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Multiple positive solutions of fractional differential equations with improper integral boundary conditions on the half-line

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Abstract

This paper investigates the existence of positive solutions for a class of fractional boundary value problems involving an improper integral and the infinite-point on the half-line by making use of properties of the Green function and Avery–Peterson fixed point theorem. In addition, an example is presented to illustrate the applicability of our main result.

Keywords: Fractional differential equation; Avery–Peterson fixed point theorem; Improper integral; Half-line; Multiple positive solutions

1 Introduction

Fractional differential equations describe various phenomena in diverse areas of natural science such as physics, polymer rheology, biology, mechanics, epidemiology, and other fields, see [1–6]. Over the last few decades, the study of fractional calculus and fractional differential equations had been gaining more and more attention because researchers have found that fractional-order models are more suitable than integer-order models for some realistic problems due to their excellent description of the memory and hereditary properties of numerous materials and processes. Compared with classical integer-order models, the main advantage of fractional differential equations is the accuracy of description of the real world.

Recently, lots of papers on fractional differential equations with finite domain have appeared [7–12]. By means of many methods, such as the variational method, the upper and lower solution technique, Legett–Williams fixed point theorem, and so on, the existence results of solutions for boundary value problems of fractional differential equations have been obtained. While much of the work on fractional calculus deals with finite domain, there is a considerable development on the topics involving an unbounded domain [13–19].

In [20], the authors proved the existence and uniqueness of a positive solution to the following problem:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + \mu(f(t,u(t)) + q(t)g(x(t))) = 0, \quad t \in (0,+\infty), \\ x(0) = x'(0) = 0, \\ D_{0^+}^{\alpha-1}x(\infty) = \beta \int_0^{\eta} x(s)\,ds + \lambda, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ and $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative. $\beta, \eta > 0$, and $\Gamma(\alpha+1) > \beta\eta^{\alpha}$; $\mu, \lambda \geq 0$ are called the eigenvalue and disturbance parameters, respectively.

In [21], the authors considered the fractional differential equation with integral boundary value condition on the half line:

$$\begin{cases} {}^{H}D_{1^+}^{\alpha}x(t) + a(t)f(t,x(t)) = 0, \quad t \in (1,+\infty), \\ x(1) = x'(1) = 0, \\ {}^{H}D_{1^+}^{\alpha-1}x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^{H}I_{1^+}^{\beta_i}x(\eta) + \rho \sum_{j=1}^{n} \sigma_j x(\xi_j), \end{cases} \tag{1.2}$$

where ${}^{H}D_{1^+}^{\alpha}$ is the Hadamard-type fractional derivative, $2 < \alpha < 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$. They got the existence of at least three positive solutions from the generalized Avery–Henderson fixed point theorem.

Through the discussions of (1.1) and (1.2), an interesting question is proposed: whether the positive solution still exists and what kind of properties it has for a fractional boundary value problem with $f(t,x(t),x'(t))$ and the infinite-point? As far as we know, there is no answer to this question, which inspired us to study the following problem on an infinite interval:

$$\begin{cases} D_{0^+}^{\beta}x(t) + a(t)f(t,x(t),x'(t)) = 0, \quad t \in [0,+\infty), \\ x(0) = x'(0) = 0, \\ \lim_{t\to+\infty} D_{0^+}^{\beta-1}x(t) = \int_0^{+\infty} h(t)x'(t)\,dt + \sum_{i=1}^{\infty} \eta_i D_{0^+}^{\gamma}x(\xi_i), \end{cases} \tag{1.3}$$

where $2 < \beta \leq 3$, $0 \leq \gamma \leq \beta - 1$, and $D_{0^+}^{\beta}$ is the standard Riemann–Liouville fractional derivative; $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \xi_{i+1} < \cdots < +\infty$, $\eta_i > 0$, $i = 1, 2, \ldots$.

In this paper, we make the following assumptions:

(H_1) $f \in C([0,+\infty) \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $f(t,0,0) \not\equiv 0$ on any subinterval of $(0,+\infty)$ and $f(t,(1+t^{\beta-1})x,(1+t^{\beta-1})y)$ is bounded when x, y are bounded.

(H_2) $a, h \in C([0,+\infty),[0,+\infty))$ are not identical zero on any closed subinterval of $[0,+\infty)$ and

$$\int_0^{+\infty} a(s)\,ds < +\infty.$$

(H_3) $\Delta = \Gamma(\beta) - (\beta-1)\int_0^{+\infty} \tau^{\beta-2}h(\tau)\,d\tau - \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty} \eta_i \xi_i^{\beta-\gamma-1} > 0$.

In the study of radially symmetric solutions of nonlinear elliptic equations and gas pressure models in semiinfinite porous media, the problem of boundary values on the half-line arises naturally. It is well known that there are not many studies of fractional differential systems on an infinite interval, although it is necessary to do so.

In this paper, we aim to obtain the existence of positive solutions for system (1.3) on an infinite interval. In contrast to the existing research, we study the system with an improper integral and infinite-point boundary value conditions on the half-line, which is more general than those of multipoint boundary value conditions in the known papers. What is more, the method which we use in this paper is Avery–Peterson fixed point theorem, and multiple positive solutions are obtained for the system (1.3).

The remainder of the paper is arranged as follows. In Sect. 2, we introduce and derive several key definitions, lemmas, and properties. In Sect. 3, we investigate the existence and multiplicity of positive solutions to boundary value problem (1.3). In Sect. 4, an example is displayed to demonstrate the applicability of our main results. Finally, we conclude this paper.

2 Preliminaries

For the convenience of the reader, we introduce here some indispensable definitions and properties which will play an important role in the following sections.

Definition 1 ([1]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds.$$

Definition 2 ([1]) The Riemann–Liouville fractional derivative of order $\beta > 0$ of a function $h \in C((0,+\infty),\mathbb{R})$ is defined as

$$D_{0^+}^{\beta} h(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\beta-1} h(s)\, ds, \quad n = [\beta] + 1.$$

Lemma 1 ([22]) *Assume that $h \in C(0,1) \cap L^1(0,1)$ is such that $D_{0^+}^{\alpha} h \in C(0,1) \cap L^1(0,1)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, $n = [\alpha] + 1$.

