Xing et al. *Boundary Value Problems* (2023) 2023:110
https://doi.org/10.1186/s13661-023-01799-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and multiplicity of periodic solutions for a nonlinear resonance equation with singularities

Xiumei Xing[1], Lingling Wang[2] and Shaoyong Lai[2*]

*Correspondence:
Laishaoy@swufe.edu.cn
[2]The School of Math., Southwestern University of Finance and Economics, Wenjiang, 611130, Chengdu, China
Full list of author information is available at the end of the article

Abstract

We investigate a second-order periodic system with singular potential and resonance. Utilizing the main integral method and fixed point theorems, we establish the existence and multiplicity of periodic solutions with respect to time under certain assumptions on the unbounded or oscillatory term.

Mathematics Subject Classification: 34C11; 34C28

Keywords: Periodic solution; Singular potential; Existence; Multiplicity

1 Introduction and main result

Many scholars have investigated the singular second-order equation

$$\ddot{x} + V'(x) + g(x) = p(t) \tag{1.1}$$

with functions $V(x)$, $g(x)$, and $p(t)$ satisfying certain restrictions. The multiplicity and existence of periodic solutions for Eq. (1.1) are discussed by utilizing the topology degree theory [1]. Qian and Torres [2] use the Poincaré–Birkhoff twist theorem to find the dynamical features of Eq. (1.1). Jiang [3] employs invariant curves and Moser's small twist theorem to discuss the boundedness of solutions for Eq. (1.1). The unbounded and periodic solutions of Eq. (1.1) may coexist [4]. Assuming that $g(x) = 0$ and $p(t + \pi) = p(t)$, Capietto et al. [5] consider Eq. (1.1) with

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1 - x_-^2)^\nu} - 1, \tag{1.2}$$

where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$, and $\nu > 0$ is an integer. Using the Moser twist theorem and the Lazer–Leach assumption

$$1 + \frac{1}{2}\int_0^\pi p(t_0 + \theta)\sin\theta\, d\theta > 0 \quad \forall t_0 \in \mathbb{R},$$

Capietto et al. [5] investigated the boundedness of solutions and quasi-periodic solutions for Eq. (1.1). Following the ideas in [5], Jiang [3] and Liu [6] and Wang and Jiang [7] discussed the boundedness of solutions for Eq. (1.1) under different conditions.

To clearly understand the objective of our work, we recall the main result about Eq. (1.1) in Lazer and Leach [8]. Suppose that $g(x)$ is smooth and bounded and that $V(x)$ satisfies the conditions

$$\lim_{x\to a_+} V(x) = +\infty, \qquad \lim_{x\to+\infty} \frac{2V(x)}{x^2} = \frac{m^2}{4},$$

where the domain of $V(x)$ is $(a,+\infty)$, $m>0$ is an integer, and the constant a belongs to $\in(-\infty,0)$. Let

$$g_*(\rho) = \int_0^{2\pi} g\big(\rho\big|\sin(mt/2)\big|\big)\big|\sin(mt/2)\big|\,dt, \qquad p_*(\theta) = \int_0^{2\pi} p(t+\theta)\big|\sin(mt/2)\big|\,dt.$$

Applying the analysis of phase plane and topological degree methods, it is proved in [8] that Eq.(1.1) possesses at least one 2π-periodic solution, provided that there is $g_0\in[g_*^-,g_*^+]$ ($g_*^- = \liminf_{\rho\to+\infty} g_*(\rho), g_*^+ = \limsup_{\rho\to+\infty} g_*(\rho)$) (i.e.. p_* has a regular value of g_0) and the number of zeros of p_*-g_0 in $[0,2\pi/m]$ is not 2.

In 2019, Ma [9] considered the periodic solution of Eq. (1.1). Suppose that $V'(x) = n^2x, p(t)\in C^2(\mathbb{R}/2\pi\mathbb{Z})$, and $g(x)\in C^1(\mathbb{R})$ with the restrictions

$$\lim_{|x|\to+\infty} x^{-\frac{1}{2}}\big|g(x)\big| = 0, \qquad \lim_{x\to+\infty} x^{\frac{1}{2}}\big|g'(x)\big| < +\infty.$$

Ma [9] obtained that Eq. (1.1) possesses at least one 2π solution, provided that

$$\left|\int_0^{2\pi} p(t)e^{-int}\,dt\right| < \limsup_{\rho\to+\infty}\left|\int_0^{2\pi} g(\rho\cos\theta)\cos\theta\,d\theta\right|,$$

and Eq. (1.1) possesses an unbounded sequence of 2π-periodic solutions if

$$\begin{aligned}\left|\int_0^{2\pi} p(t)e^{-int}\,dt\right| < \min\Bigg\{&\limsup_{\rho\to+\infty}\Big|\int_0^{2\pi} g(\rho\cos\theta)\cos\theta\,d\theta,\\ &-\liminf_{\rho\to+\infty}\left|\int_0^{2\pi} g(\rho\cos\theta)\cos\theta\,d\theta\right|\Bigg\}.\end{aligned}$$

We observe that the function $V(x)$ is globally defined in $\mathbb{R}$ and the function $g(x)$ is unbounded or oscillatory in [9]. Note that in [8], $V(x)$ possesses a repulsive singularity at a, and $g(x)$ is bounded. Also, in [5], $V(x)$ possesses a repulsive singularity at -1, and $g(x)=0$.

When the function $V(x)$ is of the form (1.2), a natural question is to find restrictions imposed on unbounded function $g(x)$ to make Eq. (1.1) have at least one π-periodic solution and possess an unbounded sequence of π-periodic solutions. The objective of this work is to handle this problem. Precisely speaking, we investigate the existence and multiplicity of periodic solutions of the problem

$$\begin{cases}\ddot{x} + V'(x) + g(x) = p(t),\\ V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\nu} - 1, \qquad p(t)\in C^2(\mathbb{S}^1),\end{cases} \tag{1.3}$$

where $\mathbb{S}^1 = \mathbb{R}/\pi\mathbb{Z}$ and $\lim_{x\to+\infty} x^{k-\frac{1}{2}} g^{(k)}(x) = 0$, $k = 0, 1$. Here we state that the function $g(x)$ considered in our work is different from those in [5, 8] and is the same as that in [9]. The novelty of our work is that the function $V(x)$ is of the the form (1.2), which is different from those in [8, 9].

The auxiliary equation of Eq. (1.1) takes the form

$$\ddot{x} + V'(x) + g(x) = 0. \tag{1.4}$$

The Hamiltonian function of (1.4) has the expression

$$H_0(x, y) = \frac{1}{2}y^2 + V(x) + G(x),$$

where $G(x) = \int_0^x g(s)\,ds$. For $H > 0$, we denote by $\tau(H)$ the least positive period of the orbit $\Gamma_H : H_0(x, y) = H, \bar{p} = \int_0^\pi p(s)e^{is}\,ds$. Set

$$\Gamma(H) = \sqrt{H}\big(\tau(H) - \pi\big). \tag{1.5}$$

Equation (1.4) possesses the following autonomous Hamiltonian system:

$$x' = y, \qquad y' = -V'(x) - g(x).$$

Now we state the main result of our work.

Theorem 1.1 *Suppose that* $g(x) \in C^1(\mathbb{R})$, $p(t) \in C^2(\mathbb{R}/\pi\mathbb{Z})$, *and*

$$\lim_{x\to+\infty} x^{k-\frac{1}{2}} g^{(k)}(x) = 0, \quad k = 0, 1. \tag{1.6}$$

Then

(*i*) *Problem* (1.3) *possesses at least one* π*-periodic solution if*

$$|\bar{p}| < \sqrt{\frac{2}{\pi}} \limsup_{H\to+\infty} \Gamma(H). \tag{1.7}$$

(*ii*) *Problem* (1.3) *possesses an unbounded sequence of* π*-periodic solutions if*

$$|\bar{p}| < \sqrt{\frac{2}{\pi}} \min\Big\{\limsup_{H\to+\infty} \Gamma(H), -\liminf_{H\to+\infty} \Gamma(H)\Big\}. \tag{1.8}$$

In Sect. 2, we present several lemmas, and in Sect. 3, we provide the proof of Theorem 1.1.

