Sun et al. *Boundary Value Problems* (2024) 2024:5
https://doi.org/10.1186/s13661-023-01804-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

# Existence and uniqueness of solutions for multi-order fractional differential equations with integral boundary conditions

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**Abstract**

In this paper, we consider the existence and uniqueness of solutions for the following nonlinear multi-order fractional differential equation with integral boundary conditions

$$\begin{cases} ({}^{C}D_{0+}^{\alpha}u)(t)+\sum_{i=1}^{m}\lambda_i(t)({}^{C}D_{0+}^{\alpha_i}u)(t)+\sum_{j=1}^{n}\mu_j(t)({}^{C}D_{0+}^{\beta_j}u)(t) \\ \quad +\sum_{k=1}^{p}\xi_k(t)({}^{C}D_{0+}^{\gamma_k}u)(t)+\sum_{l=1}^{q}\omega_l(t)({}^{C}D_{0+}^{\delta_l}u)(t) \\ \quad +\sigma(t)u(t)+f(t,u(t))=0, \quad t\in[0,1], \\ u''(0)=u'''(0)=0, \quad u'(0)=\eta_1\int_0^1 u(s)\,ds, \quad u(1)=\eta_2\int_0^1 u(s)\,ds, \end{cases}$$

where $0<\delta_1<\delta_2<\cdots<\delta_q<1<\gamma_1<\gamma_2<\cdots<\gamma_p<2<\beta_1<\beta_2<\cdots<\beta_n<3<\alpha_1<\alpha_2<\cdots<\alpha_m<\alpha<4$ and $\eta_1+2(1-\eta_2)\neq 0$. Using a fixed point theorem and Banach contractive mapping principle, we obtain some existence and uniqueness results.

**Keywords:** Multi-order fractional differential equation; Integral boundary condition; Existence; Uniqueness

## 1 Introduction

Boundary value problems (BVPs) for ordinary or partial differential equations have received wide attention due to their importance in engineering, physics, material mechanics, and chemotaxis mechanisms; see, for example, [1–6] and the references therein.

Since fractional-order models are more accurate than integer-order models, fractional differential equations, which have profound physical backgrounds and rich theoretical connotations, have attracted much attention. Recently, many scholars have studied the properties of solutions to some BVPs of fractional differential equations using the Banach contraction mapping principle, Leray-Schauder nonlinear alternative, Guo-Krasnoselskii fixed point theorem, monotone iterative technique, and so on [7–16].

As stated in [17], BVPs with integral boundary conditions have various applications in applied fields, such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, and population dynamics. Recent results on BVPs of fractional differential equations with integral boundary conditions can be seen in [17–22]. However, it is necessary to point out that all the equations in the papers mentioned above only include a single fractional derivative.

In 2014, Choudhary and Daftardar-Gejji [23] discussed the antiperiodic BVP of nonlinear multi-order fractional differential equation

$$\begin{cases} \sum_{i=0}^{n} \lambda_i {}^C D^{\alpha_i} u(t) = f(t, u(t)), \quad t \in [0, T], \\ u(0) = -u(T), \end{cases} \tag{1.1}$$

where $\lambda_i \in \mathbb{R}$, $i = 0, 1, \dots, n$, $\lambda_n \neq 0$, $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_n < 1$. They proved the existence and uniqueness of solutions to the BVP (1.1) in terms of the two-parametric functions of the Mittag-Leffler type. It is worth mentioning that the equation in (1.1) is a generalization of the classical relaxation equation and governs some fractional relaxation processes.

In 2020, Choi et al. [24] investigated the existence and uniqueness of solutions to the BVP of nonlinear multi-order fractional differential equation

$$\begin{cases} ({}^C D_{0+}^{\alpha} u)(t) + \sum_{i=1}^{n} \lambda_i(t)({}^C D_{0+}^{\alpha_i} u)(t) + \sum_{i=1}^{m} \mu_i(t)({}^C D_{0+}^{\beta_i} u)(t) \\ \quad + \sigma(t)u(t) = f(t, u(t)), \quad t \in [0, 1], \\ u(1) = \mu \int_0^1 u(s)\, ds, \qquad u'(0) + u'(1) = 0, \end{cases}$$

where $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha \le 2$, $0 < \beta_1 < \beta_2 < \cdots < \beta_m < 1$, $0 < \mu < 1$, $\lambda_i \in C[0, 1]$ $(i = 1, 2, \dots, n)$, $\mu_i \in C[0, 1]$ $(i = 1, 2, \dots, m)$, $\sigma \in C[0, 1]$.

Motivated by the aforementioned works, in this paper, we consider the existence and uniqueness of solutions to the following BVP of nonlinear multi-order fractional differential equation with integral boundary conditions

$$\begin{cases} ({}^C D_{0+}^{\alpha} u)(t) + \sum_{i=1}^{m} \lambda_i(t)({}^C D_{0+}^{\alpha_i} u)(t) \\ \quad + \sum_{j=1}^{n} \mu_j(t)({}^C D_{0+}^{\beta_j} u)(t) + \sum_{k=1}^{p} \xi_k(t)({}^C D_{0+}^{\gamma_k} u)(t) \\ \quad + \sum_{l=1}^{q} \omega_l(t)({}^C D_{0+}^{\delta_l} u)(t) + \sigma(t)u(t) + f(t, u(t)) = 0, \quad t \in [0, 1], \\ u''(0) = u'''(0) = 0, \qquad u'(0) = \eta_1 \int_0^1 u(s)\, ds, \qquad u(1) = \eta_2 \int_0^1 u(s)\, ds. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume that $0 < \delta_1 < \delta_2 < \cdots < \delta_q < 1 < \gamma_1 < \gamma_2 < \cdots < \gamma_p < 2 < \beta_1 < \beta_2 < \cdots < \beta_n < 3 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha < 4$, $\eta_1 + 2(1 - \eta_2) \neq 0$, $\lambda_i, \mu_j, \xi_k, \omega_l, \sigma : [0, 1] \to \mathbb{R}$ and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous.

