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Short note on a solution with large amplitude for the limiting system arising from the competition-diffusion system

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Abstract

This paper is concerned with positive solutions of the limiting system arising from the Shigesada–Kawasaki–Teramoto model with large interspecific competition rate. It has previously been suggested that the limiting system has positive solutions with large amplitude for a certain value of parameters. As a first step, the purpose of this paper is to discuss the precise spatial profile of such solutions by employing formal calculations by the singular perturbation technique.

Keywords: Competition-diffusion system; Limiting system; Positive solution with large amplitude; Spatial profile

1 Introduction

In 1979, Shigesada et al. [7] proposed the competition-diffusion system with nonlinear diffusion effect to model the segregation of interacting species. The following is the stationary problem of their proposed system:

$$\begin{cases} 0 = \varepsilon\Delta[(1+\alpha z)w] + (1-w-\theta z)w, \\ 0 = d\varepsilon\Delta[(1+\beta w)z] + (1-\gamma\theta w - z)z, \quad x \in \Omega, \\ \partial_\nu w = 0, \qquad \partial_\nu z = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where the variables w and z mean the population density of two competing species, the parameters ε, d, θ, and γ are positive, the cross-diffusion coefficients α and β are nonnegative, the habitat Ω of two competing species is a bounded domain in $\mathbb{R}^\ell$ with smooth boundary $\partial\Omega$, the integer ℓ is the space dimension of Ω, the vector $\nu = \nu(x)$ is the outward unit normal vector on $x \in \partial\Omega$, the notation $\partial_\xi f$ denotes the derivative of the function f with respect to ξ. We remark that the values θ and $\gamma\theta$ correspond to the interspecific competition rate. Since $w(x)$ and $z(x)$ mean the population density, we restrict our discussion to the positive solution of problem (1), where we say that $(w,z)(x)$ is *positive* if $(w,z)(x)$ is in the positive quadrant for any x in the closure $\mathrm{Cl}\,\Omega$ of Ω. Since then, many mathematicians have researched the solution structure of positive solution for problem

(1) to understand what kinds of phenomena occur by the cross-diffusion effect (for example, see Jüngel [3] and Yamada [8, 9] as surveys of mathematical study related to problem (1)).

Since the change of variables $\mathbf{p} = (\psi_p, \psi_q)(\mathbf{w})$ with $\mathbf{w} = (w, z)$, $\mathbf{p} = (p, q)$,

$$p = \psi_p(\mathbf{w}) = (1 + \alpha z)w, \qquad q = \psi_q(\mathbf{w}) = (1 + \beta w)z$$

is a bijective map from the positive quadrant to itself, it turns out that there exists an inverse map $\mathbf{w} = (\Psi_w, \Psi_z)(\mathbf{p})$ of $\mathbf{p} = (\psi_u, \psi_v)(\mathbf{w})$ in the positive quadrant. After simple calculations, we obtain

$$w = \Psi_w(\mathbf{p}) = \frac{\beta p - \alpha q - 1 + \sqrt{(\beta p - \alpha q - 1)^2 + 4\beta p}}{2\beta},$$
$$z = \Psi_z(\mathbf{p}) = \frac{-\beta p + \alpha q - 1 + \sqrt{(\beta p - \alpha q - 1)^2 + 4\beta p}}{2\alpha}$$

for any $\mathbf{p}$ in the positive quadrant, and then we represent problem (1) as

$$\begin{cases} 0 = \varepsilon \Delta p + f(\mathbf{p}), \\ 0 = d\varepsilon \Delta q + g(\mathbf{p}), \quad x \in \Omega, \\ \partial_\nu p = 0, \qquad \partial_\nu q = 0, \quad x \in \partial\Omega, \end{cases} \tag{2}$$

which is a stationary problem of reaction-diffusion system with linear diffusion term, where

$$f(\mathbf{p}) = \big(1 - \Psi_w(\mathbf{p}) - \theta\Psi_z(\mathbf{p})\big)\Psi_w(\mathbf{p}),$$
$$g(\mathbf{p}) = \big(1 - \gamma\theta\Psi_w(\mathbf{p}) - \Psi_z(\mathbf{p})\big)\Psi_z(\mathbf{p}).$$

We remark that

$$\lim_{\theta\to+\infty} (\Psi_w, \Psi_z)\big(\theta^{-1}\mathbf{p}\big) = \mathbf{0}, \qquad \lim_{\theta\to+\infty} \big\{\theta(\Psi_w, \Psi_z)\big(\theta^{-1}\mathbf{p}\big)\big\} = \mathbf{p}$$

hold true for any $\mathbf{p}$ in the positive quadrant.

In the previous paper [5], it was shown that problem (2) as $\theta \to +\infty$ has two kinds of limiting systems. One limiting system is represented as

$$\begin{cases} 0 = \varepsilon \Delta p + (1 - p)p, \quad x \in \Omega_+, \\ 0 = d\varepsilon \Delta q + (1 - q)q, \quad x \in \Omega_-, \\ p = 0, \qquad q = 0, \qquad \gamma\partial_\mu p = -d\partial_\mu q, \quad x \in \Gamma, \\ \partial_\nu p = 0, \qquad \partial_\nu q = 0, \quad x \in \partial\Omega, \end{cases} \tag{3}$$

by applying the argument in Dancer et al. [2], where $\Omega = \Omega_- \cup \Omega_+ \cup \Gamma$, Γ is the interface which separates two subregions

$$\Omega_+ = \big\{x \in \Omega : p(x) > 0 = q(x)\big\}, \qquad \Omega_- = \big\{x \in \Omega : p(x) = 0 < q(x)\big\},$$

