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Positive solutions for a class of fractional differential equations with infinite-point boundary conditions on infinite intervals

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Abstract

In this paper, the existence of the multiple positive solutions for a class of higher-order fractional differential equations on infinite intervals with infinite-point boundary value conditions is mainly studied. First, we construct the Green function and analyze its properties, and then by using the Leggett–Williams fixed point theorem, some new results on the existence of positive solutions for the boundary value problem are obtained. Finally, we illustrate the application of our conclusion by an example.

Keywords: Fractional differential equation; Infinite intervals; Multiple positive solutions; Infinite-point

1 Introduction

Boundary value problems of fractional differential equations have always been of great interest to researchers and are of great importance in the fields of physics, biology, chemistry, control theory, fluid mechanics, aerodynamics, complex medium electrodynamics, and other areas of engineering and science [1–6]. The Guo–Krasnoselskii fixed point theorem, Avery–Peterson fixed point theorem, Leggett–Williams fixed point theorem, etc., are important research tools in solving fractional differential equations of boundary value conditions [7–9]. In recent years, the study of finite multipoint boundary value problems for fractional differential equations on finite intervals has yielded more significant results [10–17]. However, the existence of multiple positive solutions for fractional differential equations with infinite multipoint boundary conditions on infinite intervals is relatively rare.

In [10], the authors investigated the fractional differential equation boundary value problems at resonance:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + \lambda f(t, x(t), D_{0^+}^{\beta}x(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \\ D_{0^+}^{\beta}x(1) = \sum_{i=1}^{m} \eta_i D_{0^+}^{\beta}x(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $\alpha - 2 > \beta > 0$, $\eta_i > 0$, $0 < \xi_1 < \cdots < \xi_m < 1$ with $\sum_{i=1}^{m} \eta_i \xi_i^{\alpha-\beta-1} = 1$, and $D_{0^+}^{\alpha}$ denotes the Riemann–Liouville derivative.

In [11], the authors investigated the following m-point p-Laplacian fractional boundary value problem involving Riemann–Liouville fractional integral boundary conditions on the half-line:

$$\begin{cases} D_{0^+}^{\gamma}(\phi_p(D_{0^+}^{\alpha}u(t))) + a(t)f(t,u(t),u'(t)) = 0, \quad t \in [0,+\infty), \\ u(0) = u'(0) = 0, \\ \lim_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2} \eta_i I_{0^+}^{\beta} u'(\xi_i), \qquad D_{0^+}^{\alpha}u(0) = 0, \end{cases}$$

where $D_{0^+}^{\gamma}$ and $D_{0^+}^{\alpha}$ are the standard Riemann–Liouville fractional derivatives and $I_{0^+}^{\beta}$ is the standard Riemann–Liouville fractional integral, $0 < \gamma \leq 1$, $2 < \alpha \leq 3$, $\beta > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\eta_i > 0$, $i = 1,2,\ldots,m-2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Motivated by the above papers, in this work, we consider the following fractional differential equations with infinite-point boundary conditions on an infinite interval:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + a(t)f(t,u(t),u'(t)) = 0, \quad t \in [0,+\infty), \\ u^{(j)}(0) = 0, \quad j = 0,1,2,\ldots,n-2, \\ \lim_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = \sum_{i=1}^{\infty} \eta_i I_{0^+}^{\beta}(D_{0^+}^{\delta}u(t))(\xi_i) + \sum_{i=1}^{\infty} \rho_i D_{0^+}^{\sigma}u(\xi_i), \end{cases} \tag{1}$$

where $D_{0^+}^{\epsilon}$ is the Riemann–Liouville fractional derivative, $\epsilon \in \{\alpha,\delta,\sigma\}$, $I_{0^+}^{\beta}$ is the Riemann–Liouville fractional integral, $n-1 < \alpha \leq n$, $n > 3$, $l, n \in \mathbb{N}^+$, $\beta > 0$, $\delta > 0$, $\sigma \geq 0$, and $\sigma < \alpha - 1$, $\delta < \alpha + \beta - 1$, $\delta < \alpha$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < +\infty$, η_i, $\rho_i \geq 0$, $i = 1,2,\ldots$.

For boundary value problem (1), we will first construct the Green function and then use some properties of the Green function to obtain at least three positive solutions to boundary value problem (1) by using the Leggett–Williams fixed point theorem.

The research in this paper is different from the existing studies. In [11], the boundary condition contained finite integral terms, and the authors obtained existence of one positive solution by using the Leray–Schauder nonlinear alternative theorem. In this paper, the boundary condition of boundary value problem (1) contains infinite integral terms and infinite points, and the order of the fractional derivative is higher. The method which we use is the Leggett–Williams fixed point theorem, and we get the existence of three positive solutions. The new results of this paper can be considered as a contribution to this field.

The organization of this paper is as follows. In Sect. 2, we show some necessary definitions and lemmas from fractional calculus theory. In Sect. 3, we prove the existence of multiple positive solutions of boundary value problem (1). In Sect. 4, we will give an example to illustrate the applicability of our conclusions.

Now we list some conditions for convenience:

(H_1) $\varkappa := \Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma) - \Gamma(\alpha-\sigma)\sum_{i=1}^{\infty} \eta_i \xi_i^{\alpha+\beta-\delta-1} - \Gamma(\alpha+\beta-\delta)\sum_{i=1}^{\infty} \rho_i \xi_i^{\alpha-\sigma-1} > 0$;

(H_2) $f \in C([0,+\infty) \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$, and when u, v are bounded, $f(t,(1+t^{\alpha-1})u, (1+t^{\alpha-1})v)$ is bounded;

(H_3) $a \in C([0,+\infty),[0,+\infty))$ is not constant to 0 on any subinterval of $[0,+\infty)$, and

$$\int_0^{+\infty} a(s)\,ds < +\infty.$$

2 Preliminaries

In this section, some definitions and lemmas are introduced.

Definition 2.1 The Riemann–Liouville fractional integral of order γ ($\gamma > 0$) for a function $f : (t_0, +\infty) \to \mathbb{R}$ is defined as

$$I_{t_0^+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} f(s)\, ds, \quad t \geq t_0.$$

Definition 2.2 The Riemann–Liouville fractional derivative of order γ ($\gamma > 0$) for a function $f : (t_0, +\infty) \to \mathbb{R}$ is defined as

$$D_{t_0^+}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-\gamma-1} f(s)\, ds, \quad t > t_0,$$

where $n = [\gamma] + 1$, where $[\gamma]$ denotes the integer part of the real number γ.

Lemma 2.1 ([11]) *Suppose that* $u \in C(0,1) \cap L^1(0,1)$, $\alpha > 0$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R}, i = 1, 2, \ldots, N,$$

where N *is the smallest integer greater than or equal to* α.

Lemma 2.2 ([11]) *If* $\alpha, \beta > 0$, $f \in L^1[a,b]$, *then* $I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t) = I_{0^+}^{\beta} I_{0^+}^{\alpha} f(t)$, $D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t)$, $\forall t \in [a,b]$.

Lemma 2.3 ([11]) *If* $\alpha, \beta > 0$, *then*

$$\begin{cases} D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, & \beta > n, \\ D_{0^+}^{\alpha} t^{k} = 0, & k = 0, 1, 2, \ldots, n-1, \end{cases}$$

where n *is the smallest integer greater than or equal to* α.

