# A coupled complex mKdV equation and its N -soliton solutions via the Riemann-Hilbert approach 

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#### Abstract

This paper concerns the initial value problem of a coupled complex mKdV (CCMKDV) equations $$
\begin{aligned} & u_{t}+u_{x x x}+6\left(|u|^{2}+|v|^{2}\right) u_{x}+6 u\left(|v|^{2}\right)_{x}=0, \\ & v_{t}+v_{x x x}+6\left(|u|^{2}+|v|^{2}\right) v_{x}+6 v\left(|u|^{2}\right)_{x}=0, \end{aligned}
$$ proposed by Yang (Nonlinear Waves in Integrable and Nonintegrable Systems, 2010), which is associated with a $4 \times 4$ scattering problem. Based on matrix spectral analysis, a fourth-order matrix Riemann-Hilbert problem is formulated. By solving a specific nonregular Riemann-Hilbert problem with zeros, we present the N -soliton solutions for the CCMKDV system. Moreover, the single-soliton solutions are displayed graphically.


Keywords: Riemann-Hilbert approach; N-soliton solutions; Lax pair; Scattering data

## 1 Introduction

In nonlinear wave theory, integrable nonlinear equations play an important role, these include the KdV equation, nonlinear Schrödinger equation, Sine-Gordon equation, modified KdV equation, etc. These integrable systems often have a rich mathematical structure, such as the Lax pair formulation and an abundance of conservation laws. To explore the exact solutions of these models, a bulk of effective methods have been developed, such as the inverse scattering transform (IST) method [2, 3], the Hirota bilinear method [4], the Bäcklund transformation method [5], the Dressing method [6], etc. In the initial version of the IST method, one has to solve the Gel'fand-Levitan-Marchenko (GLM) integral equations, which is not an easy task. Later, a simplified version of IST was developed, known as the Riemann-Hilbert (RH) approach [1, 7]. Thereafter, many researchers utilized the RH approach to find the soliton solutions of some physically important integrable partial differential equations [8-18]. In recent years, the RH approach has also proved to be applicable to study the asymptotic behavior of solutions to initial boundary value problem

[^0]of some integrable nonlinear equations [19-23]. For the recent construction of soliton solutions concerning high-order poles by the RH approach, see [24, 25].
The mKdV equation
\[

$$
\begin{equation*}
u_{t}+u_{x x x}+6|u|^{2} u_{x}=0 \tag{1}
\end{equation*}
$$

\]

is an important integrable model in physics, which is initially proposed to describe the acoustic wave in some anharmonic lattices and also the Alfven wave in the cold collision-free plasma [26]. Subsequently, the multi-solitons, conserved qunatities, algebrogeometric solutions, rogue waves for the mKdV equation, and some coupled mKdV equations have been studied by many researchers via generalized Darboux transformation method, IST method, the Hirota bilinear method, and other methods [11, 12, 27-31]. The researchers also investigated the initial-boundary value problem for a coupled mKdV equations by use of the Fokas unified transform method [21, 22, 32], and the perturbation theory for the vector mKdV equation [33]. Recently, the long-time asymptotic property for the coupled mKdV equation was studied by the nonlinear steepest descent method [34]. For the classifications of the solition solutions, see [35].
The main purpose in this paper is to study a CCMKDV system [1]

$$
\begin{align*}
& u_{t}+u_{x x x}+6\left(|u|^{2}+|v|^{2}\right) u_{x}+6 u\left(|v|^{2}\right)_{x}=0  \tag{2}\\
& u_{t}+u_{x x x}+6\left(|u|^{2}+|v|^{2}\right) v_{x}+6 v\left(|u|^{2}\right)_{x}=0
\end{align*}
$$

in the RH formulation, where $u=u(x, t), v=v(x, t)$ represent complex field envelopes. The terms involving $u_{x x x}$ and $v_{x x x}$ account for dispersive effects, which influence the spread of wave packets over time. Meanwhile, the coupling terms describe interactions between the two wave components $u(x, t)$ and $v(x, t)$. It is easy to see that when $v=0$, this system reduces to the $m K d V$ equation (1). The soliton solutions for a vectorial $m K d V$ system have been studied by the Hirota bilinear method [36] and the IST method [29]. Some other coupled mKdV systems have also been investigated using the RH formulation [30, 31], but we note that this CCMKDV System (2) cannot be covered by [30, 31]. As far as we know, the RH problem for the CCMKDV System (2) has not been studied before.