and $\mu = \mu(x)$ is a unit vector normal to Γ. We note that the solution of limiting system (3) approximates the bounded solution of problem (1) for sufficiently large $\theta > 0$. By employing the change of variables $u = \gamma\theta p$ and $v = \theta q$, and taking the limit as $\theta \to +\infty$, we obtain the other limiting system

$$\begin{cases} 0 = \varepsilon\Delta u + (1-v)u, \\ 0 = d\varepsilon\Delta v + (1-u)v, \quad x \in \Omega, \\ \partial_\nu u = 0, \partial_\nu v = 0, \quad x \in \partial\Omega, \end{cases} \tag{4}$$

of problem (2). Since the positive constant solution of problem (1) in the positive quadrant is

$$\hat{\mathbf{w}} = \left(\frac{\theta - 1}{\gamma\theta^2 - 1}, \frac{\gamma\theta - 1}{\gamma\theta^2 - 1} \right),$$

the corresponding constant solution of problem (4) becomes

$$\hat{\mathbf{e}} = \lim_{\theta \to +\infty} \left\{ \theta\left(\gamma\psi_p(\hat{\mathbf{w}}), \psi_q(\hat{\mathbf{w}})\right) \right\} = (1, 1).$$

Hence, it turns out that the solution of limiting system (4) approximates the solution with small amplitude of problem (1) in a neighborhood of $\mathbf{w} = \hat{\mathbf{w}}$ for sufficiently large $\theta > 0$. We note that the solution structure of limiting system (3) has been already investigated (for example, see Dancer et al. [2]). However, although limiting system (4) is simple in a class of 2-component reaction-diffusion systems, we have not yet understood enough solution structure of positive solution for limiting system (4).

As a first step to approach the solution structure of problem (4), we assume that the habitat Ω is the ball with center origin and radius π, and we restrict our discussion to the radially symmetric positive solution, that is, we study positive solutions of the problem

$$\begin{cases} 0 = \varepsilon\mathcal{L}u + (1-v)u, \\ 0 = d\varepsilon\mathcal{L}v + (1-u)v, \quad r \in (0, \pi), \\ \partial_r u = 0, \qquad \partial_r v = 0, \quad r = 0, \pi, \end{cases} \tag{5}$$

where $\mathcal{L}u = r^{1-\ell}\partial_r[r^{\ell-1}\partial_r u]$. In the previous paper [5], it was suggested that problem (5) has a solution with very large amplitude for some $\varepsilon > 0$. In this paper, to discuss precise spatial profile of such a solution of problem (5), we construct the solution for some $\varepsilon > 0$. To do this, we employ formal calculations by the singular perturbation technique because the rigorous proof on the existence theorem of solution requires lengthy arguments (for example, see Ito [4]).

2 Preliminary

Let $(u, v)(r)$ be an arbitrary bounded positive solution of problem (5) for $\varepsilon > 0$ and $d > 0$. When $u(r) > 1$ (respectively, $v(r) > 1$) is satisfied for any $r \in [0, \pi]$, we have

$$0 = d\varepsilon \int_0^\pi \partial_r\left[r^{\ell-1}\partial_r v(r)\right] dr = \int_0^\pi v(r)\left(u(r) - 1\right) r^{\ell-1}\, dr > 0$$

$$\left(\begin{aligned}\text{respectively, } 0 = \varepsilon \int_0^{\pi} \partial_r\left[r^{\ell-1}\partial_r u(r)\right] dr \\ = \int_0^{\pi} u(r)\big(v(r)-1\big)r^{\ell-1}\,dr > 0\end{aligned}\right)$$

because of the boundary condition. This contradiction implies that the following holds true.

Lemma 1 *Let $(u,v)(r)$ be an arbitrary bounded positive solution of problem (5) for $\varepsilon > 0$ and $d > 0$. Then there exist r_u and $r_v \in [0,\pi]$ such that $u(r_u) \le 1$ and $v(r_v) \le 1$ are satisfied.*

Setting

$$D(u) = \begin{cases} u & \text{if } u \ge 0, \\ du & \text{if } u < 0, \end{cases}$$

we consider solutions of the problem

$$\begin{cases} 0 = \varepsilon\mathcal{L}[D(u)] + u, & r \in (0,\pi), \\ \partial_r u = 0, & r = 0,\pi \end{cases} \tag{6}$$

with $\langle \partial_r[D(u)]\rangle(\tau) = 0$ for each $\tau \in (0,\pi)$ such that $u(\tau) = 0$ is satisfied, where

$$\langle f\rangle(\tau) = \lim_{r\searrow\tau} f(r) - \lim_{r\nearrow\tau} f(r).$$

We denote by $u^+(r)$ (respectively, $u^-(r)$) the solution of the problem

$$0 = \mathcal{L}\big[D(u)\big] + u, \quad r > 0$$

with $u(0) = 1$ (respectively, $u(0) = -1$) and $\partial_r u(0) = 0$. The phase plane analysis leads to the following.

Lemma 2 *With $\sigma \in \{-,+\}$, there exist sequences $\{\xi_k^\sigma\}_{k=0}^\infty$ and $\{\eta_k^\sigma\}_{k=0}^\infty$ such that*

$$\begin{aligned} &0 < \xi_k^\sigma < \eta_k^\sigma < \xi_{k+1}^\sigma, \qquad u^\sigma\left(\xi_k^\sigma\right) = 0, \qquad \partial_r u^\sigma\left(\eta_k^\sigma\right) = 0, \\ &\sigma(-1)^k u^\sigma(r) < 0 \quad \text{in } r \in \left(\xi_{k-1}^\sigma, \xi_k^\sigma\right) \end{aligned}$$

for any $k \in \mathbb{N}$, where $\xi_0^\sigma = 0$ and $\eta_0^\sigma = 0$.