Lemma 2.4 *Suppose* (H_1) *holds and let* $h \in C[0, +\infty)$. *Then the fractional differential equation boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0, & t \in [0, +\infty), \\ u^{(j)}(0) = 0, & j = 0, 1, 2, \ldots, n-2, \\ \lim_{t \to +\infty} D_{0^+}^{\alpha-1} u(t) = \sum_{i=1}^{\infty} \eta_i I_{0^+}^{\beta} (D_{0^+}^{\delta} u(t))|_{t=\xi_i} + \sum_{i=1}^{\infty} \rho_i D_{0^+}^{\sigma} u(\xi_i) \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t,s) h(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)} [\frac{p(s)}{p(0)} t^{\alpha-1} - (t-s)^{\alpha-1}], & 0 \leq s \leq t < +\infty, \\ \frac{p(s)}{\Gamma(\alpha) p(0)} t^{\alpha-1}, & 0 \leq t \leq s < +\infty, \end{cases} \tag{3}$$

$$\begin{aligned} p(s) &= \Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma) - \Gamma(\alpha-\sigma)\sum_{s\le\xi_i}\eta_i(\xi_i-s)^{\alpha+\beta-\delta-1} \\ &\quad - \Gamma(\alpha+\beta-\delta)\sum_{s\le\xi_i}\rho_i(\xi_i-s)^{\alpha-\sigma-1}. \end{aligned} \tag{4}$$

Proof In view of Lemma 2.1, applying $I_{0^+}^{\alpha}$ to both sides of $D_{0^+}^{\alpha}u(t)+h(t)=0$, we have

$$u(t) = -I_{0^+}^{\alpha}h(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_{n-1}t^{\alpha-n+1} + c_nt^{\alpha-n}, \quad c_i\in\mathbb{R}, i=1,2,\dots,n.$$

By $u^{(j)}(0)=0$, $j=0,1,2,\dots,n-2$, we can know $c_2=c_3=\cdots=c_n=0$, so $u(t)=-I_{0^+}^{\alpha}h(t)+c_1t^{\alpha-1}$.

We have

$$\begin{aligned} D_{0^+}^{\alpha-1}u(t) &= -D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}h(t) + c_1D_{0^+}^{\alpha-1}t^{\alpha-1} \\ &= -\int_0^t h(s)\,ds + c_1\frac{\Gamma(\alpha)}{\Gamma(1)}t^{\alpha-1-\alpha+1} = -\int_0^t h(s)\,ds + c_1\Gamma(\alpha), \\ I_{0^+}^{\beta}D_{0^+}^{\delta}u(t) &= -I_{0^+}^{\beta}D_{0^+}^{\delta}I_{0^+}^{\alpha}h(t) + c_1I_{0^+}^{\beta}D_{0^+}^{\delta}t^{\alpha-1} \\ &= -I_{0^+}^{\beta}D_{0^+}^{\delta}\left(D_{0^+}^{-\delta}I_{0^+}^{\alpha-\delta}\right)h(t) + c_1I_{0^+}^{\beta}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}\right) \\ &= -I_{0^+}^{\beta}I_{0^+}^{\alpha-\delta}h(t) + c_1\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta)}\left(\frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha+\beta-\delta)}t^{\alpha+\beta-\delta-1}\right)\right) \\ &= -I_{0^+}^{\alpha+\beta-\delta}h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta-\delta)}t^{\alpha+\beta-\delta-1}, \\ D_{0^+}^{\sigma}u(t) &= -I_{0^+}^{\alpha-\sigma}h(t) + c_1D_{0^+}^{\sigma}t^{\alpha-1} \\ &= -I_{0^+}^{\alpha-\sigma}h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}t^{\alpha-\sigma-1}. \end{aligned}$$

By the boundary condition $\lim_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)=\sum_{i=1}^{\infty}\eta_iI_{0^+}^{\beta}(D_{0^+}^{\delta}u(t))|_{\xi_i}+\sum_{i=1}^{\infty}\rho_iD_{0^+}^{\sigma}u(\xi_i)$, we obtain

$$\begin{aligned} -\int_0^{+\infty}h(s)\,ds + c_1\Gamma(\alpha) &= \sum_{i=1}^{\infty}\eta_i\left(-I_{0^+}^{\alpha+\beta-\delta}h(\xi_i) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta-\delta)}\xi_i^{\alpha+\beta-\delta-1}\right) \\ &\quad + \sum_{i=1}^{\infty}\rho_i\left(-I_{0^+}^{\alpha-\sigma}h(\xi_i) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\xi_i^{\alpha-\sigma-1}\right). \end{aligned}$$

Then

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha) - \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta-\delta)}\sum_{i=1}^{\infty}\eta_i\xi_i^{\alpha+\beta-\delta-1} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\sum_{i=1}^{\infty}\rho_i\xi_i^{\alpha-\sigma-1}} \\ &\quad \times\left(\int_0^{+\infty}h(s)\,ds - \frac{1}{\Gamma(\alpha+\beta-\delta)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\beta-\delta-1}h(s)\,ds\right. \\ &\quad \left.- \frac{1}{\Gamma(\alpha-\sigma)}\sum_{i=1}^{\infty}\rho_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\sigma-1}h(s)\,ds\right) \end{aligned}$$

$$= \frac{\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)}{\Gamma(\alpha)\varkappa}\left(\int_0^{+\infty} h(s)\,ds - \frac{1}{\Gamma(\alpha+\beta-\delta)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\beta-\delta-1}h(s)\,ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha-\sigma)}\sum_{i=1}^{\infty}\rho_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\sigma-1}h(s)\,ds\right).$$

Thereby

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{t^{\alpha-1}\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)}{\Gamma(\alpha)\varkappa}\int_0^{+\infty} h(s)\,ds \\
&\quad - \frac{t^{\alpha-1}\Gamma(\alpha-\sigma)}{\Gamma(\alpha)\varkappa}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha+\beta-\delta-1}h(s)\,ds \\
&\quad - \frac{t^{\alpha-1}\Gamma(\alpha+\beta-\delta)}{\Gamma(\alpha)\varkappa}\sum_{i=1}^{\infty}\rho_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-\sigma-1}h(s)\,ds \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}\frac{p(s)}{p(0)}h(s)\,ds \\
&= \int_0^{+\infty} G(t,s)h(s)\,ds. \qquad \square
\end{aligned}$$

Lemma 2.5 *Suppose* (H_1) *holds and* $p(0) > 0$. *Then* $p(s) > 0$, $\forall s \in [0,+\infty)$.