2 Preliminaries

2.1 Action-angle coordinates

To use action-angle variables, we write the auxiliary equations

$$x' = \frac{\partial H_1}{\partial y}, \qquad y' = -\frac{\partial H_1}{\partial x} \tag{2.1}$$

with the Hamiltonian function

$$H_1(x,y) = \frac{1}{2}y^2 + V(x).$$

Let $T_0(h)$ denote the time period of the integral curve Γ_h of (2.1) with $H_1(x,y) = h$. Denote by $I = I_0(h)$ the area enclosed by the closed curve Γ_h for $h > 0$. Assume that $-1 < -\alpha_h < 0 < \beta_h$ satisfy $V(-\alpha_h) = V(\beta_h) = h$ and

$$I_-(h) = 2\int_0^{\alpha_h} \sqrt{2\big(h - V(s)\big)}\,ds, \qquad T_-(h) = 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\,ds.$$

Thus we have

$$I_0(h) = \pi h + 2\int_0^{\alpha_h} \sqrt{2\big(h - V(-s)\big)}\,ds = \pi h + I_-(h), \tag{2.2}$$

$$T_0(h) = I_0'(h) = \pi + 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}\,ds = \pi + T_-(h). \tag{2.3}$$

For conciseness in the following discussions, we always use c or C to represent positive constants.

We acquire $c\sqrt{h} < I_-(h) < C\sqrt{h}$. Let $h = h_0(I)$ be the inverse function of $I = I_0(h)$. Using the expression of $V(x)$ and the definition of $I_0(h)$, we derive that

$$cI < h_0(I) < CI, \qquad \left|I^k h_0^{(k)}(I)\right| < Ch(I) \quad \text{for } k = 1,2.$$

Lemma 2.1 [3] *For $n = 0,1,2$, we have*

$$\frac{d^n T_-(h)}{dh^n} = (-1)^n \frac{(2n-1)!!}{2^n} \frac{\sqrt{2}}{h^{(2n+1)/2}} + o\left(\frac{1}{h^{(2n+1)/2}}\right), \quad h \to +\infty,$$

where

$$(2n-1)!! = \begin{cases} 1\cdot 3\cdot \ldots \cdot (2n-1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

For $(x,y) \in (-1,+\infty) \times \mathbb{R}$, we define the transformation $\Psi_1 : (x,y) \to (\theta, I)$ by

$$\theta(x,y) = \begin{cases} \frac{\pi}{T_0(h(x,y))}(\frac{T_-(h)}{2} + \arcsin\frac{x}{\sqrt{2h(x,y)}}) & \text{if } x > 0, y > 0, \\ \frac{\pi}{T_0(h(x,y))}(\frac{T_-(h)}{2} + \pi - \arcsin\frac{x}{\sqrt{2h(x,y)}}) & \text{if } x > 0, y < 0, \\ \frac{\pi}{T_0(h(x,y))}(\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y))+1-(1-s^2)^{-\gamma}}}\,ds) & \text{if } x < 0, y > 0, \\ \frac{\pi}{T_0(h(x,y))}(T_0(h(x,y)) - \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y))+1-(1-s^2)^{-\gamma}}}\,ds), & x < 0, y < 0, \end{cases} \tag{2.4}$$

and

$$I(x,y) = I_0\big(h(x,y)\big) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2\big(h(x,y) - V(s)\big)}\,ds.$$

Note that the first equation in (1.3) possesses the Hamiltonian system

$$x' = \frac{\partial H}{\partial y}, \qquad y' = -\frac{\partial H}{\partial x}$$

associated with

$$H(x,y,t) = \frac{1}{2}y^2 + V(x) + G(x) - xp(t).$$

In the new variables (θ, I), we write the Hamiltonian systems of (1.3) and (1.4) in the forms

$$\theta' = \frac{\partial H}{\partial I}, \qquad I' = -\frac{\partial H}{\partial \theta} \tag{2.5}$$

and

$$\theta' = \frac{\partial H_0}{\partial I}, \qquad I' = -\frac{\partial H_0}{\partial \theta}, \tag{2.6}$$

respectively, where

$$H(\theta, I, t) = \pi h_0(I) + \pi G\bigl(x(I,\theta)\bigr) - \pi x(I,\theta)p(t) \tag{2.7}$$

and

$$H_0(\theta, I, t) = \pi h_0(I) + \pi G\bigl(x(I,\theta)\bigr). \tag{2.8}$$

For $x > 0$, using (2.4) gives rise to

$$x = \sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right). \tag{2.9}$$

Using formulas (2.5)–(2.9), we want to obtain estimates of the function $G(x(I,\theta))$. We need the following lemma.

Lemma 2.2 [3] *For sufficiently large I, if $x > 0$, then*

$$\left| I^k \frac{\partial^k x(I,\theta)}{\partial I^k} \right| \le C\sqrt{I}, \quad k \le 2.$$

For sufficiently large I, if $x < 0$, then

$$\left| I^k \frac{\partial^k x(I,\theta)}{\partial I^k} \right| \le C(1+x) \le C, \quad k \le 2.$$

To transform the first equation in problem (1.3) into a nearly integrable equation, we introduce the transformation $\Psi_2 : (I,\theta,t) \to (H,\varphi,\eta)$,

$$H(\theta, I, t) = \pi h_0(I) + \pi G\bigl(x(I,\theta)\bigr) - \pi x(I,\theta)p(t), \qquad \varphi = t, \qquad \eta = \theta,$$

that is, the time and energy are the new angular and action variables, respectively.

As $I \to +\infty$, from (1.6), (2.2), and Lemma 2.2 we conclude that $H/I \to 1$ and $\frac{\partial H}{\partial I} > 0$. Applying the implicit function theorem, we derive that there exists a function $R_2(H,t,\theta)$ belonging to C^2 with $|R_2| < H/2\pi$ such that the function

$$I = I_0\left(\frac{H}{\pi} + R_2(H,t,\theta)\right) \tag{2.10}$$

satisfies (2.7). Thus, under the transform Ψ_2, the Hamiltonian function (2.7) is transformed into the Hamiltonian function (2.10).

Note that the inverse function of I_0 is h_0. Using (2.7), we acquire

$$R_2(H,t,\theta) = x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)p(t) - G\left(x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)\right). \tag{2.11}$$

Utilizing (1.6), Lemma 2.2, and $|R_2| < H/2\pi$ gives rise to

$$\left|R_2(H,t,\theta)\right| \le \epsilon(H)H^{\frac{3}{4}},$$

where (here and further) $\epsilon(H)$ stands for a nonnegative function satisfying $\lim_{H\to+\infty}\epsilon(H) = 0$.

Similarly, we can prove that there is a function $R_1(H,\theta)$ in space C^2 with $|R_1| < H/2\pi$ such that

$$I = I_0\left(\frac{H}{\pi} + R_1(H,t)\right) \tag{2.12}$$

satisfies

$$H = \pi h_0(I) + \pi G\big(x(I,\theta)\big). \tag{2.13}$$

Thus we have

$$R_1(H,\theta) = -G\left(x\left(I_0\left(\frac{H}{\pi} + R_1\right),\theta\right)\right) \tag{2.14}$$

and

$$\left|R_1(H,\theta)\right| \le \epsilon(H)H^{\frac{3}{4}}.$$

Let $\Xi = \{\theta \in \mathbb{S}^1 : \sin(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}) = 0\}$. Obviously, by Lemma 2.1 the measure of Ξ is zero.

Lemma 2.3 *If the function $R(H,t,\theta)$ belongs to C^2 and $|R(H,t,\theta)| \le \epsilon(H)H$, provided that $h = \frac{H}{\pi} + R(H,t,\theta)$, then*

$$\lim_{H\to+\infty}(\sqrt{2h})^{k-\frac{1}{2}}g^{(k)}\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right)\sin^k\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) = 0$$

for $k = 0$ and $\theta \in \mathbb{S}^1$ and for $k = 1$ and $\theta \in \mathbb{S}^1 \setminus \Xi$.