For each $k \in \mathbb{N}$ and $\sigma \in \{-,+\}$, setting

$$\bar{u}_k^\sigma(r) = u^\sigma\left(\frac{\eta_k^\sigma r}{\pi}\right), \qquad \varepsilon_k^\sigma = \left(\frac{\pi}{\eta_k^\sigma}\right)^2, \qquad \mathbf{u}_k^\sigma(r) = \left(u_k^\sigma, v_k^\sigma\right)(r),$$

$$u_k^\sigma(r) = \begin{cases} \bar{u}_k^\sigma(r) & \text{if } \bar{u}_k^\sigma(r) \ge 0, \\ 0 & \text{if } \bar{u}_k^\sigma(r) < 0, \end{cases}$$

$$v_k^\sigma(r) = \begin{cases} 0 & \text{if } \bar{u}_k^\sigma(r) > 0, \\ -\bar{u}_k^\sigma(r) & \text{if } \bar{u}_k^\sigma(r) \le 0, \end{cases}$$

we find out that $\bar{u}_k^-(r)$ and $\bar{u}_k^+(r)$ are solutions of problem (6) for $\varepsilon = \varepsilon_k^-$ and $\varepsilon = \varepsilon_k^+$, respectively, and that $\mathbf{u}_k^-(r)$ and $\mathbf{u}_k^+(r)$ are quasi-positive solutions of the problem

$$\begin{cases} 0 = \varepsilon\mathcal{L}u + u, \quad r \in I_+, \\ 0 = d\varepsilon\mathcal{L}v + v, \quad r \in I_-, \\ u = 0, \qquad v = 0, \qquad \partial_r u = -d\partial_r v, \quad r \in I_0 \\ \partial_r u = 0, \qquad \partial_r v = 0, \qquad r = 0, \pi, \end{cases} \tag{7}$$

where $(0,\pi) = I_- \cup I_0 \cup I_+$, I_0 is the interface that separates two subintervals

$$I_+ = \{r \in (0,\pi) : u(r) > 0 = v(r)\}, \qquad I_- = \{r \in (0,\pi) : u(r) = 0 < v(r)\},$$

and we say that $(u,v)(r)$ is *quasi-positive* if $u(r) \ge 0$, $v(r) \ge 0$, and $(u,v)(r) \not\equiv \mathbf{0}$ are satisfied on $r \in [0,\pi]$. We remark that problem (7) is a linearized one for problem (5) around $(u,v) = \mathbf{0}$. Moreover, in the previous paper [5], it was suggested that positive solutions with large amplitude for problem (5) appear in a neighborhood of $\varepsilon = \varepsilon_1^-$ and $\varepsilon = \varepsilon_1^+$.

3 Solutions with large amplitude

Setting

$$\omega = \Big\{ \max_{x\in[0,\pi]} \big(u(r), v(r)\big) \Big\}^{-1}, \qquad u = \frac{\hat{u}}{\omega}, \qquad v = \frac{\hat{v}}{\omega},$$

we represent problem (5) as follows:

$$\begin{cases} E_u \equiv \omega(\varepsilon\mathcal{L}\hat{u} + \hat{u}) - \hat{u}\hat{v} = 0, \\ E_v \equiv \omega(d\varepsilon\mathcal{L}\hat{v} + \hat{v}) - \hat{u}\hat{v} = 0, \quad r \in (0,\pi), \\ \partial_r \hat{u} = 0, \qquad \partial_r \hat{v} = 0, \qquad r = 0, \pi. \end{cases} \tag{8}$$

We remark that for each $k \in \mathbb{N}$, $\mathbf{u}_k^-(r)$ and $\mathbf{u}_k^+(r)$ are quasi-positive solutions of problem (8) with $\varepsilon = \varepsilon_k^-$ and $\varepsilon = \varepsilon_k^+$, respectively, except for points $\tau \in [0,\pi]$ with $u(\tau) = v(\tau) = 0$. We only discuss the construction of positive solution for problem (8) in a neighborhood of $\varepsilon = \varepsilon_1^+$ by formal calculations employing the singular perturbation technique, because we can similarly discuss that in a neighborhood of $\varepsilon = \varepsilon_1^-$.

We set

$$\varepsilon_0 = \varepsilon_1^+, \qquad \tau = \frac{\pi\xi_1^+}{\eta_1^+} \in (0,\pi), \qquad (u_0, v_0)(r) = \mathbf{u}_1^+(r).$$