Proof By (4),

$$\begin{aligned}
p'(s) &= (\alpha+\beta-\delta-1)\Gamma(\alpha-\sigma)\sum_{s\le\xi_i}\eta_i(\xi_i-s)^{\alpha+\beta-\delta-2} \\
&\quad + (\alpha-\sigma-1)\Gamma(\alpha+\beta-\delta)\sum_{s\le\xi_i}\rho_i(\xi_i-s)^{\alpha-\sigma-2} \\
&> 0.
\end{aligned}$$

Then $p(s)$ is a monotonically increasing function in $[0,+\infty)$. By $p(0) > 0$, we get $p(s) > 0$. $\square$

Lemma 2.6 *The function* $G(t,s)$ *in Lemma* 2.4 *satisfies the following properties*:

(i) $G(t,s)$ *and* $\frac{\partial G(t,s)}{\partial t}$ *are continuous on* $[0,+\infty)$;

(ii) $G(t,s) \ge 0$, $\frac{\partial G(t,s)}{\partial t} \ge 0$, $\forall s,t \in [0,+\infty)$;

(iii) $\frac{G(t,s)}{1+t^{\alpha-1}} \le \frac{p(s)}{\Gamma(\alpha)p(0)}$, $\frac{\partial G(t,s)}{\partial t}(1+t^{\alpha-1})^{-1} \le \frac{p(s)(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}$;

(iv) *for* $k > 1$, *we have* $\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\alpha-1}} \ge g_1(s)$, $\min_{\frac{1}{k}\le t\le k}\frac{\partial G(t,s)}{\partial t}(1+t^{\alpha-1})^{-1} \ge g_2(s)$, *where*

$$g_1(s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left(\frac{1}{k^{\alpha-1}} - \left(\frac{1}{k}-s\right)^{\alpha-1}\right)\frac{1}{1+k^{1-\alpha}}, & s \in [0,\frac{1}{k}), \\ \frac{1}{\Gamma(\alpha)k^{\alpha-1}(1+k^{1-\alpha})}, & s \in [\frac{1}{k},+\infty), \end{cases} \tag{5}$$

$$g_2(s) = \begin{cases} \frac{1}{\Gamma(\alpha-1)}\left(\frac{1}{k^{\alpha-2}} - \left(\frac{1}{k}-s\right)^{\alpha-2}\right)\frac{1}{1+k^{1-\alpha}}, & s \in [0,\frac{1}{k}), \\ \frac{1}{\Gamma(\alpha-1)k^{\alpha-2}(1+k^{1-\alpha})}, & s \in [\frac{1}{k},+\infty). \end{cases} \tag{6}$$

Proof By calculation we can give

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}\frac{1}{\Gamma(\alpha-1)}[\frac{p(s)}{p(0)}t^{\alpha-2}-(t-s)^{\alpha-2}], & 0\le s\le t<+\infty,\\ \frac{p(s)}{\Gamma(\alpha-1)p(0)}t^{\alpha-2}, & 0\le t\le s<+\infty.\end{cases}$$

According to the definition of $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$, it is clear that (i) and (ii) hold. Next we will prove that (iii) and (iv) hold.

For all $t,s\in[0,+\infty)$, $\frac{G(t,s)}{1+t^{\alpha-1}}\le\frac{p(s)t^{\alpha-1}}{\Gamma(\alpha)p(0)(1+t^{\alpha-1})}$. Let $\Phi(t)=\frac{t^{\alpha-1}}{1+t^{\alpha-1}}$. Then

$$\Phi'(t)=\frac{(\alpha-1)t^{\alpha-2}(1+t^{\alpha-1})-(\alpha-1)t^{\alpha-2}t^{\alpha-1}}{(1+t^{\alpha-1})^2}=\frac{(\alpha-1)t^{\alpha-2}}{(1+t^{\alpha-1})^2}.$$

As $\Phi'(t)=0$, we have $t=0$. Therefore, $\Phi(t)$ is monotonically increasing on $[0,+\infty)$. Moreover,

$$\Phi(t)_{\max_{0\le t\le+\infty}}=\lim_{t\to+\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1.$$

It follows that $\frac{G(t,s)}{1+t^{\alpha-1}}\le\frac{p(s)}{\Gamma(\alpha)p(0)}$.

For all $t,s\in[0,+\infty)$, $\frac{\partial G(t,s)}{\partial t}(1+t^{\alpha-1})^{-1}\le\frac{p(s)t^{\alpha-2}}{\Gamma(\alpha-1)p(0)(1+t^{\alpha-1})}$. Let $\Psi(t)=\frac{t^{\alpha-2}}{1+t^{\alpha-1}}$. Then

$$\Psi'(t)=\frac{(\alpha-2)t^{\alpha-3}(1+t^{\alpha-1})-(\alpha-1)t^{\alpha-2}t^{\alpha-2}}{(1+t^{\alpha-1})^2}=\frac{(\alpha-2)t^{\alpha-3}-t^{2\alpha-4}}{(1+t^{\alpha-1})^2}.$$

When $\Psi'(t)=0$, we get $t=0$ or $t=(\alpha-2)^{\frac{1}{\alpha-1}}$. Because $\Psi(0)=0$ and $\Psi((\alpha-2)^{\frac{1}{\alpha-1}})>0$, we know that $\Psi(t)$ is monotonically increasing on $[0,(\alpha-2)^{\frac{1}{\alpha-1}})$ and monotonically decreasing on $[(\alpha-2)^{\frac{1}{\alpha-1}},+\infty)$. Therefore,

$$\Psi(t)_{\max 0\le t\le+\infty}=\frac{[(\alpha-2)^{\frac{1}{\alpha-1}}]^{\alpha-2}}{1+[(\alpha-2)^{\frac{1}{\alpha-1}}]^{\alpha-1}}=\frac{(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\alpha-1}.$$

Thus, $\frac{\partial G(t,s)}{\partial t}(1+t^{\alpha-1})^{-1}\le\frac{p(s)(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}$. That is to say, (iii) is certified.

By Lemma 2.5, if $s\in[0,\frac{1}{k})$,

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\alpha-1}}&=\min_{\frac{1}{k}\le t\le k}\frac{1}{\Gamma(\alpha)}\left(\frac{p(s)}{p(0)}t^{\alpha-1}-(t-s)^{\alpha-1}\right)\frac{1}{1+t^{\alpha-1}}\\&\ge\frac{1}{\Gamma(\alpha)}\left(\frac{1}{k^{\alpha-1}}-\left(\frac{1}{k}-s\right)^{\alpha-1}\right)\frac{1}{1+k^{1-\alpha}};\end{aligned}$$

if $s\in[\frac{1}{k},k)$,

$$\begin{aligned}\min_{\frac{1}{k}\le t\le s}\frac{G(t,s)}{1+t^{\alpha-1}}&=\min_{\frac{1}{k}\le t\le s}\frac{p(s)t^{\alpha-1}}{\Gamma(\alpha)p(0)(1+t^{\alpha-1})}\ge\frac{1}{\Gamma(\alpha)k^{\alpha-1}(1+k^{1-\alpha})},\\ \min_{s\le t\le k}\frac{G(t,s)}{1+t^{\alpha-1}}&=\min_{s\le t\le k}\frac{1}{\Gamma(\alpha)}\left(\frac{p(s)}{p(0)}t^{\alpha-1}-(t-s)^{\alpha-1}\right)\frac{1}{1+t^{\alpha-1}}\\&\ge\frac{s^{\alpha-1}}{\Gamma(\alpha)(1+s^{\alpha-1})},\end{aligned}$$

so

$$\min_{\frac{1}{k}\leq t\leq k}\frac{G(t,s)}{1+t^{\alpha-1}}\geq\frac{1}{\Gamma(\alpha)k^{\alpha-1}(1+k^{1-\alpha})};$$

if $s\in[k,+\infty)$,

$$\min_{\frac{1}{k}\leq t\leq k}\frac{G(t,s)}{1+t^{\alpha-1}}=\min_{\frac{1}{k}\leq t\leq k}\frac{p(s)t^{\alpha-1}}{\Gamma(\alpha)p(0)(1+t^{\alpha-1})}\geq\frac{1}{\Gamma(\alpha)k^{\alpha-1}(1+k^{1-\alpha})}.$$

In summary, $\min_{\frac{1}{k}\leq t\leq k}\frac{G(t,s)}{1+t^{\alpha-1}}\geq g_1(s)$.