*Remark* 1 If we let $\lambda_i(t) = \mu_j(t) = \xi_k(t) = \omega_l(t) = \sigma(t) = 0$ for $t \in [0, 1]$ and $\eta_1 = \eta_2 = \eta$ in (1.2), then the BVP (1.2) is reduced to the model in [20].

*Remark* 2 This paper is motivated greatly by [24]. Compared with [24], the existence results in this paper are established under new and simpler conditions, indicating that our works are not a trivial generalization of [24].

The main tools used in this paper are the following theorems.

**Theorem 1** (see [25]) *Let $X$ be a Banach space. Assume that $\Omega$ is an open bounded subset of $X$ with $\theta \in \Omega$, and let $T : \bar{\Omega} \to X$ be a compact operator such that*

$$\|Tu\| \leq \|u\|, \quad u \in \partial\Omega. \tag{1.3}$$

*Then $T$ has a fixed point in $\bar{\Omega}$.*

**Theorem 2** (see [26]) *Let $(X, d)$ be a complete metric space and $T : X \to X$ be contractive. Then $T$ has a unique fixed point in $X$.*

## 2 Preliminaries

First of all, for the reader's convenience, we mainly introduce some definitions and lemmas of the Riemann-Liouville fractional integrals and fractional derivatives and the Caputo fractional derivatives on a finite interval of the real line. For details, one can refer to [27, 28].

In this section, we always assume that $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mu > 0$ and $[\mu]$ denotes the integer part of $\mu$.

**Definition 1** (see [27]) The Riemann-Liouville fractional integrals $I^{\mu}_{0+}u$ and $I^{\mu}_{1-}u$ of order $\mu$ on $[0, 1]$ are defined by

$$\left(I^{\mu}_{0+}u\right)(t) := \frac{1}{\Gamma(\mu)} \int_0^t \frac{u(s)\,ds}{(t-s)^{1-\mu}}$$

and

$$\left(I^{\mu}_{1-}u\right)(t) := \frac{1}{\Gamma(\mu)} \int_t^1 \frac{u(s)\,ds}{(s-t)^{1-\mu}},$$

respectively, where

$$\Gamma(\mu) = \int_0^{+\infty} s^{\mu-1} e^{-s}\,ds.$$

**Definition 2** (see [27]) The Riemann-Liouville fractional derivatives $D^{\mu}_{0+}u$ and $D^{\mu}_{1-}u$ of order $\mu$ on $[0, 1]$ are defined by

$$\begin{aligned}\left(D^{\mu}_{0+}u\right)(t) &:= \left(\frac{d}{dt}\right)^n \left(I^{n-\mu}_{0+}u\right)(t) \\ &= \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)\,ds}{(t-s)^{\mu-n+1}}\end{aligned}$$

and

$$\begin{aligned}\left(D^{\mu}_{1-}u\right)(t) &:= \left(-\frac{d}{dt}\right)^n \left(I^{n-\mu}_{1-}u\right)(t) \\ &= \frac{1}{\Gamma(n-\mu)} \left(-\frac{d}{dt}\right)^n \int_t^1 \frac{u(s)\,ds}{(s-t)^{\mu-n+1}},\end{aligned}$$

respectively, where $n = [\mu] + 1$.

**Definition 3** (see [27]) Let $D^{\mu}_{0+}[u(s)](t) \equiv (D^{\mu}_{0+}u)(t)$ and $D^{\mu}_{1-}[u(s)](t) \equiv (D^{\mu}_{1-}u)(t)$ be the Riemann-Liouville fractional derivatives of order $\mu$. Then the Caputo fractional derivatives ${}^{C}D^{\mu}_{0+}u$ and ${}^{C}D^{\mu}_{1-}u$ of order $\mu$ on $[0,1]$ are defined by

$$\big({}^{C}D^{\mu}_{0+}u\big)(t) := \left(D^{\mu}_{0+}\left[u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}s^k\right]\right)(t)$$

and

$$\big({}^{C}D^{\mu}_{1-}u\big)(t) := \left(D^{\mu}_{1-}\left[u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(1)}{k!}(1-s)^k\right]\right)(t),$$

respectively, where

$$n = [\mu] + 1 \quad \text{for } \mu \notin \mathbb{N}; \qquad n = \mu \quad \text{for } \mu \in \mathbb{N}. \tag{2.1}$$

**Lemma 1** (see [27]) *Let $n$ be given by (2.1) and $u \in C^n[0,1]$. Then*

$$\big(I^{\mu}_{0+}\,{}^{C}D^{\mu}_{0+}u\big)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

*where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$.*

**Lemma 2** (see [28]) *Let $\nu > \mu$. Then the equation $({}^{C}D^{\mu}_{0+}I^{\nu}_{0+}u)(t) = (I^{\nu-\mu}_{0+}u)(t)$, $t \in [0,1]$ is satisfied for $u \in C[0,1]$.*

**Lemma 3** (see [27]) *Let $n$ be given by (2.1). Then the following relations hold:*
(1) *For $k \in \{0,1,2,\ldots,n-1\}$, ${}^{C}D^{\mu}_{0+}t^k = 0$;*
(2) *If $\nu > n$, then ${}^{C}D^{\mu}_{0+}t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}t^{\nu-\mu-1}$.*

## 3 Results

Let $C[0,1]$ be the Banach space of all continuous functions defined on $[0,1]$ with the norm

$$\|u\| = \max_{0\le t\le 1}\big|u(t)\big|.$$

**Lemma 4** *Let $y \in C[0,1]$ be a given function. Then the BVP*

$$\begin{cases} ({}^{C}D^{\alpha}_{0+}v)(t) + y(t) = 0, & t \in [0,1], \\ v''(0) = v'''(0) = 0, \\ v'(0) = \eta_1\int_0^1 v(s)\,ds, & v(1) = \eta_2\int_0^1 v(s)\,ds \end{cases} \tag{3.1}$$