Since our purpose of this paper is to seek classical solutions of problem (8), we may impose

$$\langle \partial_r \hat{u} \rangle(\tau) = 0, \qquad \langle \partial_r \hat{v} \rangle(\tau) = 0 \tag{9}$$

on the condition at $r = \tau$. In an analogous way to the argument in Ito [4], we introduce δ as an auxiliary parameter and assume the following estimate:

$$\begin{cases} \xi = (r-\tau)/\delta, \qquad \omega = \delta^3, \qquad \varepsilon = \varepsilon_0 + \varepsilon_1\delta + \varepsilon_2\delta^2 + o(\delta^2), \\ \hat{u}(r) = u_0(r) + u_1(\xi)\delta + u_2(r,\xi)\delta^2 + u_3(r,\xi)\delta^3 + o(\delta^3), \\ \hat{v}(r) = v_0(r) + v_1(\xi)\delta + v_2(r,\xi)\delta^2 + v_3(r,\xi)\delta^3 + o(\delta^3) \end{cases} \tag{10}$$

as $\delta \to 0$, where each function is bounded and continuous. Setting

$$u_{0,k}^{\pm} = \lim_{h \searrow 0} \frac{\partial_r^k u_0(\tau \pm h)}{k!}, \qquad v_{0,k}^{\pm} = \lim_{h \searrow 0} \frac{\partial_r^k v_0(\tau \pm h)}{k!}$$

for each $k \in \mathbb{N}$ and $\gamma_0 = v_{0,1}^+$, we have

$$u_{0,k}^+ = 0, \qquad v_{0,k}^- = 0 \quad \text{for any } k \in \mathbb{N},$$

and

$$u_0(r) = u_0(\tau + \delta\xi) = \begin{cases} u_{0,1}^-\xi\delta + u_{0,2}^-\xi^2\delta^2 + o(\delta^2) & \text{if } r < \tau, \\ 0 & \text{if } r > \tau, \end{cases}$$

$$v_0(r) = v_0(\tau + \delta\xi) = \begin{cases} 0 & \text{if } r < \tau, \\ v_{0,1}^+\xi\delta + v_{0,2}^+\xi^2\delta^2 + o(\delta^2) & \text{if } r > \tau \end{cases}$$

as $\delta \to 0$ for any compact subset of $\mathbb{R}$. The definition of $u_0(r)$ and $v_0(r)$ implies that

$$\begin{aligned} &v_{0,1}^+ = \gamma_0 > 0, \qquad u_{0,1}^- = -dv_{0,1}^+ = -d\gamma_0 < 0, \\ &u_{0,2}^- = -\frac{(\ell-1)u_{0,1}^-}{2\tau} = \frac{d(\ell-1)\gamma_0}{2\tau} > 0, \\ &v_{0,2}^+ = -\frac{(\ell-1)v_{0,1}^+}{2\tau} = -\frac{(\ell-1)\gamma_0}{2\tau} < 0 \end{aligned}$$

hold true. Substituting the above estimate (10) into problem (8), we have

$$\begin{aligned} E_u = {} & \big(\varepsilon_0\partial_\xi^2 u_1(\xi) - u_1(\xi)v_1(\xi) - v_{0,1}^{\pm}\xi u_1(\xi) - u_{0,1}^{\pm}\xi v_1(\xi)\big)\delta^2 \\ & + \big\{\varepsilon_0\partial_\xi^2 u_2(\tau,\xi) - \big(v_1(\xi) + v_{0,1}^{\pm}\xi\big)u_2(\tau,\xi) - \big(u_1(\xi) + u_{0,1}^{\pm}\xi\big)v_2(\tau,\xi) \\ & + \varepsilon_1\partial_\xi^2 u_1(\xi) + \varepsilon_0(\ell-1)\tau^{-1}\partial_\xi u_1(\xi) \\ & - v_{0,2}^{\pm}\xi^2 u_1(\xi) - u_{0,2}^{\pm}\xi^2 v_1(\xi)\big\}\delta^3 + o\big(\delta^3\big), \\ E_v = {} & \big(d\varepsilon_0\partial_\xi^2 v_1(\xi) - u_1(\xi)v_1(\xi) - v_{0,1}^{\pm}\xi u_1(\xi) - u_{0,1}^{\pm}\xi v_1(\xi)\big)\delta^2 \\ & + \big\{d\varepsilon_0\partial_\xi^2 v_2(\tau,\xi) - \big(v_1(\xi) + v_{0,1}^{\pm}\xi\big)u_2(\tau,\xi) - \big(u_1(\xi) + u_{0,1}^{\pm}\xi\big)v_2(\tau,\xi) \\ & + d\varepsilon_1\partial_\xi^2 v_1(\xi) + d\varepsilon_0(\ell-1)\tau^{-1}\partial_\xi v_1(\xi) \\ & - v_{0,2}^{\pm}\xi^2 u_1(\xi) - u_{0,2}^{\pm}\xi^2 v_1(\xi)\big\}\delta^3 + o\big(\delta^3\big) \end{aligned}$$

as $\delta \to 0$ for any compact subset of $\mathbb{R}_\pm$, where $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (0, +\infty)$. Moreover, we see from condition (9) that

$$0 = \langle \partial_r u_0 \rangle(\tau) + \langle \partial_\xi u_1 \rangle(0), \qquad 0 = \big\langle \partial_\xi u_2(\tau, \cdot)\big\rangle(0),$$
$$0 = \langle \partial_r v_0 \rangle(\tau) + \langle \partial_\xi v_1 \rangle(0), \qquad 0 = \big\langle \partial_\xi v_2(\tau, \cdot)\big\rangle(0)$$

are satisfied.