Similarly, we can obtain the following: if $s\in[0,\frac{1}{k})$,

$$\begin{aligned}\min_{\frac{1}{k}\leq t\leq k}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}&=\min_{\frac{1}{k}\leq t\leq k}\frac{1}{\Gamma(\alpha-1)}\left(\frac{p(s)}{p(0)}t^{\alpha-2}-(t-s)^{\alpha-2}\right)\\&\quad\times\frac{1}{1+t^{\alpha-1}}\\&\geq\frac{1}{\Gamma(\alpha-1)}\left(\frac{1}{k^{\alpha-2}}-\left(\frac{1}{k}-s\right)^{\alpha-2}\right)\frac{1}{1+k^{1-\alpha}};\end{aligned}$$

if $s\in[\frac{1}{k},k)$,

$$\begin{aligned}\min_{\frac{1}{k}\leq t\leq s}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}&=\min_{\frac{1}{k}\leq t\leq s}\frac{p(s)t^{\alpha-2}}{\Gamma(\alpha-1)p(0)(1+t^{\alpha-1})}\\&\geq\frac{1}{\Gamma(\alpha-1)k^{\alpha-2}(1+k^{1-\alpha})},\\\min_{s\leq t\leq k}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}&=\min_{s\leq t\leq k}\frac{1}{\Gamma(\alpha-1)}\left(\frac{p(s)}{p(0)}t^{\alpha-2}-(t-s)^{\alpha-2}\right)\\&\quad\times\frac{1}{1+t^{\alpha-1}}\\&\geq\frac{s^{\alpha-2}}{\Gamma(\alpha-1)(1+s^{\alpha-1})},\end{aligned}$$

so

$$\min_{\frac{1}{k}\leq t\leq k}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}\geq\frac{1}{\Gamma(\alpha-1)k^{\alpha-2}(1+k^{1-\alpha})};$$

if $s\in[k,+\infty)$,

$$\begin{aligned}\min_{\frac{1}{k}\leq t\leq k}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}&=\min_{\frac{1}{k}\leq t\leq k}\frac{p(s)t^{\alpha-2}}{\Gamma(\alpha-1)p(0)(1+t^{\alpha-1})}\\&\geq\frac{1}{\Gamma(\alpha-1)k^{\alpha-2}(1+k^{1-\alpha})}.\end{aligned}$$

In conclusion, $\min_{\frac{1}{k}\leq t\leq k}\frac{\partial G(t,s)}{\partial t}(1+t^{\alpha-1})^{-1}\geq g_2(s)$. Therefore, (iv) is proved. □

Let $\mathbb{E}_\infty = \{u \in C^1([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty, \lim_{t\to+\infty} \frac{|u'(t)|}{1+t^{\alpha-1}} < +\infty\}$, endowed with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}}$, $\|u'\|_\infty = \sup_{t\geq 0} \frac{|u'(t)|}{1+t^{\alpha-1}}$. It is clear that $\mathbb{E}_\infty$ is a Banach space [11].

Lemma 2.7 ([11]) *Set $\mathbb{O} = \{u \in \mathbb{E}_\infty, \|u\| < \rho, \text{where } \rho > 0\}$, $\mathbb{O}(t) = \{\frac{u(t)}{1+t^{\alpha-1}}, u \in \mathbb{O}\}$, $\mathbb{O}'(t) = \{\frac{u'(t)}{1+t^{\alpha-1}}, u \in \mathbb{O}\}$. Then $\mathbb{O}$ is relatively compact in $\mathbb{E}_\infty$ if $\mathbb{O}(t)$ and $\mathbb{O}'(t)$ are equicontinuous on any finite subinterval of $[0,+\infty)$ and equiconvergent at $+\infty$, that is, for any $\epsilon > 0$ there exists $\zeta = \zeta(\epsilon) > 0$ such that*

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \epsilon, \qquad \left|\frac{u'(t_1)}{1+t_1^{\alpha-1}} - \frac{u'(t_2)}{1+t_2^{\alpha-1}}\right| < \epsilon, \quad \forall u \in \mathbb{O}, t_1, t_2 > \zeta.$$

Lemma 2.8 ([12]) *Let $\mathbb{P}$ be a cone in a real Banach space. Assume that there exists a concave nonnegative continuous functional θ on $\mathbb{P}$, with $\theta(u) \leq \|u\|$, $\forall u \in \overline{\mathbb{P}_c}$. Letting $a, b, c > 0$ be constants, we define*

$$\mathbb{P}_d = \{u \in \mathbb{P} : \|u\| < d\}, \qquad \overline{\mathbb{P}_d} = \{u \in \mathbb{P} : \|u\| \leq d\},$$
$$\mathbb{P}(\theta, a, b) = \{u \in \mathbb{P} : \theta(u) \geq a, \|u\| \leq b\}.$$

Let $T : \overline{\mathbb{P}_c} \to \overline{\mathbb{P}_c}$ be a completely continuous operator. Suppose that there exist constants $0 < a < b < d \leq c$ such that the following conditions hold:

(i) *$\{u \in \mathbb{P}(\theta, b, d) : \theta(u) > b\} \neq \emptyset$ and $\theta(Tu) > b$, $\forall u \in \mathbb{P}(\theta, b, d)$;*

(ii) *$\|Tu\| < a$, $\forall u \in \overline{\mathbb{P}_a}$;*

(iii) *$\theta(Tu) > b$ for $u \in \mathbb{P}(\theta, b, c)$, with $\|Tu\| > d$.*

Then T has at least three fixed points u_1, u_2, and u_3 in $\overline{\mathbb{P}_c}$. Furthermore, $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

3 Main results

Define a cone $\mathbb{P} \subset \mathbb{E}_\infty$ by $\mathbb{P} = \{u \in \mathbb{E}_\infty : u(t) \geq 0, u'(t) \geq 0\}$. We introduce an operator $T : \mathbb{P} \to \mathbb{E}_\infty$ as follows:

$$Tu(t) = \int_0^{+\infty} G(t,s)a(s)f\bigl(s, u(s), u'(s)\bigr)\, ds. \tag{7}$$

By Lemma 2.4, we can know that the fixed point of T is the solution of the boundary value problem (1) and vice versa.

Now we make the following assumption:

(H_4) $\int_0^{+\infty} a(s)p(s)\, ds < +\infty$, where $p(s)$ is defined as (4).

Lemma 3.1 *Suppose that (H_1)–(H_4) hold. Then $T : \mathbb{P} \to \mathbb{P}$ is a completely continuous operator.*

Proof First of all, we will show that $T : \mathbb{P} \to \mathbb{P}$.

In view of the properties of $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ and the nonnegativity of f, it is easy to know that $Tu(t) \geq 0$, $Tu'(t) \geq 0$, $\forall t \in [0,+\infty)$.