Lemma 2 ([1, 22])

(1) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} h(t) = h(t)$, *where $h \in C(0,1) \cap L^1(0,1)$;*

(2) *If $h \in L^1(0,1)$, $\alpha > \beta > 0$, then $D_{0^+}^{\beta} I_{0^+}^{\alpha} h(t) = I_{0^+}^{\alpha-\beta} h(t)$;*

(3) *If $\lambda > -1$, then*

$$D_{0^+}^{\beta} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+1)} t^{\lambda-\beta},$$

and $D_{0^+}^{\beta} t^{\beta-m} = 0$, $m = 1, 2, \dots, n$, where $n = [\beta] + 1$.

Lemma 3 *Suppose that $g \in C([0,+\infty),[0,+\infty))$, $2 < \beta \le 3$, then the solution of boundary value problem*

$$\begin{cases} D_{0^+}^{\beta} x(t) + g(t) = 0, \\ x(0) = x'(0) = 0, \\ \lim_{t\to+\infty} D_{0^+}^{\beta-1} x(t) = \int_0^{+\infty} h(t)x'(t)\,dt + \sum_{i=1}^{\infty} \eta_i D_{0^+}^{\gamma} x(\xi_i) \end{cases} \tag{2.1}$$

is

$$x(t) = \int_0^{+\infty} G(t,s)g(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)z(0)} z(s) - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}, & 0 \le s \le t < +\infty, \\ \frac{t^{\beta-1}}{\Gamma(\beta)z(0)} z(s), & 0 \le t \le s < +\infty, \end{cases} \tag{2.2}$$

$$z(s) = 1 - \frac{1}{\Gamma(\beta-1)} \int_s^{+\infty} (\tau - s)^{\beta-2} h(\tau)\,d\tau - \frac{1}{\Gamma(\beta-\gamma)} \sum_{s\le\xi_i} \eta_i (\xi_i - s)^{\beta-\gamma-1}. \tag{2.3}$$

Proof Considering $D_{0^+}^{\beta} x(t) + g(t) = 0$ and Lemma 1, we have

$$x(t) = -I_{0^+}^{\beta} g(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + c_3 t^{\beta-3}.$$

Due to $x(0) = x'(0) = 0$, we get $c_2 = c_3 = 0$, which implies that

$$x(t) = -I_{0^+}^{\beta} g(t) + c_1 t^{\beta-1} = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)\,ds + c_1 t^{\beta-1}. \tag{2.4}$$

Thus

$$\begin{aligned} x'(t) &= -\frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} g(s)\,ds + c_1(\beta-1)t^{\beta-2}, \\ D_{0^+}^{\beta-1} x(t) &= D_{0^+}^{\beta-1}\left[-I_{0^+}^{\beta} g(t) + c_1 t^{\beta-1}\right] = -I_{0^+} g(t) + c_1 D_{0^+}^{\beta-1} t^{\beta-1} \\ &= -\int_0^t g(s)\,ds + c_1 \Gamma(\beta), \end{aligned}$$

and

$$\begin{aligned} D_{0^+}^{\gamma} x(t) &= D_{0^+}^{\gamma}\left[-I_{0^+}^{\beta} g(t) + c_1 t^{\beta-1}\right] = -I_{0^+}^{\beta-\gamma} g(t) + c_1 D_{0^+}^{\gamma} t^{\beta-1} \\ &= -\frac{1}{\Gamma(\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma-1} g(s)\,ds + c_1 \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} t^{\beta-\gamma-1}. \end{aligned}$$

In view of the boundary condition $\lim_{t\to+\infty} D_{0^+}^{\beta-1} x(t) = \int_0^{+\infty} h(t)x'(t)\,dt + \sum_{i=1}^{\infty} \eta_i D_{0^+}^{\gamma} x(\xi_i)$, we obtain

$$\begin{aligned} &-\int_0^{+\infty} g(s)\,ds + c_1 \Gamma(\beta) \\ &\quad= \int_0^{+\infty} h(\tau)\left[-\frac{1}{\Gamma(\beta-1)} \int_0^{\tau} (\tau-s)^{\beta-2} g(s)\,ds + c_1(\beta-1)\tau^{\beta-2}\right] d\tau \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\eta_i\left[-\frac{1}{\Gamma(\beta-\gamma)}\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds+c_1\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\xi_i^{\beta-\gamma-1}\right]\\
&=-\frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}h(\tau)\int_0^{\tau}(\tau-s)^{\beta-2}g(s)\,ds\,d\tau+c_1(\beta-1)\int_0^{+\infty}\tau^{\beta-2}h(\tau)\,d\tau\\
&\quad-\frac{1}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds+c_1\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\xi_i^{\beta-\gamma-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left[\Gamma(\beta)-(\beta-1)\int_0^{+\infty}\tau^{\beta-2}h(\tau)\,d\tau-\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\xi_i^{\beta-\gamma-1}\right]c_1\\
&\quad=\int_0^{+\infty}g(s)\,ds-\frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}h(\tau)\int_0^{\tau}(\tau-s)^{\beta-2}g(s)\,ds\,d\tau\\
&\qquad-\frac{1}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
c_1=\frac{1}{\Delta}\biggl[&\int_0^{+\infty}g(s)\,ds-\frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}h(\tau)\int_0^{\tau}(\tau-s)^{\beta-2}g(s)\,ds\,d\tau\\
&-\frac{1}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds\biggr].
\end{aligned}
$$

Substituting c_1 into (2.4), we get

$$
\begin{aligned}
x(t)&=-\frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}g(s)\,ds+c_1t^{\beta-1}\\
&=-\frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}g(s)\,ds\\
&\quad+\frac{t^{\beta-1}}{\Delta}\biggl[\int_0^{+\infty}g(s)\,ds-\frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}h(\tau)\int_0^{\tau}(\tau-s)^{\beta-2}g(s)\,ds\,d\tau\\
&\quad-\frac{1}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds\biggr]\\
&=-\frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}g(s)\,ds\\
&\quad+\frac{t^{\beta-1}}{\Delta}\biggl[\int_0^{+\infty}g(s)\,ds-\frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}\biggl(\int_s^{+\infty}(\tau-s)^{\beta-2}h(\tau)\,d\tau\biggr)g(s)\,ds\\
&\quad-\frac{1}{\Gamma(\beta-\gamma)}\int_0^{+\infty}\sum_{s\le\xi_i}\eta_i(\xi_i-s)^{\beta-\gamma-1}g(s)\,ds\biggr]\\
&=-\frac{1}{\Gamma(\beta)}\int_0^{t}(t-s)^{\beta-1}g(s)\,ds+\frac{t^{\beta-1}}{\Delta}\int_0^{+\infty}\biggl[1-\frac{1}{\Gamma(\beta-1)}\int_s^{+\infty}(\tau-s)^{\beta-2}h(\tau)\,d\tau\\
&\quad-\frac{1}{\Gamma(\beta-\gamma)}\sum_{s\le\xi_i}\eta_i(\xi_i-s)^{\beta-\gamma-1}\biggr]g(s)\,ds
\end{aligned}
$$