3.1 Solution $(u_1, v_1)(\xi)$

In this subsection, we seek a bounded solution $(u_1, v_1)(\xi)$ of the problem

$$\begin{cases} 0 = \varepsilon_0 \partial_\xi^2 u_1(\xi) - u_1(\xi) v_1(\xi) + d\gamma_0 \xi v_1(\xi), \\ 0 = d\varepsilon_0 \partial_\xi^2 v_1(\xi) - u_1(\xi) v_1(\xi) + d\gamma_0 \xi v_1(\xi), \quad \xi \in \mathbb{R}_-, \\ 0 = \varepsilon_0 \partial_\xi^2 u_1(\xi) - u_1(\xi) v_1(\xi) - \gamma_0 \xi u_1(\xi), \\ 0 = d\varepsilon_0 \partial_\xi^2 v_1(\xi) - u_1(\xi) v_1(\xi) - \gamma_0 \xi u_1(\xi), \quad \xi \in \mathbb{R}_+, \\ u_1(0) = p_1, \qquad \langle \partial_\xi u_1 \rangle(0) = -d\gamma_0, \\ v_1(0) = q_1, \qquad \langle \partial_\xi v_1 \rangle(0) = -\gamma_0 \end{cases} \tag{11}$$

with $(u_1, v_1)(\pm\infty) = \mathbf{0}$, where p_1 and q_1 are positive constants to be determined, and $f(\pm\infty) = \lim_{\xi \to \pm\infty} f(\xi)$. We set

$$w_1(\xi) = u_1(\xi) - d v_1(\xi), \qquad \gamma_1 = p_1 - d q_1.$$

From

$$\langle \partial_\xi w_1 \rangle(0) = \langle \partial_\xi u_1 \rangle(0) - d \langle \partial_\xi v_1 \rangle(0) = 0,$$

it follows that $w_1(\xi)$ is a C^1-class bounded solution of the problem

$$\begin{cases} 0 = \varepsilon_0 \partial_\xi^2 w_1(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+, \\ w_1(0) = \gamma_1, \qquad w_1(\pm\infty) = 0. \end{cases}$$

Hence, we obtain $\gamma_1 = 0$ and $u_1(\xi) = d v_1(\xi)$ for any $\xi \in \mathbb{R}$.

Lemma 3 (Lemma 4.2 in [1]) *The problem*

$$0 = \partial_t^2 u + (t - u)u, \qquad u > \max(0, t), \quad t \in \mathbb{R}$$

has a unique solution $\mathcal{H}(t)$ such that the following holds true:

(1) *$\mathcal{H}(t)$ is a strictly increasing convex function, and*
(2) *$\mathcal{H}(t) = O(e^t)$ as $t \to -\infty$, $\mathcal{H}(t) - t = O(e^{-t})$ as $t \to +\infty$, and $\mathcal{H}(t) = t + \mathcal{H}(-t)$ for any $t \in \mathbb{R}$ are satisfied.*

We note that problem (11) can be represented as

$$0 = \varepsilon_0 \partial_\xi^2 v_1(\xi) - \big(\gamma_0 |\xi| + v_1(\xi)\big) v_1(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+.$$

After simple calculations, it turns out that the bounded solution $(u_1, v_1)(\xi)$ of problem (11) with $u_1(\xi) = dv_1(\xi)$ for any $\xi \in \mathbb{R}$ can be represented as

$$u_1(\xi) = dv_1(\xi), \qquad v_1(\xi) = \begin{cases} \varepsilon_0\alpha_0^2\mathcal{H}(\alpha_0\xi) & \text{if } \xi \in \mathbb{R}_-, \\ \varepsilon_0\alpha_0^2\mathcal{H}(-\alpha_0\xi) & \text{if } \xi \in \mathbb{R}_+, \end{cases}$$

where $\alpha_0 = (\gamma_0/\varepsilon_0)^{1/3} > 0$. Hence, we take p_1 and q_1 as satisfying

$$p_1 = d\varepsilon_0\alpha_0^2\mathcal{H}(0), \qquad q_1 = \varepsilon_0\alpha_0^2\mathcal{H}(0).$$

Moreover, we can check

$$\frac{\langle u_1\rangle(0)}{d} = \langle v_1\rangle(0) = -2\varepsilon_0\alpha_0^3\partial_t\mathcal{H}(0) = -\gamma_0$$

because of $\partial_t\mathcal{H}(0) = 1/2$, that is, the difference between left-hand and right-hand differential coefficients in problem (11) always holds true. Employing $v_1(\pm\infty) = 0$, we calculate the asymptotic expansion of $v_1(\xi)$ as $|\xi| \to +\infty$, and then we have

$$v_1(\xi) = C_\pm^0 \operatorname{Ai}\bigl(\alpha_0|\xi|\bigr)\bigl(1 + o(1)\bigr)$$

as $\xi \to \pm\infty$, where $\operatorname{Ai}(z)$ is the Airy function of the first kind, and C_-^0 and C_+^0 are suitable positive constants.

3.2 Constant ε_1

In this subsection, setting $w(\xi) = u_2(\tau, \xi)$ and $z(\xi) = v_2(\tau, \xi)$, we seek a bounded solution $(w, z)(\xi)$ of the problem