By (H_2)–(H_4) and Lemma 2.6, for any $u \in \mathbb{P}$ and $t \in [0,+\infty)$, we have $\frac{u(t)}{1+t^{\alpha-1}} \leq \|u\|_\infty$, $\frac{u'(t)}{1+t^{\alpha-1}} \leq \|u'\|_\infty$, and there exists $\mu_u > 0$ such that

$$\begin{aligned}
&\sup_{t\in[0,+\infty)} \frac{|Tu(t)|}{1+t^{\alpha-1}} \\
&\quad= \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\quad\leq \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{p(s)}{\Gamma(\alpha)p(0)} a(s)f\biggl(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}},\bigl(1+s^{\alpha-1}\bigr)\frac{u'(s)}{1+s^{\alpha-1}}\biggr)\,ds \\
&\quad\leq \frac{\mu_u}{\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)a(s)\,ds < +\infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sup_{t\in[0,+\infty)} \frac{|T'u(t)|}{1+t^{\alpha-1}} \\
&\quad= \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{\partial G(t,s)}{\partial t} \bigl(1+t^{\alpha-1}\bigr)^{-1} a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\quad\leq \sup_{t\in[0,+\infty)} \frac{(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)} \\
&\qquad\times \int_0^{+\infty} p(s)a(s)f\biggl(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}},\bigl(1+s^{\alpha-1}\bigr)\frac{u'(s)}{1+s^{\alpha-1}}\biggr)\,ds \\
&\quad\leq \frac{(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}\mu_u}{\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)a(s)\,ds < +\infty.
\end{aligned}$$

Therefore, $T(\mathbb{P}) \subset \mathbb{P}$.

Secondly, we will prove that $T : \mathbb{P} \to \mathbb{P}$ is continuous.

Let $u_n \to u$ as $n \to +\infty$ in $\mathbb{P}$, that is, $\|u_n - u\| \to 0$ $(n \to \infty)$. By Lemma 2.6,

$$\begin{aligned}
&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}} - \frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&\quad= \left|\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\,ds - \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\right| \\
&\quad\leq \frac{1}{\Gamma(\alpha)p(0)} \left|\int_0^{+\infty} p(s)a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\,ds - \int_0^{+\infty} p(s)a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\right| \\
&\quad\leq \frac{1}{\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)\biggl|a(s)f\biggl(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u_n(s)}{1+s^{\alpha-1}},\bigl(1+s^{\alpha-1}\bigr)\frac{u_n'(s)}{1+s^{\alpha-1}}\biggr) \\
&\qquad- a(s)f\biggl(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}},\bigl(1+s^{\alpha-1}\bigr)\frac{u'(s)}{1+s^{\alpha-1}}\biggr)\biggr|\,ds.
\end{aligned}$$

It can be known from Lebesgue's dominated convergence theorem and the continuity of f that

$$\|Tu_n - Tu\|_\infty = \sup_{t\in[0,+\infty)} \left|\frac{Tu_n(t)}{1+t^{\alpha-1}} - \frac{Tu(t)}{1+t^{\alpha-1}}\right| \to 0, \quad n \to +\infty.$$

In the same way, we have

$$\begin{aligned}
&\left|\frac{(Tu_n(t))'}{1+t^{\alpha-1}}-\frac{(Tu(t))'}{1+t^{\alpha-1}}\right| \\
&\quad=\left|\int_0^{+\infty}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}a(s)f\left(s,u_n(s),u_n'(s)\right)ds\right. \\
&\qquad\left.-\int_0^{+\infty}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}a(s)f\left(s,u(s),u'(s)\right)ds\right| \\
&\quad\le\frac{(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}\int_0^{+\infty}p(s)\left|a(s)f\left(s,\left(1+s^{\alpha-1}\right)\frac{u_n(s)}{1+s^{\alpha-1}},\left(1+s^{\alpha-1}\right)\frac{u_n'(s)}{1+s^{\alpha-1}}\right)\right. \\
&\qquad\left.-a(s)f\left(s,\left(1+s^{\alpha-1}\right)\frac{u(s)}{1+s^{\alpha-1}},\left(1+s^{\alpha-1}\right)\frac{u'(s)}{1+s^{\alpha-1}}\right)\right|ds.
\end{aligned}$$

It is known from Lebesgue's dominated convergence theorem and the continuity of f that

$$\left\|(Tu_n)'-(Tu)'\right\|_\infty=\sup_{t\in[0,+\infty)}\left|\frac{(Tu_n(t))'}{1+t^{\alpha-1}}-\frac{(Tu(t))'}{1+t^{\alpha-1}}\right|\to 0,\quad n\to+\infty.$$

Hence, $T:\mathbb{P}\to\mathbb{P}$ is continuous.

Now let $\Omega\subset\mathbb{P}$ be bounded. Then there exists a positive constant κ such that $\|u\|\le\kappa$, $\forall u\in\Omega$. Next we will prove that $T(\Omega)$ is bounded.

In view of (H_2), let the positive constant

$$r=\sup\left\{f\left(t,\left(1+t^{\alpha-1}\right)u,\left(1+t^{\alpha-1}\right)v\right):(t,u,v)\in[0,+\infty)\times[0,\kappa]\times[0,\kappa]\right\}.$$

For any $u\in\Omega$, by Lemma 2.6, we obtain

$$\begin{aligned}
\|Tu\|_\infty&=\sup_{t\in[0,+\infty)}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&=\sup_{t\in[0,+\infty)}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)f\left(s,u(s),u'(s)\right)ds \\
&\le\sup_{t\in[0,+\infty)}\frac{1}{\Gamma(\alpha)p(0)}\int_0^{+\infty}p(s)a(s)f\left(s,\left(1+s^{\alpha-1}\right)\frac{u(s)}{1+s^{\alpha-1}},\left(1+s^{\alpha-1}\right)\frac{u'(s)}{1+s^{\alpha-1}}\right)ds \\
&\le\frac{r}{\Gamma(\alpha)p(0)}\int_0^{+\infty}p(s)a(s)\,ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\|(Tu)'\right\|_\infty&=\sup_{t\in[0,+\infty)}\left|\frac{(Tu(t))'}{1+t^{\alpha-1}}\right| \\
&=\sup_{t\in[0,+\infty)}\int_0^{+\infty}\frac{\partial G(t,s)}{\partial t}\left(1+t^{\alpha-1}\right)^{-1}a(s)f\left(s,u(s),u'(s)\right)ds \\
&\le\sup_{t\in[0,+\infty)}\frac{(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}
\end{aligned}$$

$$\times \int_0^{+\infty} p(s)a(s)f\left(s, \left(1+s^{\alpha-1}\right)\frac{u(s)}{1+s^{\alpha-1}}, \left(1+s^{\alpha-1}\right)\frac{u'(s)}{1+s^{\alpha-1}}\right) ds$$
$$\le \frac{r(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)a(s)\,ds.$$

Therefore, because $\|Tu\| = \max_{u\in\Omega}\{\|Tu\|_\infty, \|(Tu)'\|_\infty\}$, we get that $T(\Omega)$ is bounded.

Next we will show that $\{\frac{Tu(t)}{1+t^{\alpha-1}} : u \in \Omega\}$ and $\{\frac{(Tu(t))'}{1+t^{\alpha-1}} : u \in \Omega\}$ are equi-continuous on any finite subinterval of $[0, +\infty)$.

