sigma_2\right)(\|l\|+\|m\|)\\
&+Q_1\left(\frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)}+\sigma_1\right)+Q_2\sigma_2.
\end{aligned}
$$

Hence, when taking the norm for $x\in J$, we arrive at

$$
\begin{aligned}
\left\|\Pi_1(l,m)\right\| \le{}&\left(\mathcal{U}\left(\frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)}+\sigma_1\right)+\mathcal{V}\sigma_2\right)(\|l\|+\|m\|)\\
&+Q_1\left(\frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)}+\sigma_1\right)+Q_2\sigma_2.
\end{aligned}
\tag{18}
$$

Similar to this, given $(l,m)\in A_r$, one can obtain

$$
\begin{aligned}
\left\|\Pi_2(l,m)\right\| \le{}&\left(\mathcal{U}\sigma_1+\mathcal{V}\left(\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+\sigma_2\right)\right)(\|l\|+\|m\|)\\
&+Q_2\left(\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+\sigma_2\right)+Q_1\sigma_1.
\end{aligned}
$$

Consequently, we have for any $(l,m)\in A_r$

$$
\begin{aligned}
\left\|\Pi(l,m)\right\| ={}&\left\|\Pi_1(l,m)\right\|+\left\|\Pi_2(l,m)\right\|\\
\le{}&\left(\mathcal{U}\left(\frac{(\psi(\varpi)-\psi(0))^{\rho}}{\Gamma(\rho+1)}+2\sigma_1\right)+\mathcal{V}\left(\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+2\sigma_2\right)\right)(\|l\|+\|m\|)\\
&+Q_1\left(\frac{(\psi(\varpi)-\psi(0)^{\rho}}{\Gamma(\rho+1)}+2\sigma_1\right)+Q_2\left(\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+2\sigma_2\right)<r,
\end{aligned}
$$

proving that Π maps A_r into itself.

Let $(l_1,m_1),(l_2,m_2)\in\xi$, $x\in[0,1]$ to prove the operator Π is a contraction. Then, in view of $(M2)$, we have

$$
\begin{aligned}
&\left|\Pi_1(l_1,m_1)(x)-\Pi_1(l_2,m_2)(x)\right|\\
&\le\frac{1}{2}\left\{\frac{1}{2}\left(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\left|\eta(g,l_1(g),m_1(g))-\eta(g,l_2(g),m_2(g))\right|dg\right.\right.\\
&\quad\left.+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\left|\zeta(g,l_1(g),m_1(g))-\zeta(g,l_2(g),m_2(g))\right|dg\right)\\
&\quad+\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\left(\int_0^g\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\right.\\
&\quad\times\left|\zeta(\mu,l_1(\mu),m_1(\mu))-\zeta(\mu,l_2(\mu),m_2(\mu))\right|d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\big|\eta\big(\mu,l_1(\mu),m_1(\mu)\big)-\eta\big(\mu,l_2(\mu),m_2(\mu)\big)\big|\,d\mu\bigg)\,dg\bigg\}\\
&+\int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\big|\eta\big(g,l_1(g),m_1(g)\big)-\eta\big(g,l_2(g),m_2(g)\big)\big|\,dg\\
&\leq \bigg(\mathcal{U}\bigg(\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}+\sigma_1\bigg)+\mathcal{V}\sigma_2\bigg)\big(\|l\|+\|m\|\big).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\big|\Pi_2(l_1,m_1)(x)-\Pi_2(l_2,m_2)(x)\big|\\
&\leq \frac{1}{2}\bigg\{\frac{1}{2}\bigg(\int_0^\varpi \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\big|\eta\big(g,l_1(g),m_1(g)\big)-\eta\big(g,l_2(g),m_2(g)\big)\big|\,dg\\
&+\int_0^\varpi \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\big|\zeta\big(g,l_1(g),m_1(g)\big)-\zeta\big(g,l_2(g),m_2(g)\big)\big|\bigg)\\
&+\frac{1}{\varsigma-\varrho}\int_\varrho^\varsigma\bigg(\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\\
&\times\big|\zeta\big(\mu,l_1(\mu),m_1(\mu)\big)-\zeta\big(\mu,l_2(\mu),m_2(\mu)\big)\big|\,d\mu\\
&+\int_0^g \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\big|\eta\big(\mu,l_1(\mu),m_1(\mu)\big)-\eta\big(\mu,l_2(\mu),m_2(\mu)\big)\big|\,d\mu\bigg)\,dg\bigg\}\\
&+\int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\big|\zeta\big(g,l_1(g),m_1(g)\big)-\zeta\big(g,l_2(g),m_2(g)\big)\big|\,dg\\
&\leq \bigg(\mathcal{U}\sigma_1+\mathcal{V}\bigg(\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}+\sigma_2\bigg)\bigg)\big(\|l\|+\|m\|\big).
\end{aligned}
$$