*has a unique solution*

$$v(t) = \int_0^1 G(t,s)y(s)\,ds, \quad t \in [0,1],$$

*where*

$$G(t,s) = \frac{[(2t-1)\eta_1 + 2]\alpha(1-s)^{\alpha-1} - 2[(t-1)\eta_1 + \eta_2](1-s)^{\alpha}}{[\eta_1 + 2(1-\eta_2)]\Gamma(\alpha+1)}$$

$$-\begin{cases}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1.\end{cases}$$

*Proof* In view of the equation in (3.1) and Lemma 1, we have

$$v(t)=-\big(I_{0+}^{\alpha}y\big)(t)-c_0-c_1t-c_2t^2-c_3t^3,\quad t\in[0,1],$$

which, together with the boundary conditions in (3.1), shows that

$$v(t)=-\big(I_{0+}^{\alpha}y\big)(t)+\big[(t-1)\eta_1+\eta_2\big]\int_0^1 v(s)\,ds+\big(I_{0+}^{\alpha}y\big)(1),\quad t\in[0,1]. \tag{3.2}$$

From (3.2), we get

$$\begin{aligned}&\int_0^1 v(s)\,ds\\ &\quad=\frac{2}{\eta_1+2(1-\eta_2)}\biggl[\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)\,ds-\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)\,ds\biggr].\end{aligned} \tag{3.3}$$

Therefore, it follows from (3.2) and (3.3) that the BVP (3.1) has a unique solution

$$\begin{aligned}&v(t)\\ &\quad=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+\frac{(2t-1)\eta_1+2}{[\eta_1+2(1-\eta_2)]\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)\,ds\\ &\qquad-\frac{2[(t-1)\eta_1+\eta_2]}{[\eta_1+2(1-\eta_2)]\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}y(s)\,ds\\ &\quad=\int_0^t\biggl\{-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{[(2t-1)\eta_1+2]\alpha(1-s)^{\alpha-1}-2[(t-1)\eta_1+\eta_2](1-s)^{\alpha}}{[\eta_1+2(1-\eta_2)]\Gamma(\alpha+1)}\biggr\}y(s)\,ds\\ &\qquad+\int_t^1\biggl\{\frac{[(2t-1)\eta_1+2]\alpha(1-s)^{\alpha-1}-2[(t-1)\eta_1+\eta_2](1-s)^{\alpha}}{[\eta_1+2(1-\eta_2)]\Gamma(\alpha+1)}\biggr\}y(s)\,ds\\ &\quad=\int_0^1 G(t,s)y(s)\,ds,\quad t\in[0,1].\end{aligned}$$

□

*Remark* 3 Since $G(t,s)$ defined in Lemma 4 is continuous on $[0,1]\times[0,1]$, there exists a constant $M>0$ such that

$$\big|G(t,s)\big|\le M\quad\text{for }(t,s)\in[0,1]\times[0,1]. \tag{3.4}$$

**Lemma 5** *If $y\in C[0,1]$ is a solution of the equation*

$$\begin{aligned}y(t)={}&f\biggl(t,\int_0^1 G(t,s)y(s)\,ds\biggr)-\sum_{i=1}^{m}\lambda_i(t)\big(I_{0+}^{\alpha-\alpha_i}y\big)(t)\\ &-\sum_{j=1}^{n}\mu_j(t)\big(I_{0+}^{\alpha-\beta_j}y\big)(t)-\sum_{k=1}^{p}\xi_k(t)\big(I_{0+}^{\alpha-\gamma_k}y\big)(t)\\ &-\sum_{l=1}^{q}\omega_l(t)\biggl\{\big(I_{0+}^{\alpha-\delta_l}y\big)(t)-\frac{2\eta_1}{[\eta_1+2(1-\eta_2)]\Gamma(2-\delta_l)}\end{aligned} \tag{3.5}$$

$$\cdot \big[ \big(I_{0+}^{\alpha} y\big)(1) - \big(I_{0+}^{\alpha+1} y\big)(1) \big] t^{1-\delta_l} \Big\}$$
$$+ \sigma(t) \int_0^1 G(t,s) y(s)\, ds, \quad t \in [0,1],$$

*then* $u(t) := \int_0^1 G(t,s)y(s)\,ds$, $t \in [0,1]$ *is a solution of the BVP* (1.2) *in* $C^{\alpha}[0,1]$. *Conversely, if* $u \in C^{\alpha}[0,1]$ *is a solution of the BVP* (1.2), *then* $y(t) := -({}^C D_{0+}^{\alpha} u)(t)$, $t \in [0,1]$ *is a solution of the equation* (3.5) *in* $C[0,1]$, *where* $G(t,s)$ *is defined as in Lemma* 4.

*Proof* First, suppose that $y \in C[0,1]$ is a solution of the equation (3.5). We prove that the function $u(t) = \int_0^1 G(t,s)y(s)\,ds$, $t \in [0,1]$ is a solution of the BVP (1.2) in $C^{\alpha}[0,1]$.

In fact, by Lemma 4, we know

$$\big({}^C D_{0+}^{\alpha} u\big)(t) + y(t) = 0, \quad t \in [0,1] \tag{3.6}$$

and

$$u''(0) = u'''(0) = 0, \qquad u'(0) = \eta_1 \int_0^1 u(s)\,ds, \qquad u(1) = \eta_2 \int_0^1 u(s)\,ds. \tag{3.7}$$

In view of Lemma 1 and (3.6), we obtain

$$\big(I_{0+}^{\alpha} y\big)(t) = -u(t) - c_0 - c_1 t - c_2 t^2 - c_3 t^3, \quad t \in [0,1],$$

which, together with (3.7), implies that

$$\begin{aligned} \big(I_{0+}^{\alpha} y\big)(t) = {} & -u(t) + \frac{(2t-1)\eta_1 + 2}{\eta_1 + 2(1-\eta_2)} \big(I_{0+}^{\alpha} y\big)(1) - \frac{2[(t-1)\eta_1 + \eta_2]}{\eta_1 + 2(1-\eta_2)} \big(I_{0+}^{\alpha+1} y\big)(1), \\ & t \in [0,1]. \end{aligned} \tag{3.8}$$

