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# Analytical mechanics methods in finite element analysis of multibody elastic system

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**Abstract**

The study of multibody systems with elastic elements involves at the moment the reevaluation of the classical methods of analysis offered by analytical mechanics. Modeling this system with the finite element method requires obtaining the motion equation for an element in the circumstances imposed by a multibody system. The paper aims to present the main analysis methods used by researchers, to make a comparative analysis, and to show the advantages or disadvantages offered by different methods. For the presentation of the main methods (namely Lagrange's equations, Gibbs–Appell's equations, Maggi's formalism, Kane's equations, and Hamilton's equations) a unified notation is used. The paper provides a critical evaluation of the studied applications that involved some of these methods, highlighting the reason why it was decided to use them. Also, the paper identifies potential research areas to explore.

**Keywords:** MBS; FEM; Maggi's formalism; Lagrange's equations; Energy of acceleration; Hamilton's equations; Kane's equations; Gibbs–Appel method; Analytical methods

## 1 Introduction

In the last decades, the development of technology and fabrication of machines and equipment that work at ever higher speeds, using ever greater forces in difficult operating conditions, have led to the need for a more detailed study of these systems, developing a new field of research, namely that of multibody systems (MBS). Since in the previously described situation the elasticity of the bodies manifests itself in such a way that it influences the mechanical behavior of the systems, a study of MBS with elastic elements has become necessary. Absolutely naturally, this development being in a stage of maximum development of the finite element method (FEM), the use of the method in the study of MBS with elastic elements is the optimal solution to approach these systems. Modeling systems using FEM requires a stage of modeling and writing the equations of motion, and this implies the use of currently less used mechanics chapters, such as the analytical mechanics chapter [1].

The main advantage of analytical mechanics methods in the study of any mechanical system is the generality of the applied procedures. This allows the easy creation of efficient algorithms and software. The current mechanical systems found in different indus-

trial fields are complex and subject to extreme operating conditions. The use of analytical mechanics methods offers the possibility of unified treatment of systems, regardless of their size and the degrees of freedom (DOF) that describe their behavior [2–4]. The notions used are generally familiar to the researchers (mechanical energy, kinetic, potential, work, momentum, Hamiltonian, etc.). Analytical mechanics methods generalize the way to analyze a mechanical system. We mention that within analytical mechanics there are equivalent formulations of the general principles, a researcher can choose among them the method he considers the most appropriate for the specific case he is studying. Also, the method can very well describe the constraints that appear in a particular system. In this framework, Lagrange's equations represent the established method of analyzing complex problems, bringing together several advantages for the researcher: the use of well-known notions (kinetic energy, potential energy, work), the use of scalar quantities avoiding vector ones, the possibility of algorithmizing the procedure, the existence a large number of examples treated with this method in the literature [5–8]. Lagrange's equations can highlight the existence of some constants in the case of conservative systems that can facilitate the analysis [9].

Analytical mechanics offers other possibilities to obtain the equations of motion which, in certain circumstances, can bring major advantages in the process of dynamic analysis of a real mechanical system. During the last decades, researchers used different formulas to obtain the equations of motion, observing certain peculiarities in the parametric description of the systems, which were successfully used to facilitate the modeling and shorten the analysis time [10–17].

The introduction of equivalent forms for the description of a system also required the development of specific calculation methods. The mathematical bases of system modeling were developed. The final goal is to offer the researcher the possibility of easy modeling and use of existing commercial software as far as possible, so that the most complex systems can be studied. The established method for analyzing these systems is FEM, and researchers have tried to develop methods that involve FEM as a numerical method for solving problems [18–20].

Modeling using alternative methods from analytical mechanics represents an important aid in the analysis of engineering applications with minimal costs, but their effective application involves the development of numerical methods oriented towards the respective formulations. For this reason, the potential of using these alternative forms is relatively low. Numerical analysis methods that support the reduction of calculation time and cost reduction are provided in [21]. Another integration procedure where the symbolic formalism is used is described in [22]. The valability of the procedure is illustrated by comparing the results with those obtained using other methods.

The bodies that make up an MBS can have rigid movements, but the elasticity of some of them can cause elastic deformations to appear that cannot be neglected. In [23, 24] such an analysis is made that very well suits the methodology applied in the general case. Thus, the construction of the matrix coefficients that appear in the equations is done outside the finite element code. Some engineering applications built using this philosophy are described in [25–29].

The use of FEM for modeling, together with the use of classical MBS analysis methods, involves significant computing resources and high costs. In order to reduce the size of the studied systems, reduced-order models (ROM) have been recently used. An exam-

ple of the application of this strategy is given in [30]. The presented application is for an nPlan system, but the expansion in 3D does not involve problems except for the volume of calculation. In this way, the effort and calculation time are significantly reduced.

In order to simplify the symbolic writing of the equations of motion, the topological methods of describing the systems were used. In this way, the time required for modeling is reduced [31]. Along with this technique, similar or auxiliary techniques have been imagined and described [32–37].

## 2 Finite element analysis of an elastic MBS

In the previous section, research was presented that uses FEM for modeling, insisting especially on the analytical methods proposed by the researchers. This section is focused on the presentation of the context of FEM use, the level reached in the use of the method in MBS applications, and the problems involved. The finite element method represents an approximation method to determine the deformation field of an elastic body. For this, the elastic body is discretized into independent bodies, linked together by nodes, which ensure the transmission of forces between elements. Each node is defined by parameters that represent the independent coordinates of the element. The principle of the method is the approximation of the displacement field of the element with known polynomial functions. Each type of finite element chosen for the study is characterized by specific interpolation functions (shape functions). In this way, the displacement of each point belonging to the finite element is defined by the displacements or rotations of the nodes of the studied element. In this way, the differential equations of the mechanics of the continuous medium can be applied, considering the functions that determine the known displacements to be known analytical functions. For a single finite element, the evolution equations of the element can then be written, which are second-order differential equations with constant coefficients. To obtain these equations, established methods in analytical mechanics are used, such as, for example, the method of Lagrange's equations. The matrix coefficients of the obtained equations are determined by the shape functions chosen to define the finite element.