Proof From (1.6), for any $\epsilon > 0$, there exist constants $M_k > 0$ such that

$$x^{k-\frac{1}{2}} g^{(k)}(x) < \epsilon, \quad x > M_k.$$

Since $\lim_{H\to+\infty} h(H,t,\theta) \to +\infty$ uniformly for t and $\theta \setminus \Xi$, there exist positive numbers $M_k > 0$ such that for $H > M_k$,

$$\frac{1}{\sqrt[4]{2h}} \max_{|x|\le M_k} \left| x^k g^{(k)}(x) \right| < \epsilon.$$

Thus, for $|\sqrt{2h}\sin(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2})| > M_k$, we have

$$\left| (\sqrt{2h})^{k-\frac{1}{2}} g^{(k)}\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right) \sin^k\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \right| \le \epsilon,$$

and for $|\sqrt{2h}\sin(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2})| \le M_k$, we have

$$\begin{aligned} &\left| (\sqrt{2h})^{k-\frac{1}{2}} g^{(k)}\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right) \sin^k\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \right| \\ &\quad \le \frac{1}{\sqrt[4]{2h}} \max_{|x|\le M_k} \left| x^k g^{(k)}(x) \right| < \epsilon, \end{aligned}$$

which ends the proof. □

For $r \in \mathbb{R}$, when $H \gg 1$, we write $u = u(H,t,\theta) \in C^2(r,\epsilon)$ if $|\partial_H^k u| \le \epsilon(H) H^{r-k}$ for $k = 0,1,2$ and $|\partial_H^k \partial_t u| \le C H^{\frac{1}{2}-k}$ for $k = 0,1$.

Lemma 2.4 *Let $h = \frac{H}{\pi} + u(H,t,\theta)$ with $u \in C(\frac{3}{4},\epsilon)$. Then*

$$\left| g^{(k)}\big(x\big(I_0(h),\theta\big)\big)\big(\partial_H x\big(I_0(h),\theta\big)\big)^{k+1} \right| \le \epsilon(H) H^{\frac{3}{4}-(k+1)} \tag{2.15}$$

for $k = 0,1$ and $\theta \in \mathbb{S}^1 \setminus \Xi$.

Proof When $x > 0$, by a direct computation we have

$$\begin{aligned} &g(x)\partial_H x \\ &\quad = g\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right) \frac{1}{\sqrt{2h}}\left(\frac{1}{\pi} + \partial_H u\right) \sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \\ &\qquad + g\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right) \sqrt{2h}\cos\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \\ &\qquad \times \left[\frac{1}{\pi} T_0'(h)\theta + \frac{T_-'(h)}{2}\right] \frac{1}{\pi} + \partial_H u) \end{aligned}$$

and

$$\begin{aligned} &g'(x)(\partial_H x)^2 \\ &\quad = g'\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right) \frac{1}{2h}\left(\frac{1}{\pi} + \partial_H u\right)^2 \sin^2\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&+ g'\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right)2h\cos^2\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \\
&\times \left[\frac{1}{\pi}T_0'(h)\theta + \frac{T_-'(h)}{2}\right]^2\left(\frac{1}{\pi} + \partial_H u\right)^2 \\
&+ 2g'\left(\sqrt{2h}\sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\right)\left(\frac{1}{\pi} + \partial_H u\right)^2 \sin\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right) \\
&\times \cos\left(\frac{T_0(h)}{\pi}\theta - \frac{T_-(h)}{2}\right)\cdot\left[\frac{1}{\pi}T_0'(h)\theta + \frac{T_-'(h)}{2}\right].
\end{aligned}
$$

From (2.3), Lemma 2.1, and Lemma 2.3 we obtain (2.15). When $x < 0$, using Lemma 2.2, we acquire that (2.15) holds. The proof is finished. □

Lemma 2.5 *For all $\epsilon > 0$, t, and $\theta \in \mathbb{S}^1 \setminus \Xi$, as $H \to +\infty$, we have*

$$
\begin{aligned}
&\left|\partial_H^k R_2(H,t,\theta)\right| \le \epsilon(H)H^{\frac{3}{4}-k} \quad \text{for } k \le 2, \\
&\left|\partial_H^k \partial_t^l R_2(H,t,\theta)\right| \le CH^{\frac{1}{2}-k} \quad \text{for } k + l \le 2, l \ge 1.
\end{aligned}
$$

Proof (i) When $k + l = 0$, the conclusion follows from (1.6), (2.2), (2.11), and Lemma 2.2.

(ii) When $k + l = 1$, define

$$
\begin{aligned}
\triangle = 1 &- \partial_I x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)T_0\left(\frac{H}{\pi} + R_2\right)p(t) \\
&+ g(x)\partial_I x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)T_0\left(\frac{H}{\pi} + R_2\right).
\end{aligned}
$$

For $|\triangle| \ge \frac{1}{2}$ and $H \gg 1$, we get

$$
\triangle \cdot \partial_H R_2(H,t,\theta) = -\frac{1}{\pi}(\triangle - 1), \tag{2.16}
$$

$$
\triangle \cdot \partial_t R_2(H,t,\theta) = x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)p'(t). \tag{2.17}
$$

Using Lemma 2.2 and (1.6) yields

$$
\left|-\frac{1}{\pi}(\triangle - 1)\right| \le \epsilon(H)H^{\frac{3}{4}-1}.
$$

Applying Lemma 2.2 gives rise to

$$
\left|x\left(I_0\left(\frac{H}{\pi} + R_2\right),\theta\right)p'(t)\right| \le C\sqrt{H}.
$$

Thus $|\partial_H R_2(H,t,\theta)| \le \epsilon(H)H^{\frac{3}{4}-1}$ and $|\partial_t R_2(H,t,\theta)| \le CH^{\frac{1}{2}}$.

(iii) When $k + l = 2$, differentiating both sides of (2.16) with respect to H and t, respectively, we acquire

$$
\partial_H \triangle \cdot \partial_H R_2 + \triangle \partial_H^2 R_2 = -\frac{1}{\pi}\partial_H \triangle,
$$

$$\partial_t\triangle\cdot\partial_H R_2+\triangle\partial_H\partial_t R_2=-\frac{1}{\pi}\partial_t\triangle.$$

Differentiating both sides of (2.17) about t gives

$$\begin{aligned}\partial_t\triangle\cdot\partial_t R_2+\triangle\partial_t^2 R_2&=\partial_I x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)T_0\left(\frac{H}{\pi}+R_2\right)\partial_t R_2 p'(t)\\&\quad+x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)p''(t).\end{aligned}$$

A direct computation yields

$$\begin{aligned}\partial_H\Delta&=\partial_I^2 x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)T_0^2\left(\frac{H}{\pi}+R_2\right)\left(\frac{1}{\pi}+\partial_H R_2\right)p(t)\\&\quad+\partial_I x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)\partial_H T_-\left(\frac{H}{\pi}+R_2\right)\left(\frac{1}{\pi}+\partial_H R_2(H,t,\theta)\right)p(t)\\&\quad+g'\left(x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)\right)\left(\partial_I x\left(I_0\left(\frac{H}{\pi}+R_2\right)T_0\left(\frac{H}{\pi}+R_2\right)\right)\right)^2\left(\frac{1}{\pi}+\partial_H R_2\right)\\&\quad+g\left(x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)\right)\left(\partial_I^2 x(I_0\left(\frac{H}{\pi}+R_2\right)T_0^2\left(\frac{H}{\pi}+R_2\right)\left(\frac{1}{\pi}+\partial_H R_2\right)\right.\\&\quad+g\left(x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)\right)\left(\partial_I x(I_0\left(\frac{H}{\pi}+R_2\right)\partial_H T_-\left(\frac{H}{\pi}+R_2\right)\left(\frac{1}{\pi}+\partial_H R\right).\end{aligned}$$

From Lemmas 2.1–2.5 we have $|\partial_H\Delta|<\epsilon(H)H^{-\frac{5}{4}}$ and $|\partial_H R_2(H,t,\theta)|<\epsilon(H)H^{\frac{3}{4}-1}$.

