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# Ground-state sign-changing homoclinic solutions for a discrete nonlinear $p$ -Laplacian equation with logarithmic nonlinearity

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## Abstract

By using a direct non-Nehari manifold method from (Tang and Cheng in *J. Differ. Equ.* 261:2384–2402, 2016), we obtain an existence result of ground-state sign-changing homoclinic solutions that only changes sign once and ground-state homoclinic solutions for a class of discrete nonlinear  $p$ -Laplacian equations with logarithmic nonlinearity. Moreover, we prove that the sign-changing ground-state energy is larger than twice the ground-state energy.

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**Keywords:** Discrete  $p$ -Laplacian equations; Ground-state sign-changing homoclinic solutions; Ground-state homoclinic solutions; Non-Nehari manifold; Logarithmic nonlinearity

## 1 Introduction

The existence of solutions for the discrete nonlinear  $p$ -Laplacian equations by variational methods has been a hot topic in the last twenty years and we refer readers to [4, 13, 15, 19, 20] for example. In particular, in [4], Chen-Tang considered the following discrete  $p$ -Laplacian system:

$$\begin{cases} \Delta(\varphi_p(\Delta u(n-1))) - a(n)\varphi_p(u(n)) + \nabla W(n, u(n)) = 0, & n \in \mathbb{Z}, \\ \lim_{n \rightarrow \pm\infty} u(n) = 0, \end{cases} \quad (1.1)$$

where  $p > 1$ ,  $\varphi_p$  is the  $p$ -Laplace operator,  $u \in \mathbb{R}^N$ ,  $a : \mathbb{Z} \rightarrow \mathbb{R}$  and  $W : \mathbb{Z} \times \mathbb{R}^N \rightarrow \mathbb{R}$ . When  $W(n, x)$  is an odd function in  $x$ , continuously differentiable, and satisfies other suitable conditions, they obtained that the system has an unbounded sequence of homoclinic solutions using the symmetric mountain-pass theorem. When  $p = 2$ , (1.1) reduces to the discrete nonlinear Schrödinger (DNLS) equation. The DNLS equation is one of the most important inherently discrete models and plays a crucial role in modeling various phenomena from solid-state and condensed-matter physics to biology [7–10]. In recent years, the existence of standing-wave solutions for the DNLS equation has attracted some attention (see [3, 5, 11, 14, 23]). In particular, in [5], Chen-Tang-Yu studied the following DNLS

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equation:

$$\begin{cases} -\Delta^2 u(n-1) + \varepsilon(n)(u(n)) - \omega u(n) = f(n, u(n)), & n \in \mathbb{Z}, \\ \lim_{n \rightarrow \pm\infty} u(n) = 0. \end{cases}$$

When  $f$  satisfies the superquadratic growth condition and the monotonicity condition, using the method in [6] and [21], they obtained that the equation has a ground-state solution and a least-energy sign-changing solution, which changes sign exactly once. Furthermore, they obtained that the energy of the sign-changing solution is twice that of the ground-state solution. Next, we recall two studies [2] and [21] that inspire our work partially. In [2], Chang-Wang-Yan studied the following logarithmic Schrödinger equation on a locally finite graph  $G = (V, E)$ :

$$-\Delta u + a(x)u = u \log u^2, \quad x \in V,$$

where  $a : V \rightarrow \mathbb{R}$ . When  $a$  is bounded from below and the volume of set  $\{x \in V : a(x) \leq M\}$  is finite, they used the Nehari manifold method to obtain that the equation has a ground-state solution. Moreover, when  $a$  is bounded from below and  $1/a(x)$  is a Lebesgue integrable function on the set  $\{x \in V : a(x) > M_0\}$ , they also found that the equation has a ground-state solution by using the mountain-pass theorem. In [21], Tang-Cheng investigated the following Kirchhoff-type problem:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N = 1, 2, 3$ . When  $f$  satisfies the supercubic growth and the monotonicity condition, they used a new energy inequality, the deformation lemma, Miranda's theorem, and the non-Nehari manifold method to obtain the same result as in [5].