The equations are obtained, for a finite element, generally in a local reference frame. The next stage is the assembly of all the motion equations, written for each individual element, into a system of differential equations that characterizes the motion and deformations of the entire system. To achieve this, it is necessary to write all these equations of motion, each related to a local reference system, in a unique global reference system. At this level, all the mentioned operations are well documented and verified by commercial software.

An important problem is obtaining the equations of motion for a single element using a method chosen in such a way that the number of operations required to solve the problem is minimal.

The study of this problem has been done for a long time, the research carried out being presented in a rich literature. The first researches dealt with mechanical elements in motion that can be discretized by one-dimensional finite elements and the motion was considered planar [38–40]. A two-dimensional element is presented in [41]. More complex and sophisticated elements were used in the last decade [42–47].

In the analysis of complex mechanical systems, Lagrange's equations remain the main method for the dynamic analysis of an MBS with elastic elements. Writing the equations of motion for a finite element, the major step in modeling a system, requires the application

of a method known from analytical mechanics, and Lagrange's equations have proven their effectiveness over time. They are currently probably the most used method in this analysis.

Analytical mechanics also offers alternative, equivalent study methods, along with Lagrange's equations. The disadvantage of these methods is the use of concepts and notions rarely used in applications, which is why they are used less. The industrial applications, which at the moment are of particular complexity, still make these methods interesting because they offer, for certain types of applications, advantages that can lead to easier description and reduction of modeling costs. Of all these available methods, namely the Gibbs–Appell formalism, Maggi's equations, Kane's equations, and Hamilton's equations, stood out as being used more often.

The predominant use of Lagrange's equations is also due to the fact that the generalized coordinates make possible a unified treatment of the studied problems, and also efficient algorithms can be made for the numerical solution of the problems. Lagrange's multipliers can be eliminated by writing these equations, which makes solving problems significantly easier. The number of unknowns is thus reduced to the number of generalized independent coordinates that describe the system. In FEM, where the number of degrees of freedom (DOF) is very high, this property can lead to the significant decrease of the operations that must be performed.

Gibs–Appell's method introduces a new notion, namely the energy of accelerations, less familiar to researchers. Instead, the number of derivation operations in this case is lower than in the case of Lagrange's method and there are cases in which the use of this method can lead to savings in solving time [48, 49]. A disadvantage is the fact that five matrix terms appear in the expression of acceleration, while only four in the expression of speed. The lower number of differentiation operations compensates for this disadvantage [50–52]. Gibs–Appell's equations represent a less used method but which has recently been reconsidered for its advantages, in the context of current industrial development [53–57].

Regarding Hamilton's equations, there is a limited literature dealing with this subject. They are starting to be reconsidered, in the context of the development of the use of numerical methods. In this method, the second-order differential equations will be replaced with first-order differential equations, but their number is doubled. However, the fact that we are not obliged to pass within the numerical procedures from the second- to first-order equations by introducing additional variables can be an advantage [58–61].

Other methods and contributions to the development of the field are presented in [62–67].

## 3 Main notions and notations

### 3.1 Kinematics

In the following, some notations will be introduced, most of them devoted to mechanics and FEM, which will be used in the presentation of alternative analysis methods [49]. We will analyze a single finite element that is in general motion together with the body, having at the same time an elastic deformation. This element is referred to as a local (mobile) coordinate system. The motion of this local system is known. The elasticity of the elements is manifested for all cases studied by the elastic deformation that the chosen point can take. The motion of this local reference system in relation to a global reference system is considered known. So the velocity $\bar{v}_o$ and the acceleration $\bar{a}_o$ of the origin of the local reference frame and the angular velocity $\bar{\omega}$ and the angular acceleration $\bar{\varepsilon}$ of the reference

frame are known. The indices L (from local) and G (from global) will indicate the sizes coresponding to local/global reference frame. An orthonormal operator $[R] = [r_{ij}]$, $i,j = 1,2,3\ldots$ transforms the components of a vector from the local system to the global one, $a_{i,G} = r_{ij}a_{j,L}$, $i,j = 1,2,3$; $[R]$ will represent the rotation matrix,

$$[R] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}. \tag{1}$$

The vector $\bar{r}_O$ has the components $(X_{O,1}, X_{O,2}, X_{O,3})$ in the global reference frame and $(x_{O,1}, x_{O,2}, x_{O,3})$ in the local reference frame. Similarly, $\bar{r}_M$ has the components $(X_{M,1}, X_{M,2}, X_{M,3})$ and $(x_{M,1}, x_{M,2}, x_{M,3})$, while $\bar{r}_{M'}$ has the components $(X_{M',1}, X_{M',2}, X_{M',3})$ and $(x_{M',1}, x_{M',2}, x_{M',3})$, respectively. The position vector of the point $M$ considering the origin $O$ is $\bar{r}$ with the components $(X_1, X_2, X_3)$ and $(x_1, x_2, x_3)$ in the two reference frames and the displacement vector of the point $M$, $\bar{u}$, has the components $(u_1, u_2, u_3)$. Here $\bar{r}_M$ and $\bar{r}_{M'}$ represent the position vectors of the points $M$ and $M'$, respectively, and $\bar{r}_O$ is the position vector of the origin.

The arbitrary point $M$ of the finite element becomes, after deformation, the point $M'$. Its coordinates are (in the global reference frame)

$$X_{M',i} = X_{O,i} + \alpha_{ij}(x_j + u_j); \quad i,j = 1,2,3. \tag{2}$$

The dependence between independent nodal displacements and the displacement of a current point are approximated in FEA through the relation (due to the linear elasticity of the material)

$$u_i = N_{ij}\delta_j \quad i = 1,2,3; j = \overline{1,p}, \tag{3}$$

where

$$\{\delta\}_L = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_p \end{Bmatrix} \tag{4}$$

is the vector of the independent coordinates and $\delta_1, \delta_2, \ldots, \delta_p$ are the indepenent coordinates. In the FEM, the dispacements are approximated using known shape functions. The position of the point $M'$ is, considering Eq. (3), given by

$$X_{M',i} = X_{O,i} + \alpha_{ij}x_j + \alpha_{ij}N_{jk}u_k; \quad i,j = 1,2,3; k = \overline{1,p}. \tag{5}$$