This paper is arranged as follows. In Sect. 2, we first deal with the $4 \times 4$ Lax pair of the CCMKDV System (2), after some spectral analysis, we present the analytical property of the Jost solutions for the spectral equation of x-part. In Sect. 3, we shall formulate the corresponding RH problem for this CCMKDV system. In Sect. 4, we shall solve the RH problem with simple zeros. By restricting to the reflectionless case and reconstructing the potentials, we will construct the N -soliton solutions of Eq. (2). By choosing suitable parameters, we shall graphically show the behavior of single-soliton solutions for the CCMKDV system. The last section dealsThis paper is arranged as follows. In Sect. 2, we treat firstly with the $4 \times 4$ Lax pair of the CCMKDV System (2), after some spectral analysis, we present the analytical property of the Jost solutions for the spectral equation of x-part. In Sect. 3, we shall formulate the corresponding RH problem for this CCMKDV system. In Sect. 4, we shall solve the RH problem with simple zeros, and by restricting to the reflectionless case and reconstructing the potentials, the N -soliton solutions of of Eq. (2) will be constructed. By choosing suitable parameters, we shall graphically show the behavior of single-soliton solutions for the CCMKDV system. The last section is concerned with the conclusions. with the conclusions.

## 2 Spectral analysis

In this section, we focus on the scattering problem for the CCMKDV System (2) and study its matrix Jost solutions.

### 2.1 Spectral problem

As is pointed out by Yang [1], the CCMKDV System (2) is Lax integrable in the following sense

$$
\begin{align*}
& Y_{x}=U Y=(-i \omega \sigma+\tilde{U}) Y  \tag{3}\\
& Y_{t}=V Y=\left(-4 i \omega^{3} \sigma+\tilde{V}\right) Y \tag{4}
\end{align*}
$$

where $\omega$ is a spectral parameter and $Y(x, t, \omega)$ is a $4 \times 1$ matrix function. The matrices $\sigma$, $\tilde{U}, Q, \tilde{V}$ are defined as follows

$$
\begin{array}{ll}
\sigma=\left(\begin{array}{cc}
I_{2} & 0 \\
0 & -I_{2}
\end{array}\right), & \tilde{U}=\left(\begin{array}{cc}
0 & Q \\
-Q^{\dagger} & 0
\end{array}\right), \\
Q=\left(\begin{array}{cc}
u & v \\
v^{*} & u^{*}
\end{array}\right), & \tilde{V}=\left(\begin{array}{cc}
\tilde{V}_{11} & \tilde{V}_{12} \\
\tilde{V}_{21} & \tilde{V}_{22}
\end{array}\right),
\end{array}
$$

in which

$$
\begin{aligned}
& \tilde{V}_{11}=2 i \omega Q Q^{\dagger}-Q_{x} Q^{\dagger}+Q Q_{x}^{\dagger}, \\
& \tilde{V}_{12}=4 \omega^{2} Q+2 i \omega Q_{x}-Q_{x x}-2 Q Q^{\dagger} Q, \\
& \tilde{V}_{21}=-4 \omega^{2} Q^{\dagger}+2 i \omega Q_{x}^{\dagger}+Q_{x x}^{\dagger}+2 Q^{\dagger} Q Q^{\dagger}, \\
& \tilde{V}_{22}=-2 i \omega Q^{\dagger} Q+Q^{\dagger} Q_{x}-Q_{x}^{\dagger} Q,
\end{aligned}
$$

where $\dagger$ represents the Hermitian conjugate of a matrix. It is easy to find that $\tilde{U}$ satisfies the following symmetry conditions

$$
\begin{equation*}
\tilde{U}^{\dagger}=-\tilde{U} \tag{5}
\end{equation*}
$$

In what follows, we always assume that the potential functions satisfy the zero boundary conditions, that is

$$
u(x, t) \rightarrow 0, \quad v(x, t) \rightarrow 0, \quad x \rightarrow \pm \infty
$$

To study the localized solutions for Eq. (2), we simply introduce a new matrix spectral function

$$
\begin{equation*}
J(x, t, \omega)=Y(x, t, \omega) e^{i \omega \sigma x+4 i \omega^{3} \sigma t} \tag{6}
\end{equation*}
$$

Then, (3) and (4) can be rewritten as

$$
\begin{equation*}
J_{x}=-i \omega[\sigma, J]+\tilde{U} J, \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
J_{t}=-4 i \omega^{3}[\sigma, J]+\tilde{V} J, \tag{8}
\end{equation*}
$$

where $[\sigma, J]=\sigma J-J \sigma$.
In the scattering problem, we firstly investigate the Jost solutions $J_{ \pm}(x, \omega)$ of Eq. (7) satisfying

$$
\begin{align*}
& J_{+}(x, \omega) \rightarrow \mathrm{I}, \quad x \rightarrow+\infty,  \tag{9}\\
& J_{-}(x, \omega) \rightarrow \mathrm{I}, \quad x \rightarrow-\infty,
\end{align*}
$$

where I denotes the $4 \times 4$ unit matrix, the subscripts in $J_{ \pm}$indicate at which end of the x -axis the boundary conditions are set. It is well known that

$$
\begin{equation*}
(\operatorname{det} J)_{x}=\operatorname{det} J \cdot \operatorname{tr}\left(J_{x} J^{-1}\right) \tag{10}
\end{equation*}
$$

where $\operatorname{det}(J)$ denotes the determinant of matrix $J, \operatorname{tr}(\cdot)$ represents the trace of a matrix. In virtue of $\operatorname{tr}(\tilde{U})=0$, which yields