So, it follows from (3.8) and Lemmas 2 and 3 that

$$\big(I_{0+}^{\alpha-\alpha_i} y\big)(t) = -\big({}^C D_{0+}^{\alpha_i} u\big)(t), \quad i = 1,2,\dots,m, t \in [0,1], \tag{3.9}$$

$$\big(I_{0+}^{\alpha-\beta_j} y\big)(t) = -\big({}^C D_{0+}^{\beta_j} u\big)(t), \quad j = 1,2,\dots,n, t \in [0,1], \tag{3.10}$$

$$\big(I_{0+}^{\alpha-\gamma_k} y\big)(t) = -\big({}^C D_{0+}^{\gamma_k} u\big)(t), \quad k = 1,2,\dots,p, t \in [0,1] \tag{3.11}$$

and

$$\begin{aligned} \big(I_{0+}^{\alpha-\delta_l} y\big)(t) = {} & -\big({}^C D_{0+}^{\delta_l} u\big)(t) \\ & + \frac{2\eta_1}{[\eta_1 + 2(1-\eta_2)]\Gamma(2-\delta_l)} \big[ \big(I_{0+}^{\alpha} y\big)(1) - \big(I_{0+}^{\alpha+1} y\big)(1) \big] t^{1-\delta_l}, \\ & l = 1,2,\dots,q, t \in [0,1]. \end{aligned} \tag{3.12}$$

Substituting (3.9)-(3.12), $y(t) = -({}^CD_{0+}^{\alpha}u)(t)$ and $\int_0^1 G(t,s)y(s)\,ds = u(t)$ into (3.5), we get

$$\begin{aligned}&\left({}^CD_{0+}^{\alpha}u\right)(t) + \sum_{i=1}^{m}\lambda_i(t)\left({}^CD_{0+}^{\alpha_i}u\right)(t) + \sum_{j=1}^{n}\mu_j(t)\left({}^CD_{0+}^{\beta_j}u\right)(t) + \sum_{k=1}^{p}\xi_k(t)\left({}^CD_{0+}^{\gamma_k}u\right)(t)\\&\quad+\sum_{l=1}^{q}\omega_l(t)\left({}^CD_{0+}^{\delta_l}u\right)(t) + \sigma(t)u(t) + f\left(t,u(t)\right) = 0, \quad t\in[0,1],\end{aligned}$$

which, together with (3.7), shows that $u(t) = \int_0^1 G(t,s)y(s)\,ds$, $t\in[0,1]$ is a solution of the BVP (1.2).

Next, suppose that $u \in C^{\alpha}[0,1]$ is a solution of the BVP (1.2), that is, $u\in C^{\alpha}[0,1]$ satisfies the equation

$$\begin{aligned}&\left({}^CD_{0+}^{\alpha}u\right)(t) + \sum_{i=1}^{m}\lambda_i(t)\left({}^CD_{0+}^{\alpha_i}u\right)(t) + \sum_{j=1}^{n}\mu_j(t)\left({}^CD_{0+}^{\beta_j}u\right)(t) + \sum_{k=1}^{p}\xi_k(t)\left({}^CD_{0+}^{\gamma_k}u\right)(t)\\&\quad+\sum_{l=1}^{q}\omega_l(t)\left({}^CD_{0+}^{\delta_l}u\right)(t) + \sigma(t)u(t) + f\left(t,u(t)\right) = 0, \quad t\in[0,1]\end{aligned} \tag{3.13}$$

and the boundary conditions

$$u''(0) = u'''(0) = 0, \qquad u'(0) = \eta_1\int_0^1 u(s)\,ds, \qquad u(1) = \eta_2\int_0^1 u(s)\,ds.$$

We prove that the function $y(t) = -({}^CD_{0+}^{\alpha}u)(t)$, $t\in[0,1]$ is a solution of the equation (3.5) in $C[0,1]$.

In fact, in view of Lemma 4, we know

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \quad t\in[0,1]. \tag{3.14}$$

Furthermore, by the expression of $G(t,s)$, we may obtain

$$u(t) = -\left(I_{0+}^{\alpha}y\right)(t) + \frac{(2t-1)\eta_1+2}{\eta_1+2(1-\eta_2)}\left(I_{0+}^{\alpha}y\right)(1) - \frac{2[(t-1)\eta_1+\eta_2]}{\eta_1+2(1-\eta_2)}\left(I_{0+}^{\alpha+1}y\right)(1), \quad t\in[0,1],$$

which, together with Lemmas 2 and 3, implies that

$$\left({}^CD_{0+}^{\alpha_i}u\right)(t) = -\left(I_{0+}^{\alpha-\alpha_i}y\right)(t), \quad i=1,2,\dots,m, t\in[0,1], \tag{3.15}$$

$$\left({}^CD_{0+}^{\beta_j}u\right)(t) = -\left(I_{0+}^{\alpha-\beta_j}y\right)(t), \quad j=1,2,\dots,n, t\in[0,1], \tag{3.16}$$

$$\left({}^CD_{0+}^{\gamma_k}u\right)(t) = -\left(I_{0+}^{\alpha-\gamma_k}y\right)(t), \quad k=1,2,\dots,p, t\in[0,1] \tag{3.17}$$

and

$$\begin{aligned}&\left({}^CD_{0+}^{\delta_l}u\right)(t) = -\left(I_{0+}^{\alpha-\delta_l}y\right)(t) + \frac{2\eta_1}{[\eta_1+2(1-\eta_2)]\Gamma(2-\delta_l)}\left[\left(I_{0+}^{\alpha}y\right)(1) - \left(I_{0+}^{\alpha+1}y\right)(1)\right]t^{1-\delta_l},\\&\quad l=1,2,\dots,q, t\in[0,1].\end{aligned} \tag{3.18}$$