The components of the velocity vector of $M'$ are

$$\dot{X}_{M',i} = \dot{X}_{O,i} + \dot{\alpha}_{ij}x_j + \dot{\alpha}_{ij}N_{jk}\delta_k + \alpha_{ij}N_{jk}\dot{\delta}_k; \quad i,j = 1,2,3; k = \overline{1,p}, \tag{6}$$

and the componenets of the acceleration are

$$\begin{aligned}&\ddot{X}_{M',i} = \ddot{X}_{O,i} + \ddot{\alpha}_{ij}x_j + \ddot{\alpha}_{ij}N_{jk}\delta_k + 2\dot{\alpha}_{ij}N_{jk}\dot{\delta}_k + \alpha_{ij}N_{jk}\ddot{\delta}_k;\\ &i,j = 1,2,3; \quad k = \overline{1,p}.\end{aligned} \tag{7}$$

If we are interested in expressing the velocity in the local coordinate system, we have:

$$x_{M',i} = x_{O,i} + x_j + N_{jk}u_k; \quad i,j = 1,2,3; k = \overline{1,p}, \tag{8}$$

$$\begin{aligned}\dot{x}_{M',i} &= \alpha_{ji}\dot{X}_{M',j} = \alpha_{ji}\dot{X}_{O,j} + \alpha_{ji}\dot{\alpha}_{jk}x_k + \alpha_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r + \alpha_{ji}\alpha_{jk}N_{kr}\dot{\delta}_r\\ &= \dot{x}_{O,i} + \alpha_{ji}\dot{\alpha}_{jk}x_k + \alpha_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r + N_{ir}\dot{\delta}_r; \quad i,j,k = 1,2,3; r = \overline{1,p},\end{aligned} \tag{9}$$

since $\alpha_{ji}\alpha_{jk} = \delta_{ik}$, where $\delta_{ik}$ is the Kronecker delta.

### 3.2 Kinetic energy

The kinetic energy has the expression

$$\begin{aligned}E_C &= \frac{1}{2}\int_V \rho(\dot{x}_{M',i})^2\,dV\\ &= \frac{1}{2}\int_V \rho(\dot{x}_{O,i} + \alpha_{ij}\dot{\alpha}_{jk}x_k + \alpha_{ij}\dot{\alpha}_{jk}N_{kr}\delta_r + N_{ir}\dot{\delta}_r)^2\,dV\\ &= \frac{1}{2}\int_V \rho(\dot{x}_{O,i}\dot{x}_{O,i})\,dV + \frac{1}{2}\int_V \rho\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}x_k x_m\,dV\\ &\quad + \frac{1}{2}\int_V \rho\alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}N_{kr}N_{nt}\delta_r\delta_t\,dV\\ &\quad + \frac{1}{2}\int_V \rho N_{ir}N_{it}\dot{\delta}_r\dot{\delta}_t\,dV + \int_V \rho\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}x_k\,dV\\ &\quad + \int_V \rho\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}N_{kr}\delta_l\,dV + \int_V \rho\dot{x}_{O,i}N_{ir}\dot{\delta}_r\,dV\\ &\quad + \int_V \rho\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}x_k N_{mr}\delta_r\,dV + \int_V \rho\alpha_{ij}\dot{\alpha}_{jk}x_k N_{ir}\dot{\delta}_r\,dV\\ &\quad + \int_V \rho\alpha_{ij}\dot{\alpha}_{jk}N_{kr}N_{it}\delta_r\dot{\delta}_t\,dV, \quad i,j,l,k,m,n = 1,2,3; r,t = \overline{1,p}.\end{aligned} \tag{10}$$

In the appendix, the expanded expression of the kinetic energy can be found.
Using the notations

$$m = \int_V \rho\,dV; \qquad J_{km} = \int_V \rho x_k x_m\,dV, \tag{11}$$

$$m_{kr,nt} = \int_V \rho N_{kr}N_{nt}\,dV; \qquad m_{rt} = \int_V \rho N_{ir}N_{it}\,dV, \tag{12}$$

$$S_k = \int_V \rho x_k\,dV; \qquad m^I_{O,kr} = \int_V \rho N_{kr}\,dV, \tag{13}$$

$$m_{k,mr} = \int_V \rho x_k N_{mr}\,dV; \qquad \alpha_{ij}\alpha_{jk} = \delta_{ik}, \tag{14}$$

one obtains

$$\begin{aligned} Ec = {} & \frac{1}{2}m(\dot{x}_{O,i}\dot{x}_{O,i}) + \frac{1}{2}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}J_{km} + \frac{1}{2}\alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}m_{kr,nt}\delta_r\delta_t + \frac{1}{2}m_{rt}\dot{\delta}_r\dot{\delta}_t \\ & + S_k\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk} + m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}\delta_r + m_{O,ir}\dot{x}_{O,i}\dot{\delta}_r \\ & + m_{k,mr}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}\delta_r + m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_r\dot{\delta}_t. \end{aligned} \tag{15}$$

### 3.3 Potential energy

The internal work for a single finite element (potential energy) due to elastic deformation is obtained via the consecrated relation

$$E_p = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,dV, \tag{16}$$

where $\varepsilon_{ij}$ represents the strain tensor and $\sigma_{ij}$ the stress vector. The generalized Hooke's law can be written in the form

$$\sigma_{ij} = H_{ijkl}\varepsilon_{kl}. \tag{17}$$

Taking into account Eq. (2), the strains are [6]

$$\varepsilon_{ij} = b_{ijk}u_k = b_{ijk}N_{kr}\delta_r. \tag{18}$$

Using Eqs. (17)–(18), one obtains

$$E_p = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,dV = \frac{1}{2}\int_V H_{ijkl}b_{klm}b_{ijp}N_{mt}N_{pr}\delta_r\delta_t\,dV. \tag{19}$$