$$
\begin{equation*}
\operatorname{det} J_{ \pm}=1 \tag{11}
\end{equation*}
$$

### 2.2 Analytic properties of Jost solutions

Before the construction of RH problem, it suffices to establish the analytic properties of the Jost solutions $J_{ \pm}$and the scattering matrix $S(\omega)$. To this end, let us split $J_{ \pm}$into column vectors, i.e., $J_{ \pm}=\left(\left[J_{ \pm}\right]_{1},\left[J_{ \pm}\right]_{2},\left[J_{ \pm}\right]_{3},\left[J_{ \pm}\right]_{4}\right)$. By the Volterra integration, one can rewrite the first column $\left[J_{-}\right]_{1}$ as follows:

$$
\begin{align*}
& {\left[J_{-}\right]_{11}=1+\int_{-\infty}^{x}\left(u\left[J_{-}\right]_{31}+v\left[J_{-}\right]_{41}\right)(y) d y} \\
& {\left[J_{-}\right]_{21}=\int_{-\infty}^{x}\left(v^{*}\left[J_{-}\right]_{31}+u^{*}\left[J_{-}\right]_{41}\right)(y) d y} \\
& {\left[J_{-}\right]_{31}=-\int_{-\infty}^{x}\left(u^{*}\left[J_{-}\right]_{11}+v\left[J_{-}\right]_{21}\right)(y) e^{2 i \omega(x-y)} d y}  \tag{12}\\
& {\left[J_{-}\right]_{41}=-\int_{-\infty}^{x}\left(v^{*}\left[J_{-}\right]_{11}+u\left[J_{-}\right]_{21}\right)(y) e^{2 i \omega(x-y)} d y}
\end{align*}
$$

By similar arguments on the analytic properties of the Jost solution [1, 8], one knows that $\left[J_{-}\right]_{1}$ is analytic in the upper half-plane $\omega \in \mathbb{C}_{+}$and continuous on the real axis. In a similar way, $\left[J_{-}\right]_{2},\left[J_{+}\right]_{3},\left[J_{+}\right]_{4}$ are analytic in the upper half-plane $\omega \in \mathbb{C}_{+}$and continuous on the real axis, while $\left[J_{+}\right]_{1},\left[J_{+}\right]_{2},\left[J_{-}\right]_{3},\left[J_{-}\right]_{4}$ are analytic in the lower half-plane $\omega \in \mathbb{C}_{-}$and continuous on the real axis.
Denote $E=e^{-i \omega \sigma x}$, it is easy to see that both $J_{+} E$ and $J_{-} E$ are fundamental solutions of (3), hence they are linearly related by the scattering matrix $S(\omega)$. That is

$$
\begin{equation*}
J_{-}=J_{+} E S(\omega) E^{-1}, \quad \omega \in \mathbb{R} \tag{13}
\end{equation*}
$$

It follows from (11) and (13) that

$$
\begin{equation*}
\operatorname{det} S(\omega)=1 \tag{14}
\end{equation*}
$$

From the relation (13), we get

$$
\begin{equation*}
S(\omega)=E^{-1} J_{+}^{-1} J_{-} E, \quad \omega \in \mathbb{R} \tag{15}
\end{equation*}
$$

which indicates that we need to study the analytic properties of $J_{+}^{-1}$ before deriving the analytic properties of the entries of $S(\omega)$. To this end, we start with the adjoint spectral equation of (7)

$$
\begin{equation*}
K_{x}=-i \omega[\sigma, K]-K \tilde{U} \tag{16}
\end{equation*}
$$

one can simply find that $J_{ \pm}^{-1}$ satisfy Eq. (16), where $J_{ \pm}^{-1}$ are partitioned into rows

By similar arguments, we find that $\left[J_{+}^{-1}\right]^{1},\left[J_{+}^{-1}\right]^{2},\left[J_{-}^{-1}\right]^{3},\left[J_{-}^{-1}\right]^{4}$ are analytic in $\omega \in \mathbb{C}_{+}$, while $\left[J_{-}^{-1}\right]^{1},\left[J_{-}^{-1}\right]^{2},\left[J_{+}^{-1}\right]^{3},\left[J_{+}^{-1}\right]^{4}$ are analytic for $\omega \in \mathbb{C}_{-}$. Thanks to the analytic property of $J_{+}^{-1}$ and $J_{-}$, it follows that $s_{11}, s_{12}, s_{21}, s_{22}$ are analytic in the upper half-plane $\omega \in \mathbb{C}_{+}, s_{33}, s_{34}$, $s_{43}, s_{44}$ are analytic in $\mathbb{C}_{-}, s_{13}, s_{14}, s_{23}, s_{24}, s_{31}, s_{32}, s_{42}$ are only defined and continuous for $\omega \in \mathbb{R}$.