$$= \int_0^{+\infty} G(t,s)g(s)\,ds,$$

where $G(t,s)$ is defined by (2.2). The proof is completed. □

Lemma 4 *If* (H_3) *holds, then the function* $0 < z(s) < 1$, $s \in [0,+\infty)$, *and* $z(s)$ *is nondecreasing on* $[0,+\infty)$.

Proof From hypothesis (H_1) and (2.3), we have

$$z(0) = 1 - \frac{1}{\Gamma(\beta-1)}\int_0^{+\infty}(\tau - s)^{\beta-2}h(\tau)\,d\tau - \frac{1}{\Gamma(\beta-\gamma)}\sum_{i=1}^{\infty}\eta_i(\xi_i)^{\beta-\gamma-1} = \frac{\Delta}{\Gamma(\beta)} > 0$$

and $z(s) < 1$. On the other hand,

$$z'(s) = \frac{1}{\Gamma(\beta-2)}\int_s^{+\infty}(\tau - s)^{\beta-3}h(\tau)\,d\tau + \frac{1}{\Gamma(\beta-\gamma-1)}\sum_{s\le\xi_i}\eta_i(\xi_i - s)^{\beta-\gamma-2}.$$

Consequently, $z(s)$ is nondecreasing on $[0,+\infty)$ and $0 < z(s) < 1$, $s \in [0,+\infty)$. The proof is completed. □

Lemma 5 *If* (H_3) *holds, the function* $G(t,s)$ *in Lemma* 3 *satisfies the following properties*:
(1) $G(t,s)$ *and* $\frac{\partial}{\partial t}G(t,s)$ *are continuous on* $[0,+\infty)\times[0,+\infty)$;
(2) $G(t,s) \ge 0$ *and* $\frac{\partial}{\partial t}G(t,s) \ge 0$ *for all* $t,s \in [0,+\infty)$;
(3) $\frac{G(t,s)}{1+t^{\beta-1}} < L$, $\frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}} < (\beta-1)L$ *for all* $t,s \in [0,+\infty)$, *where* $L = \frac{1}{\Delta}$;
(4) *Let* $k > 1$, *then*

$$\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}} \ge \begin{cases} 0, & 0 \le s < \frac{1}{k}, \\ \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})}, & s \ge \frac{1}{k}. \end{cases}$$

Proof In view of (2.2), it is obvious that

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac{t^{\beta-2}}{\Gamma(\beta-1)z(0)}z(s) - \frac{1}{\Gamma(\beta-1)}(t-s)^{\beta-2}, & 0 \le s \le t < +\infty, \\ \frac{t^{\beta-2}}{\Gamma(\beta-1)z(0)}z(s), & 0 \le t \le s < +\infty. \end{cases} \tag{2.5}$$

(1) It is evident to see that $G(t,s)$ and $\frac{\partial}{\partial t}G(t,s)$ are continuous on $[0,+\infty)\times[0,+\infty)$.
(2) For $0 \le s \le t < +\infty$,

$$\begin{aligned} G(t,s) &= \frac{t^{\beta-1}}{\Gamma(\beta)z(0)}z(s) - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} \\ &\ge \frac{t^{\beta-1}}{\Gamma(\beta)z(0)}z(0) - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} \\ &= \frac{t^{\beta-1}}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} \ge 0. \end{aligned}$$

For $0 \le t \le s < +\infty$, it is easy to show that $G(t,s) \ge 0$.
In the same way as for $G(t,s)$, we obtain $\frac{\partial}{\partial t}G(t,s) \ge 0$ for all $t,s \in [0,+\infty)$.

(3) For $0 \le s \le t < +\infty$,

$$\begin{aligned}\frac{G(t,s)}{1+t^{\beta-1}} &= \frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})}z(s) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})} \\ &\le \frac{z(s)}{\Gamma(\beta)z(0)} < \frac{1}{\Gamma(\beta)z(0)} = \frac{1}{\Delta} = L, \\ \frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}} &\le \frac{t^{\beta-2}}{\Gamma(\beta-1)z(0)(1+t^{\beta-1})}z(s) \le \frac{(\beta-1)t^{\beta-2}}{\Delta(1+t^{\beta-1})} < \frac{\beta-1}{\Delta} = (\beta-1)L.\end{aligned}$$

Indeed, $\frac{t^{\beta-2}}{1+t^{\beta-1}} < 1$. Denote $h(t) = \frac{t^{\beta-2}}{(1+t^{\beta-1})}$, $t \in [0,+\infty)$, then $h'(t) = \frac{t^{\beta-3}(\beta-2-t^{\beta-1})}{(1+t^{\beta-1})^2}$. Let $h'(t) = 0$, we get $t = (\beta-2)^{\frac{1}{\beta-1}}$. So $h_{\max} = \frac{(\beta-2)^{\frac{\beta-2}{\beta-1}}}{\beta-1} < \frac{1}{\beta-1} < 1$.