It follows from the preceding inequalities that

$$
\begin{aligned}
&\big\|\Pi(l_1,m_1)(x)-\Pi(l_2,m_2)(x)\big\|\\
&=\big\|\Pi_1(l_1,m_1)-\Pi_1(l_2,m_2)\big\|+\big\|\Pi_2(l_1,m_1)-\Pi_2(l_2,m_2)\big\|\\
&\leq\bigg\{\mathcal{U}\bigg(\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}+2\sigma_1\bigg)+\mathcal{V}\bigg(\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}+2\sigma_2\bigg)\bigg\}\\
&\quad\times\big\|(l_1-l_2,m_1-m_2)\big\|.
\end{aligned}
$$

In this manner, Π is a contraction mapping based on (17). As a result, according to the Banach contraction mapping, Π possesses a unique fixed point. Hence, we may conclude that there is unique solution to problem (2) on J. The proof is concluded. □

4 Hyers–Ulam stability of the coupled system

Definition 4 If there exist positive constants $F_i > 0$ $(i = 1,2)$, then the coupled system of Hammerstein-type integral equations is Hyers–Ulam stable. Also, the following assertions hold: for $\Phi_i > 0$, $i = 1,2$, if

$$
\begin{aligned}
&\bigg|l(x)-\frac{1}{2}\bigg\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\bigg(\int_0^\varpi \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}H(g)\,dg\\
&+\int_0^\varpi \frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}Z(g)\,dg\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \biggl(\int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\tau-1}}{\Gamma(\tau)} Z(\mu)\, d\mu \\
&- \int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\rho-1}}{\Gamma(\rho)} H(\mu)\, d\mu \biggr)\, dg \biggr\} \\
&+ \int_0^x \frac{\psi'(g)(\psi(x) - \psi(g))^{\rho-1}}{\Gamma(\rho)} H(g)\, dg \biggr| \le \Phi_1,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\biggl| m(x) - \frac{1}{2} \biggl\{ - \frac{B}{\varsigma - \varrho} + \frac{1}{2} \biggl(\int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\rho-1}}{\Gamma(\rho)} H(g)\, dg \\
&+ \int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\tau-1}}{\Gamma(\tau)} Z(g)\, dg \biggr) \\
&+ \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \biggl(\int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\tau-1}}{\Gamma(\tau)} Z(\mu)\, d\mu \\
&- \int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\rho-1}}{\Gamma(\rho)} H(\mu)\, d\mu \biggr)\, dg \biggr\} \\
&+ \int_0^x \frac{\psi'(g)(\psi(x) - \psi(g))^{\tau-1}}{\Gamma(\tau)} H(g)\, dg \biggr| \le \Phi_2,
\end{aligned}
\tag{20}
$$