## 3 Riemann-Hilbert problem

Let us begin with the symmetry conditions for $J_{ \pm}$and $S(\omega)$. Firstly, it follows easily from (7) and (5) that

$$
\begin{equation*}
J_{ \pm, x}^{\dagger}\left(x, \omega^{*}\right)=-i \omega\left[\sigma, J_{ \pm}^{\dagger}\left(x, \omega^{*}\right)\right]-J_{ \pm}^{\dagger}\left(x, \omega^{*}\right) \tilde{U}, \tag{18}
\end{equation*}
$$

hence both $J_{ \pm}^{\dagger}\left(x, \omega^{*}\right)$ and $J_{ \pm}^{-1}(x, \omega)$ solve the adjoint spectral problem (16), and they also tend to the unit matrix as $x \rightarrow \pm \infty$. Thus,

$$
\begin{equation*}
J_{ \pm}^{\dagger}\left(x, \omega^{*}\right)=J_{ \pm}^{-1}(x, \omega) . \tag{19}
\end{equation*}
$$

By similar arguments, one gets

$$
\begin{equation*}
S^{\dagger}\left(\omega^{*}\right)=S^{-1}(\omega) \tag{20}
\end{equation*}
$$

Next, we shall formulate the RH problem. To accomplish this, we introduce the following matrix function

$$
\begin{equation*}
\Psi^{+}=J_{-} H_{1}+J_{+} H_{2}=\left(\left[J_{-}\right]_{1},\left[J_{-}\right]_{2},\left[J_{+}\right]_{3},\left[J_{+}\right]_{4}\right), \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{1}=\operatorname{diag}(1,1,0,0), \quad H_{2}=\operatorname{diag}(0,0,1,1) . \tag{22}
\end{equation*}
$$

It is evident that $\Psi^{+}$is analytic in $\mathbb{C}_{+}$and solves the spectral equation (7). Moreover,

$$
\begin{align*}
\Psi^{+} & =J_{+} E\left(S(\omega) H_{1}+H_{2}\right) E^{-1}=J_{+} E\left(\begin{array}{llll}
s_{11}(\omega) & s_{12}(\omega) & 0 & 0 \\
s_{21}(\omega) & s_{22}(\omega) & 0 & 0 \\
s_{31}(\omega) & s_{32}(\omega) & 1 & 0 \\
s_{41}(\omega) & s_{42}(\omega) & 0 & 1
\end{array}\right) E^{-1}  \tag{23}\\
& =J_{-} E\left(H_{1}+S^{\dagger}(\omega) H_{2}\right) E^{-1}=J_{-} E\left(\begin{array}{llll}
1 & 0 & s_{31}^{*}(\omega) & s_{41}^{*}(\omega) \\
0 & 1 & s_{32}^{*}(\omega) & s_{42}^{*}(\omega) \\
0 & 0 & s_{33}^{*}(\omega) & s_{43}^{*}(\omega) \\
0 & 0 & s_{34}^{*}(\omega) & s_{44}^{*}(\omega)
\end{array}\right) E^{-1} .
\end{align*}
$$

Therefore

$$
\begin{equation*}
\operatorname{det}\left(\Psi^{+}\right)=M_{1}=M_{2}^{*} \tag{24}
\end{equation*}
$$

in which $M_{1}=s_{11} s_{22}-s_{12} s_{21}, M_{2}=s_{33} s_{44}-s_{43} s_{34}$. According to (23), we get $\Psi^{+}(x, \omega) \rightarrow I$, $\omega \in \mathbb{C}_{+} \rightarrow \infty$. To acquire the analytic counterpart of $\Psi^{+}$in $\mathbb{C}_{-}$, we investigate the following matrix function,

$$
\Psi^{-}=H_{1} J_{-}^{-1}+H_{2} J_{+}^{-1}=\left(\begin{array}{l}
{\left[J_{-}^{-1}\right]^{[1]}}  \tag{25}\\
{\left[J_{-}^{-1}\right]^{[2]}} \\
{\left[J_{+}^{-1}\right]^{[3]}} \\
{\left[J_{+}^{-1}\right]^{[4]}}
\end{array}\right)
$$

From (23) it is clear that $\Psi^{-}$is analytic in $\mathbb{C}_{-}$and also solves the spectral equation (7). Moreover, one has

$$
\begin{align*}
\Psi^{-} & =E\left(H_{1} S^{\dagger}+H_{2}\right) E^{-1} J_{+}^{-1}=E\left(\begin{array}{cccc}
s_{11}^{*} & s_{21}^{*} & s_{31}^{*} & s_{41}^{*} \\
s_{12}^{*} & s_{22}^{*} & s_{23}^{*} & s_{24}^{*} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) E^{-1} J_{+}^{-1} \\
& =E\left(H_{1}+H_{2} S\right) E^{-1} J_{-}^{-1}=E\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
s_{31} & s_{32} & s_{33} & s_{34} \\
s_{41} & s_{42} & s_{43} & s_{44}
\end{array}\right) E^{-1} J_{-}^{-1} . \tag{26}
\end{align*}
$$