For any $\lambda > 0$, $t_1, t_2 \in [0, \lambda]$, without loss of generality, we may assume that $t_2 \ge t_1$. For all $u \in \Omega$, by Lemma 2.7, we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\quad= \left|\int_0^{+\infty} \frac{G(t_2,s)}{1+t_2^{\alpha-1}} a(s)f\big(s,u(s),u'(s)\big)\,ds - \int_0^{+\infty} \frac{G(t_1,s)}{1+t_1^{\alpha-1}} a(s)f\big(s,u(s),u'(s)\big)\,ds\right| \\
&\quad= \left|\int_0^{+\infty} \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right) a(s)f\big(s,u(s),u'(s)\big)\,ds\right. \\
&\qquad \left.+ \int_0^{+\infty} \left(\frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right) a(s)f\big(s,u(s),u'(s)\big)\,ds\right| \\
&\quad\le \int_0^{+\infty} \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\qquad + \int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{+\infty} \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\quad\le \int_0^{t_1} \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\qquad + \int_{t_1}^{t_2} \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\qquad + \int_{t_2}^{+\infty} \left|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\quad\le \frac{r}{\Gamma(\alpha)}\left[\int_0^{t_1} \frac{\frac{p(s)}{p(0)}t_2^{\alpha-1} - \frac{p(s)}{p(0)}t_1^{\alpha-1} + (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} a(s)\,ds\right. \\
&\qquad + \int_{t_1}^{t_2} \frac{\frac{p(s)}{p(0)}t_2^{\alpha-1} - (t_2-s)^{\alpha-1} - \frac{p(s)}{p(0)}t_1^{\alpha-1}}{1+t_2^{\alpha-1}} a(s)\,ds \\
&\qquad \left.+ \int_{t_2}^{+\infty} \frac{\frac{p(s)}{p(0)}t_2^{\alpha-1} - \frac{p(s)}{p(0)}t_1^{\alpha-1}}{1+t_2^{\alpha-1}} a(s)\,ds\right] \to 0, \quad t_1 \to t_2, \\
&\int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}\right| a(s)f\big(s,u(s),u'(s)\big)\,ds \\
&\quad= \int_0^{t_1} \frac{G(t_1,s)|t_2^{\alpha-1} - t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})} a(s)f\big(s,u(s),u'(s)\big)\,ds
\end{aligned}$$

$$+\int_{t_1}^{+\infty}\frac{G(t_1,s)|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds$$
$$\le\frac{r}{\Gamma(\alpha)}\left[\int_0^{t_1}\frac{(\frac{p(s)}{p(0)}t_1^{\alpha-1}-(t_1-s)^{\alpha-1})|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}a(s)\,ds\right.$$
$$\left.+\int_{t_1}^{+\infty}\frac{\frac{p(s)}{p(0)}t_1^{\alpha-1}|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}a(s)\,ds\right]\to 0,\quad t_1\to t_2.$$

Thus, $|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}|\to 0$ as $t_1\to t_2$.

Similarly, we can prove that $|\frac{(Tu)'(t_2)}{1+t_2^{\alpha-1}}-\frac{(Tu)'(t_1)}{1+t_1^{\alpha-1}}|\to 0$ uniformly for $u\in\Omega$ when $t_1\to t_2$. So $\{\frac{Tu(t)}{1+t^{\alpha-1}}:u\in\Omega\}$ and $\{\frac{(Tu(t))'}{1+t^{\alpha-1}}:u\in\Omega\}$ are equicontinuous on any finite subinterval of $[0,+\infty)$.

At last we will prove that $\{\frac{Tu(t)}{1+t^{\alpha-1}}:u\in\Omega\}$ and $\{\frac{(Tu(t))'}{1+t^{\alpha-1}}:u\in\Omega\}$ are equiconvergent at $t\to+\infty$.

For any $u\in\Omega$, by (H_4), we get

$$\begin{aligned}&\int_0^{+\infty}p(s)a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\\&\quad=\int_0^{+\infty}p(s)a(s)f\left(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}},\bigl(1+s^{\alpha-1}\bigr)\frac{u'(s)}{1+s^{\alpha-1}}\right)ds\\&\quad\le r\int_0^{+\infty}p(s)a(s)\,ds<+\infty.\end{aligned}$$

It can be known from Lemma 2.6 that

$$\begin{aligned}\lim_{t\to+\infty}\left|\frac{Tu(t)}{1+t^{\alpha-1}}\right|&=\lim_{t\to+\infty}\left|\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\right|\\&\le\frac{r}{\Gamma(\alpha)p(0)}\int_0^{+\infty}p(s)a(s)\,ds<+\infty,\\\lim_{t\to+\infty}\left|\frac{(Tu(t))'}{1+t^{\alpha-1}}\right|&=\lim_{t\to+\infty}\left|\int_0^{+\infty}\frac{\partial G(t,s)}{\partial t}\bigl(1+t^{\alpha-1}\bigr)^{-1}a(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\right|\\&\le\frac{r(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}\int_0^{+\infty}p(s)a(s)\,ds<+\infty.\end{aligned}$$

Therefore, $\{\frac{Tu(t)}{1+t^{\alpha-1}}:u\in\Omega\}$ and $\{\frac{(Tu(t))'}{1+t^{\alpha-1}}:u\in\Omega\}$ are equiconvergent at $t\to+\infty$. By Lemma 2.7, we can know that $T\Omega$ is relatively compact. So T is completely continuous. □

Now in the following part of the paper, we take $k=(\alpha-2)^{\frac{1}{\alpha-1}}$. Then $k>1$ and $\min_{\frac{1}{k}\le t\le k}\frac{(\alpha-1)t^{\alpha-2}}{1+t^{\alpha-1}}=(\alpha-2)^{\frac{1}{\alpha-1}}$. We will use the Leggett–Williams fixed point theorem to prove that there are at least three positive solutions to boundary value problem (1). For convenience, we denote

$$\begin{aligned}K_1&=\frac{\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)}{\Gamma(\alpha)p(0)}\int_0^{+\infty}a(s)\,ds,\\K_2&=\frac{\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{\Gamma(\alpha)p(0)}\int_0^{+\infty}a(s)\,ds,\end{aligned}$$

$$K_1^* = \int_{\frac{1}{k}}^{k} g_1(s)a(s)\,ds, \qquad K_2^* = \int_{\frac{1}{k}}^{k} g_2(s)a(s)\,ds,$$
$$K = \max\{K_1, K_2\}, \qquad K^* = \min\{K_1^*, K_2^*\}.$$

We denote a nonnegative concave functional on $\mathbb{P}$ by

$$\theta(u) = \min\left\{\min_{\frac{1}{k}\le t\le k} \frac{u(t)}{1+t^{\alpha-1}}, \min_{\frac{1}{k}\le t\le k} \frac{u'(t)}{1+t^{\alpha-1}}\right\}.$$

Theorem 3.1 *Assume that (H_1)–(H_4) hold. Let $0 < a < b < d = c$, let $b < [2(\alpha-1)^{\frac{2-\alpha}{\alpha-1}} - 1]d$, and suppose that the function f satisfies the following conditions*:

$$(C_1)\quad f\bigl(t, \bigl(1+t^{\alpha-1}\bigr)u, \bigl(1+t^{\alpha-1}\bigr)v\bigr) < \frac{c}{K}, \quad 0 \le t < +\infty, 0 \le u \le c, 0 \le v \le c,$$
$$(C_2)\quad f\bigl(t, \bigl(1+t^{\alpha-1}\bigr)u, \bigl(1+t^{\alpha-1}\bigr)v\bigr) > \frac{b}{K^*}, \quad \frac{1}{k} \le t \le k, b \le u \le c, b \le v \le c,$$
$$(C_3)\quad f\bigl(t, \bigl(1+t^{\alpha-1}\bigr)u, \bigl(1+t^{\alpha-1}\bigr)v\bigr) < \frac{a}{K}, \quad 0 \le t < +\infty, 0 \le u \le a, 0 \le v \le a.$$

Then boundary value problem (1) *has at least three positive solutions $u_1, u_2, u_3 \in \overline{\mathbb{P}_c}$ such that $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.*

Proof We will show that all conditions of Lemma 2.8 are satisfied for T defined by (7). For all $u \in \overline{\mathbb{P}_c}$, we have $\|u\| \le c$, that is, $0 \le \frac{u(t)}{1+t^{\alpha-1}} \le c$, $0 \le \frac{u'(t)}{1+t^{\alpha-1}} \le c$, $\forall t \in [0,+\infty)$. By using assumption (C_1), we can get $f(t,u(t),u'(t)) = f(t,(1+t^{\alpha-1})\frac{u(t)}{1+t^{\alpha-1}},(1+t^{\alpha-1})\frac{u'(t)}{1+t^{\alpha-1}}) < \frac{c}{K}$, $t \in [0,+\infty)$.