Matrix $[k]$ represents the stiffnes matrix and has the elements

$$k_{nr} = \frac{1}{2}\int_V H_{ijkl}b_{klm}b_{ijp}N_{mn}N_{pr}\,dV. \tag{20}$$

With this notation, Eq. (19) becomes

$$E_p = \frac{1}{2}k_{ij}\delta_i\delta_j. \tag{21}$$

### 3.4 Work

The generalized concentrated forces $q_{iL}$, $i = \overline{1,p}$, and generalized volume (distributed) forces $q_{iL}$, $i = \overline{1,p}$, produce respectively work

$$W^c = q_i\delta_i; \quad i = \overline{1,p} \tag{22}$$

and

$$W^{\mathrm{d}} = q_i^*\delta_i; \quad i = \overline{1,p}. \tag{23}$$

The total work of these forces is

$$W = \left(W^c + W^{\mathrm{d}}\right) = \left(q_i + q_i^*\right)\delta_i; \quad i = \overline{1,p}. \tag{24}$$

### 3.5 Lagrangian

The Lagrangian for an element is [48]

$$L = E_c - E_p + W + W^c. \tag{25}$$

Using Eqs. (16) and (21)–(23), the Lagrangian takes the form

$$\begin{aligned} L = {} & \frac{1}{2}m(\dot{x}_{O,i}\dot{x}_{O,i}) + \frac{1}{2}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}J_{km} + \frac{1}{2}\alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}m_{kr,nt}\delta_r\delta_t + \frac{1}{2}m_{rt}\dot{\delta}_r\dot{\delta}_t \\ & + S_k\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk} + m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}\delta_r + m_{O,ir}\dot{x}_{O,i}\dot{\delta}_r \\ & + m_{k,mr}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}\delta_r + m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_r\dot{\delta}_t \\ & - k_{rt}\delta_r\delta_t + q_r\delta_r + q_r^*\delta_r; \quad i,j,k,l,m = 1,2,3; r,t = \overline{1,p}. \end{aligned} \tag{26}$$

### 3.6 Momentum

To determine the generalized momenta, we use the relation

$$p_{r,L} = \frac{\partial L}{\partial \dot{\delta}_r}. \tag{27}$$

It results in the following:

$$p_{r,L} = m_{rt}\dot{\delta}_t + m_{O,ir}\dot{x}_{O,i} + m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk} + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_t; \quad r = \overline{1,p}. \tag{28}$$

Considering now the matrix $m^*_{ur}$ chosen so that

$$m^*_{ur}m_{rt} = \delta_{ut}; \quad u,r,t = \overline{1,p} \tag{29}$$

and premultiplying Eq. (28) with $m^*_{ur}$ results in

$$m^*_{ur}p_{r,L} = \dot{\delta}_u + m^*_{ur}m_{O,ir}\dot{x}_{O,i} + m^*_{ur}m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk} + m^*_{ur}m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_t; \quad r = \overline{1,p}, \tag{30}$$

wherefrom we obtain

$$\dot{\delta}_u = m^*_{ur}p_{r,L} - m^*_{ur}m_{O,ir}\dot{x}_{O,i} - m^*_{ur}m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk} - m^*_{ur}m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_t; \quad u,r,t = \overline{1,p}. \tag{31}$$

### 3.7 Hamiltonian

The Hamiltonian is defined by the relation

$$H = \frac{\partial L}{\partial \dot{\delta}_r}\dot{\delta}_r - L, \tag{32}$$

where for the Lagrangian Eq. (26) has been used.

### 3.8 Energy of accelerations

The energy of acceleration is a notion that can be used to obtain the motion equation with the Gibs–Appell equations. Considering a solid body, the energy of acceleration becomes

$$E_a = \frac{1}{2}\int_V \rho a^2\, dV. \tag{33}$$

We have, differentiating (7) the following relations:

$$\begin{aligned}&\ddot{x}_{M',i} = \ddot{x}_{O,i} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k + \alpha_{ji}\ddot{\alpha}_{jk}x_k + \dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kl}\delta_l + \alpha_{ji}\ddot{\alpha}_{jk}N_{kl}\delta_l + 2\alpha_{ji}\dot{\alpha}_{jk}Nr_{kr}\dot{\delta}_r + N_{ir}\ddot{\delta}_r;\\ &i,j,k = 1,2,3; \quad r = \overline{1,p}.\end{aligned} \tag{34}$$

Above we have used

$$\alpha_{ji}\ddot{X}_{O,j} = \ddot{x}_{O,i}; \qquad \alpha_{ji}\dot{X}_{O,j} = \dot{x}_{O,i}; \qquad \alpha_{ji}\dot{X}_{O,j} = \dot{x}_{O,i}; \qquad \alpha_{ji}\alpha_{jk} = \delta_{ik}, \tag{35}$$

where $\delta_{ik}$ is the Kronecker delta.

Using for the acceleration the relation offered by Eq. (34), Eq. (33) becomes

$$\begin{aligned}E_a &= \frac{1}{2}\int_V \rho a^2\,dV\\ &= \frac{1}{2}\int_V \rho\ddot{x}_{M',i}\ddot{x}_{M',i}\,dV\\ &= \frac{1}{2}\int_V \rho(\ddot{x}_{O,i} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k + \alpha_{ji}\ddot{\alpha}_{jk}x_k + \dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r + \alpha_{ji}\ddot{\alpha}_{jk}N_{kr}\delta_r\\ &\quad + 2\alpha_{ji}\dot{\alpha}_{jk}N_{kr}\dot{\delta}_r + N_{ir}\ddot{\delta}_r)^2\,dV.\end{aligned} \tag{36}$$

All the terms of the energy of acceleration are presented in the appendix. More comments concerning this notion are presented in [65].