For $0 \le t \le s < +\infty$,

$$\begin{aligned}\frac{G(t,s)}{1+t^{\beta-1}} &= \frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})}z(s) < \frac{1}{\Delta} = L, \\ \frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}} &= \frac{t^{\beta-2}}{\Gamma(\beta-1)z(0)(1+t^{\beta-1})}z(s) < (\beta-1)L.\end{aligned}$$

(4) For $0 \le s < \frac{1}{k}$, combining with the increasingness of $z(s)$, we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}} &= \min_{\frac{1}{k}\le t\le k}\left[\frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})}z(s) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})}\right] \\ &\ge \min_{\frac{1}{k}\le t\le k}\frac{t^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})} \\ &\ge \frac{\frac{1}{k^{\beta-1}} - (\frac{1}{k}-s)^{\beta-1}}{\Gamma(\beta)(1+k^{\beta-1})} \\ &\ge 0.\end{aligned}$$

For $\frac{1}{k} \le s \le k$,

$$\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}} = \min\left\{\min_{\frac{1}{k}\le t<s}\frac{G(t,s)}{1+t^{\beta-1}}, \min_{s\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}}\right\}.$$

By simple analysis, we find

$$\begin{aligned}\min_{\frac{1}{k}\le t<s}\frac{G(t,s)}{1+t^{\beta-1}} &= \min_{\frac{1}{k}\le t\le s}\frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})}z(s) \\ &\ge \min_{\frac{1}{k}\le t<s}\frac{t^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})} \\ &= \frac{1}{\Gamma(\beta)(k^{\beta-1}+1)}\end{aligned}$$

and

$$\min_{s\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}} = \min_{s\le t\le k}\left[\frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})}z(s) - \frac{(t-s)^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})}\right]$$

$$\begin{aligned}
&\geq \min_{s\leq t\leq k} \frac{t^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})}\\
&\geq \frac{s^{\beta-1}}{\Gamma(\beta)(1+k^{\beta-1})}\\
&\geq \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})}.
\end{aligned}$$

Thus

$$\min_{\frac{1}{k}\leq t\leq k} \frac{G(t,s)}{1+t^{\beta-1}} \geq \min_{s\in[\frac{1}{k},k]} \left\{ \frac{1}{\Gamma(\beta)(k^{\beta-1}+1)}, \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})} \right\},$$

that is,

$$\min_{\frac{1}{k}\leq t\leq k} \frac{G(t,s)}{1+t^{\beta-1}} \geq \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})}.$$

For $s > k$,

$$\begin{aligned}
\min_{\frac{1}{k}\leq t\leq k} \frac{G(t,s)}{1+t^{\beta-1}} &= \min_{\frac{1}{k}\leq t\leq k} \frac{t^{\beta-1}}{\Gamma(\beta)z(0)(1+t^{\beta-1})} z(s)\\
&\geq \min_{\frac{1}{k}\leq t\leq k} \frac{t^{\beta-1}}{\Gamma(\beta)(1+t^{\beta-1})}\\
&\geq \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})}.
\end{aligned}$$

In conclusion,

$$\min_{\frac{1}{k}\leq t\leq k} \frac{G(t,s)}{1+t^{\beta-1}} \geq \begin{cases} 0, & 0\leq s<\frac{1}{k},\\ \frac{1}{\Gamma(\beta)k^{\beta-1}(1+k^{\beta-1})}, & s\geq \frac{1}{k}. \end{cases}$$

The proof is completed. □

Now, we consider the space E defined by

$$E = \left\{ x\in C^1\big([0,+\infty),\mathbb{R}\big) : \lim_{t\to+\infty} \frac{|x(t)|}{1+t^{\beta-1}} < +\infty, \lim_{t\to+\infty} \frac{|x'(t)|}{1+t^{\beta-1}} < +\infty \right\}$$

endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \sup_{t\geq 0} \frac{|x(t)|}{1+t^{\beta-1}}$. It is not difficult to see that E is a Banach space.

Lemma 6 ([23]) *Let* $U = \{x\in E, \|x\| < l, \textit{where } l > 0\}$, $U(t) = \{\frac{x(t)}{1+t^{\beta-1}}, x\in U\}$, $U'(t) = \{\frac{x'(t)}{1+t^{\beta-1}}, x\in U\}$. *The set* U *is relatively compact in* E *if* $U(t)$ *and* $U'(t)$ *are both equicontinuous on any finite subinterval of* $\mathbb{R}^+$ *and equiconvergent at* ∞, *that is, for any* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon) > 0$ *such that*

$$\left| \frac{x(t_1)}{1+t_1^{\beta-1}} - \frac{x(t_2)}{1+t_2^{\beta-1}} \right| < \epsilon, \qquad \left| \frac{x'(t_1)}{1+t_1^{\beta-1}} - \frac{x'(t_2)}{1+t_2^{\beta-1}} \right| < \epsilon, \quad \forall x\in U, t_1, t_2 > \delta.$$

Lemma 7 ([24]) *Let K be a cone in a real Banach space E. Let ω and μ be nonnegative continuous convex functionals on K, θ be a nonnegative continuous concave functional on K, and ϕ be a nonnegative continuous functional on K satisfying $\phi(\epsilon x) \leq \epsilon\phi(x)$ for all $0 \leq \epsilon \leq 1$, such that for some numbers $N > 0$ and $C > 0$,*

$$\theta(x) \leq \phi(x), \qquad \|x\| \leq N\omega(x),$$

for all $x \in \overline{K(\omega, C)}$. Let $l, b, r > 0$ and define the following convex sets:

$$\begin{aligned} &K(\omega, C) = \{x \in K | \omega(x) < C\}, \\ &K(\omega, \theta, b, C) = \{x \in K | b \leq \theta(x), \omega(x) \leq C\}, \\ &K(\omega, \mu, \theta, b, r, C) = \{x \in K | b \leq \theta(x), \mu(x) \leq r, \omega(x) \leq C\}, \end{aligned}$$

and a closed set

$$Q(\omega, \phi, l, C) = \{x \in K | l \leq \phi(x), \omega(x) \leq C\}.$$

Suppose

$$\mathcal{T} : \overline{K(\omega, C)} \to \overline{K(\omega, C)}$$

is completely continuous and there exist some numbers $l, b, r > 0$ with $l < b$ such that
(I_1) $\{x \in K(\omega, \mu, \theta, b, r, C) | \theta(x) > b\} \neq \emptyset$ *and* $\theta(\mathcal{T}x) > b$ *for* $x \in K(\omega, \mu, \theta, b, r, C)$;
(I_2) $\theta(\mathcal{T}x) > b$ *for* $x \in K(\omega, \theta, b, C)$ *with* $\mu(\mathcal{T}x) > r$;
(I_3) $0 \notin Q(\omega, \phi, l, C)$ *and* $\phi(\mathcal{T}x) < l$ *for* $x \in Q(\omega, \phi, l, C)$ *with* $\phi(x) = l$.
Then $\mathcal{T}$ has at least three fixed points $x_l, x_2, x_3 \in \overline{K(\omega, C)}$ such that

$$\begin{aligned} &\omega(x_i) \leq C, \quad i = 1, 2, 3; \\ &\theta(x_1) > b; \\ &l < \phi(x_2) \quad \text{with } \theta(x_2) < b; \\ &\phi(x_3) < l. \end{aligned}$$

3 Main results

Define a cone $K = \{x \in E, x(t) \geq 0, x'(t) \geq 0, t \in [0, +\infty)\}$ and the operator $\mathcal{T} : K \to E$ as follows:

$$\mathcal{T}x(t) = \int_0^{+\infty} G(t,s)a(s)f\big(s, x(s), x'(s)\big)\, ds.$$

We can deduce that the fixed point of the operator $\mathcal{T}$ is a solution of the boundary value problem (1.3) from Lemma 3.