Similarly to the above estimates, we have

$$\begin{aligned}&\left|\partial_H^2 R_2(H,t,\theta)\right|<\epsilon(H)H^{\frac{3}{4}-2},\qquad\left|\partial_H\partial_t R_2(H,t,\theta)\right|<CH^{\frac{1}{2}-1},\\&\left|\partial_t^2 R_2(H,t,\theta)\right|<CH^{\frac{1}{2}-1}.\end{aligned}$$

The proof is finished. □

Using (2.14) and (2.15), similarly to the proof of Lemma 2.5, we acquire the conclusion.

Lemma 2.6 *For all $\epsilon>0$, t, and $\theta\in\mathbb{S}^1\setminus\Xi$, as $H\to+\infty$, we have*

$$\left|\partial_H^k R_1(H,\theta)\right|\le\epsilon(H)H^{\frac{3}{4}-k}\quad\text{for } k\le 2.\tag{2.18}$$

Next we rewrite (2.5) with new variables as a nearly integrable system. To handle this process, we apply

$$\begin{aligned}&R_{12}=R_2(H,t,\theta)-R_1(H,\theta),\qquad R_\mu=\mu R_2(H,t,\theta)+(1-\mu)R_1(H,\theta),\\&h_1=\frac{H}{\pi}+R_\mu(H,t,\theta),\qquad h_2=\frac{H}{\pi}+\nu R_2(H,t,\theta)\end{aligned}$$

to express

$$R(H,t,\theta)=I_0\left(\frac{H}{\pi}+R_2(H,t,\theta)\right)-I_0\left(\frac{H}{\pi}+R_1(H,\theta)\right)-\pi x\left(I_0\left(\frac{H}{\pi}\right),\theta\right)p(t).\tag{2.19}$$

Thus we obtain

$$\begin{aligned}
&R(H,t,\theta)\\
&\quad= \pi(R_2-R_1)+I_-\left(\frac{H}{\pi}+R_2\right)-I_-\left(\frac{H}{\pi}+R_1(H,\theta)\right)-\pi x\left(I_0\left(\frac{H}{\pi}\right),\theta\right)p(t)\\
&\quad= \pi\left[x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)-x\left(I_0\left(\frac{H}{\pi}\right),\theta\right)\right]p(t)+\pi\left[G\left(x\left(I_0\left(\frac{H}{\pi}+R_1\right),\theta\right)\right)\right.\\
&\qquad\left.-G\left(x\left(I_0\left(\frac{H}{\pi}+R_2\right),\theta\right)\right)\right]+I_-\left(\frac{H}{\pi}+R_2\right)-I_-\left(\frac{H}{\pi}+R_1(H,\theta)\right)\\
&\quad= R^1(H,t,\theta)+R^2(H,t,\theta)+R^3(H,t,\theta),
\end{aligned}$$

where

$$\begin{aligned}
R^1 &= \pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0(h_2)R_2\,d\nu\right]p(t),\\
R^2 &= \pi\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I x\big(I_0(h_1),\theta\big)T_0(h_1)R_{12}\,d\mu,\\
R^3 &= \int_0^1 T_-(h_1)R_{12}\,d\mu.
\end{aligned}$$

Lemma 2.7 *For all $\epsilon>0$, t, and $\theta\in\mathbb{S}^1\setminus\Xi$, as $H\to+\infty$, the function $R(H,t,\theta)$ possesses the property*

$$\left|\partial_H^k\partial_t^l R(H,t,\theta)\right|\le\epsilon(H)H^{\frac{1}{4}-k}\quad \textit{for } k+l\le 1. \tag{2.20}$$

Proof (i) When $k+l=0$, using Lemmas 2.1 and 2.5 and (2.18), we obtain (2.20).

(ii) When $k=1$ and $l=0$,

$$\begin{aligned}
\partial_H R^1 &= \pi\left[\int_0^1 \partial_I^2 x\big(I_0(h_2),\theta\big)T_0^2(h_2)\left(\frac{1}{\pi}+\nu\partial_H R_2\right)R_2\,d\nu\right]p(t)\\
&\quad+\pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0'(h_2)\left(\frac{1}{\pi}+\nu\partial_H R_2\right)R_2\,d\nu\right]p(t)\\
&\quad+\pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0(h_2)\partial_H R_2)\,d\nu\right]p(t),\\
\partial_H R^2 &= \pi\left[\int_0^1 g'\big(x\big(I_0(h_1),\theta\big)\big)\big(\partial_I x\big(I_0(h_1),\theta\big)T_0(h_1)\big)^2\left(\frac{1}{\pi}+\partial_H R_\mu\right)R_{12}\,d\mu\right]\\
&\quad+\pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I x\big(I_0(h_1),\theta\big)T_0'(h_1)\left(\frac{1}{\pi}+\partial_H R_\mu\right)R_{12}\,d\mu\right]\\
&\quad+\pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I x\big(I_0(h_1),\theta\big)T_0(h_1)\partial_H R_{12}\,d\mu\right]\\
&\quad+\pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I^2 x\big(I_0(h_1),\theta\big)T_0^2(h_1)\left(\frac{1}{\pi}+\partial_H R_\mu\right)R_{12}\,d\mu\right],
\end{aligned}$$

and

$$\partial_H R^3 = \left[\int_0^1 T'_-(h_1)\left(\frac{1}{\pi} + \partial_H R_\mu\right)R_{12}\,d\mu + \int_0^1 T_-(h_1)\partial_H R_{12}\,d\mu\right].$$

We derive (2.20) from Lemmas 2.2 and 2.4–2.6.

(iii) When $k = 0$ and $l = 1$,

$$\begin{aligned}
\partial_t R^1 &= \pi\left[\int_0^1 \partial_I^2 x\big(I_0(h_2),\theta\big)T_0^2(h_2)(\nu\partial_t R_2)R_2\,d\nu\right]p(t)\\
&\quad + \pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0'(h_2)(\nu\partial_t R_2)R_2\,d\nu\right]p(t)\\
&\quad + \pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0(h_2)\partial_t R_2\,d\nu\right]p(t)\\
&\quad + \pi\left[\int_0^1 \partial_I x\big(I_0(h_2),\theta\big)T_0(h_2)R_2\,d\nu\right]p'(t),\\
\partial_t R^2 &= \pi\left[\int_0^1 g'\big(x\big(I_0(h_1),\theta\big)\big)\big[\partial_I x\big(I_0(h_1),\theta\big)T_0(h_1)\big]^2(\partial_t R_\mu)R_{12}\,d\mu\right]\\
&\quad + \pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I x\big(I_0(h_1),\theta\big)T_0'(h_1)(\partial_t R_\mu)R_{12}\,d\mu\right]\\
&\quad + \pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I x\big(I_0(h_1),\theta\big)T_0(h_1)\partial_t R_{12}\,d\mu\right]\\
&\quad + \pi\left[\int_0^1 g\big(x\big(I_0(h_1),\theta\big)\big)\partial_I^2 x\big(I_0(h_1),\theta\big)T_0^2(h_1)\partial_t R_\mu R_{12}\,d\mu\right],
\end{aligned}$$

and

$$\partial_t R^3 = \int_0^1 T'_-(h_1)\partial_t R_\mu R_{12}\,d\mu + \int_0^1 T_-(h_1)\partial_t R_{12}\,d\mu.$$

We obtain inequality (2.20) from Lemmas 2.1–2.2 and 2.4–2.6. □

Now we rewrite (2.5) with the variables H, t, θ. Utilizing (2.19) yields

$$\begin{aligned}
I &= I_0\left(\frac{H}{\pi} + R_1(H,\theta)\right) + \pi x\left(I_0\left(\frac{H}{\pi}\right),\theta\right)p(t) + R(H,t,\theta)\\
&= I_0\left(\frac{H}{\pi} + R_1(H,\theta)\right) + \pi x(H,\theta)p(t) + \tilde{R}(H,t,\theta),
\end{aligned} \tag{2.21}$$

where

$$\tilde{R}(H,t,\theta) = \pi\int_0^1 \partial_I x\left(H + \mu I_-\left(\frac{H}{\pi}\right),\theta\right)p(t)I_-\left(\frac{H}{\pi}\right)d\mu + R(H,t,\theta).$$