It follows easily that

$$
\begin{equation*}
\operatorname{det} \Psi^{-}=M_{2}=M_{1}^{*} \tag{27}
\end{equation*}
$$

and $\Psi^{-}(x, \omega) \rightarrow I, \omega \in \mathbb{C}_{-} \rightarrow \infty$.
Now we are ready to formulate the RH problem with the aid of $\Psi^{+}$and $\Psi^{-}$, that is

$$
\begin{equation*}
\Psi^{-}(x, \omega) \Psi^{+}(x, \omega)=G(x, \omega), \quad \omega \in \mathbb{R} \tag{28}
\end{equation*}
$$

where the jump matrix reads as follows

$$
G(x, \omega)=E\left(\begin{array}{cccc}
1 & 0 & s_{31}^{*} & s_{41}^{*}  \tag{29}\\
0 & 1 & s_{32}^{*} & s_{42}^{*} \\
s_{31} & s_{32} & 1 & 0 \\
s_{41} & s_{42} & 0 & 1
\end{array}\right) E^{-1} .
$$

Moreover, it follows from (19), (23), and (26) that

$$
\begin{equation*}
\left(\Psi^{+}\right)^{\dagger}\left(x, \omega^{*}\right)=\Psi^{-}(x, \omega) \tag{30}
\end{equation*}
$$

## $4 \mathbf{N}$-soliton solutions

We shall solve the RH problem (28) with simple zeros, that is, assume that $\operatorname{det} \Psi^{+}(\omega)=$ $M_{1}(\omega)$ and $\operatorname{det} \Psi^{-}(\omega)=M_{1}^{*}\left(\omega^{*}\right)$ admit simple zeros, for the multiple zero case, we refer to [37, 38]. It follows from (24) and (27) that, the number of the zeros of $\operatorname{det} \Psi^{+}$and $\operatorname{det} \Psi^{-}$is the same, moreover, if $\omega_{1} \in \mathbb{C}_{+}$is a simple zero of $\operatorname{det} \Psi^{+}$, then $\omega_{1}^{*} \in \mathbb{C}_{-}$must be a simple zero of $\operatorname{det} \Psi^{-}$. Hence, it is supposed that $\left\{\omega_{k} \in \mathbb{C}_{+}, 1 \leq k \leq N\right\}$ are simple zeros of $M_{1}(\omega)$, then we can simply denote the zeros of $M_{1}^{*}\left(\omega^{*}\right)$ by $\left\{\omega_{k}^{*} \in \mathbb{C}_{-}, 1 \leq k \leq N\right\}$. Then there must exist a column vector $\mu_{k}$ fulfilling

$$
\begin{equation*}
\Psi^{+}\left(\omega_{k}\right) \mu_{k}=0, \quad 1 \leq k \leq N . \tag{31}
\end{equation*}
$$

Taking the Hermitian conjugate to (31) yields

$$
\begin{equation*}
\left(\mu_{k}\right)^{\dagger}\left(\Psi^{+}\left(\omega_{k}\right)\right)^{\dagger}=0, \quad 1 \leq k \leq N \tag{32}
\end{equation*}
$$

In virtue of (30), we have

$$
\begin{equation*}
\left(\mu_{k}\right)^{\dagger} \Psi^{-}\left(\omega_{k}^{*}\right)=0, \quad 1 \leq k \leq N . \tag{33}
\end{equation*}
$$

For simplicity, we denote

$$
\begin{equation*}
\hat{\mu}_{k}:=\left(\mu_{k}\right) \dagger, \quad 1 \leq k \leq N . \tag{34}
\end{equation*}
$$

Next, we shall find the explicit expressions for $\mu_{k}$ and $\hat{\mu}_{k}$. It follows from (7), (21), and (31) that

$$
\begin{equation*}
\Psi^{+}\left(\omega_{k}\right)\left(\mu_{k, x}+i \omega_{k} \sigma \mu_{k}\right)=0 \tag{35}
\end{equation*}
$$

By similar argument, we have

$$
\begin{equation*}
\Psi^{+}\left(\omega_{k}\right)\left(\mu_{k, t}+4 i \omega_{k}^{3} \sigma \mu_{k}\right)=0 \tag{36}
\end{equation*}
$$

thus

$$
\begin{equation*}
\mu_{k}=e^{-i \omega_{k} \sigma x-4 i \omega_{k}^{3} \sigma t} \mu_{k 0}, \quad 1 \leq k \leq N, \tag{37}
\end{equation*}
$$

in which $\mu_{k 0}$ is a complex constant vector.