Lemma 8 *If (H_1), (H_2), and (H_3) hold, then the operator $\mathcal{T} : K \to K$ is completely continuous.*

Proof To complete the proof, we divide it into the following five steps:

Step 1. We will show that $\mathcal{T}: K \to K$.

Clearly,

$$\mathcal{T}x'(t) = \int_0^{+\infty} \frac{\partial}{\partial t} G(t,s)a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds.$$

Due to the continuity and nonnegativity of $G(t,s)$, $\frac{\partial}{\partial t}G(t,s)$, $a(t)$, and $f(t,x,y)$, we know that $\mathcal{T}x(t) \geq 0$ and $\mathcal{T}x'(t) \geq 0$ are continuous with respect to $t \in [0,+\infty)$. Applying (3) of Lemma 5 and (H_1), (H_2), for any fixed $x \in K$, we get $\frac{x(t)}{1+t^{\beta-1}} \leq \|x\|$, $\frac{x'(t)}{1+t^{\beta-1}} \leq \|x\|$, $t \in [0,+\infty)$, and then there exists $\alpha_x > 0$ such that

$$\begin{aligned}
\lim_{t\to+\infty} \frac{|\mathcal{T}x(t)|}{1+t^{\beta-1}} &= \lim_{t\to+\infty} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\beta-1}} a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&\leq L \int_0^{+\infty} a(s)f\left(s, \bigl(1+s^{\beta-1}\bigr)\frac{x(s)}{1+s^{\beta-1}}, \bigl(1+s^{\beta-1}\bigr)\frac{x'(s)}{1+s^{\beta-1}}\right) ds \\
&\leq L\alpha_x \int_0^{+\infty} a(s)\,ds < +\infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to+\infty} \frac{|\mathcal{T}x'(t)|}{1+t^{\beta-1}} &= \lim_{t\to+\infty} \int_0^{+\infty} \frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}} a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&\leq (\beta-1)L \int_0^{+\infty} a(s)f\left(s, \bigl(1+s^{\beta-1}\bigr)\frac{x(s)}{1+s^{\beta-1}}, \bigl(1+s^{\beta-1}\bigr)\frac{x'(s)}{1+s^{\beta-1}}\right) ds \\
&\leq (\beta-1)L\alpha_x \int_0^{+\infty} a(s)\,ds < +\infty.
\end{aligned}$$

As a result, $\mathcal{T}(K) \subset K$.

Step 2. We will check the continuity of $\mathcal{T}$.

Let $\{x_n\} \subset E$ with $x_n \to x$, $x_n' \to x'$ as $n \to +\infty$. Hence, there exists a positive constant r_0 such that

$$\max\Bigl\{\|x\|_\infty, \sup_{n\in\mathbb{N}} \|x_n\|_\infty\Bigr\} < r_0, \qquad \max\Bigl\{\bigl\|x'\bigr\|_\infty, \sup_{n\in\mathbb{N}} \bigl\|x_n'\bigr\|_\infty\Bigr\} < r_0.$$

With the help of Lemma 5, the continuity of f, and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\|\mathcal{T}x_n - \mathcal{T}x\|_\infty &= \sup_{t\geq 0} \left| \frac{\mathcal{T}x_n(t)}{1+t^{\beta-1}} - \frac{\mathcal{T}x(t)}{1+t^{\beta-1}} \right| \\
&= \sup_{t\geq 0} \left| \int_0^{+\infty} \frac{G(t,s)}{1+t^{\beta-1}} a(s)\bigl[f\bigl(s,x_n(s),x_n'(s)\bigr) - f\bigl(s,x(s),x'(s)\bigr)\bigr]\,ds \right| \\
&\leq L \int_0^{+\infty} a(s)\bigl|f\bigl(s,x_n(s),x_n'(s)\bigr) - f\bigl(s,x(s),x'(s)\bigr)\bigr|\,ds \to 0 \quad (n\to\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mathcal{T}x_n' - \mathcal{T}x'\right\|_\infty \\
&\quad = \sup_{t\geq 0}\left|\frac{\mathcal{T}x_n'(t)}{1+t^{\beta-1}} - \frac{\mathcal{T}x'(t)}{1+t^{\beta-1}}\right| \\
&\quad = \sup_{t\geq 0}\left|\int_0^{+\infty}\frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}}a(s)\left[f\left(s,x_n(s),x_n'(s)\right) - f\left(s,x(s),x'(s)\right)\right]ds\right| \\
&\quad \leq (\beta-1)L\int_0^{+\infty}a(s)\left|f\left(s,x_n(s),x_n'(s)\right) - f\left(s,x(s),x'(s)\right)\right|ds \to 0 \quad (n\to\infty).
\end{aligned}$$

Therefore

$$\|\mathcal{T}x_n - \mathcal{T}x\| \to 0, \quad n\to\infty,$$

which implies that $\mathcal{T}$ is continuous.

Step 3. Let $P\subset K$ be a bounded set, then there exists a positive constant k_1 such that $\|x\|\leq k_1$ for any $x\in P$. By (H_1), let

$$k_2 = \sup\left\{f\left(t,\left(1+t^{\beta-1}\right)x,\left(1+t^{\beta-1}\right)y\right),(t,x,y)\in[0,+\infty)\times[0,k_1]\times[0,k_1]\right\}.$$

Next, we will prove $\mathcal{T}(P)$ is bounded.