Substituting (3.14)-(3.18) and $({}^C D_{0+}^{\alpha} u)(t) = -y(t)$ into (3.13), we get (3.5). This indicates that $y(t) = -({}^C D_{0+}^{\alpha} u)(t)$, $t \in [0,1]$ is a solution of equation (3.5). □

Now, we define an operator $T : C[0,1] \to C[0,1]$ by

$$\begin{aligned}
(Ty)(t) = {} & f\left(t, \int_0^1 G(t,s)y(s)\,ds\right) - \sum_{i=1}^{m} \lambda_i(t)\big(I_{0+}^{\alpha-\alpha_i} y\big)(t) \\
& - \sum_{j=1}^{n} \mu_j(t)\big(I_{0+}^{\alpha-\beta_j} y\big)(t) - \sum_{k=1}^{p} \xi_k(t)\big(I_{0+}^{\alpha-\gamma_k} y\big)(t) \\
& - \sum_{l=1}^{q} \omega_l(t)\bigg\{\big(I_{0+}^{\alpha-\delta_l} y\big)(t) - \frac{2\eta_1}{[\eta_1 + 2(1-\eta_2)]\Gamma(2-\delta_l)} \\
& \cdot \big[\big(I_{0+}^{\alpha} y\big)(1) - \big(I_{0+}^{\alpha+1} y\big)(1)\big] t^{1-\delta_l}\bigg\} \\
& + \sigma(t) \int_0^1 G(t,s)y(s)\,ds, \quad t \in [0,1].
\end{aligned}$$

Obviously, if $y$ is a fixed point of $T$, then $u(t) = \int_0^1 G(t,s)y(s)\,ds$, $t \in [0,1]$ is a solution of the BVP (1.2).

For convenience, in the remainder of this paper, we denote

$$C_1 = \left|\frac{2\eta_1}{\eta_1 + 2(1-\eta_2)}\right| \frac{\alpha+2}{\Gamma(\alpha+2)}$$

and

$$\begin{aligned}
C_2 = {} & \sum_{i=1}^{m} \frac{\|\lambda_i\|}{\Gamma(\alpha-\alpha_i+1)} + \sum_{j=1}^{n} \frac{\|\mu_j\|}{\Gamma(\alpha-\beta_j+1)} + \sum_{k=1}^{p} \frac{\|\xi_k\|}{\Gamma(\alpha-\gamma_k+1)} \\
& + \sum_{l=1}^{q} \|\omega_l\| \left[\frac{1}{\Gamma(\alpha-\delta_l+1)} + \frac{C_1}{\Gamma(2-\delta_l)}\right] + \|\sigma\| M.
\end{aligned}$$

**Theorem 3** *Assume that there exists a constant $r > 0$ such that*

$$\max_{(t,x)\in[0,1]\times[-Mr,Mr]} \big|f(t,x)\big| \le (1-C_2)r. \tag{3.19}$$

*Then the BVP (1.2) has at least one solution.*

*Proof* Let $\Omega = \{y \in C[0,1] : \|y\| < r\}$. Then, for any $y \in \bar{\Omega}$, by (3.4), we know

$$\left|\int_0^1 G(t,s)y(s)\,ds\right| \le Mr, \quad t \in [0,1]. \tag{3.20}$$

On the one hand, for any $y \in \bar{\Omega}$, in view of (3.20), we get

$$\begin{aligned}
& \big|(Ty)(t)\big| \\
& \quad = \bigg| f\left(t, \int_0^1 G(t,s)y(s)\,ds\right) - \sum_{i=1}^{m} \lambda_i(t)\big(I_{0+}^{\alpha-\alpha_i} y\big)(t)
\end{aligned}$$

$$
\begin{aligned}
& - \sum_{j=1}^{n} \mu_j(t)\bigl(I_{0+}^{\alpha-\beta_j} y\bigr)(t) - \sum_{k=1}^{p} \xi_k(t)\bigl(I_{0+}^{\alpha-\gamma_k} y\bigr)(t) \\
& - \sum_{l=1}^{q} \omega_l(t)\biggl\{\bigl(I_{0+}^{\alpha-\delta_l} y\bigr)(t) - \frac{2\eta_1}{[\eta_1 + 2(1-\eta_2)]\Gamma(2-\delta_l)} \\
& \cdot \bigl[\bigl(I_{0+}^{\alpha} y\bigr)(1) - \bigl(I_{0+}^{\alpha+1} y\bigr)(1)\bigr]t^{1-\delta_l}\biggr\} + \sigma(t)\int_0^1 G(t,s)y(s)\,ds\biggr| \\
\leq{}& \biggl|f\biggl(t, \int_0^1 G(t,s)y(s)\,ds\biggr)\biggr| + \sum_{i=1}^{m}\bigl|\lambda_i(t)\bigr|\frac{1}{\Gamma(\alpha-\alpha_i)}\int_0^t (t-s)^{\alpha-\alpha_i-1}\bigl|y(s)\bigr|\,ds \\
& + \sum_{j=1}^{n}\bigl|\mu_j(t)\bigr|\frac{1}{\Gamma(\alpha-\beta_j)}\int_0^t (t-s)^{\alpha-\beta_j-1}\bigl|y(s)\bigr|\,ds \\
& + \sum_{k=1}^{p}\bigl|\xi_k(t)\bigr|\frac{1}{\Gamma(\alpha-\gamma_k)}\int_0^t (t-s)^{\alpha-\gamma_k-1}\bigl|y(s)\bigr|\,ds \\
& + \sum_{l=1}^{q}\bigl|\omega_l(t)\bigr|\biggl\{\frac{1}{\Gamma(\alpha-\delta_l)}\int_0^t (t-s)^{\alpha-\delta_l-1}\bigl|y(s)\bigr|\,ds \\
& + \biggl|\frac{2\eta_1}{\eta_1 + 2(1-\eta_2)}\biggr|\frac{1}{\Gamma(2-\delta_l)}\biggl[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\bigl|y(s)\bigr|\,ds \\
& + \frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}\bigl|y(s)\bigr|\,ds\biggr]\biggr\} + \bigl|\sigma(t)\bigr|\biggl|\int_0^1 G(t,s)y(s)\,ds\biggr| \\
\leq{}& \max_{(t,x)\in[0,1]\times[-Mr,Mr]}\bigl|f(t,x)\bigr| + \Biggl\{\sum_{i=1}^{m}\frac{\|\lambda_i\|}{\Gamma(\alpha-\alpha_i+1)} + \sum_{j=1}^{n}\frac{\|\mu_j\|}{\Gamma(\alpha-\beta_j+1)} \\
& + \sum_{k=1}^{p}\frac{\|\xi_k\|}{\Gamma(\alpha-\gamma_k+1)} + \sum_{l=1}^{q}\|\omega_l\|\biggl[\frac{1}{\Gamma(\alpha-\delta_l+1)} + \frac{C_1}{\Gamma(2-\delta_l)}\biggr] + \|\sigma\|M\Biggr\}\|y\| \\
\leq{}& \max_{(t,x)\in[0,1]\times[-Mr,Mr]}\bigl|f(t,x)\bigr| + C_2 r, \quad t\in[0,1],
\end{aligned}
\tag{3.21}
$$