For all $u \in \overline{\mathbb{P}_c}$,

$$\begin{aligned}
\|Tu\|_\infty &= \sup_{t\in[0,+\infty)} \left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&= \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&< \frac{c}{K\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)a(s)\,ds \\
&\le \frac{\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)c}{K\Gamma(\alpha)p(0)} \int_0^{+\infty} a(s)\,ds = \frac{cK_1}{K} \le c, \\
\|Tu'\|_\infty &= \sup_{t\in[0,+\infty)} \left|\frac{Tu'(t)}{1+t^{\alpha-1}}\right| \\
&= \sup_{t\in[0,+\infty)} \int_0^{+\infty} \frac{\partial G(t,s)}{\partial t} \bigl(1+t^{\alpha-1}\bigr)^{-1} a(s) f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&< \frac{c(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{K\Gamma(\alpha)p(0)} \int_0^{+\infty} p(s)a(s)\,ds \\
&\le \frac{\Gamma(\alpha+\beta-\delta)\Gamma(\alpha-\sigma)c(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{K\Gamma(\alpha)p(0)} \int_0^{+\infty} a(s)\,ds = \frac{cK_2}{K} \le c,
\end{aligned}$$

we have $\|Tu\| < c$. Thus, $T : \overline{\mathbb{P}_c} \to \overline{\mathbb{P}_c}$. By Lemma 3.1, we can know that T is completely continuous. Using the above argument, it follows from assumption (C_3) that if $u \in \overline{\mathbb{P}_a}$, then $\|Tu\| < a$. Hence, condition (ii) of Lemma 2.8 holds.

Next we will show that condition (i) of Lemma 2.8 holds.

To check that, we choose $u^*(t) = \frac{b+d}{2}(1 + t^{\alpha-1})$, $0 \le t < +\infty$. Obviously, $u^* \in \mathbb{P}$. By the proof of condition (iii) of Lemma 2.6,

$$\|u^*\|_\infty = \frac{b+d}{2} < d, \qquad \|u^{*'}\|_\infty = \sup_{t\in[0,+\infty)} \frac{(b+d)(\alpha-1)t^{\alpha-2}}{2(1+t^{\alpha-1})} = \frac{(b+d)(\alpha-2)^{\frac{\alpha-2}{\alpha-1}}}{2} < d.$$

Thus, $\|u^*\| < d$. Also, because

$$\min_{\frac{1}{k}\le t\le k} \frac{u^*(t)}{1+t^{\alpha-1}} = \frac{b+d}{2} > b,$$

$$\min_{\frac{1}{k}\le t\le k} \frac{u^{*'}(t)}{1+t^{\alpha-1}} = \min_{\frac{1}{k}\le t\le k} \frac{(b+d)(\alpha-1)t^{\alpha-2}}{2(1+t^{\alpha-1})} = \frac{(b+d)(\alpha-2)^{\frac{1}{\alpha-1}}}{2} > \frac{b+d}{2} > b,$$

we have $\theta(u^*) > b$. Therefore, $u^* \in \{u \in \mathbb{P}(\theta, b, d) : \theta(u) > b\} \neq \emptyset$.

For all $u \in \mathbb{P}(\theta, b, d)$, $\forall t \in [\frac{1}{k}, k]$, we have $b \le \frac{u(t)}{1+t^{\alpha-1}} \le c$, $b \le \frac{u'(t)}{1+t^{\alpha-1}} \le c$. By (C_2), it follows that $f(t, u(t), u'(t)) = f(t, (1+t^{\alpha-1})\frac{u(t)}{1+t^{\alpha-1}}, (1+t^{\alpha-1})\frac{u'(t)}{1+t^{\alpha-1}}) > \frac{b}{K^*}$. For any $t \in [\frac{1}{k}, k]$, we get

$$\begin{aligned}
\min_{\frac{1}{k}\le t\le k} \frac{Tu(t)}{1+t^{\alpha-1}} &= \min_{\frac{1}{k}\le t\le k} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s, u(s), u'(s)\bigr)\,ds \\
&\ge \int_0^{+\infty} \min_{\frac{1}{k}\le t\le k} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s, u(s), u'(s)\bigr)\,ds \\
&> \frac{b}{K^*} \int_{\frac{1}{k}}^{k} g_1(s) a(s)\,ds \\
&\ge \frac{bK_1^*}{K^*} \ge b, \\
\min_{\frac{1}{k}\le t\le k} \frac{(Tu(t))'}{1+t^{\alpha-1}} &= \min_{\frac{1}{k}\le t\le k} \int_0^{+\infty} \frac{\partial G(t,s)}{\partial t} \bigl(1+t^{\alpha-1}\bigr)^{-1} a(s) f\bigl(s, u(s), u'(s)\bigr)\,ds \\
&> \frac{b}{K^*} \int_{\frac{1}{k}}^{k} g_2(s) a(s)\,ds = \frac{bK_2^*}{K^*} \ge b.
\end{aligned}$$

Thus, $\theta(Tu) > b$, $\forall u \in \mathbb{P}(\theta, b, d)$, that is, condition (i) of Lemma 2.8 holds.

At last, we assume that $u \in \mathbb{P}(\theta, b, c)$ with $\|Tu\| > d$. Then $\|u\| \le c$, $b \le \frac{u(t)}{1+t^{\alpha-1}} \le c$, and $b \le \frac{u'(t)}{1+t^{\alpha-1}} \le c$ for $t \in [\frac{1}{k}, k]$. By assumption (C_2), we have $\theta(Tu) > b$. Hence, condition (iii) of Lemma 2.8 is satisfied.

To sum up, all hypotheses of Lemma 2.8 are satisfied. So we get that the boundary value problem (1) has at least three positive solutions u_1, u_2, and u_3, such that $\|u_1\| < a$, $b < \theta(u_2)$, and $a < \|u_3\|$ with $\theta(u_3) < b$. □

4 An example

Let $\alpha = 3.2$, $\beta = 0.6$, $\delta = 1.2$, $\sigma = 0.7$, $\xi_i = 1 - \frac{1}{2i+2}$, $\eta_i = \frac{1}{4^i}$, $\rho_i = \frac{1}{5^i}$, and $a(t) = 4e^{-t}$.