For the new perturbation $\tilde{R}$, from Lemmas 2.1–2.2 and 2.7 we have

$$\left|\partial_H^k\partial_t^l\tilde{R}(H,t,\theta)\right| \le \epsilon(H)H^{\frac{1}{4}-k} \quad \text{for } k + l \le 1. \tag{2.22}$$

Lemma 2.8 *If* (1.6) *holds, then*

$$\lim_{H\to+\infty} H^{-\frac{1}{4}+k}\Gamma^{(k)}(H) = 0, \quad k = 0, 1. \tag{2.23}$$

Proof Using (2.12) and (2.13), we get

$$\begin{aligned}
&\tau(H) - \pi \\
&\quad = \int_0^\pi \partial_H I_0\left(\frac{H}{\pi} + R_1(H,\theta)\right) d\theta - \pi \\
&\quad = \int_0^\pi \frac{1}{\partial_I[\pi h_0(I) + \pi G(x(I,\theta))]} d\theta - \pi \\
&\quad = \int_0^\pi \frac{1}{\frac{1}{1+\pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)} + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]} d\theta - \pi \\
&\quad = \int_0^\pi \frac{1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)}{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2} d\theta - \pi.
\end{aligned}$$

Applying (1.5), (2.2), and the definition of $h_0(I)$ gives rise to

$$\Gamma(H) = \sqrt{H}\int_0^\pi \left[\frac{\pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1) - g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2}{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2}\right] d\theta. \tag{2.24}$$

Thus

$$\begin{aligned}
&H^{-\frac{1}{4}}\Gamma(H) \\
&\quad = H^{\frac{1}{4}}\int_0^\pi \left[\frac{\pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1) - g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2}{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2}\right] d\theta.
\end{aligned}$$

For $k = 0$, we obtain (2.23) from Lemmas 2.1, 2.4, and 2.6. Note that

$$\begin{aligned}
\tau'(H) = &\int_0^\pi \frac{\pi\partial_H^2 R_1 + \partial_H^2 I_-(\frac{H}{\pi}+R_1)(\frac{1}{\pi}+\partial_H R_1)}{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2} d\theta \\
&- \int_0^\pi \frac{g'(x)(\partial_I x)^2[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^4}{\{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2\}^2} d\theta \\
&- \int_0^\pi \frac{g(x)(\partial_I^2 x)[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^4}{\{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2\}^2} d\theta \\
&- \int_0^\pi \frac{2[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2 g(x)\partial_I x[\pi\partial_H^2 R_1 + \partial_H^2 I_-(\frac{H}{\pi}+R_1)]}{\{1 + g(x)\partial_I x[1 + \pi\partial_H R_1 + \partial_H I_-(\frac{H}{\pi}+R_1)]^2\}^2} d\theta.
\end{aligned}$$

Using Lemmas 2.1, 2.4, and 2.6, we get $H^{\frac{5}{4}}\tau'(H) \le \epsilon(H)$. Thus we obtain

$$H^{\frac{3}{4}}\Gamma'(H) = \frac{1}{2}H^{-\frac{1}{4}}\Gamma(H) + H^{\frac{5}{4}}\tau'(H) \le \epsilon(H).$$

Hence (2.23) holds for $k = 1$. The proof is finished. □

2.2 Canonical transformations

Lemma 2.9 *If* (1.6) *holds, then there exists a canonical transform*

$$\Psi_3 : H = \rho, \qquad t = \tau + T(\rho,\theta)$$

associated with $T(\rho,\theta+\pi) = T(\rho,\theta)$ *such that the transformed system of* (2.21) *takes the form*

$$\frac{d\rho}{d\theta} = -\partial_\tau \hat{I}(\rho,\tau,\theta), \qquad \frac{d\tau}{d\theta} = \partial_\rho \hat{I}(\rho,\tau,\theta), \tag{2.25}$$

where

$$\hat{I}(\rho,\tau,\theta) = J(\rho) + \pi x(\rho,\theta)p(\tau) + \hat{R}(\rho,\tau,\theta)$$

and

$$J(\rho) = \frac{1}{\pi}\int_0^\pi I_0\left(\frac{\rho}{\pi} + R_1(\rho,\theta)\right) d\theta.$$

For the new perturbation $\hat{R}$ *and for all* $\epsilon > 0$, t, *and* $\theta \in \mathbb{S}^1 \setminus \Xi$, *if* $H \to +\infty$ *and* $k + l \le 1$, *then*

$$\left|\partial_\rho^k \partial_\tau^l \hat{R}(\rho,\tau,\theta)\right| \le \epsilon(\rho)\rho^{\frac{1}{4}-k}. \tag{2.26}$$

Proof Define Ψ_3 implicitly by

$$\rho = H + \partial_\tau S(H,\tau,\theta), \qquad t = \tau + \partial_H S(H,\tau,\theta),$$

where the function $S = S(H,\tau,\theta)$ will be determined later. Using Ψ_3, (2.21) becomes

$$\tilde{I}(\rho,\tau,\theta) = I_0\left(\frac{H}{\pi} + R_1(H,\theta)\right) + \pi x(H,\theta)p(t) + \tilde{R}(H,t,\theta) - \frac{\partial S}{\partial \theta}.$$

Now we choose

$$S = \int_0^\theta \left[I_0\left(\frac{H}{\pi} + R_1(H,\theta)\right) - J(H)\right] d\theta.$$

Therefore $\rho = H$. Assuming that $T(H,\theta) = \partial_H S(H,\theta)$, we know that Ψ_3 takes the form

$$H = \rho, \qquad t = \tau + T(\rho,\theta).$$

and the function $\tilde{I}$ reads as

$$\hat{I}(\rho,\tau,\theta) = J(\rho) + \pi x(\rho,\theta)p(\tau) + \hat{R}(\rho,\tau,\theta),$$

where

$$\hat{R}(\rho,\tau,\theta) = \tilde{R}\big(\rho,\tau + T(\rho,\theta),\theta\big) + \pi x(\rho,\theta)\int_0^1 p'\big(\tau + \mu T(\rho,\theta)\big)T(\rho,\theta)\,d\mu.$$

By a direct computation, (2.26) is derived from Lemmas 2.1, 2.2, and 2.6–2.8. □

3 Proof of main result

Now we introduce a small parameter $\delta > 0$ satisfying

$$\Psi_4: \quad \rho = \delta^{-2}\nu, \quad \nu \in [a,b],$$

where a and b such that $b > a > 0$ do not depend on $\delta > 0$.

In the new variables (ν,τ), (2.25) takes the form

$$\frac{d\nu}{d\theta} = -\frac{\partial}{\partial\tau}\hat{I}(\nu,\tau,\theta,\delta), \qquad \frac{d\tau}{d\theta} = \frac{\partial}{\partial\nu}\hat{I}(\nu,\tau,\theta,\delta), \tag{3.1}$$

where

$$\hat{I}(\nu,\tau,\theta,\delta) = \delta^2\bigl[J\bigl(\delta^{-2}\nu\bigr) + \pi x\bigl(\delta^{-2}\nu,\theta\bigr)p(\tau) + \hat{R}\bigl(\delta^{-2}\nu,\tau,\theta\bigr)\bigr].$$

Denote $\hat{R}(\nu,\tau,\theta,\delta) = \delta^2\hat{R}(\delta^{-2}\nu,\tau,\theta)$. From (2.22) we derive that for $k + l \le 1$,

$$\delta^{-1}\bigl|\partial_\nu^k\partial_\tau^l\hat{R}(\nu,\tau,\theta,\delta)\bigr| \le \epsilon\bigl(\delta^{-2}\nu\bigr)\nu^{\frac{1}{4}-k} \quad \text{as } \delta \to 0^+. \tag{3.2}$$