$$\begin{cases} 0 = \varepsilon_0\partial_\xi^2 w(\xi) - v_1(\xi)w(\xi) - (u_1(\xi) - d\gamma_0\xi)z(\xi) \\ \qquad + \varepsilon_1\partial_\xi^2 u_1(\xi) + \varepsilon_0(\ell - 1)\tau^{-1}\partial_\xi u_1(\xi) - u_{0,2}^-\xi^2 v_1(\xi), \\ 0 = d\varepsilon_0\partial_\xi^2 z(\xi) - v_1(\xi)w(\xi) - (u_1(\xi) - d\gamma_0\xi)z(\xi) \\ \qquad + d\varepsilon_1\partial_\xi^2 v_1(\xi) + d\varepsilon_0(\ell - 1)\tau^{-1}\partial_\xi v_1(\xi) - u_{0,2}^-\xi^2 v_1(\xi), \quad \xi \in \mathbb{R}_-, \\ 0 = \varepsilon_0\partial_\xi^2 w(\xi) - (v_1(\xi) + \gamma_0\xi)w(\xi) - u_1(\xi)z(\xi) \\ \qquad + \varepsilon_1\partial_\xi^2 u_1(\xi) + \varepsilon_0(\ell - 1)\tau^{-1}\partial_\xi u_1(\xi) - v_{0,2}^+\xi^2 u_1(\xi), \\ 0 = d\varepsilon_0\partial_\xi^2 z(\xi) - (v_1(\xi) + \gamma_0\xi)w(\xi) - u_1(\xi)z(\xi) \\ \qquad + d\varepsilon_1\partial_\xi^2 v_1(\xi) + d\varepsilon_0(\ell - 1)\tau^{-1}\partial_\xi v_1(\xi) - v_{0,2}^+\xi^2 u_1(\xi), \quad \xi \in \mathbb{R}_+, \\ w(0) = p_2, \qquad \langle\partial_\xi w\rangle(0) = 0, \qquad z(0) = q_2, \qquad \langle\partial_\xi z\rangle(0) = 0 \end{cases} \tag{12}$$

with $(w, z)(\pm\infty) = \mathbf{0}$, where p_2 and q_2 are suitable constants to be determined. From the condition at $\xi = 0$, we remark that $w(\xi)$ and $z(\xi)$ are of C^1-class in $\xi \in \mathbb{R}$.

Since $\hat{w}(\xi) = w(\xi) - dz(\xi)$ is a C^1-class solution of the problem

$$\begin{cases} 0 = \varepsilon_0\partial_\xi^2\hat{w}(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+, \\ \hat{w}(0) = p_2 - dq_2, \qquad \hat{w}(\pm\infty) = 0, \end{cases}$$

we have $\hat{w}(\xi) = 0$ for any $\xi \in \mathbb{R}$, which implies $p_2 = dq_2$ and $w(\xi) = dz(\xi)$ for any $\xi \in \mathbb{R}$. Hence, we find out that $z(\xi)$ is a C^1-class solution of the problem

$$\begin{cases} \mathcal{K}z(\xi) = \varepsilon_1 F_+(\xi) + F_-(\xi), & \xi \in \mathbb{R}_- \cup \mathbb{R}_+, \\ z(0) = q_2, \qquad \partial_\xi z(0) = \hat{q}_2, \end{cases} \tag{13}$$

where

$$A(\xi) = 2\nu_1(\xi) + \gamma_0|\xi|, \qquad \mathcal{K}z(\xi) = \varepsilon_0 \partial_\xi^2 z(\xi) - A(\xi)z(\xi),$$
$$F_+(\xi) = -\partial_\xi^2 \nu_1(\xi), \qquad F_-(\xi) = -\varepsilon_0(\ell - 1)\tau^{-1}\partial_\xi \nu_1(\xi) + \nu_{0,2}^+ \xi|\xi|\nu_1(\xi),$$

and $\hat{q}_2$ is a constant to be determined. By Lemma 3 and the expression of $\nu_1(\xi)$, we have $A(\xi) > 0$ for any $\xi \in \mathbb{R}$. Since $\nu_1(\xi)$ is an even function in $\xi \in \mathbb{R}$, it turns out that $F_-(\xi)$ (respectively, $F_+(\xi)$) is a bounded odd (respectively, even) function in $\xi \in \mathbb{R}$.

Let $\psi(\xi)$ be the solution of the problem

$$0 = \mathcal{K}\psi(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+ \tag{14}$$

with $\psi(0) = 1$ and $\partial_\xi \psi(0) = 0$. Since $\nu_1(\xi)$ and $A(\xi)$ are even functions in $\xi \in \mathbb{R}$ and $A(\xi) > 0$ holds true for any $\xi \in \mathbb{R}$, we see that $\psi(\xi)$ is positive and increasing in $\xi \in \mathbb{R}_+$ and satisfies $\psi(\xi) = \psi(-\xi)$ for any $\xi \in \mathbb{R}$. From

$$\psi(\xi) = 1 + \frac{1}{\varepsilon_0}\int_0^\xi \int_0^t A(s)\psi(s)\,ds\,dt \geq 1 + \frac{\gamma_0}{\varepsilon_0}\int_0^\xi \int_0^t s\,ds\,dt = 1 + \frac{\gamma_0 \xi^3}{6\varepsilon_0}$$

in $\xi \in \mathbb{R}_+$, it follows that

$$\phi_-(\xi) = \psi(\xi)\int_\xi^{+\infty} \frac{1}{\{\psi(s)\}^2}\,ds, \quad \xi \in \mathbb{R},$$

is a positive solution of problem (14) and satisfies

$$\phi_-(0) = \int_{\mathbb{R}_+} \frac{1}{\{\psi(s)\}^2}\,ds > 0, \qquad \partial_\xi \phi_-(0) = -1, \qquad \lim_{\xi \to -\infty} \phi_-(\xi) = +\infty.$$

Moreover, it follows that $\phi_+(\xi)$ defined by $\phi_+(\xi) = \phi_-(-\xi)$ for any $\xi \in \mathbb{R}$ is a positive solution of problem (14) in $\xi \in \mathbb{R}$, and that

$$\begin{aligned} \lim_{\xi \to -\infty} \phi_+(\xi) &= \lim_{\xi \to +\infty} \phi_-(\xi) \\ &= \lim_{\xi \to +\infty} \left\{ \psi(\xi)\int_\xi^{+\infty} \frac{1}{\{\psi(s)\}^2}\,ds \right\} = \lim_{\xi \to +\infty} \frac{1}{\partial_\xi \psi(\xi)} = 0 \end{aligned}$$

holds true because of L'Hôpital's rule. Summarizing the above argument, we see that $\{\phi_-(\xi), \phi_+(\xi)\}$ is a fundamental set of solution for problem (14).