which shows that $T(\bar{\Omega})$ is uniformly bounded.

On the other hand, for any $y \in \bar{\Omega}$ and $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$, we have

$$
\begin{aligned}
& \bigl|(Ty)(t_2) - (Ty)(t_1)\bigr| \\
& \leq \biggl|f\biggl(t_2, \int_0^1 G(t_2,s)y(s)\,ds\biggr) - f\biggl(t_1, \int_0^1 G(t_1,s)y(s)\,ds\biggr)\biggr| \\
& + \biggl|\sum_{i=1}^{m}\bigl[\lambda_i(t_1)\bigl(I_{0+}^{\alpha-\alpha_i} y\bigr)(t_1) - \lambda_i(t_2)\bigl(I_{0+}^{\alpha-\alpha_i} y\bigr)(t_2)\bigr]\biggr| \\
& + \biggl|\sum_{j=1}^{n}\bigl[\mu_j(t_1)\bigl(I_{0+}^{\alpha-\beta_j} y\bigr)(t_1) - \mu_j(t_2)\bigl(I_{0+}^{\alpha-\beta_j} y\bigr)(t_2)\bigr]\biggr| \\
& + \biggl|\sum_{k=1}^{p}\bigl[\xi_k(t_1)\bigl(I_{0+}^{\alpha-\gamma_k} y\bigr)(t_1) - \xi_k(t_2)\bigl(I_{0+}^{\alpha-\gamma_k} y\bigr)(t_2)\bigr]\biggr|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\sum_{l=1}^{q}\left\{\left[\omega_l(t_1)\left(I_{0+}^{\alpha-\delta_l}y\right)(t_1)-\omega_l(t_2)\left(I_{0+}^{\alpha-\delta_l}y\right)(t_2)\right]\right.\right.\\
&+\left.\left.\frac{2\eta_1}{[\eta_1+2(1-\eta_2)]\Gamma(2-\delta_l)}\left[\left(I_{0+}^{\alpha}y\right)(1)-\left(I_{0+}^{\alpha+1}y\right)(1)\right]\left[\omega_l(t_2)t_2^{1-\delta_l}-\omega_l(t_1)t_1^{1-\delta_l}\right]\right\}\right|\\
&+\left|\sigma(t_2)\int_0^1 G(t_2,s)y(s)\,ds-\sigma(t_1)\int_0^1 G(t_1,s)y(s)\,ds\right|\\
\le{}&\left|f\left(t_2,\int_0^1 G(t_2,s)y(s)\,ds\right)-f\left(t_1,\int_0^1 G(t_2,s)y(s)\,ds\right)\right|\\
&+\left|f\left(t_1,\int_0^1 G(t_2,s)y(s)\,ds\right)-f\left(t_1,\int_0^1 G(t_1,s)y(s)\,ds\right)\right|\\
&+\sum_{i=1}^{m}\left[\left|\lambda_i(t_1)\right|\left|\left(I_{0+}^{\alpha-\alpha_i}y\right)(t_1)-\left(I_{0+}^{\alpha-\alpha_i}y\right)(t_2)\right|+\left|\left(I_{0+}^{\alpha-\alpha_i}y\right)(t_2)\right|\left|\lambda_i(t_1)-\lambda_i(t_2)\right|\right]\\
&+\sum_{j=1}^{n}\left[\left|\mu_j(t_1)\right|\left|\left(I_{0+}^{\alpha-\beta_j}y\right)(t_1)-\left(I_{0+}^{\alpha-\beta_j}y\right)(t_2)\right|+\left|\left(I_{0+}^{\alpha-\beta_j}y\right)(t_2)\right|\left|\mu_j(t_1)-\mu_j(t_2)\right|\right]\\
&+\sum_{k=1}^{p}\left[\left|\xi_k(t_1)\right|\left|\left(I_{0+}^{\alpha-\gamma_k}y\right)(t_1)-\left(I_{0+}^{\alpha-\gamma_k}y\right)(t_2)\right|+\left|\left(I_{0+}^{\alpha-\gamma_k}y\right)(t_2)\right|\left|\xi_k(t_1)-\xi_k(t_2)\right|\right]\\
&+\sum_{l=1}^{q}\left\{\left[\left|\omega_l(t_1)\right|\left|\left(I_{0+}^{\alpha-\delta_l}y\right)(t_1)-\left(I_{0+}^{\alpha-\delta_l}y\right)(t_2)\right|+\left|\left(I_{0+}^{\alpha-\delta_l}y\right)(t_2)\right|\left|\omega_l(t_1)-\omega_l(t_2)\right|\right]\right.\\
&+\left|\frac{2\eta_1}{\eta_1+2(1-\eta_2)}\right|\frac{1}{\Gamma(2-\delta_l)}\left[\left|\left(I_{0+}^{\alpha}y\right)(1)\right|+\left|\left(I_{0+}^{\alpha+1}y\right)(1)\right|\right]\\
&\left.\cdot\left[\left|\omega_l(t_2)\right|\left(t_2^{1-\delta_l}-t_1^{1-\delta_l}\right)+\left|\omega_l(t_2)-\omega_l(t_1)\right|t_1^{1-\delta_l}\right]\right\}\\
&+\left|\sigma(t_2)\right|\left|\int_0^1 G(t_2,s)y(s)\,ds-\int_0^1 G(t_1,s)y(s)\,ds\right|\\
&+\left|\sigma(t_2)-\sigma(t_1)\right|\left|\int_0^1 G(t_1,s)y(s)\,ds\right|\\
\le{}&\left|f\left(t_2,\int_0^1 G(t_2,s)y(s)\,ds\right)-f\left(t_1,\int_0^1 G(t_2,s)y(s)\,ds\right)\right|\\
&+\left|f\left(t_1,\int_0^1 G(t_2,s)y(s)\,ds\right)-f\left(t_1,\int_0^1 G(t_1,s)y(s)\,ds\right)\right|\\
&+\sum_{i=1}^{m}\frac{r}{\Gamma(\alpha-\alpha_i+1)}\left[2\|\lambda_i\|(t_2-t_1)^{\alpha-\alpha_i}+\left|\lambda_i(t_1)-\lambda_i(t_2)\right|\right]\\
&+\sum_{j=1}^{n}\frac{r}{\Gamma(\alpha-\beta_j+1)}\left\{\|\mu_j\|\left[t_2^{\alpha-\beta_j}-t_1^{\alpha-\beta_j}+2(t_2-t_1)^{\alpha-\beta_j}\right]+\left|\mu_j(t_1)-\mu_j(t_2)\right|\right\}\\
&+\sum_{k=1}^{p}\frac{r}{\Gamma(\alpha-\gamma_k+1)}\left[\|\xi_k\|\left(t_2^{\alpha-\gamma_k}-t_1^{\alpha-\gamma_k}\right)+\left|\xi_k(t_1)-\xi_k(t_2)\right|\right]\\
&+\sum_{l=1}^{q}\left\{\frac{r}{\Gamma(\alpha-\delta_l+1)}\left[\|\omega_l\|\left(t_2^{\alpha-\delta_l}-t_1^{\alpha-\delta_l}\right)+\left|\omega_l(t_1)-\omega_l(t_2)\right|\right]\right.
\end{aligned}
$$