Because of

$$\tau\bigl(\delta^{-2}\nu\bigr) = \pi + \delta\nu^{-\frac{1}{2}}\Gamma\bigl(\delta^{-2}\nu\bigr), \qquad J'\bigl(\delta^{-2}\nu\bigr) = \frac{1}{\pi}\tau\bigl(\delta^{-2}\nu\bigr),$$

we write system (3.1) in the form

$$\begin{cases} \frac{d\nu}{d\theta} = -\pi\delta^2 x(\delta^{-2}\nu,\theta)p'(\tau) - \partial_\tau\hat{R}(\nu,\tau,\theta,\delta), \\ \frac{d\tau}{d\theta} = 1 + \frac{\delta}{\pi}\nu^{-\frac{1}{2}}\Gamma(\delta^{-2}\nu) + \delta^2\pi\,\partial_\nu x(\delta^{-2}\nu,\theta)p(\tau) + \partial_\nu\hat{R}(\nu,\tau,\theta,\delta). \end{cases} \tag{3.3}$$

Let $(\nu(\theta,\nu_0,\tau_0),\tau(\theta,\nu_0,\tau_0))$ denote the solution of (3.3) associated with the initial data

$$\bigl(\nu(0,\nu_0,\tau_0),\tau(0,\nu_0,\tau_0)\bigr) = (\nu_0,\tau_0).$$

Utilizing (3.2), we conclude that if $\delta \ll 1$, then a solution of (3.3) exists in $[0,2\pi]$ for any $(\nu_0,\tau_0) \in [a,b] \times [0,\pi]$. Moreover,

$$0 < \frac{1}{2}a \le \nu(\theta,\nu_0,\tau_0) \le 2b \quad \forall\theta \in [0,2\pi].$$

Assume that the solution $(\nu(\theta,\nu_0,\tau_0),\tau(\theta,\nu_0,\tau_0))$ is of the form

$$\nu(\theta,\nu_0,\tau_0) = \nu_0 + \delta F_2(\theta,\nu_0,\tau_0)), \qquad \tau(\theta,\nu_0,\tau_0) = \tau_0 + \theta + \delta F_1(\theta,\nu_0,\tau_0).$$

Then the Poincaré map of (3.3), represented by P, possesses the expression

$$P(\nu_0,\tau_0) = \bigl(\nu_0 + \delta F_2(\pi,\nu_0,\tau_0),\tau_0 + \pi + \delta F_1(\pi,\nu_0,\tau_0)\bigr).$$

Since $(\nu(\theta,\nu_0,\tau_0),\tau(\theta,\nu_0,\tau_0))$ is a solution of (3.3), we acquire

$$\begin{cases} \frac{dF_1}{d\theta} = \frac{1}{\pi}(\nu_0+\delta F_2)^{-\frac{1}{2}}\Gamma(\delta^{-2}(\nu_0+\delta F_2)) \\ \qquad + \pi\delta\partial_\nu x(\delta^{-2}(\nu_0+\delta F_2),\theta)p(\tau) + \delta^{-1}\partial_\nu\hat{R}(\nu,\tau,\theta,\delta), \\ \frac{dF_2}{d\theta} = -\pi\delta x(\delta^{-2}(\nu_0+\delta F_2),\theta)p'(\tau) - \delta^{-1}\partial_\tau\hat{R}(\nu,\tau,\theta,\delta). \end{cases} \tag{3.4}$$

From (3.2) and (3.4) we derive that

$$|F_1| \le C, \qquad |F_2| \le C$$

uniformly in $\theta \in \mathbb{S}^1 \setminus \Xi$.

Lemma 2.9, (3.2), (3.3), and (3.4) yield

$$\begin{aligned} F_1(\pi,\nu_0,\tau_0) &= \int_0^\pi \Big[\frac{1}{\pi}(\nu_0+\delta F_2)^{-\frac{1}{2}}\Gamma\big(\delta^{-2}(\nu_0+\delta F_2)\big) \\ &\qquad + \delta\pi\,\partial_\nu x\big(\delta^{-2}(\nu_0+\delta F_2),\theta\big)p(\tau)\Big]\,d\theta + o(1), \\ F_2(\pi,\nu_0,\tau_0) &= -\pi\int_0^\pi \delta x\big(\delta^{-2}(\nu_0+\delta F_2),\theta\big)p'(\tau)\,d\theta + o(1). \end{aligned}$$

Thus the Poincaré map P of (3.3) takes the form

$$P: \begin{cases} \tau_1 = \tau_0 + \pi + \delta l_1(\nu_0,\tau_0,\delta) + \delta o(1), \\ \nu_1 = \nu_0 + \delta l_2(\nu_0,\tau_0,\delta) + \delta o(1), \end{cases}$$

where

$$\begin{aligned} l_1(\nu_0,\tau_0,\delta) &= \int_0^\pi \Big[\frac{1}{\pi}(\nu_0+\delta F_2)^{-\frac{1}{2}}\Gamma\big(\delta^{-2}(\nu_0+\delta F_2)\big) \\ &\qquad + \delta\pi\,\partial_\nu x\big(\delta^{-2}(\nu_0+\delta F_2),\theta\big)p(\tau)\Big]\,d\theta \\ &= \nu_0^{-\frac{1}{2}}\Gamma\big(\delta^{-2}\nu_0\big) + \int_0^\pi \delta\pi\,\partial_\nu x\big(\delta^{-2}(\nu_0+\delta F_2),\theta\big)p(\tau)\,d\theta + o(1) \\ &= \nu_0^{-\frac{1}{2}}\Gamma\big(\delta^{-2}\nu_0\big) + \int_0^\pi \delta\pi\,\partial_\nu x\big(\delta^{-2}\nu_0,\theta\big)p(\tau_0+\theta)\,d\theta + o(1) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} l_2(\nu_0,\tau_0,\delta) &= -\pi\int_0^\pi \delta x\big(\delta^{-2}(\nu_0+\delta F_2),\theta\big)p'(\tau)\,d\theta \\ &= -\pi\int_0^\pi \delta x\big(\delta^{-2}\nu_0,\theta\big)p'(t_0+\theta)\,d\theta + o(1). \end{aligned} \tag{3.6}$$

Applying arguments similar to those in [5], we obtain the following estimates:

$$\operatorname{mes}\left\{\theta \in [0,\pi], x\left(\frac{\nu_0}{\delta^2},\theta\right) > 0\right\} = \pi + \delta O(1),$$

$$\operatorname{mes}\left\{\theta \in [0,\pi], x\left(\frac{\nu_0}{\delta^2},\theta\right) < 0\right\} = \delta O(1).$$

When $x < 0$, we have that

$$\left|x\left(\frac{\nu_0}{\delta^2}\right)\right| = O(1), \qquad \left|\partial_I x\left(\frac{\nu_0}{\delta^2}\right)\right| = \delta^2 O(1).$$

When $x > 0$, from the definition of θ it follows that

$$x\left(\frac{\nu_0}{\delta^2},\theta\right) = \sqrt{\frac{2\rho_0}{\delta^2\pi}}\sin\theta + O(1), \qquad \partial_I x\left(\frac{\nu_0}{\delta^2},\theta\right) = \sqrt{\frac{\delta^2}{2\pi\rho_0}}\sin\theta + \delta^2 O(1).$$

Thus from (3.5) we obtain

$$\begin{aligned}
&l_1(\nu_0,\tau_0,\delta)\\
&\quad= \nu_0^{-\frac{1}{2}}\Gamma\left(\delta^{-2}\nu_0\right) + \delta\pi\int_{\{\theta\in[0,\pi]:x>0\}} \partial_\nu x\left(\delta^{-2}\nu_0,\theta\right)p(\tau_0+\theta)\,d\theta + o(1)\\
&\quad= \nu_0^{-\frac{1}{2}}\Gamma\left(\delta^{-2}\nu_0\right) + \delta\pi\int_{\{\theta\in[0,\pi]:x>0\}} \delta^{-2}\sqrt{\frac{\delta^2}{2\pi\nu_0}}\sin\theta p(\tau_0+\theta)\,d\theta + o(1)\\
&\quad= \nu_0^{-\frac{1}{2}}\Gamma\left(\delta^{-2}\nu_0\right) + \pi\int_0^\pi \sqrt{\frac{1}{2\pi\nu_0}}\sin\theta p(\tau_0+\theta)\,d\theta + o(1)\\
&\quad= \nu_0^{-\frac{1}{2}}\left[\Gamma\left(\delta^{-2}\nu_0\right) + \sqrt{\frac{\pi}{2}}\int_0^\pi \sin\theta p(\tau_0+\theta)\,d\theta\right] + o(1).
\end{aligned} \tag{3.7}$$