Employing $\nu_1(\pm\infty) = 0$, we calculate the asymptotic expansion of $\phi_-(\xi)$ and $\phi_+(\xi)$ as $|\xi| \to +\infty$, and then we have

$$\phi_-(\xi) = \begin{cases} C_-^- \operatorname{Bi}(\alpha_0|\xi|)(1 + o(1)) & \text{as } \xi \to -\infty, \\ C_+^- \operatorname{Ai}(\alpha_0|\xi|)(1 + o(1)) & \text{as } \xi \to +\infty, \end{cases}$$

$$\phi_+(\xi) = \begin{cases} C_-^+ \operatorname{Ai}(\alpha_0|\xi|)(1 + o(1)) & \text{as } \xi \to -\infty, \\ C_+^+ \operatorname{Bi}(\alpha_0|\xi|)(1 + o(1)) & \text{as } \xi \to +\infty, \end{cases}$$

where $\operatorname{Bi}(t)$ is the Airy function of the second kind, and $C_\pm^-$ and $C_\pm^+$ are suitable nonzero constants. Employing the asymptotic expansion of $\operatorname{Ai}(t)$ and $\operatorname{Bi}(t)$ as $t \to +\infty$, we have

$$\frac{\operatorname{Ai}(t)}{\partial_t \operatorname{Ai}(t)} = -\frac{1}{\sqrt{t}}\big(1 + o(1)\big), \qquad \frac{\operatorname{Bi}(t)}{\partial_t \operatorname{Bi}(t)} = \frac{1}{\sqrt{t}}\big(1 + o(1)\big),$$
$$\operatorname{Ai}(t)\operatorname{Bi}(t) = \frac{1}{2\pi\sqrt{t}}\big(1 + o(1)\big),$$

as $t \to +\infty$, and then we obtain

$$\frac{\{\phi_-(\xi)\}^2\phi_+(\xi)}{\partial_\xi\phi_-(\xi)} = -\frac{C_\pm^- C_\pm^+}{2\pi\alpha_0^2|\xi|}\big(1 + o(1)\big) \to 0,$$
$$\frac{\phi_-(\xi)\{\phi_+(\xi)\}^2}{\partial_\xi\phi_+(\xi)} = \frac{C_\pm^- C_\pm^+}{2\pi\alpha_0^2|\xi|}\big(1 + o(1)\big) \to 0$$

as $\xi \to \pm\infty$. From the above estimate and L'Hôpital's rule, it follows that for any bounded function $f(\xi)$ from $\mathbb{R}$ to itself, the functions

$$\int_{-\infty}^{\xi} \phi_-(\xi)\phi_+(s)f(s)\,ds, \qquad \int_{\xi}^{+\infty} \phi_-(s)\phi_+(\xi)f(s)\,ds$$

from $\mathbb{R}$ to itself are bounded and satisfy

$$\lim_{\xi\to\pm\infty}\int_{-\infty}^{\xi} \phi_-(\xi)\phi_+(s)f(s)\,ds = 0, \qquad \lim_{\xi\to\pm\infty}\int_{\xi}^{+\infty} \phi_-(s)\phi_+(\xi)f(s)\,ds = 0.$$

Let $f(\xi)$ be an arbitrary bounded function from $\mathbb{R}$ to itself. After simple calculations, we see that the bounded solution $W(\xi)$ of the problem

$$\mathcal{K}W(\xi) = f(\xi), \quad \xi \in \mathbb{R}$$

can be represented as

$$W(\xi) = -\frac{1}{\varepsilon_0\Phi_0}\left(\int_{-\infty}^{\xi} \phi_-(\xi)\phi_+(s)f(s)\,ds + \int_{\xi}^{+\infty} \phi_-(s)\phi_+(\xi)f(s)\,ds\right) \tag{15}$$

for any $\xi \in \mathbb{R}$, where

$$\Phi_0 = \det\begin{pmatrix} \phi_-(0) & \phi_+(0) \\ \partial_\xi\phi_-(0) & \partial_\xi\phi_+(0) \end{pmatrix} = 2\int_{\mathbb{R}_+} \frac{1}{\{\psi(s)\}^2}\,ds > 0.$$

We denote by $\mathcal{K}^{-1}f(\xi)$ the solution $W(\xi)$ given in equation (15). Since

$$W(-\xi) = -\frac{1}{\varepsilon_0 \Phi_0}\left(\int_{-\infty}^{\xi} \phi_-(\xi)\phi_+(s)f(-s)\,ds + \int_{\xi}^{+\infty} \phi_-(s)\phi_+(\xi)f(-s)\,ds\right)$$

holds true because of $\phi_-(\xi) = \phi_+(-\xi)$ for any $\xi \in \mathbb{R}$, we obtain $W(-\xi) = W(\xi)$ (respectively, $W(-\xi) = -W(\xi)$)for any $\xi \in \mathbb{R}$ if $f(\xi)$ is an even (respectively, odd) function in $\xi \in \mathbb{R}$