For all $x\in P$, from Lemma 5, we get

$$\begin{aligned}
\|\mathcal{T}x\|_\infty &= \sup_{t\geq 0}\left|\frac{\mathcal{T}x(t)}{1+t^{\beta-1}}\right| = \sup_{t\geq 0}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\beta-1}}a(s)f\left(s,x(s),x'(s)\right)ds \\
&\leq L\int_0^{+\infty}a(s)f\left(s,\left(1+s^{\beta-1}\right)\frac{x(s)}{1+s^{\beta-1}},\left(1+s^{\beta-1}\right)\frac{x'(s)}{1+s^{\beta-1}}\right)ds \\
&\leq Lk_2\int_0^{+\infty}a(s)\,ds < +\infty.
\end{aligned}$$

In a similar manner, we establish

$$\left\|\mathcal{T}x'\right\|_\infty \leq (\beta-1)Lk_2\int_0^{+\infty}a(s)\,ds < +\infty$$

for all $x\in P$. It follows that $\mathcal{T}(P)$ is uniformly bounded.

Step 4. We will prove that $\{\frac{\mathcal{T}x(t)}{1+t^{\beta-1}}, x\in P\}$, $\{\frac{\mathcal{T}x'(t)}{1+t^{\beta-1}}, x\in P\}$ are equicontinuous on any finite subinterval of $[0,+\infty)$.

For any $\rho>0$ and $t_1,t_2\in[0,\rho]$, without loss of generality, we assume that $t_2>t_1$. For all $x\in P$, we obtain

$$\begin{aligned}
\left|\frac{\mathcal{T}x(t_2)}{1+t_2^{\beta-1}} - \frac{\mathcal{T}x(t_1)}{1+t_1^{\beta-1}}\right| &\leq \int_0^{+\infty}\left|\frac{G(t_2,s)}{1+t_2^{\beta-1}} - \frac{G(t_1,s)}{1+t_1^{\beta-1}}\right|a(s)f\left(s,x(s),x'(s)\right)ds \\
&\leq \int_0^{+\infty}\left|\frac{G(t_2,s)}{1+t_2^{\beta-1}} - \frac{G(t_1,s)}{1+t_2^{\beta-1}}\right|a(s)f\left(s,x(s),x'(s)\right)ds \\
&\quad + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_2^{\beta-1}} - \frac{G(t_1,s)}{1+t_1^{\beta-1}}\right|a(s)f\left(s,x(s),x'(s)\right)ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} \left| \frac{G(t_2,s) - G(t_1,s)}{1 + t_2^{\beta-1}} \right| a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\quad + \int_0^{+\infty} \frac{G(t_1,s)|t_2^{\beta-1} - t_1^{\beta-1}|}{(1 + t_1^{\beta-1})(1 + t_2^{\beta-1})} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\le \int_0^{t_1} \frac{|G(t_2,s) - G(t_1,s)|}{1 + t_2^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\quad + \int_{t_1}^{t_2} \frac{|G(t_2,s) - G(t_1,s)|}{1 + t_2^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\quad + \int_{t_2}^{+\infty} \frac{|G(t_2,s) - G(t_1,s)|}{1 + t_2^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\quad + L \int_0^{+\infty} \frac{t_2^{\beta-1} - t_1^{\beta-1}}{1 + t_2^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \\
&\to 0 \quad (t_1 \to t_2).
\end{aligned}
$$

Similarly, we have

$$
\left| \frac{\mathcal{T}x'(t_2)}{1 + t_2^{\beta-1}} - \frac{\mathcal{T}x'(t_1)}{1 + t_1^{\beta-1}} \right| \to 0 \quad (t_1 \to t_2).
$$

Hence, $\{\frac{\mathcal{T}x(t)}{1+t^{\beta-1}}, x \in P\}$, $\{\frac{\mathcal{T}x'(t)}{1+t^{\beta-1}}, x \in P\}$ are equicontinuous on any finite subinterval of $[0, +\infty)$.

Step 5. We will prove that the sets $\{\frac{\mathcal{T}x(t)}{1+t^{\beta-1}}, x \in P\}$ and $\{\frac{\mathcal{T}x'(t)}{1+t^{\beta-1}}, x \in P\}$ are equiconvergent at $t \to +\infty$.

For all $x \in U$,

$$
\begin{aligned}
\lim_{t \to +\infty} \left| \frac{\mathcal{T}x(t)}{1 + t^{\beta-1}} \right| &= \lim_{t \to +\infty} \left| \int_0^{+\infty} \frac{G(t,s)}{1 + t^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\,ds \right| \\
&\le Lk_2 \int_0^{+\infty} a(s)\,ds < +\infty.
\end{aligned}
$$

Moreover, we get

$$
\lim_{t \to +\infty} \left| \frac{\mathcal{T}x'(t)}{1 + t^{\beta-1}} \right| < +\infty.
$$

Accordingly, $\{\frac{\mathcal{T}x(t)}{1+t^{\beta-1}}, x \in P\}$ and $\{\frac{\mathcal{T}x'(t)}{1+t^{\beta-1}}, x \in P\}$ are equiconvergent at $t \to +\infty$. As a result, $\mathcal{T} : K \to K$ is completely continuous by Lemma 6. The proof is completed. □

Next, we will prove the existence of at least three positive solutions by making use of Avery–Peterson theorem. For convenience, we denote

$$
\begin{aligned}
&L_1 = \frac{1}{\Gamma(\beta) k^{\beta-1} (1 + k^{\beta-1})}, \\
&M = (\beta - 1) L \int_0^{+\infty} a(s)\,ds, \\
&m = L_1 \int_{\frac{1}{k}}^{k} a(s)\,ds,
\end{aligned}
$$

and define

$$\omega(x) = \mu(x) = \|x\|, \qquad \phi(x) = \sup_{t\geq 0} \frac{|x(t)|}{1+t^{\beta-1}}, \qquad \theta(x) = \min_{\frac{1}{k}\leq t\leq k} \frac{x(t)}{1+t^{\beta-1}},$$

where $x \in K$.