It is well known that by eliminating the zeros and the Plemelj's formula [39], the irregular RH problem (28) can be solved as follows [1]

$$
\begin{equation*}
\Psi^{-}=I+\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\mu_{k} \hat{\mu}_{l}\left(M^{-1}\right)_{k l}}{\omega-\omega_{l}}, \quad \Psi^{+}=I-\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\mu_{k} \hat{\mu}_{l}\left(M^{-1}\right)_{k l}}{\omega-\omega_{l}^{*}}, \tag{38}
\end{equation*}
$$

in which $M$ is an invertible $N \times N$ matrix with elements $M_{k l}$ defined as follows

$$
\begin{equation*}
M_{k l}=\frac{\hat{\mu}_{k} \mu_{l}}{\omega_{l}-\omega_{k}^{*}}, \quad 1 \leq k, l \leq N \tag{39}
\end{equation*}
$$

To obtain the soliton solutions for the coupled mKdV Equation (2), it suffices to expand the solutions $\Psi^{+}$when $\omega \rightarrow \infty$,

$$
\begin{equation*}
\Psi^{+}=I+\frac{\Psi_{1}^{+}}{\omega}+\frac{1}{\omega^{2}} \Psi_{2}^{+}+O\left(\frac{1}{\omega^{3}}\right) . \tag{40}
\end{equation*}
$$

Notice that the soliton solutions correspond to the case when the scattering data vanish, that is, $s_{31}=s_{32}=s_{41}=s_{42}=0$. Substituting (40) into the spectral Equation (7), one arrives at

$$
\begin{equation*}
\tilde{U}=-i\left[\sigma, \Psi_{1}^{+}\right], \tag{41}
\end{equation*}
$$

which yields

$$
\begin{equation*}
u=-2 i\left[\Psi_{1}^{+}\right]_{13}, \quad v=-2 i\left[\Psi_{1}^{+}\right]_{14}, \tag{42}
\end{equation*}
$$

where $\left[\Psi_{1}^{+}\right]_{13}$ represents the $(1,3)$-th element of the matrix $\left[\Psi_{1}^{+}\right]$. By direct computation, it follows from (38) that

$$
\begin{equation*}
\Psi_{1}^{+}=-\sum_{k=1}^{N} \sum_{l=1}^{N} \mu_{k} \hat{\mu}_{l}\left(M^{-1}\right)_{k l} . \tag{43}
\end{equation*}
$$

Take $\Theta_{k}=i \omega_{k} x+4 i \omega_{k}^{3} t$ and $\mu_{k 0}=\left(a_{k}^{1}, a_{k}^{2}, a_{k}^{3}, a_{k}^{4}\right)^{\mathrm{T}}, 1 \leq k \leq N$, therefore, the N -soliton solutions for the coupled mKdV Equation (2) read

$$
\begin{equation*}
u=2 i \sum_{k=1}^{N} \sum_{l=1}^{N} a_{k}^{1}\left(a_{l}^{3}\right)^{*} e^{-\Theta_{k}+\Theta_{l}^{*}}\left(M^{-1}\right)_{k l}, \quad v=2 i \sum_{k=1}^{N} \sum_{l=1}^{N} a_{k}^{1}\left(a_{l}^{4}\right)^{*} e^{-\Theta_{k}+\Theta_{l}^{*}}\left(M^{-1}\right)_{k l} \tag{44}
\end{equation*}
$$

where $a_{k}^{2}\left(a_{l}^{3}\right)^{*}=\left(a_{k}^{1}\right)^{*} a_{l}^{4}$ and $a_{k}^{2}\left(a_{l}^{4}\right)^{*}=\left(a_{k}^{1}\right)^{*} a_{l}^{3}$.
Specifically, when $N=1$, we can obtain the explicit expressions of $u, v$. To this end, we introduce the following notations

$$
\begin{align*}
& \omega_{1}=\delta_{1}+i \eta_{1}, \quad \Theta_{1}=i \omega_{1} x+4 i \omega_{1}^{3} t, \quad \mu_{10}=\left(a_{1}, a_{2}, a_{3}, a_{4}\right)^{T}, \\
& \rho_{1}=\ln \left(\frac{\left|a_{3}\right|}{\left|a_{1}\right|}\right), \quad \tau_{1}=\arg a_{1}-\arg a_{3},  \tag{45}\\
& \kappa_{1}=\arg a_{3}-\arg a_{4}, \quad z_{1}=\Theta_{1}+\Theta_{1}^{*}+\rho_{1},
\end{align*}
$$

in which $a_{i}(i=1,2,3,4)$ are complex constants, $\arg A$ denotes the argument of $A$. It follows from (37) and (45) that

$$
\begin{align*}
& \mu_{1}=\left(a_{1} e^{-\Theta_{1}}, a_{2} e^{-\Theta_{1}}, a_{3} e^{\Theta_{1}}, a_{4} e^{\Theta_{1}}\right)^{\mathrm{T}},  \tag{46}\\
& \hat{\mu}_{1}=\left(a_{1}^{*} e^{-\Theta_{1}^{*}}, a_{2}^{*} e^{-\Theta_{1}^{*}}, a_{3}^{*} e^{\Theta_{1}^{*}}, a_{4} e^{\Theta_{1}^{*}}\right) .
\end{align*}
$$