Similarly, we derive from (3.6) that

$$\begin{aligned}
l_2(\nu_0,\tau_0,\delta) &= -\pi\int_0^\pi \delta x\left(\delta^{-2}\nu_0,\theta\right)p'(\tau_0+\theta)\,d\theta + o(1)\\
&= -\pi\delta\int_{\{\theta\in[0,\pi]:x>0\}} x\left(\delta^{-2}\nu_0,\theta\right)p'(\tau_0+\theta)\,d\theta + o(1)\\
&= -\pi\delta\sqrt{\frac{2\rho_0}{\delta^2\pi}}\int_{\{\theta\in[0,\pi]:x>0\}} p'(\tau_0+\theta)\sin\theta\,d\theta + o(1)\\
&= -\sqrt{2\pi\rho_0}\int_0^\pi p'(\tau_0+\theta)\sin\theta\,d\theta + o(1).
\end{aligned} \tag{3.8}$$

We conclude that the Poincaré map P reads as

$$P: \begin{cases} \tau_1 = \tau_0 + \pi + \delta l_1(\nu_0,\tau_0,\delta) + \delta o(1),\\ \nu_1 = \nu_0 + \delta l_2(\nu_0,\tau_0,\delta) + \delta o(1). \end{cases} \tag{3.9}$$

Using

$$\left|\int_0^\pi \sin\theta p(\tau_0+\theta)\,d\theta\right| = \left|\int_{\tau_0}^{\tau_0+\pi} p(s)(\sin s\cos\tau_0 - \cos\theta\sin\tau_0)\,ds\right| \le |\bar{p}| \tag{3.10}$$

and noticing (1.7), we acquire

$$\limsup_{H\to+\infty}\Gamma(H) > \sqrt{\frac{\pi}{2}}|\bar{p}|. \tag{3.11}$$

Let $\varpi > 0$ and

$$\varpi \le \limsup_{H\to+\infty}\Gamma(H) - \sqrt{\frac{\pi}{2}}|\bar{p}|.$$

Since $\sqrt{H}\Gamma'(H) \to 0$ as $H \to +\infty$, there exists a number $\bar{H} > \frac{1}{b} > 0$ satisfying

$$\left|\sqrt{H}\Gamma'(H)\right| \le \min\left\{\frac{b^{-\frac{1}{2}}\varpi}{4(a^{-1}-b^{-1})}, \frac{\varpi}{16}\right\} \tag{3.12}$$

for $H \ge \bar{H}$. Utilizing (3.11), we can choose a sequence $\{H_m^1\}_{m=1}^{\infty}$ with $\bar{H} \le H_m^1 \to +\infty$ such that

$$H_{m+1}^1 > \frac{b}{a}H_m^1 \tag{3.13}$$

and

$$\Gamma\left(H_m^1\right) > \frac{3}{4}\varpi + \sqrt{\frac{\pi}{2}}|\bar{p}|. \tag{3.14}$$

Take $\delta_{1m} = (bH_m^1)^{-\frac{1}{2}}$. Then we have $\delta_{1m} \to 0$ as $m \to +\infty$. It follows from (3.13) and (3.14) that

$$\delta_{1m}^{-2}\left(a^{-1}-b^{-1}\right) = H_m^1\left(\frac{b}{a}-1\right) < H_{m+1}^1,$$

and hence $[\delta_{1m}^{-2}b^{-1}, \delta_{1m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{1m}^{-1}] \subset [\delta_{1m}^{-2}b^{-1}, \delta_{1m}^{-2}a^{-1}] \subset [H_m^1, H_{m+1}^1]$.

For any $H \in [\delta_{1m}^{-2}b^{-1}, \delta_{1m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{1m}^{-1}]$, we claim that

$$\Gamma(H) > \frac{1}{4}\varpi + \sqrt{\frac{\pi}{2}}|\bar{p}|. \tag{3.15}$$

Indeed, suppose that there is $H_m^{1*} \in [\delta_{1m}^{-2}b^{-1}, \delta_{1m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{1m}^{-1}]$ such that

$$\Gamma\left(H_m^{1*}\right) \le \frac{1}{4}\varpi + \sqrt{\frac{\pi}{2}}|\bar{p}|.$$

Using (3.12), we have

$$\begin{aligned}
\frac{1}{2}\varpi &\le \left|\Gamma\left(H_m^{1*}\right) - \Gamma\left(H_m^1\right)\right| \\
&= \left|\Gamma'\left(H_m^1 + \mu\left(H_m^{1*} - H_m^1\right)\right)\right|\left(H_m^{1*} - H_m^1\right)\left(\mu \in [0,1]\right) \\
&= \frac{|\sqrt{H_m^1 + \mu(H_m^{1*} - H_m^1)}\Gamma'(H_m^1 + \mu(H_m^{1*} - H_m^1))|(H_m^{1*} - H_m^1)}{\sqrt{H_m^1 + \mu(H_m^{1*} - H_m^1)}}
\end{aligned}$$

$$\leq \frac{b^{-\frac{1}{2}}\varpi((a^{-1}-b^{-1})\delta_m^{-1})}{4(a^{-1}-b^{-1})\delta_m^{-1}b^{-\frac{1}{2}}},$$
$$\leq \frac{\varpi}{4},$$

which is a contradiction. Thus (3.15) holds.

Let

$$D_1 = \{(\rho,\tau) : \rho \in [\delta_{1m}^{-2}b^{-1}, \delta_{1m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{1m}^{-1}]\}.$$

Taking $B_1 = \{(x,y) : \Psi_4\Psi_3\Psi_2\Psi_1(x,y) \subset D_1\}$ and using the fixed point theorem in Ding [10], we derive that (1.7) in Theorem 1.1 is a consequence of (3.7)–(3.9) and (3.10)–(3.11).

Considering (1.8), we assume that

$$\liminf_{H\to+\infty}\Gamma(H) < -\sqrt{\frac{2}{\pi}}|\bar{p}|. \tag{3.16}$$

Let $\hat{\omega} > 0$ and

$$\hat{\omega} \leq -\sqrt{\frac{2}{\pi}}|\bar{p}| - \liminf_{H\to+\infty}\Gamma(H).$$

Since $\sqrt{H}\Gamma'(H) \to 0$ as $H \to +\infty$, there exists a number $\hat{H} > \frac{1}{b} > 0$ such that

$$\left|\sqrt{H}\Gamma'(H)\right| \leq \min\left\{\frac{b^{-\frac{1}{2}}\hat{\omega}}{4(a^{-1}-b^{-1})}, \frac{\hat{\omega}}{16}\right\} \tag{3.17}$$

for $H \geq \hat{H}$. Combining (3.16), we can find a sequence $\{H_m^2\}_{m=1}^{\infty}$ associated with $\hat{H} \leq H_m^2 \to +\infty$ satisfying

$$H_{m+1}^2 > \frac{b}{a}H_m^2 \tag{3.18}$$

and

$$\Gamma(H_m^2) \leq -\frac{3}{4}\hat{\omega} - \sqrt{\frac{2}{\pi}}|\bar{p}|. \tag{3.19}$$

Taking $\delta_{2m} = (bH_m^2)^{-\frac{1}{2}}$, we get $\delta_{2m} \to 0$ as $m \to +\infty$. It follows from (3.18) that

$$\delta_{2m}^{-2}(a^{-1}-b^{-1}) = H_m^2\left(\frac{b}{a}-1\right) < H_{m+1}^2,$$

and hence $H \in [\delta_{2m}^{-2}b^{-1}, \delta_{2m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{2m}^{-1}] \subset [H_m^2, H_{m+1}^2]$.