then there exist $(l^*(x), m^*(x))$, satisfying

$$
\begin{aligned}
l^*(x) &= \frac{1}{2} \biggl\{ \frac{B}{\varsigma - \varrho} + \frac{1}{2} \biggl(\int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\rho-1}}{\Gamma(\rho)} H(g)\, dg \\
&\quad + \int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\tau-1}}{\Gamma(\tau)} Z(g)\, dg \biggr) \\
&\quad - \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \biggl(\int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\tau-1}}{\Gamma(\tau)} Z(\mu)\, d\mu \\
&\quad - \int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\rho-1}}{\Gamma(\rho)} H(\mu)\, d\mu \biggr)\, dg \biggr\} \\
&\quad - \int_0^x \frac{\psi'(g)(\psi(x) - \psi(g))^{\rho-1}}{\Gamma(\rho)} H(g)\, dg, \\
m^*(x) &= \frac{1}{2} \biggl\{ - \frac{B}{\varsigma - \varrho} + \frac{1}{2} \biggl(\int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\rho-1}}{\Gamma(\rho)} H(g)\, dg \\
&\quad + \int_0^{\varpi} \frac{\psi'(g)(\psi(\varpi) - \psi(g))^{\tau-1}}{\Gamma(\tau)} Z(g)\, dg \biggr) \\
&\quad + \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \biggl(\int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\tau-1}}{\Gamma(\tau)} Z(\mu)\, d\mu \\
&\quad - \int_0^g \frac{\psi'(\mu)(\psi(g) - \psi(\mu))^{\rho-1}}{\Gamma(\rho)} H(\mu)\, d\mu \biggr)\, dg \biggr\} \\
&\quad - \int_0^x \frac{\psi'(g)(\psi(x) - \psi(g))^{\tau-1}}{\Gamma(\tau)} H(g)\, dg,
\end{aligned}
$$

such that

$$
\bigl| l(x) - l^*(x) \bigr| \le F_1 \Phi_1, \quad x \in [0, \varpi],
\tag{21}
$$

$$\left|m(x)-m^{*}(x)\right| \leq F_{2}\Phi_{2}, \quad x \in [0,\varpi]. \tag{22}$$

Based on our observations, Hyers–Ulam stability of the solution to the problem is determined in this section.

Theorem 3 *By the assumption that $\eta,\zeta:[0,1]\times\mathbb{R}^{2}\to\mathbb{R}$ are continuous functions and there exist constants U_i, V_i, $i=1,2$ such that for all $x\in[0,\varpi]$ and $l,m,h_1,h_2\in\mathbb{R}$,*

$$\left|\eta(x,l,h_{1})-\eta(x,m,h_{2})\right| \leq U_{1}|l-m|+U_{2}|h_{1}-h_{2}|, \quad \forall x\in J, U_{i}\in\mathbb{R}, i=1,2, \tag{23}$$

$$\left|\zeta(x,l,h_{1})-\zeta(x,m,h_{2})\right| \leq V_{1}|l-m|+V_{2}|h_{1}-h_{2}|, \quad \forall x\in J, V_{i}\in\mathbb{R}, i=1,2 \tag{24}$$

system (2) *is Hyers–Ulam stable.*

Proof Consider $(l(x),m(x))$ to be the exact solution and $(l^{*}(x),m^{*}(x))$ to be any other solution of system (3) according to Theorem 2 and Definition 4. Following that, using (6) and (7), we have