## 4 Equivalent formulations (FEA for MBS)

### 4.1 Lagrange's equations

The use of Lagrange's equations offers the possibility of unified solution of MBS dynamics problems. To solve a dynamics problem, the same steps apply. In addition, vector quantities are abandoned, using only scalar quantities. One must first establish the independent generalized coordinates, calculate the kinetic and potential energies, and determine the generalized forces. The method is more than 200 years old, but it has not lost its importance, yet. The classic Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_i}\right) - \frac{\partial L}{\partial \delta_i} = 0; \quad i = \overline{1,p}. \tag{37}$$

Considering the Lagrangian expressed in Eq. (26), one successively obtains:

$$\begin{aligned}\frac{\partial L}{\partial \dot{\delta}_r} &= m_{rt}\dot{\delta}_t + m_{O,ir}\dot{x}_{O,i} + m_{k,ir}\alpha_{ij}\dot{\alpha}_{jk}\\ &\quad + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\delta_r; \quad i,j,k,l,m = 1,2,3; r,t = \overline{1,p},\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}_r} &= m_{rt}\ddot{\delta}_t + m_{O,ir}\ddot{x}_{O,i} + m_{k,ir}\dot{\alpha}_{ij}\dot{\alpha}_{jk} + m_{k,ir}\alpha_{ij}\ddot{\alpha}_{jk} + m_{ik,rt}\dot{\alpha}_{ij}\dot{\alpha}_{jk}\delta_r\\ &\quad + m_{ik,rt}\alpha_{ij}\ddot{\alpha}_{jk}\delta_r + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r; \quad i,j,k,l,m = 1,2,3; r,t = \overline{1,p},\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{\partial L}{\partial \delta_r} &= \alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}m_{kr,nt}\delta_t + m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk} + m_{k,mr}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}\\ &\quad + m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_t - k_{rt}\delta_t + q_r + q_r^*;\end{aligned}$$

$$i,j,k,l,m,n = 1,2,3; \quad r,t = \overline{1,p}, \tag{40}$$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}_r} - \frac{\partial L}{\partial \delta_r} \\
&\quad = m_{rt}\ddot{\delta}_t + 2m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r + (k_{rt} - \dot{\alpha}_{jk}\dot{\alpha}_{in}m_{kn,rt} + m_{ik,rt}\alpha_{ij}\ddot{\alpha}_{jk})\delta_r \\
&\qquad - m_{O,ir}\ddot{x}_{O,i} + m_{k,ir}\alpha_{ij}\ddot{\alpha}_{jk} + m_{ik,rt}\dot{\alpha}_{ij}\dot{\alpha}_{jk}\delta_r \\
&\qquad - m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk} - q_r - q_r^* = 0; \quad i,j,k,l,m,n = 1,2,3; r,t = \overline{1,p}.
\end{aligned} \tag{41}$$

The final form of the equations becomes

$$\begin{aligned}
&m_{rt}\ddot{\delta}_t + 2m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r + (k_{rt} - \dot{\alpha}_{jk}\dot{\alpha}_{in}m_{kn,rt} + m_{ik,rt}\alpha_{ij}\ddot{\alpha}_{jk})\delta_r \\
&\quad = q_r + q_r^* - m_{O,ir}\ddot{x}_{O,i} - m_{k,ir}\alpha_{ij}\ddot{\alpha}_{jk} - m_{ik,rt}\dot{\alpha}_{ij}\dot{\alpha}_{jk}\delta_r - m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}; \\
&i,j,k,l,m,n = 1,2,3; \quad r,t = \overline{1,p}.
\end{aligned} \tag{42}$$

### 4.2 Gibbs–Appell's equations

Gibbs–Appell's equations represent an alternative to Lagrange's equations. To use these, it is necessary know the energy of acceleration, obtained in Eq. (33). The Gibbs–Appell's equations are [49, 52, 53, 55]

$$\frac{\partial E_a}{\partial \ddot{\delta}_r} = Q_r \quad r = \overline{1,p}. \tag{43}$$

The equations (33) have in their components the following terms:

- $E_{a2}$ contains the quadratic terms in accelerations,

$$E_{a2} = \frac{1}{2}m_{rt}\ddot{\delta}_r\ddot{\delta}_t \quad r,t = \overline{1,p}; \tag{44}$$

- $E_{a1}$ contains the linear terms in accelerations,

$$\begin{aligned}
E_{a1} = {} & \ddot{x}_{O,i}\ddot{\delta}_r m^I_{O,ir} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk})\ddot{\delta}_r m_{k,mr} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk})\delta_r\ddot{\delta}_t m_{kr,mt} \\
& + 2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r\ddot{\delta}_t m_{kr,mt}\, dV;
\end{aligned} \tag{45}$$

- $E_{a0}$ contains no terms with generalized accelerations that play no role in obtaining the equations. These terms are not interesting for us.

So, the energy of acceleration can be written as

$$E_a = E_{ao} + E_{a1} + E_{a2}. \tag{46}$$

Equation (41) can be written as

$$\frac{\partial(E_{a1} + E_{a2})}{\partial \ddot{\delta}_r} = Q_r \quad r = \overline{1,p}. \tag{47}$$

The generalized force vector is

$$Q_{r,L} = k_{rt}\delta_t + \delta_r + \delta_r^*. \tag{48}$$

If we differentiate, it results in

$$\frac{\partial E_{a2}}{\partial \ddot{\delta}_r} = m_{rt}\ddot{\delta}_t \quad r,t = \overline{1,p}; \tag{49}$$

$$\frac{\partial E_{a1}}{\partial \ddot{\delta}_r} = \ddot{x}_{O,i}m^I_{O,ir} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ji}\ddot{\alpha}_{jk})m_{k,mr} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ji}\ddot{\alpha}_{jk})\delta_r m_{kr,mt} + 2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r m_{kr,mt}. \tag{50}$$

Obviously,

$$\frac{\partial E_{a0}}{\partial \delta_r} = 0; \quad r = \overline{1,p}, \tag{51}$$

and performing all the calculations yields Eq. (42).

Applying the Lagrange's equations, three differentiations $\frac{\partial L}{\partial \delta_r}, \frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}_r}, \frac{\partial L}{\partial \ddot{\delta}_r}$ must be done. Using Gibbs–Appell's equations, it is necessary to perform only a single differentiation $\frac{\partial E_a}{\partial \ddot{\delta}_r}$ [47].

Since it requires a smaller number of differentials, the number of calculations decreases and therefore the time and cost involved in modeling is reduced. Obviously, the use of FEM implies in current applications a very large number of finite elements and, as a consequence, a very large number of calculations. The reduction of operations using Gibs–Appell's method instead of Lagrange's classic method can lead to a significant reduction in computer time.