For any $H \in [\delta_{2m}^{-2}b^{-1}, \delta_{2m}^{-2}b^{-1} + (a^{-1}-b^{-1})\delta_{2m}^{-1}]$, we claim that

$$\Gamma(H) < -\frac{1}{4}\hat{\omega} - \sqrt{\frac{2}{\pi}}|\bar{p}|. \tag{3.20}$$

Indeed, assume that there is $H_m^{2*} \in [\delta_{2m}^{-2} b^{-1}, \delta_{2m}^{-2} b^{-1} + (a^{-1} - b^{-1})\delta_{2m}^{-1}]$ such that

$$\Gamma\left(H_m^{2*}\right) \geq -\frac{1}{4}\varpi - \sqrt{\frac{2}{\pi}}|\bar{p}|.$$

Then using (3.17), (3.18), and (3.19) we get

$$\begin{aligned}
\frac{1}{2}\hat{\omega} &\leq \left|\Gamma\left(H_m^{2*}\right) - \Gamma\left(H_m^2\right)\right| \\
&= \left|\Gamma'\left(H_m^2 + \mu\left(H_m^{2*} - H_m^2\right)\right)\right|\left(H_m^{2*} - H_m^2\right)\left(\mu \in [0,1]\right) \\
&= \frac{|\sqrt{H_m^2 + \mu(H_m^{2*} - H_m^2)}\Gamma'(H_m^2 + \mu(H_m^{2*} - H_m^2))|(H_m^* - H_m^2)}{\sqrt{H_m^2 + \mu(H_m^{2*} - H_m^2)}} \\
&\leq \frac{b^{-\frac{1}{2}}\hat{\omega}((a^{-1} - b^{-1})\delta_m^{-1})}{4(a^{-1} - b^{-1})\delta_m^{-1} b^{-\frac{1}{2}}} \\
&\leq \frac{\hat{\hat{\omega}}}{4},
\end{aligned}$$

which is a contradiction. Thus (3.20) is valid.

Let

$$D_2 = \left\{(\rho, \tau) : \rho \in \left[\delta_{2m}^{-2} b^{-1}, \delta_{2m}^{-2} b^{-1} + \left(a^{-1} - b^{-1}\right)\delta_{2m}^{-1}\right]\right\}$$

and $B_2 = \{(x,y) : \Psi_4\Psi_3\Psi_2\Psi_1(x,y) \subset D_2\}$.

Now we choose $\{H_{1m_k}\}_1^{+\infty}$ and $\{H_{2m_k}\}_1^{+\infty}$ such that $H_{1m_k} < H_{1m_k+1} < H_{2m_k} < H_{2m_k+1}$ and let

$$D_3 = \left\{(\rho, \tau) : \rho \in \left[\delta_{1m_k}^{-2} b^{-1}, \delta_{2m_k}^{-2} b^{-1} + \left(a^{-1} - b^{-1}\right)\delta_{2m_k}^{-1}\right]\right\}.$$

Setting $B_3 = \{(x,y) : \Psi_4\Psi_3\Psi_2\Psi_1(x,y) \subset D_3\}$ and using the twist theorem in Ding [11], we obtain that inequality (1.8) in Theorem 1.1 is a consequence of (3.10), (3.15), (3.7), and (3.20). The proof of Theorem 1.1 is finished.

To verify the given conditions and understand our main result, we give the following remark.

Remark 3.1 Using Lemmas 2.1, 2.4, and 2.6, Eq. (2.24) takes the form

$$\Gamma(H) = \sqrt{2}\pi^{\frac{3}{2}} + \sqrt{H}\int_0^\pi \left(\pi\,\partial_H R_1 - g(x)\partial_I x\right) d\theta + o(1).$$

Combining (2.14) with Lemma 2.4, we have $\partial_H R_1 = -g(x)\partial_I x + o(\frac{1}{\sqrt{H}})$. Thus we obtain

$$\Gamma(H) = \sqrt{2}\pi^{\frac{3}{2}} - \sqrt{H}\int_0^\pi (\pi + 1)g(x)\partial_I x)\, d\theta + o(1).$$

By the results in [5], $x = \sqrt{\frac{2H}{\pi}}\sin\theta + O(1)$, $\partial_I x = \sqrt{\frac{1}{2\pi H}}\sin\theta + O(\frac{1}{H})$, and

$$\Gamma(H) = \sqrt{2}\pi^{\frac{3}{2}} - \frac{1}{\sqrt{2\pi}}\int_0^\pi (\pi + 1)g\left(\sqrt{\frac{2H}{\pi}}\sin\theta\right)\sin\theta\, d\theta + o(1).$$

For $g(x)$ and $V(x)$ in problem (1.3), if $g(x)$ satisfies (1.6) and $p(t)$ satisfies (1.7), then we know that the equation

$$\ddot{x} + V'(x) + \sin\ln\left(1 + x^2\right) = p(t)$$

has at least one π-periodic solutions. Letting $p(t)$ satisfy (1.8), we conclude that the equation

$$\ddot{x} + V'(x) + \ln\left(1 + x^2\right)\sin\ln\left(1 + x^2\right) = p(t)$$

has an unbounded sequence of π-periodic solutions.

Acknowledgements
The authors would like to thank the anonymous referees for their valuable and helpful comments, which led to a meaningful improvement of the paper.

Funding
This work is supported by National Natural Science Foundation of China (No. 12361042) and the 14th Five Year Key Discipline of Xinjiang Autonomous Region (78756342).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Dr. Xing and Dr. Wang give all the computaions and derivations of the paper. Lai checks the whole paper and corrects the paper and gives some suggeations. All authors contribute equally in this works.

Author details
[1]The School of Math. and Statis., Yili Normal University, 835000, Yining, China. [2]The School of Math., Southwestern University of Finance and Economics, Wenjiang, 611130, Chengdu, China.

Received: 16 July 2023 Accepted: 11 November 2023 Published online: 22 November 2023

References

1. Fonda, A., Toder, R.: Periodic orbits of radially symmetric Keplerian-like systems: a topological degree approach. J. Differ. Equ. **244**, 3235–3264 (2008)
2. Qian, D., Torres, P.J.: Bouncing solutions of an equation with attractive singularity. Proc. R. Soc. Edinb. A **134**, 201–214 (2004)
3. Jiang, S.: Boundedness of solutions for a class of second-order differential equation with singularity. Bound. Value Probl. **84**, 1–15 (2013)
4. Liu, Q., Torres, P.J., Qian, D.: Periodic, quasiperiodic and unbounded solutions of radially symmetric systems with repulsive singularities at resonance. Nonlinear Differ. Equ. Appl. **22**, 1115–1142 (2015)
5. Capietto, A., Dambrosio, W., Liu, B.: On the boundedness of solutions to a nonlinear singular oscillator. Z. Angew. Math. Phys. **60**, 1007–1034 (2009)
6. Liu, B.: Quasi-periodic solutions of forced isochronous oscillators at resonance. J. Differ. Equ. **246**, 3471–3495 (2009)
7. Wang, H.Y., Jiang, S.J.: Boundedness of solutions of forced isochronous oscillators with singularity at resonance. Adv. Differ. Equ. **2014**, 55 (2014)
8. Lazer, A.C., Leach, D.E.: Bounded perturbations of forced harmonic oscillators at resonance. Ann. Mat. Pura Appl. **82**, 49–68 (1969)
9. Ma, S.: Existence and multiplicity of periodic solutions for semilinear Duffing equations at resonance. Int. J. Qual. Theory Differ. Equ. Appl. **5**, 33–44 (2019)
10. Ding, T.R.: Nonlinear oscillation at the point of resonance. Sci. Sin., Ser. A **1**, 1–13 (1982)
11. Ding, W.Y.: A generalization of Poincaré–Birkhoff theorem. Proc. Am. Math. Soc. **88**, 341–346 (1983)

Publisher's Note