$$\begin{aligned}
\left|l(x)-l^{*}(x)\right| \leq{}& \left|\frac{1}{2}\left\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\left(\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(g,l(g),m(g)\bigr)\,dg\right.\right.\right.\\
&+\left.\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\bigl(g,l(g),m(g)\bigr)\,dg\right)\\
&-\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\left(\int_{0}^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\zeta\bigl(\mu,l(\mu),m(\mu)\bigr)\,d\mu\right.\\
&-\left.\left.\int_{0}^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(\mu,l(\mu),m(\mu)\bigr)\,d\mu\right)dg\right\}\\
&-\int_{0}^{x}\frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(g,l(g),m(g)\bigr)\,dg\\
&-\frac{1}{2}\left\{\frac{B}{\varsigma-\varrho}+\frac{1}{2}\left(\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(g,l^{*}(g),m^{*}(g)\bigr)\,dg\right.\right.\\
&+\left.\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\bigl(g,l^{*}(g),m^{*}(g)\bigr)\,dg\right)\\
&-\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\left(\int_{0}^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\zeta\bigl(\mu,l^{*}(\mu),m^{*}(\mu)\bigr)\,d\mu\right.\\
&-\left.\left.\int_{0}^{g}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(\mu,l^{*}(\mu),m^{*}(\mu)\bigr)\,d\mu\right)dg\right\}\\
&+\left.\int_{0}^{x}\frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\bigl(g,l^{*}(g),m^{*}(g)\bigr)\,dg\right|\\
\leq{}& \frac{1}{4}\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\\
&\times\bigl[\bigl|\eta\bigl(g,l(g),m(g)\bigr)-\eta\bigl(g,l^{*}(g),m^{*}(g)\bigr)\bigr|\bigr]\,dg\\
&+\frac{1}{4}\int_{0}^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\\
&\times\bigl[\bigl|\zeta\bigl(g,l(g),m(g)\bigr)-\zeta\bigl(g,l^{*}(g),m^{*}(g)\bigr)\bigr|\bigr]\,dg
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2(\varsigma-\varrho)}\int_{\varrho}^{\varsigma}\left\{\int_0^r \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\right.\\
&\left.\times\left[\left|\zeta\left(\mu,l(\mu),m(\mu)\right)-\zeta\left(\mu,l^*(\mu),m^*(\mu)\right)\right|\right]d\mu\right\}dg\\
&+\frac{1}{2(\varsigma-\varrho)}\int_{\varrho}^{\varsigma}\left\{\int_0^r \frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\right.\\
&\left.\times\left[\left|\eta\left(\mu,l(\mu),m(\mu)\right)-\eta\left(\mu,l^*(\mu),m^*(\mu)\right)\right|d\mu\right\}dg\\
&+\int_0^x \frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}dg\left[\left|\eta\left(g,l(g),m(g)\right)-\eta\left(g,l^*(g),m^*(g)\right)\right|\right]\\
\le{}&\Theta_\eta\left\{\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}+2\sigma_1\right\}\left[\left|l(x)-l^*(x)\right|+\left|m(x)-m^*(x)\right|\right],
\end{aligned}
\tag{25}
$$

which implies that

$$
\begin{aligned}
\left\|l-l^*\right\| &\le \Theta_\eta\left\{\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}+2\sigma_1\right\}\times\left[\left\|l(x)-l^*(x)\right\|+\left\|m(x)-m^*(x)\right\|\right]\\
&\le F_1\Phi_1,
\end{aligned}
\tag{26}
$$

where $F_1=\{\frac{(\psi(\varpi)-\psi(0))^{(\rho)}\Theta_\eta}{\Gamma(\rho+1)}+2\sigma_1\}$.