$$+\frac{C_1 r}{\Gamma(2-\delta_l)}\left[\|\omega_l\|\left(t_2^{1-\delta_l}-t_1^{1-\delta_l}\right)+\left|\omega_l(t_2)-\omega_l(t_1)\right|\right]\Bigg\}$$
$$+\|\sigma\|r\int_0^1\left|G(t_2,s)-G(t_1,s)\right|ds+Mr\left|\sigma(t_2)-\sigma(t_1)\right|,$$

which, together with the fact that $f(t,x)$ and $G(t,s)$ are uniformly continuous on $[0,1]\times[-Mr,Mr]$ and $[0,1]\times[0,1]$, respectively, implies that $T(\bar{\Omega})$ is equicontinuous.

Therefore, by Arzela-Ascoli theorem, we know that $T:\bar{\Omega}\to C[0,1]$ is compact.

Moreover, for any $y\in\partial\Omega$, in view of (3.19) and (3.21), we obtain

$$\left|(Ty)(t)\right|\le\max_{(t,x)\in[0,1]\times[-Mr,Mr]}\left|f(t,x)\right|+C_2r\le(1-C_2)r+C_2r=r=\|y\|,\quad t\in[0,1],$$

which shows that

$$\|Ty\|\le\|y\|,\quad y\in\partial\Omega.$$

So, it follows from Theorem 1 that the operator $T$ has a fixed point $y^*\in\bar{\Omega}$, and so, $u^*(t)=\int_0^1 G(t,s)y^*(s)\,ds$, $t\in[0,1]$ is a solution of the BVP (1.2). □

**Theorem 4** *Assume that there exists a constant* $0<L<\frac{1-C_2}{M}$ *such that*

$$\left|f(t,x_1)-f(t,x_2)\right|\le L|x_1-x_2|\quad\text{for all } t\in[0,1],x_1,x_2\in\mathbb{R}.\tag{3.22}$$