Theorem 1 *Assume that* (H_1), (H_2), *and* (H_3) *are hold. If there exist constants* $0 < l < b < r \leq C$ *such that*

(S_1) $f(t,(1+t^{\beta-1})x,(1+t^{\beta-1})y) < \frac{C}{M}$, $0 \leq t < +\infty$, $0 \leq x \leq C$, $0 \leq y \leq C$,

(S_2) $f(t,(1+t^{\beta-1})x,(1+t^{\beta-1})y) > \frac{b}{m}$, $\frac{1}{k} \leq t < k$, $b \leq x \leq r$, $0 \leq y \leq C$,

(S_3) $f(t,(1+t^{\beta-1})x,(1+t^{\beta-1})y) < \frac{l}{M}$, $0 \leq t < +\infty$, $0 \leq x \leq l$, $0 \leq y \leq C$.

Then the boundary value problem (1.3) *has at least three positive solutions* x_1, x_2, *and* x_3 *satisfying*

$$\begin{aligned}
&\omega(x_i) \leq C \quad (i = 1,2,3);\\
&\theta(x_1) > b; \qquad l < \phi(x_2), \theta(x_2) < b; \qquad \phi(x_3) < l.
\end{aligned}$$

Proof Evidently, $\theta(x) \leq \phi(x)$, $\phi(\epsilon x) \leq \epsilon\phi(x)$, and $\|x\| \leq \mu(x)$.

For all $x \in \overline{K(\omega, C)}$, we know $\omega(x) = \|x\| \leq C$. That is to say, $0 \leq \frac{x(t)}{1+t^{\beta-1}} \leq C$ and $0 \leq \frac{x'(t)}{1+t^{\beta-1}} \leq C$ for $t \in [0,+\infty)$. Applying (S_1), we find

$$f\big(t,x(t),x'(t)\big) = f\left(t,\big(1+t^{\beta-1}\big)\frac{x(t)}{1+t^{\beta-1}},\big(1+t^{\beta-1}\big)\frac{x'(t)}{1+t^{\beta-1}}\right) < \frac{C}{M}, \quad t \in [0,+\infty).$$

Then combining with Lemma 5, we obtain

$$\begin{aligned}
\|\mathcal{T}x\|_\infty &= \sup_{t\geq 0} \frac{|\mathcal{T}x(t)|}{1+t^{\beta-1}}\\
&= \sup_{t\geq 0} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\beta-1}} a(s) f\big(s,x(s),x'(s)\big)\,ds\\
&\leq L \int_0^{+\infty} a(s) f\big(s,x(s),x'(s)\big)\,ds\\
&< L \cdot \frac{C}{M} \int_0^{+\infty} a(s)\,ds\\
&= \frac{C}{\beta-1} < C
\end{aligned}$$

and

$$\begin{aligned}
\big\|\mathcal{T}x'\big\|_\infty &= \sup_{t\geq 0} \frac{|\mathcal{T}x'(t)|}{1+t^{\beta-1}}\\
&= \sup_{t\geq 0} \int_0^{+\infty} \frac{\frac{\partial}{\partial t}G(t,s)}{1+t^{\beta-1}} a(s) f\big(s,x(s),x'(s)\big)\,ds\\
&\leq (\beta-1)L \int_0^{+\infty} a(s) f\big(s,x(s),x'(s)\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&< (\beta-1)L \cdot \frac{C}{M}\int_0^{+\infty} a(s)\,ds \\
&= C,
\end{aligned}
$$

which lead to $\|\mathcal{T}x\| = \max\{\|\mathcal{T}x\|_\infty, \|\mathcal{T}x'\|_\infty\} < C$. That is, $\mathcal{T} : \overline{K(\omega, C)} \to \overline{K(\omega, C)}$.

Next, we will show that $\mathcal{T}$ satisfies the conditions of Lemma 7.

First, let $x_0(t) = \frac{b+r}{2}(1+t^{\beta-1})$, $0 \le t < +\infty$, then $x_0'(t) = \frac{(\beta-1)(b+r)}{2}t^{\beta-2}$. Clearly, $x_0 \in K$ and $\|x_0\|_\infty = \frac{b+r}{2} < r \le C$. In addition,

$$
\|x_0'\|_\infty = \sup_{t\ge 0}\frac{|x_0'(t)|}{1+t^{\beta-1}} = \sup_{t\ge 0}\frac{(\beta-1)(b+r)}{2}\frac{t^{\beta-2}}{1+t^{\beta-1}} = \frac{(b+r)}{2}(\beta-2)^{\frac{\beta-2}{\beta-1}} < r \le C.
$$

Hence, $\|x_0\| < C$, that is, $\mu(x_0) < r$, $\omega(x_0) < C$. What is more, $\theta(x_0) = \min_{\frac{1}{k}\le t\le k}\frac{x_0(t)}{1+t^{\beta-1}} = \frac{b+r}{2} > b$. Thus, $\{x \in K(\omega,\mu,\theta,b,r,C) | \theta(x) > b\} \neq \emptyset$. In view of (S_2), we have

$$
f\bigl(t, x(t), x'(t)\bigr) = f\left(t, \bigl(1+t^{\beta-1}\bigr)\frac{x(t)}{1+t^{\beta-1}}, \bigl(1+t^{\beta-1}\bigr)\frac{x'(t)}{1+t^{\beta-1}}\right) > \frac{b}{M}, \quad t \in [0, +\infty).
$$

For all $u \in K(\omega,\mu,\theta,b,r,C)$, we get $b \le \frac{x(t)}{1+t^{\beta-1}} \le r$, $t \in [\frac{1}{k}, k]$. Then

$$
\begin{aligned}
\theta(\mathcal{T}x) &= \min_{\frac{1}{k}\le t\le k}\frac{\mathcal{T}x(t)}{1+t^{\beta-1}} \\
&= \min_{\frac{1}{k}\le t\le k}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\beta-1}}a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&\ge \int_0^{+\infty}\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}}a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&> L_1 \cdot \frac{b}{m}\int_{\frac{1}{k}}^{k} a(s)\,ds = b.
\end{aligned}
$$

So, the condition (I_1) is satisfied.

Second, if $x \in K(\omega,\theta,b,C)$ and $\mu(\mathcal{T}x) > r$, by (S_2), we know

$$
\begin{aligned}
\theta(\mathcal{T}x) &= \min_{\frac{1}{k}\le t\le k}\frac{\mathcal{T}x(t)}{1+t^{\beta-1}} \\
&= \min_{\frac{1}{k}\le t\le k}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\beta-1}}a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&\ge \int_0^{+\infty}\min_{\frac{1}{k}\le t\le k}\frac{G(t,s)}{1+t^{\beta-1}}a(s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\
&> L_1 \cdot \frac{b}{m}\int_{\frac{1}{k}}^{k} a(s)\,ds = b.
\end{aligned}
$$

Therefore, the condition (I_2) holds.