### 4.3 Hamilton's method

The use of the Lagrange's formalism leads to a system of second-order differential equations. Technically, solving this system of second-order equations is done by transforming it into a system of first-order differential equations of double dimension. Hamiltonian mechanics uses $2n$ unknowns, and the system of differential equations obtained is from the beginning a system of differential equations of the first order, of size $2n$. The unknowns are the generalized coordinates and canonically conjugated moment:

$$p_{i,L} = -\frac{\partial L}{\partial \dot{\delta}_i}; \quad i = \overline{1,p}. \tag{52}$$

So the main difference between Lagrange's and Hamilton's method is the use of the canonical conjugated moment instead of the generalized velocities. The major advantage of applying the method could be precisely the direct obtaining of a system of first-order equations, which can be solved directly, using the usual commercial software.

Hamilton's equations are a first-order system of differential equations [57]. They are

$$\dot{\delta}_{r,L} = \frac{\partial H}{\partial p_{r,L}}; \qquad \dot{p}_{r,L} = -\frac{\partial H}{\delta_r}. \tag{53}$$

From Eqs. (27)–(29), one obtains

$$\dot{\delta}_r = m^*_{ru}p_{u,L} - m^*_{ru}m_{O,iu}\dot{x}_{O,i} - m^*_{ru}m_{k,iu}\alpha_{ij}\dot{\alpha}_{jk} - m^*_{ru}m_{ik,ut}\alpha_{ij}\dot{\alpha}_{jk}\delta_t; \quad u,r,t = \overline{1,p};$$

$$\dot{p}_{r,L} = \frac{\partial L}{\partial \delta_r} = \alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}m_{kr,nt}\delta_t + m_{O,kr}\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk} + m_{k,mr}\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm} \tag{54}$$
$$+ m_{ik,rt}\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_t - k_{rt}\delta_t + q_r + q_r^*; \quad i,j,k,l,m,n = 1,2,3; r,t = \overline{1,p}.$$

These represent the equations of motion sought.

The main advantage of Hamilton's method is that it gives us a system of first-order differential equations, but in which the number of unknowns to be found is double. In the case of using other methods, the differential equations obtained are of the second order. Solving techniques require their transformation into first-order differential systems, by introducing new variables. In the case of Hamilton's method, these new variables are obtained directly and have physical significance [57–61].

### 4.4 Maggi's equations

A demonstration of applying Maggi's equations to MBS can be found in [62, 67]. For a system described by the independent coordinates $q_1, q_2, \ldots, q_n$ connected with each other through $m$ linear relationships, we have

$$\sum_{j=1}^{n} a_{ij}(q_1, q_2, \ldots, q_n, t)\dot{q}_j + b_i(q_1, q_2, \ldots, q_n, t) = 0, \quad i = \overline{1,m}. \tag{55}$$

The form of the Maggi's equations is

$$\sum_{k=1}^{n} a_{kj}\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] = 0; \quad j = \overline{1,n-m}. \tag{56}$$

These $n-m$ independent equations are the Maggi's equations. It is possible now to analyze and apply the equations to a single finite element. It is clear from Eq. (52) that Maggi's equations represent another form of the Gibbs–Appell's formulation. Similar results for other generalized media can be found in [68–77].

### 4.5 Kane's equations

Starting from the known equations

$$\sum_{i=1}^{N}(\bar{F}_i - m_i\bar{a}_i)\delta\bar{r}_i = 0, \tag{57}$$

and considering that the studied mechanical system of $N$ material points is described by a number of $p$ generalized coordinates, one can write

$$\sum_{i=1}^{N}(\bar{F}_i - m_i\bar{a}_i)\frac{\partial \bar{r}_i}{\partial q_k} = 0; \quad k = \overline{1,p}. \tag{58}$$

In classical mechanics, one uses the relation [51, 64]

$$\frac{\partial \bar{r}_i}{\partial q_k} = \frac{\partial \bar{v}_i}{\partial \dot{q}_k}; \quad k = \overline{1,p}, i = \overline{1,N}. \tag{59}$$

Introducing the latter into (58) yields

$$\sum_{i=1}^{N}(\bar{F}_i - m_i\bar{a}_i)\frac{\partial\bar{v}_i}{\partial\dot{q}_k} = 0; \quad k = \overline{1,p}. \tag{60}$$

Observing that

$$\frac{\partial\bar{v}_i}{\partial\dot{q}_k} = \frac{\partial\bar{v}_i}{\partial u_k} = \bar{v}_i^{(k)}; \quad k = \overline{1,n}; i = \overline{1,N}. \tag{61}$$

Equation (55) becomes

$$\sum_{i=1}^{N}\bar{F}_i\frac{\partial\bar{v}_i}{\partial u_k} = \sum_{i=1}^{N}m_i\bar{a}_i\frac{\partial\bar{v}_i}{\partial u_k}; \quad k = \overline{1,n}; i = \overline{1,N}, \tag{62}$$

with $\bar{F}_i$being the external forces acting in nodes.

Now, for an elastic finite element considered as a solid, Eq. (62) becomes

$$\sum_{i=1}^{N}\bar{F}_i\frac{\partial\bar{v}_i}{\partial\dot{q}_k} = \int_V\bar{a}\frac{\partial\bar{v}}{\partial\dot{q}_k}\,dm, \quad k = \overline{1,n}, \tag{63}$$

where $N$ is now the number of nodes of the finite element.

The acceleration of an arbitrary point is obtained using Eq. (34). Considering Eq. (9), we have

$$\frac{\partial\dot{x}_{M',i}}{\partial\dot{\delta}_r} = N_{it}; \quad i,j,k = 1,2,3; r = \overline{1,p}, \tag{64}$$

$$\begin{aligned}\frac{\partial\dot{x}_{M',i}}{\partial\dot{\delta}_t}\ddot{x}_{M',i} = N_{it}(&\ddot{x}_{O,i} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k + \alpha_{ji}\ddot{\alpha}_{jk}x_k + \dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kl}\delta_l + \alpha_{ji}\ddot{\alpha}_{jk}N_{kl}\delta_l\\ &+ 2\alpha_{ji}\dot{\alpha}_{jk}Nr_{kr}\dot{\delta}_r + N_{ir}\ddot{\delta}_r);\\ i,j,k = 1,2,3; \quad t,r = \overline{1,p}.\end{aligned} \tag{65}$$

The generalized forces $F_r$ in our case act in nodal points of the finite element. After some calculus, we obtain Eq. (4).