*Then the BVP* (1.2) *has a unique solution.*

*Proof* For any $y_1,y_2\in C[0,1]$, in view of (3.4) and (3.22), we have

$$\begin{aligned}
&\left|(Ty_1)(t)-(Ty_2)(t)\right|\\
&\le\left|f\left(t,\int_0^1 G(t,s)y_1(s)\,ds\right)-f\left(t,\int_0^1 G(t,s)y_2(s)\,ds\right)\right|\\
&\quad+\left|\sum_{i=1}^m\lambda_i(t)\left(I_{0+}^{\alpha-\alpha_i}(y_2-y_1)\right)(t)\right|+\left|\sum_{j=1}^n\mu_j(t)\left(I_{0+}^{\alpha-\beta_j}(y_2-y_1)\right)(t)\right|\\
&\quad+\left|\sum_{k=1}^p\xi_k(t)\left(I_{0+}^{\alpha-\gamma_k}(y_2-y_1)\right)(t)\right|\\
&\quad+\left|\sum_{l=1}^q\omega_l(t)\left\{\left(I_{0+}^{\alpha-\delta_l}(y_2-y_1)\right)(t)+\frac{2\eta_1}{[\eta_1+2(1-\eta_2)]\Gamma(2-\delta_l)}\right.\right.\\
&\quad\left.\left.\cdot\left[\left(I_{0+}^{\alpha}(y_1-y_2)\right)(1)+\left(I_{0+}^{\alpha+1}(y_2-y_1)\right)(1)\right]t^{1-\delta_l}\right\}\right|\\
&\quad+\left|\sigma(t)\int_0^1 G(t,s)\left(y_1(s)-y_2(s)\right)ds\right|\\
&\le L\left|\int_0^1 G(t,s)\left(y_1(s)-y_2(s)\right)ds\right|+\sum_{i=1}^m\left|\lambda_i(t)\right|\left(I_{0+}^{\alpha-\alpha_i}|y_1-y_2|\right)(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\left|\mu_j(t)\right|\left(I_{0+}^{\alpha-\beta_j}|y_1-y_2|\right)(t)+\sum_{k=1}^{p}\left|\xi_k(t)\right|\left(I_{0+}^{\alpha-\gamma_k}|y_1-y_2|\right)(t)\\
&+\sum_{l=1}^{q}\left|\omega_l(t)\right|\left\{\left(I_{0+}^{\alpha-\delta_l}|y_1-y_2|\right)(t)\right.\\
&\left.+\left|\frac{2\eta_1}{\eta_1+2(1-\eta_2)}\right|\frac{1}{\Gamma(2-\delta_l)}\left[\left(I_{0+}^{\alpha}|y_1-y_2|\right)(1)+\left(I_{0+}^{\alpha+1}|y_1-y_2|\right)(1)\right]\right\}\\
&+\left|\sigma(t)\right|\left|\int_0^1 G(t,s)\big(y_1(s)-y_2(s)\big)\,ds\right|\\
\leq{}&\left\{LM+\sum_{i=1}^{m}\frac{\|\lambda_i\|}{\Gamma(\alpha-\alpha_i+1)}+\sum_{j=1}^{n}\frac{\|\mu_j\|}{\Gamma(\alpha-\beta_j+1)}+\sum_{k=1}^{p}\frac{\|\xi_k\|}{\Gamma(\alpha-\gamma_k+1)}\right.\\
&\left.+\sum_{l=1}^{q}\|\omega_l\|\left[\frac{1}{\Gamma(\alpha-\delta_l+1)}+\frac{C_1}{\Gamma(2-\delta_l)}\right]+\|\sigma\|M\right\}\|y_1-y_2\|\\
={}&(LM+C_2)\|y_1-y_2\|,\quad t\in[0,1],
\end{aligned}
$$

and so,

$$
\|Ty_1-Ty_2\|\leq(LM+C_2)\|y_1-y_2\|,
$$

which, together with $LM+C_2<1$, implies that $T$ is a contractive mapping. So, it follows from Theorem 2 that the operator $T$ has a unique fixed point $y^{**}\in C[0,1]$. Therefore, by Lemma 5, we know that the BVP (1.2) has a unique solution $u^{**}(t)=\int_0^1 G(t,s)y^{**}(s)\,ds$, $t\in[0,1]$. □

*Example* 1 We consider the BVP

$$
\begin{cases}
({}^CD_{0+}^{\frac{7}{2}}u)(t)+0.1\sqrt{t}({}^CD_{0+}^{\frac{5}{2}}u)(t)+0.2e^{-t^2}({}^CD_{0+}^{\frac{3}{2}}u)(t)+0.6t^2({}^CD_{0+}^{\frac{1}{2}}u)(t)\\
\quad+\frac{\sqrt{\pi}}{16}\cos tu(t)+(t-\frac{5\sqrt{\pi}}{16})\sqrt{u^2(t)+1}=0,\quad t\in[0,1],\\
u''(0)=u'''(0)=0,\qquad u'(0)=\frac{2}{3}\int_0^1 u(s)\,ds,\qquad u(1)=\frac{1}{2}\int_0^1 u(s)\,ds.
\end{cases}
\tag{3.23}
$$

Since $\alpha=\frac{7}{2}$, $\eta_1=\frac{2}{3}$ and $\eta_2=\frac{1}{2}$, we may choose $M=\frac{8}{5\sqrt{\pi}}$, and so, in view of $\beta_1=\frac{5}{2}$, $\gamma_1=\frac{3}{2}$, $\delta_1=\frac{1}{2}$, $\lambda_i(t)=0$, $\mu_1(t)=0.1\sqrt{t}$, $\xi_1(t)=0.2e^{-t^2}$, $\omega_1(t)=0.6t^2$ and $\sigma(t)=\frac{\sqrt{\pi}}{16}\cos t$ for $t\in[0,1]$, a direct computation shows that $C_2=\frac{3150\pi+1408}{7875\pi}$.

Now that $f(t,x)=(t-\frac{5\sqrt{\pi}}{16})\sqrt{x^2+1}$ for $(t,x)\in[0,1]\times\mathbb{R}$, if we let $L=\frac{5\sqrt{\pi}}{16}$, then

$$
LM+C_2\approx 0.96<1
$$

and

$$
\begin{aligned}
\left|f(t,x_1)-f(t,x_2)\right|&=\left|\left(t-\frac{5\sqrt{\pi}}{16}\right)\left(\sqrt{x_1^2+1}-\sqrt{x_2^2+1}\right)\right|\\
&\leq L\frac{|x_1^2-x_2^2|}{\sqrt{x_1^2+1}+\sqrt{x_2^2+1}}
\end{aligned}
$$

$$\leq L \frac{|x_1| + |x_2|}{\sqrt{x_1^2 + 1} + \sqrt{x_2^2 + 1}} |x_1 - x_2|$$
$$\leq L|x_1 - x_2|, \quad t \in [0,1], x_1, x_2 \in \mathbb{R},$$

which indicates that (3.22) is satisfied.

Therefore, it follows from Theorem 4 that the BVP (3.23) has a unique solution.

**Funding**
This work is supported by the National Natural Science Foundation of China (Grant 12361039).

**Data availability**
Not applicable.

## Declarations

**Competing interests**
The authors declare no competing interests.

**Author contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

Received: 21 September 2023 Accepted: 1 December 2023 Published online: 02 January 2024

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