In view of (38), (40), and (41), one gets

$$
\begin{align*}
\tilde{U}= & \frac{2 i}{e^{-\Theta_{1}-\Theta_{1}^{*}}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)+e^{\Theta_{1}+\Theta_{1}^{*}}\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)} \\
& \cdot\left(\begin{array}{cccc}
0 & 0 & e^{\Theta_{1}^{*}-\Theta_{1}} a_{1} a_{3}^{*} & e^{\Theta_{1}^{*}-\Theta_{1}} a_{1} a_{4}^{*} \\
0 & 0 & e^{\Theta_{1}^{*}-\Theta_{1}} a_{2} a_{3}^{*} & e^{\Theta_{1}^{*}-\Theta_{1}} a_{2} a_{4}^{*} \\
-e^{\Theta_{1}-\Theta_{1}^{*}} a_{1}^{*} a_{3} & -e^{\Theta_{1}-\Theta_{1}^{*}} a_{2}^{*} a_{3} & 0 & 0 \\
-e^{\Theta_{1}-\Theta_{1}^{*}} a_{1}^{*} a_{4} & -e^{\Theta_{1}-\Theta_{1}^{*}} a_{2}^{*} a_{4} & 0 & 0
\end{array}\right) . \tag{47}
\end{align*}
$$

Therefore, we have

$$
Q=\frac{2 i e^{\Theta_{1}^{*}-\Theta_{1}}}{e^{-\Theta_{1}-\Theta_{1}^{*}}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)+e^{\Theta_{1}+\Theta_{1}^{*}}\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)} \cdot\left(\begin{array}{ll}
a_{1} a_{3}^{*} & a_{1} a_{4}^{*}  \tag{48}\\
a_{2} a_{3}^{*} & a_{2} a_{4}^{*}
\end{array}\right) .
$$

The single soliton solutions of the CCMKDV System (2) thus read

$$
\begin{align*}
& u=e^{\Theta_{1}^{*}-\Theta_{1}+i \tau_{1}} \operatorname{sech} z_{1}, \\
& v=e^{\Theta_{1}^{*}-\Theta_{1}+i \tau_{1}+i \kappa_{1}} \operatorname{sech} z_{1} . \tag{49}
\end{align*}
$$

Besides, the complex constants $a_{1}, a_{2}, a_{3}, a_{4}$ satisfy

$$
a_{1} a_{4}^{*}=a_{3} a_{2}^{*}, \quad a_{1} a_{3}^{*}=a_{4} a_{2}^{*}
$$

Set $a_{1}=a_{2}=a_{3}=a_{4}=1, \delta_{1}=0, \eta_{1}=0.5$, we plot the graphics of single-soliton solutions for the CCMKDV System (2) in Fig. 1.


Figure 1 A single non-degenerate soliton in $u$ and $v$, the associated parameters are: $\omega_{1}=0.5 i$, $a_{1}=a_{2}=a_{3}=a_{4}=1$

When $N=2$, set $\mu_{10}=\left(a_{1}, a_{2}, a_{3}, a_{4}\right)^{T}$ and $\mu_{20}=\left(A_{1}, A_{2}, A_{3}, A_{4}\right)^{T}$, therefore the twosoliton solutions of the CCMKDV System (2) read

$$
\begin{align*}
u= & \frac{2 i}{|M|}\left\{\frac{1}{\omega_{2}-\omega_{2}^{*}}\left[e^{-\Theta_{2}-\Theta_{2}^{*}}\left(\left|A_{1}\right|^{2}+\left|A_{2}\right|^{2}\right)+e^{\Theta_{2}+\Theta_{2}^{*}}\left(\left|A_{3}\right|^{2}+\left|A_{4}\right|^{2}\right)\right] e^{-\Theta_{1}+\Theta_{1}^{*}} a_{1} a_{3}^{*}\right. \\
& -\frac{1}{\omega_{1}-\omega_{2}^{*}}\left[e^{-\Theta_{1}-\Theta_{2}^{*}}\left(a_{1} A_{1}^{*}+a_{2} A_{2}^{*}\right)+e^{\Theta_{1}+\Theta_{2}^{*}}\left(a_{3} A_{3}^{*}+a_{4} A_{4}^{*}\right)\right] e^{-\Theta_{1}+\Theta_{2}^{*}} a_{1} A_{3}^{*} \\
& -\frac{1}{\omega_{2}-\omega_{1}^{*}}\left[e^{-\Theta_{2}-\Theta_{1}^{*}}\left(a_{1}^{*} A_{1}+a_{2}^{*} A_{2}\right)+e^{\Theta_{1}^{*}+\Theta_{2}}\left(a_{3}^{*} A_{3}+a_{4}^{*} A_{4}\right)\right] e^{-\Theta_{2}+\Theta_{1}^{*}} A_{1} a_{3}^{*}  \tag{50}\\
& \left.+\frac{1}{\omega_{1}-\omega_{1}^{*}}\left[e^{-\Theta_{1}-\Theta_{1}^{*}}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)+e^{\Theta_{1}+\Theta_{1}^{*}}\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)\right] e^{-\Theta_{2}+\Theta_{2}^{*}} A_{1} A_{3}^{*}\right\}, \\
v= & \frac{2 i}{|M|}\left\{\frac{1}{\omega_{2}-\omega_{2}^{*}}\left[e^{-\Theta_{2}-\Theta_{2}^{*}}\left(\left|A_{1}\right|^{2}+\left|A_{2}\right|^{2}\right)+e^{\Theta_{2}+\Theta_{2}^{*}}\left(\left|A_{3}\right|^{2}+\left|A_{4}\right|^{2}\right)\right] e^{-\Theta_{1}+\Theta_{1}^{*}} a_{1} a_{4}^{*}\right. \\
& -\frac{1}{\omega_{1}-\omega_{2}^{*}}\left[e^{-\Theta_{1}-\Theta_{2}^{*}}\left(a_{1} A_{1}^{*}+a_{2} A_{2}^{*}\right)+e^{\Theta_{1}+\Theta_{2}^{*}}\left(a_{3} A_{3}^{*}+a_{4} A_{4}^{*}\right)\right] e^{-\Theta_{1}+\Theta_{2}^{*}} a_{1} A_{4}^{*} \\
& -\frac{1}{\omega_{2}-\omega_{1}^{*}}\left[e^{-\Theta_{2}-\Theta_{1}^{*}}\left(a_{1}^{*} A_{1}+a_{2}^{*} A_{2}\right)+e^{\Theta_{1}^{*}+\Theta_{2}}\left(a_{3}^{*} A_{3}+a_{4}^{*} A_{4}\right)\right] e^{-\Theta_{2}+\Theta_{1}^{*}} A_{1} a_{4}^{*}  \tag{51}\\
& \left.+\frac{1}{\omega_{1}-\omega_{1}^{*}}\left[e^{-\Theta_{1}-\Theta_{1}^{*}}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)+e^{\Theta_{1}+\Theta_{1}^{*}}\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)\right] e^{-\Theta_{2}+\Theta_{2}^{*}} A_{1} A_{4}^{*}\right\},
\end{align*}
$$