## 5 Conclusions and discussions

The main step in the case of using the FEM for the analysis of an MBS with elastic elements is writing the equations of motion. Then the operations related to the assembly of the obtained system of differential equations, their integration, and the analysis of the results can be conducted according to the classic methods used in FEM procedures. These procedures are well established and verified by numerous applications. The most important stage in such an analysis remains writing the equations of motion. The difficulty in obtaining them lies in the complexity of the system to be studied. Finding a formalism that allows obtaining these equations as easily as possible is therefore an important objective of research in this field. Analytical mechanics proposes several formalisms to be able to obtain the equations of motion, equivalent to each other, which ultimately provide the

same equations. The established method for the analysis of these systems is the method of Lagrange's equations. The analysis of the specialized literature reveals this important role played by the Lagrangian. The most important advantage is the fact that researchers in the field of mechanics are familiar with this method and with the fundamental notions used (kinetic energy, potential energy, work). Then Lagrangian mechanics operates only with scalar quantities, and can approach the most complex systems. Analytical mechanics, on the other hand, provides equivalent formulations which for certain types of systems with different particularities offer advantages in modeling and shortening the analysis time, therefore also the costs involved. These are Gibbs–Appell's equations, Hamilton's equations, Kane's equations, Maggi's equations, etc.

Analytical mechanics therefore offers a multitude of equivalent formulas, with any of these methods being able to obtain, in the end, the equations of motion for a finite element. The problem then is of choosing the most suitable method for the study of a concrete problem. This generally depends on the experience of the researcher and his familiarity with the concepts presented.

Obviously, most researchers will opt for the use of Lagrange's equations, but there may be cases in which another method proves its advantages. This happens against the background of the special development of numerical methods and the possibilities of numerical simulation of problems. In the framework of this work, several equivalent formulas used more frequently by researchers were presented, presenting the advantages and disadvantages of each method. It is up to the researcher and his experience to choose the most suitable method for a certain type of problem. We mentioned the main advantages of Lagrange's method, in particular familiarizing the researchers with the method and the fundamental notions used, as well as in its simplicity.

Gibbs–Appell's equations can be easily obtained with a smaller number of operations than Lagrange's equations. In this way, the method becomes more economical and easier to apply. The difficulty lies in introducing the notion of acceleration energy, which is less familiar to researchers.

Maggi's method is essentially equivalent to the Gibbs–Appell's method and presents some advantages in writing and in use. The most profitable method seems to be the method of Hamilton's equations, where one finally operates with a system of differential equations of the first order. The disadvantage would be that the number of unknowns is double as when solving differential equations of the second order.

If we take into account all the advantages and disadvantages of the presented methods, it can be estimated that the equivalent methods of obtaining the equations of motion offered by analytical mechanics will continue to develop in the study of dynamic systems with elastic elements, being reevaluated and improved. This will happen due to the requirements imposed by the current technological development.

## Appendix

$$E_c = \frac{1}{2}\left(\int_V \rho\, dV\right)(\dot{x}_{O,i}\dot{x}_{O,i}) + \frac{1}{2}\left(\int_V \rho x_k x_m\, dV\right)\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm} + \frac{1}{2}\left(\int_V \rho N_{kr}N_{nt}\, dV\right)\alpha_{ij}\alpha_{im}\dot{\alpha}_{jk}\dot{\alpha}_{mn}\delta_r\delta_t$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\int_V \rho N_{ir}N_{it}\,dV\right)\dot{\delta}_r\dot{\delta}_t+\left(\int_V \rho x_k\,dV\right)\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}\\
&+\left(\int_V \rho N_{kr}\,dV\right)\dot{x}_{O,i}\alpha_{ij}\dot{\alpha}_{jk}\delta_r+\left(\int_V \rho N_{ir}\,dV\right)\dot{x}_{O,i}\dot{\delta}_r\\
&+\left(\int_V \rho x_k N_{mr}\,dV\right)\alpha_{ij}\alpha_{il}\dot{\alpha}_{jk}\dot{\alpha}_{lm}\delta_r+\left(\int_V \rho x_k N_{ir}\,dV\right)\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r\\
&+\left(\int_V \rho N_{kr}N_{it}\,dV\right)\alpha_{ij}\dot{\alpha}_{jk}\delta_r\dot{\delta}_t\quad i,j,l,k,m,n=1,2,3;r,t=\overline{1,p};\\
E_a=&\frac{1}{2}(\ddot{x}_{O,i}\ddot{x}_{O,i})\left(\int_V \rho\,dV\right)+\frac{1}{2}\alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\ddot{\alpha}_{lm}\int_V \rho(x_k x_m)\,dV\\
&+\frac{1}{2}\alpha_{ji}\alpha_{il}\ddot{\alpha}_{jk}\ddot{\alpha}_{lm}\left(\int_V \rho(N_{kr}N_{mt})\,dV\right)\delta_r\delta_t\\
&+2\alpha_{ij}\dot{\alpha}_{jk}\alpha_{il}\dot{\alpha}_{lm}\left(\int_V \rho N_{kr}N_{mt}\,dV\right)\dot{\delta}_r\dot{\delta}_t+\frac{1}{2}\left(\int_V \rho N_{ir}N_{it}\,dV\right)\ddot{\delta}_r\ddot{\delta}_t\\
&+\ddot{x}_{O,i}\alpha_{ij}\ddot{\alpha}_{jk}\left(\int_V \rho x_k\,dV\right)\\
&+\ddot{x}_{O,i}\alpha_{ij}\ddot{\alpha}_{jk}\delta_r\left(\int_V \rho N_{kr}\,dV\right)+\ddot{x}_{O,i}2\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r\left(\int_V \rho N_{kr}\,dV\right)+\ddot{x}_{O,i}\ddot{\delta}_r\int_V \rho N_{ir}\,dV\\
&+\frac{1}{2}\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k\dot{\alpha}_{li}\dot{\alpha}_{lm}x_m\,dV+\ddot{x}_{O,i}\dot{\alpha}_{ji}\dot{\alpha}_{jk}\delta_r\left(\int_V \rho x_k\,dV\right)\\
&+\ddot{x}_{O,i}\alpha_{ji}\dot{\alpha}_{jk}\delta_r\left(\int_V \rho N_{kr}\,dV\right)\\
&+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k\alpha_{li}\ddot{\alpha}_m x_m\,dV+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k\dot{\alpha}_{li}\dot{\alpha}_{lm}N_{mr}\delta_r\,dV\\
&+\int_V \rho(\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k\alpha_{li}\ddot{\alpha}_{lm}N_{mr}\delta_r)\,dV\\
&+2\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k\alpha_{li}\dot{\alpha}_{lm}N_{mr}\dot{\delta}_r\,dV+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}x_k N_{ir}\ddot{\delta}\,dV\\
&+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r\dot{\alpha}_{jl}\dot{\alpha}_{jlm}N_{mr}\delta_r\,dV\\
&+\alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\ddot{\alpha}_{lm}\delta_r\left(\int_V \rho x_k N_{kr}\,dV\right)+2\alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\dot{\alpha}_{lm}\dot{\delta}_r\left(\int_V \rho x_k N_{kr}\,dV\right)\\
&+\alpha_{ji}\ddot{\alpha}_{jk}\ddot{\delta}_r\left(\int_V \rho x_k N_{ir}\,dV\right)\\
&+2\alpha_{ji}\ddot{\alpha}_{jk}\delta_r\alpha_{il}\dot{\alpha}_{lm}\dot{\delta}_t\left(\int_V \rho N_{kr}N_{mt}\,dV\right)+\alpha_{ji}\ddot{\alpha}_{jk}\delta_r\ddot{\delta}_r\left(\int_V \rho N_{kr}N_{it}\,dV\right)\\
&+2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r\ddot{\delta}_t\left(\int_V \rho N_{kr}N_{it}\right)dV\\
&+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r\alpha_{jl}\ddot{\alpha}_{lm}N_{mr}\delta_r\,dV+2\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r\alpha_{jl}\dot{\alpha}_{lm}N_{mr}\dot{\delta}_r\,dV\\
&+\int_V \rho\dot{\alpha}_{ji}\dot{\alpha}_{jk}N_{kr}\delta_r N_{it}\ddot{\delta}_t\,dV.
\end{aligned}
$$