Let $z(\xi)$ be a solution of problem (13) with $z(\pm\infty) = 0$ for $(\varepsilon_1, q_2, \hat{q}_2) \in \mathbb{R}^3$. Since $A(\xi)$ is an even function in $\xi \in \mathbb{R}$, it follows that

$$Z_+(\xi) = \frac{z(\xi) + z(-\xi)}{2}, \qquad Z_-(\xi) = \frac{z(\xi) - z(-\xi)}{2}$$

satisfy

$$\begin{cases} \mathcal{K}Z_-(\xi) = F_-(\xi), \\ \mathcal{K}Z_+(\xi) = \varepsilon_1 F_+(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+, \\ Z_-(0) = 0, \qquad \partial_\xi Z_-(0) = \hat{q}_2, \qquad Z_+(0) = q_2, \qquad \partial_\xi Z_+(0) = 0. \end{cases}$$

Since $F_-(\xi)$ and $F_+(\xi)$ are bounded in $\xi \in \mathbb{R}$, we can represent the solution $z(\xi)$ of problem (13) as

$$z(\xi) = \hat{Z}_-(\xi) + \varepsilon_1 \hat{Z}_+(\xi), \quad \xi \in \mathbb{R}_- \cup \mathbb{R}_+, \tag{16}$$

where

$$\hat{Z}_-(\xi) = \left[\mathcal{K}^{-1}F_-\right](\xi), \qquad \hat{Z}_+(\xi) = \left[\mathcal{K}^{-1}F_+\right](\xi), \quad \xi \in \mathbb{R}.$$

By integration by parts, we have

$$\begin{aligned} -\varepsilon_0 \hat{q}_2 \phi_-(0) &= \int_{\mathbb{R}_+} \left(\mathcal{K}\hat{Z}_-(s)\phi_-(s) - \hat{Z}_-(s)\mathcal{K}\phi_-(s)\right) ds = \int_{\mathbb{R}_+} F_-(s)\phi_-(s)\,ds, \\ -\varepsilon_0 \hat{Z}_+(0) &= \int_{\mathbb{R}_+} \left(\mathcal{K}\hat{Z}_+(s)\phi_-(s) - \hat{Z}_+(s)\mathcal{K}\phi_-(s)\right) ds = \int_{\mathbb{R}_+} F_+(s)\phi_-(s)\,ds \\ &= -\frac{1}{\varepsilon_0}\int_{\mathbb{R}_+} \left[\left\{\nu_1(s)\right\}^2 + \gamma_0 s \nu_1(s)\right]\phi_-(s)\,ds < 0, \end{aligned}$$

which implies that $\hat{q}_2$ and $\hat{Z}_+(0)$ are determined by the above equations. Hence, we see from $q_2 = \hat{Z}_+(0)\varepsilon_1$ that the positive solution of problem (12) is determined for any fixed $\varepsilon_1 \in \mathbb{R}$.

Lemma 4 *Let $\varepsilon_1 \in \mathbb{R}$ be arbitrarily fixed, and let $(w, z)(\xi)$ be a solution of problem* (12) *with $(w, z)(\pm\infty) = \mathbf{0}$. Then $w(\xi) = dz(\xi)$ holds true for any $\xi \in \mathbb{R}$ and $z(\xi)$ is represented as equation* (16).

Because rigorous proofs are not given in the above arguments, the summary of this paper is written down as a conjecture.

Conjecture 1 *If we take ω as $\omega = \delta^3$, then there exists $\delta_0 > 0$ such that there exist continuous functions $(u,v)(\cdot,\delta)$ and $\varepsilon(\delta)$ defined on an interval $(0,\delta_0)$ such that $(u,v)(r,\delta)$ is a positive solution of problem (8) with $\varepsilon = \varepsilon(\delta)$ for each $\delta \in (0,\delta_0)$, and*

$$\lim_{\delta \searrow 0} \varepsilon(\delta) = \varepsilon_0,$$

$$\lim_{\delta \searrow 0} (u,v)(\cdot,\delta) = (u_0,v_0)(\cdot) \quad \textit{uniformly in any compact set on } (0,\pi)$$

are satisfied.

4 Conclusion and future works

Since we can study the rough solution structure and the fine solution structure of problem (1) through limiting systems (3) and (4), respectively, we might guess that these structures lead to the global solution structure for problem (1). In this paper, we formally constructed a family of solutions for problem (8) arising from limiting system (4). To do this, we employed the singular perturbation technique such as in Ito [4].

Recently in [6] it was shown that $0 < \varepsilon_1^- < \varepsilon_1^+$ holds true for any $d > 1$. Furthermore, employing the argument as in Dancer et al. [2], it follows that limiting system (3) has no nontrivial solutions for any $\varepsilon > \varepsilon_1^+$ and has quasi-positive solutions for any $0 < \varepsilon < \varepsilon_1^+$. Although we have not yet determined the value of ε_1 in estimate (10), it is possible that we could determine its value by studying the higher order asymptotic expansion of the solution as $\delta \searrow 0$. If the validity of $\varepsilon(\delta) > \varepsilon_1^+$ can be proved, then we obtain a positive solution of problem (1), which does not appear in limiting system (3), by employing limiting system (4).

Since problem (1) has nonlinear diffusion terms, only the linearized stability is not sufficient for the stability analysis of solution for the system of evolution equations whose stationary problem is represented as problem (1), so that the stability analysis is still open also for the solution corresponding to the positive solution of limiting systems (3) and (4). Since in this paper we have obtained the asymptotic expansion of positive solutions with large amplitude for limiting system (4), it may be possible to establish the stability of the solution by using the expansion of the solution constructed in this paper.

Since the existence of the solutions constructed in this paper seems to be proved in an analogous way to the argument in Ito [4], one of our future works is that we establish a rigorous proof.

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