Finally, it is easy to see that $0 \notin Q(\omega,\phi,l,C)$ because of $\phi(0) = 0 < l$. Assume that $x \in Q(\omega,\phi,l,C)$ with $\phi(x) = l$. According to (S_3), we find

$$
\phi(\mathcal{T}x) = \sup_{t\ge 0}\frac{|\mathcal{T}x(t)|}{1+t^{\beta-1}}
$$

$$
\begin{aligned}
&= \sup_{t\geq 0} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\beta-1}} a(s) f\bigl(s, x(s), x'(s)\bigr)\, ds \\
&\leq L \int_0^{+\infty} a(s) f\bigl(s, x(s), x'(s)\bigr)\, ds \\
&< L \cdot \frac{l}{M} \int_0^{+\infty} a(s)\, ds = \frac{l}{\beta-1} < l.
\end{aligned}
$$

Thus, the condition (I_3) is satisfied.

By Lemma 7, the conclusion of Theorem 1 holds. This completes the proof. □

4 Example

Consider the following boundary value problem:

$$
\begin{cases}
D_{0^+}^{\frac{5}{2}} x(t) + a(t) f(t, x(t), x'(t)) = 0, \quad t \in [0, +\infty), \\
x(0) = x'(0) = 0, \\
\lim_{t\to+\infty} D_{0^+}^{\frac{3}{2}} x(t) = \int_0^{+\infty} h(t) x'(t)\, dt + \sum_{i=1}^{\infty} \frac{1}{4^i} D_{0^+}^{\frac{7}{6}} x(1 - \frac{1}{i+1}),
\end{cases}
\tag{4.1}
$$

where $a(t) = e^{-t}$, $h(t) = \frac{1}{4} e^{-t}$,

$$
f(t,x,y) = \begin{cases}
\frac{e^{-t}}{1000} + \frac{1}{10^4}(\frac{x}{1+t^{\frac{3}{2}}})^2 + \frac{y}{10{,}000(1+t^{\frac{3}{2}})}, \\
\quad \text{if } (t,x,y) \in [0,+\infty) \times [0, \frac{4(1+t^{\frac{3}{2}})}{5}] \times [0,+\infty), \\
\frac{e^{-t}}{1000} + \frac{1}{10^4}(\frac{x}{1+t^{\frac{3}{2}}})^2 + \frac{1000(x-\frac{4}{5}(1+t^{\frac{3}{2}}))}{(1+t^{\frac{3}{2}})^2} + \frac{y}{10{,}000(1+t^{\frac{3}{2}})}, \\
\quad \text{if } (t,x,y) \in [0,+\infty) \times (\frac{4(1+t^{\frac{3}{2}})}{5}, 1+t^{\frac{3}{2}}] \times [0,+\infty), \\
\frac{e^{-t}}{1000} + \frac{1}{10^4}(\frac{x}{1+t^{\frac{3}{2}}})^2 + \frac{200}{1+t^{\frac{3}{2}}} + \frac{y}{10{,}000(1+t^{\frac{3}{2}})}, \\
\quad \text{if } (t,x,y) \in [0,+\infty) \times (1+t^{\frac{3}{2}}, +\infty) \times [0,+\infty).
\end{cases}
$$

By direct computation, we get

$$
\begin{aligned}
&\Delta = \Gamma(\beta) - (\beta-1)\int_0^{+\infty} \tau^{\beta-2} h(\tau)\, d\tau - \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} \sum_{i=1}^{\infty} \eta_i \xi_i^{\beta-\gamma-1} \approx 0.5845, \\
&L = \frac{1}{\Delta} = 1.7109, \\
&L_1 = \frac{1}{\Gamma(\beta) k^{\beta-1}(1+k^{\beta-1})} \approx 0.0695, \\
&M = (\beta-1) L \int_0^{+\infty} a(s)\, ds = 2.5664, \\
&m = L_1 \int_{\frac{1}{k}}^{k} a(s)\, ds = 0.0327.
\end{aligned}
$$

Let $l = \frac{4}{5}$, $b = 1$, $r = C = 1000$, $k = 2$, and assume $f(t,x,y)$ satisfies

$$
f\bigl(t, \bigl(1+t^{\beta-1}\bigr)x, \bigl(1+t^{\beta-1}\bigr)y\bigr) \leq 300.101 < \frac{C}{M},
$$

$$\text{if } 0 \le t < +\infty, 0 \le x \le 1000, 0 \le y \le 1000,$$

$$f\bigl(t, \bigl(1 + t^{\beta-1}\bigr)x, \bigl(1 + t^{\beta-1}\bigr)y\bigr) \ge 53.141 > \frac{b}{m},$$

$$\text{if } \frac{1}{2} \le t < 2, 1 \le x \le 1000, 0 \le y \le 1000,$$

$$f\bigl(t, \bigl(1 + t^{\beta-1}\bigr)x, \bigl(1 + t^{\beta-1}\bigr)y\bigr) \le 0.100 < \frac{l}{M},$$

$$\text{if } 0 \le t < +\infty, 0 \le x \le \frac{4}{5}, 0 \le y \le 1000.$$

Then the boundary value problem (4.1) has at least three positive solutions x_1, x_2, and x_3 satisfying

$$\omega(x_i) \le 1000 \quad (i = 1, 2, 3);$$

$$\theta(x_1) > 1; \qquad \frac{4}{5} < \phi(x_2), \theta(x_2) < 1; \qquad \phi(x_3) < \frac{4}{5}.$$

5 Conclusions

This paper is devoted to the study of a class of fractional boundary value problems which involve an improper integral and the infinite-point on the half-line. Thanks to Avery–Peterson fixed point theorem, we have presented sufficient conditions that demonstrate the existence of at least three positive solutions. The new results generalize some existing results in the literature. From the discussion and results in this paper, we conclude that Avery–Peterson fixed point theorem is an effective method to deal with the multiplicity of positive solutions of fractional boundary value problems on the half-line.

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Declarations

Competing interests
The authors declare no competing interests.

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