Taking into account the previously defined notations (see Eqs. (12)–(14)), the expression of the energy of accelerations becomes

$$
\begin{aligned}
E_a = {} & \frac{1}{2}m(\ddot{x}_{O,i}\ddot{x}_{O,i}) + \frac{1}{2}\alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\ddot{\alpha}_{lm}J_{km} + \frac{1}{2}\alpha_{ji}\alpha_{il}\ddot{\alpha}_{jk}\ddot{\alpha}_{lm}m_{kr,mt}\delta_r\delta_t \\
& + 2\alpha_{ij}\dot{\alpha}_{jk}\alpha_{il}\dot{\alpha}_{lm}m_{kr,mt}\dot{\delta}_r\dot{\delta}_t + \frac{1}{2}m_{rt}\ddot{\delta}_r\ddot{\delta}_t + \ddot{x}_{O,i}\alpha_{ij}\ddot{\alpha}_{jk}S_k \\
& + \ddot{x}_{O,i}\alpha_{ij}\ddot{\alpha}_{jk}\delta_r m^I_{O,kr} + \ddot{x}_{O,i}2\alpha_{ij}\dot{\alpha}_{jk}\dot{\delta}_r m^I_{O,kr} + \ddot{x}_{O,i}\ddot{\delta}_r m^I_{O,ir} \\
& + \frac{1}{2}\dot{\alpha}_{ji}\dot{\alpha}_{jk}\dot{\alpha}_{li}\dot{\alpha}_{lm}J_{km} + \ddot{x}_{O,i}\dot{\alpha}_{ji}\dot{\alpha}_{jk}\delta_r S_k + \ddot{x}_{O,i}\alpha_{ji}\dot{\alpha}_{jk}\delta_r m^I_{O,kr} \\
& + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\alpha_{li}\ddot{\alpha}_m J_{km} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\dot{\alpha}_{li}\dot{\alpha}_{lm}\delta_r m_{k,mr} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\alpha_{li}\ddot{\alpha}_{lm}\delta_r m_{k,mr} \\
& + 2\dot{\alpha}_{ji}\dot{\alpha}_{jk}\alpha_{li}\dot{\alpha}_{lm}\dot{\delta}_r m_{k,mr} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\ddot{\delta}_r m_{k,ir} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\dot{\alpha}_{jl}\dot{\alpha}_{jlm}\delta_r\delta_r m_{kr,mt} \\
& + \alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\ddot{\alpha}_{lm}\delta_r m_{k,mr} + 2\alpha_{ji}\ddot{\alpha}_{jk}\alpha_{il}\dot{\alpha}_{lm}\dot{\delta}_r m_{k,mr} + \alpha_{ji}\ddot{\alpha}_{jk}\ddot{\delta}_r m_{k,ir} \\
& + 2\alpha_{ji}\ddot{\alpha}_{jk}\delta_r\alpha_{il}\dot{\alpha}_{lm}\dot{\delta}_t m_{kr,mt} + \alpha_{ji}\ddot{\alpha}_{jk}\delta_r\ddot{\delta}_r m_{kr,it} + 2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r\ddot{\delta}_t m_{kr,it} \\
& + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\alpha_{jl}\ddot{\alpha}_{lm}\delta_r\delta_t m_{kr,mt} + 2\dot{\alpha}_{ji}\dot{\alpha}_{jk}\alpha_{jl}\dot{\alpha}_{lm}\delta_r\dot{\delta}_t m_{kr,mt} + \dot{\alpha}_{ji}\dot{\alpha}_{jk}\delta_r\ddot{\delta}_t m_{kr,it}.
\end{aligned}
$$

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Not applicable.

## Declarations

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The authors declare no competing interests.

**Author contributions**
M.L.S. and S.V. coordinated the manuscript; M.M. performed the mathematical calculations; M.L.S., S.V. and M.M. supervised the entire manuscript and agreed with this final form of the manuscript.

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