Similarly, we further have

$$
\begin{aligned}
\left|m(x)-m^*(x)\right| \le{}& \left|\frac{1}{2}\left\{-\frac{B}{\varsigma-\varrho}+\frac{1}{2}\left(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\left(g,l(g),m(g)\right)dg\right.\right.\right.\\
&\left.+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\left(g,l(g),m(g)\right)dg\right)\\
&+\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\left(\int_0^g\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\zeta\left(\mu,l(\mu),m(\mu)\right)d\mu\right.\\
&\left.\left.-\int_0^g\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\eta\left(\mu,l(\mu),m(\mu)\right)d\mu\right)dg\right\}\\
&-\int_0^x\frac{\psi'(g)(\psi(x)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\eta\left(g,l(g),m(g)\right)ds\\
&-\frac{1}{2}\left\{-\frac{B}{\varsigma-\varrho}+\frac{1}{2}\left(\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\eta\left(g,l^*(g),m^*(g)\right)dg\right.\right.\\
&\left.+\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\left(g,l^*(g),m^*(g)\right)dg\right)\\
&+\frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma}\left(\int_0^g\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\zeta\left(\mu,l^*(\mu),m^*(\mu)\right)d\mu\right.\\
&\left.\left.-\int_0^g\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\eta\left(\mu,l^*(\mu),m^*(\mu)\right)d\mu\right)dg\right\}\\
&\left.+\int_0^x\frac{\psi'(g)(\psi(x)-\psi(g))^{\tau-1}}{\Gamma(\tau)}\zeta\left(g,l^*(g),m^*(g)\right)dg\right|\\
\le{}&\frac{1}{4}\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\rho-1}}{\Gamma(\rho)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \big[\big|\eta\big(g,l(g),m(g)\big)-\eta\big(g,l^*(g),m^*(g)\big)\big|\big]\,dg \\
&+\frac{1}{4}\int_0^{\varpi}\frac{\psi'(g)(\psi(\varpi)-\psi(g))^{\tau-1}}{\Gamma(\tau)} \\
&\times \big[\big|\zeta\big(g,l(g),m(g)\big)-\zeta\big(g,l^*(g),m^*(g)\big)\big|\big]\,dg \\
&+\frac{1}{2(\varsigma-\varrho)}\int_{\varrho}^{\varsigma}\left\{\int_0^{r}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\tau-1}}{\Gamma(\tau)}\right. \\
&\left.\times \big[\big|\zeta\big(\mu,l(\mu),m(\mu)\big)-\zeta\big(\mu,l^*(\mu),m^*(\mu)\big)\big|\big]\,d\mu\right\}dg+\frac{1}{2(\varsigma-\varrho)} \\
&\times \int_{\varrho}^{\varsigma}\left\{\int_0^{r}\frac{\psi'(\mu)(\psi(g)-\psi(\mu))^{\rho-1}}{\Gamma(\rho)}\right. \\
&\left.\times \big[\big|\eta\big(\mu,l(\mu),m(\mu)\big)-\eta\big(\mu,l^*(\mu),m^*(\mu)\big)\big|\,d\mu\right\}dg \\
&+\int_0^{x}\frac{\psi'(g)(\psi(x)-\psi(g))^{\rho-1}}{\Gamma(\rho)}\,ds \\
&\times \big[\big|\eta\big(g,l(g),m(g)\big)-\eta\big(g,l^*(g),m^*(g)\big)\big|\big] \\
\le{}& \Theta_{\zeta}\left\{\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+2\sigma_2\right\}\big[\big|l(x)-l^*(x)\big|+\big|m(x)-m^*(x)\big|\big], \qquad (27)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|m-m^*\| &\le \Theta_{\zeta}\left\{\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+2\sigma_2\right\}\times\big[\|l(x)-l^*(x)\|+\|m(x)-m^*(x)\|\big] \\
&\le F_2\Phi_2, \qquad (28)
\end{aligned}
$$

where $F_2=\Theta_{\zeta}\{\frac{(\psi(\varpi)-\psi(0))^{\tau}}{\Gamma(\tau+1)}+2\sigma_2\}$.

As a result, considering (25) and (27), the system of integral equations (3) is Hyers–Ulam stable, and as a result, the solution of system (2) is also Hyers–Ulam stable. □

5 Illustrative example

Here, we verify the proposed theoretical results by implementing them on a problem.

Example 1 Consider the following problem

$$
\begin{cases}
{}^{c}D^{(2/3:x^2)}l(x)=\eta(x,l,m), & x\in[0,2],\\
{}^{c}D^{(4/5:x^2)}l(x)=\zeta(x,l,m), & x\in[0,2],\\
(l+m)(0)=-(l+m)(2), \quad \int_{1/2}^{3/2}(l-m)(r)\,dg=3,
\end{cases}
\qquad (29)
$$

where $\rho=2/3$, $\tau=4/5$, $\psi(x)=x^2$, $\varrho=1/2$, $\varsigma=3/2$, $B=3$, $\varpi=2$, and $\eta(x,l,m)$ and $\zeta(x,l,m)$ will be fixed later.