in which

$$
\begin{align*}
|M|= & \frac{1}{\left(\omega_{1}-\omega_{1}^{*}\right)\left(\omega_{2}-\omega_{2}^{*}\right)}\left[e^{-\Theta_{1}-\Theta_{1}^{*}}\left(\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}\right)+e^{\Theta_{1}+\Theta_{1}^{*}}\left(\left|a_{3}\right|^{2}+\left|a_{4}\right|^{2}\right)\right] \\
& \times\left[e^{-\Theta_{2}-\Theta_{2}^{*}}\left(\left|A_{1}\right|^{2}+\left|A_{2}\right|^{2}\right)+e^{\Theta_{2}+\Theta_{2}^{*}}\left(\left|A_{3}\right|^{2}+\left|A_{4}\right|^{2}\right)\right] \\
& -\frac{1}{\left(\omega_{1}-\omega_{2}^{*}\right)\left(\omega_{2}-\omega_{1}^{*}\right)}\left[e^{-\Theta_{1}-\Theta_{2}^{*}}\left(a_{1} A_{1}^{*}+a_{2} A_{2}^{*}\right)+e^{\Theta_{1}+\Theta_{2}^{*}}\left(a_{3} A_{3}^{*}+a_{4} A_{4}^{*}\right)\right]  \tag{52}\\
& \times\left[e^{-\Theta_{2}-\Theta_{1}^{*}}\left(a_{1}^{*} A_{1}+a_{2}^{*} A_{2}\right)+e^{\Theta_{2}+\Theta_{1}^{*}}\left(a_{3}^{*} A_{3}+a_{4}^{*} A_{4}\right)\right] .
\end{align*}
$$

In virtue of (5), we have

$$
\begin{equation*}
A_{1}=A_{2}^{*}=a_{2}=a_{1}^{*}, \quad-A_{3}=A_{4}^{*}=a_{4}=a_{3}^{*} \tag{53}
\end{equation*}
$$

Moreover, the two-soliton interactions are graphically shown in Fig. 2 by choosing some suitable parameters.

## 5 Concluding remarks

A CCMKDV System (2) was investigated in this paper. By analyzing its spectral problem, we construct the associated $4 \times 4$ matrix RH problem. By eliminating the zeros and the Plemelj's formula [39], we solve the RH problem with simple zeros. Subsequently, we presented the N -soliton solutions formula for (2). Specifically, the single soliton solutions are presented explicitly and the dynamical behavior of the single-soliton solutions has been shown graphically. We note that we only treat the case when the potential functions satisfy some zero boundary conditions. For the general case when the potentials fail to obey these vanishing conditions, more general solutions could be obtained, which may be studied in the future.


Figure 2 Two-soliton interactions between $u$ and $v$ via (50)-(52), the associated parameters are: $\omega_{1}=2 i$, $\omega_{2}=i,-A_{3}=A_{4}^{*}=a_{1}=a_{2}^{*}=1+2 i, A_{1}=A_{2}^{*}=a_{3}=a_{4}^{*}=1-2 i$

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## Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Declarations

## Ethics approval and consent to participate

Not applicable.

## Competing interests

The authors declare no competing interests.

## Author contributions

This paper was written entirely by myself. The author read and approved the final manuscript.

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