Now we consider the following fractional boundary value problem on an infinite interval:

$$\begin{cases} D_{0^+}^{3.2}u(t) + 4e^{-t}f(t,u(t),u'(t)) = 0, \quad t \in [0,+\infty), \\ u^{(j)}(0) = 0, \quad j = 0,1,2, \\ \lim_{t\to+\infty} D_{0^+}^{1.2}u(t) = \sum_{i=1}^{\infty} \eta_i I_{0^+}^{0.6}(D_{0^+}^{1.2}u(t))(\xi_i) + \sum_{i=1}^{\infty} \rho_i D_{0^+}^{0.7}u(\xi_i), \end{cases} \tag{8}$$

where

$$f(t,u,v) = \begin{cases} \frac{1}{167(1+t^2)} + \frac{1}{3500}(\frac{u}{1+t^{1.2}})^5 + \frac{1}{3500}(\frac{v}{1+t^{1.2}})^5, \\ \quad (t,u,v) \in [0,+\infty) \times [0,\frac{1}{30}] \times [0,+\infty), \\ \frac{1}{167(1+t^2)} + \frac{1}{3500}(\frac{u}{1+t^{1.2}})^5 + \frac{70(u-\frac{1}{30})}{1867(1+t^{1.2})} + \frac{1}{3500}(\frac{v}{1+t^{1.2}})^5, \\ \quad (t,u,v) \in [0,+\infty) \times [\frac{1}{30},\frac{1}{15}] \times [0,+\infty), \\ \frac{1}{167(1+t^2)} + \frac{1}{3500}(\frac{u}{1+t^{1.2}})^5 + \frac{7}{5601(1+t^{1.2})} + \frac{1}{3500}(\frac{v}{1+t^{1.2}})^5, \\ \quad (t,u,v) \in [0,+\infty) \times [\frac{1}{15},+\infty] \times [0,+\infty). \end{cases}$$

A direct calculation shows that

$$\begin{aligned} \varkappa &= \Gamma(1.6)\Gamma(1.5) - \Gamma(1.5)\biggl[\frac{1}{4}\cdot\left(1-\frac{1}{4}\right)^{0.6} + \frac{1}{16}\cdot\left(1-\frac{1}{6}\right)^{0.6} + \cdots \\ &\quad + \frac{1}{4^i}\cdot\left(1-\frac{1}{2i+2}\right)^{0.6} + \cdots\biggr] \\ &\quad - \Gamma(1.6)\biggl[\frac{1}{5}\cdot\left(1-\frac{1}{4}\right)^{0.5} + \frac{1}{25}\cdot\left(1-\frac{1}{6}\right)^{0.5} + \cdots + \frac{1}{5^i}\cdot\left(1-\frac{1}{2i+2}\right)^{0.5} + \cdots\biggr] \\ &\geq \Gamma(1.6)\Gamma(1.5) \\ &\quad - \Gamma(1.5)\cdot\left(\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^i}+\cdots\right) - \Gamma(1.6)\cdot\left(\frac{1}{5}+\frac{1}{5^2}+\cdots+\frac{1}{5^i}+\cdots\right) \\ &= 0.2731 > 0, \end{aligned}$$

$\int_0^{+\infty} a(t)\,dt = \int_0^{+\infty} e^{-t}\,dt = 1.$

By Theorem 3.1, we set $a = \frac{1}{30}$, $b = \frac{1}{15}$, $c = 5$, and $k = 1.0864$. The calculation yields

$$K_1 = 2.6316, \qquad K_2 = 2.0125, \qquad K_1^* = 0.1277, \qquad K_2^* = 0.4599.$$

Then f satisfies

$$\begin{aligned} &f\bigl(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})u\bigr) \leq 1.8238 < \frac{c}{K}, \quad 0 \leq t < +\infty, 0 \leq u \leq 5, 0 \leq v \leq 5; \\ &f\bigl(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})u\bigr) \geq 0.5764 > \frac{b}{K^*}, \\ &\quad 0.9205 \leq t \leq 1.0864, \frac{1}{15} \leq u \leq 5, \frac{1}{15} \leq v \leq 5; \\ &f\bigl(t,(1+t^{\alpha-1})u,(1+t^{\alpha-1})u\bigr) \leq 0.0060 < \frac{a}{K}, \quad 0 \leq t < +\infty, 0 \leq u \leq \frac{1}{30}, 0 \leq v \leq \frac{1}{30}. \end{aligned}$$

Therefore, all assumptions of Theorem 3.1 are satisfied. Thus, the fractional boundary value problem (8) has at least three positive solutions u_1, u_2, and u_3 satisfying $\|u_1\| < \frac{1}{30}$, $\frac{1}{15} < \theta(u_2)$, and $\frac{1}{30} < \|u_3\|$ with $\theta(u_3) < \frac{1}{15}$.

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Consent for publication
The authors confirm: that the work described has not been published before (except in the form of an abstract or as part of a published lecture, review, or thesis); that it is not under consideration for publication elsewhere; that its publication has been approved by all co-authors, if any; that its publication has been approved (tacitly or explicitly) by the responsible authorities at the institution where the work is carried out.

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References

1. Adiguzel, R.S., Aksoy, U., Karapinar, E., et al.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**(2), 313–333 (2021)
2. Bai, Z., Lv, H.: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**(2), 495–505 (2005)
3. Lazreg, J.E., Abbas, S., Benchohra, M., et al.: Impulsive Caputo-Fabrizio fractional differential equations in b-matric spaces. Open Math. **19**, 363–372 (2021)
4. Ahmad, B., Ntouyas, S.K.: Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Appl. Math. Comput. **266**, 615–622 (2015)
5. Isaia, F.: On a nonlinear integral equation without compactness. Acta Math. Univ. Comen. **75**(2), 233–240 (2016)
6. Niazi, A.U.K., Wei, J., Rehman, M.U., et al.: Boundary value problem for nonlinear fractional differential equations with delay. Adv. Differ. Equ. **24**, 1–14 (2017)
7. Li, Y.: Multiple positive solutions for nonlinear mixed fractional differential equation with *p*-Laplacian operator. Adv. Differ. Equ. **112**, 1–12 (2019)
8. Kai, S., Zong, Z.: Positive solutions for fractional differential equations with integral and infinite-point boundary conditions. Math. Appl. **33**(3), 563–571 (2020)
9. Mahdjouba, A., Juan, J.N., Abdelghani, O.: System of fractional boundary value problem with *p*-Laplacian and advanced arguments. Adv. Differ. Equ. **39**(6), 1425–1434 (2016)
10. Wang, Y., Wang, H.: Triple positive solutions for fractional differential equation boundary value problems at resonance. Appl. Math. Lett. **106**, 1–14 (2020)
11. Dondu, O., Ilkay, Y.K.: Positive solutions for *m*-point *p*-Laplacian fractional boundary value problem involving Riemann-Liouville fractional integral boundary conditions on the half line. Filomat **34**(9), 3161–3173 (2020)
12. Wei, J., Zong, Z.: Positive solutions of boundary value problem for a class of high-order fractional differential equations on infinite interval. Math. Appl. **30**(4), 750–759 (2017)
13. Amjad, A., Nabeela, K., Seema, I.: Successively iterative method for a class of high-order fractional differential equations with multi-point boundary value conditions on half-line. Bound. Value Probl. **5**, 1–16 (2016)
14. Yuan, T., Zhan, B., Su, S.: Positive solutions for a boundary value problem of fractional differential equation with *p*-Laplacian operator. Adv. Differ. Equ. **349**, 1–14 (2019)
15. Dondu, O., Ilkay, Y.K.: Positive solutions for fractional-order nonlinear boundary value problems on infinite interval. Int. J. Nonlinear Anal. Appl. **12**(1), 317–335 (2021)
16. Li, G., Jing, Z., Lian, L., et al.: Existence of multiple positive solutions for a class of infinite-point singular *p*-Laplacian fractional differential equation with singular source terms. Nonlinear Anal. **27**(4), 609–629 (2022)
17. Hao, X., Wang, H., Liu, L., et al.: Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and *p*-Laplacian operator. Bound. Value Probl. **182**, 1–18 (2017)

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