The data we have available lead us to the conclusion that $\sigma_1=0.9881225695$, $\sigma_2=1.000470521$, where σ_1 and σ_2 are in (13) and (14).

To illustrate Theorem 1, we take $\eta(x,l,m)=\frac{e^{-x}}{2(\sqrt{6+x^2})}(xl+m+\cos t)$ and $\zeta(x,l,m)=\frac{1}{(2+x)^2}(l+tm+e^{-t})$.

It is clear that η and ζ are continuous and satisfy the condition $(M1)$ with

$$\varkappa_1(x) = \frac{\cos xe^{-x}}{2\sqrt{(6+x^2)}}; \qquad \varkappa_2(x) = \frac{xe^{-x}}{2\sqrt{(6+x^2)}}; \qquad \varkappa_3(x) = \frac{e^{-x}}{2\sqrt{(6+x^2)}};$$
$$\kappa_1 = \frac{e^{-x}}{(2+x)^2}; \qquad \kappa_2 = \frac{x}{(2+x)^2}; \qquad \kappa_3 = \frac{1}{(2+x)^2}.$$

Furthermore, we have

$$\|\varkappa_2\|\left(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right) + \|\kappa_2\|\left(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}\right) \approx 0.8689$$

and

$$\|\varkappa_3\|\left(2\sigma_1 + \frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)}\right) + \|\kappa_3\|\left(2\sigma_2 + \frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)}\right) \approx 0.5336.$$

Therefore, the criterion provided in Theorem 1 applies, and the problem (29) has a unique solution with $\eta(x,l,m)$ and $\zeta(x,l,m)$ given by (30) and (31).

To illustrate the application of Theorem 2, we choose

$$\begin{aligned} \eta\big(x,l(x),m(x)\big) &= \frac{1}{18(1+x^2)}\left\{\frac{l}{1+|l|} + \tan^{-1}(m)\right\}, \\ \zeta\big(x,l(x),m(x)\big) &= \frac{1}{2\sqrt{6+x^2}}\big(2\tan^{-1}(l) + \sin(m)\big). \end{aligned} \tag{30}$$

Then,

$$\begin{aligned} \big|\eta(x,l,h_1) - \eta(x,m,h_2)\big| &\le \frac{1}{40}|l-m| + \frac{1}{30}|h_1-h_2|, \\ \big|\zeta(x,l,h_1) - \zeta(x,m,h_2)\big| &\le \frac{1}{30}|l-m| + \frac{1}{40}|h_1-h_2|. \end{aligned} \tag{31}$$

Therefore, $(M2)$ holds and we have $\mathcal{U} = 0.0333$; $\mathcal{V} = 0.0333$.

Then,

$$\begin{aligned} &\max(\mathcal{U}_1,\mathcal{U}_2)\Lambda_\eta + \max(\mathcal{V}_1,\mathcal{V}_2)\Lambda_\zeta \\ &\quad= 0.0333\left\{\frac{(\psi(\varpi)-\psi(0))^\rho}{\Gamma(\rho+1)} + 2\sigma_1\right\} + 0.0333\left\{\frac{(\psi(\varpi)-\psi(0))^\tau}{\Gamma(\tau+1)} + 2\sigma_2\right\} \\ &\quad= 0.1544 + 0.1788 \\ &\quad= 0.3333 \\ &\quad< 1. \end{aligned}$$

Consequently, the coupled system (29) has a unique solution and is Hyers–Ulam stable, as it satisfies all the conditions of Theorem 2.

6 Conclusion

We have derived some novel results on the existence, uniqueness, and stability of the solution for a nonlinear coupled system with ψ-Caputo fractional derivatives. We have used the features of fixed-point theory to establish the proposed results. We have provided an illustrative example to verify the theoretical findings. The proposed analyses of existence, uniqueness, and stability in terms of the ψ-Caputo fractional derivative are novel and provide further insights into the theory of coupled FBVPs. In the future, the results will be helpful to check the qualitative behavior of the proposed types of problems. This work can be extended to any other fractional derivative.

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