In this paper, inspired by [2, 4, 5], we mainly use the method in [21] to develop the results in [5] to the following discrete nonlinear  $p$ -Laplacian equation involving logarithmic nonlinearity:

$$\begin{cases} -\Delta(a(n-1)\varphi_p(\Delta u(n-1))) + b(n)\varphi_p(u(n)) \\ = c(n)|u(n)|^{q-2}u(n) \ln |u(n)|^r, & n \in \mathbb{Z}, \\ \lim_{|n| \rightarrow \infty} u(n) = 0, \end{cases} \quad (1.2)$$

where  $1 < p < q$ ,  $\varphi_p(s) = |s|^{p-2}s$  is the  $p$ -Laplacian operator,  $\frac{p}{2} \in \mathbb{N}^*$ ,  $\mathbb{N}^*$  denotes the positive integer set,  $a, b, c : \mathbb{Z} \rightarrow (0, +\infty)$ ,  $r \geq 1$ ,  $u : \mathbb{Z} \rightarrow \mathbb{R}$ , and  $\Delta u(n) = u(n+1) - u(n)$  is the forward difference operator. Note that the nonlinear term  $c(n)|u(n)|^{q-2}u(n) \ln |u(n)|^r$  does not satisfy the monotonicity condition in [5]. Therefore, the situation we studied is different from that in [5] even if  $p = 2$ . There exist two main difficulties in studying equation (1.2). One is that the associated functional  $I$  of equation (1.2) is not well defined in  $E$ , which is caused by the logarithmic nonlinearity, and the other is that the quasilinearity of the  $p$ -Laplacian operator makes it difficult and complex to establish energy inequalities.

For the first difficulty, we mainly use the idea in [2] to establish a well-defined space  $\mathcal{D}$ , thereby avoiding the case that  $\sum_{n \in \mathbb{Z}} c(n)|u(n)|^q \ln |u(n)|^r = -\infty$ . For the second difficulty, we use the binomial theorem and the combination number formula, and then by some careful calculations and analysis, establish some useful energy inequalities. We introduce the following assumptions:

- (C<sub>1</sub>) there exists a positive constant  $b_0$  such that  $b(n) \geq b_0$  for all  $n \in \mathbb{Z}$  and  $\lim_{|n| \rightarrow +\infty} b(n) = +\infty$ ;
- (C<sub>2</sub>) there is a positive constant  $c_0$  such that  $c(n) \leq c_0$  for all  $n \in \mathbb{Z}$  and  $\sum_{n \in \mathbb{Z}} c(n) < +\infty$ .

Next, we define

$$\begin{aligned} V &= \left\{ \{u(n)\}_{n \in \mathbb{Z}} : u(n) \in \mathbb{R}, n \in \mathbb{Z} \right\}, \\ E &= \left\{ u \in V : \sum_{n \in \mathbb{Z}} [a(n)|\Delta u(n)|^p + b(n)|u(n)|^p] < +\infty \right\} \end{aligned}$$

and

$$\|u\| := \left( \sum_{n \in \mathbb{Z}} [a(n)|\Delta u(n)|^p + b(n)|u(n)|^p] \right)^{\frac{1}{p}}. \quad (1.3)$$

Then,  $E$  is a reflexive Banach space. As usual, let  $1 < p < +\infty$  and define

$$l^p(\mathbb{Z}, \mathbb{R}) = \left\{ u \in V : \sum_{n \in \mathbb{Z}} |u(n)|^p < +\infty \right\}$$

with the norm

$$\|u\|_{l^p} = \left( \sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{\frac{1}{p}}.$$

When  $p = +\infty$ , we define

$$l^\infty(\mathbb{Z}, \mathbb{R}) = \left\{ u \in V : \sup_{n \in \mathbb{Z}} |u(n)| < +\infty \right\}$$

with

$$\|u\|_{l^\infty} = \sup_{n \in \mathbb{Z}} |u(n)|.$$

Note that equation (1.2) is formally related to the energy functional  $I : E \rightarrow \mathbb{R} \cup \{+\infty\}$  that is defined by

$$\begin{aligned} I(u) &= \frac{1}{p} \sum_{n \in \mathbb{Z}} [a(n)|\Delta u(n)|^p + b(n)|u(n)|^p] + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n)|u(n)|^q \\ &\quad - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n)|u(n)|^q \ln |u(n)|^r. \end{aligned}$$

However, the functional  $I$  is not well defined in  $E$  (see Appendix 1). We discuss the functional  $I$  on the set

$$\mathcal{D} = \left\{ u \in E : \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r < +\infty \right\},$$

that is,

$$I(u) = \frac{1}{p} \|u\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r, \quad \forall u \in \mathcal{D}. \quad (1.4)$$

Note that

$$\lim_{t \rightarrow 0} \frac{t^{q-1} \ln |t|^r}{t^{p-1}} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{t^{q-1} \ln |t|^r}{t^{\zeta-1}} = 0 \quad (1.5)$$

for all  $n \in \mathbb{Z}$ , where  $\zeta \in (q, +\infty)$ . Then, by  $(C_2)$ , for any given  $\varepsilon > 0$ , there exists a positive constant  $C_\varepsilon$  such that

$$\begin{aligned} c(n) |t|^{q-1} |\ln |t|^r| &\leq c(n) \varepsilon |t|^{p-1} + c(n) C_\varepsilon |t|^{\zeta-1} \\ &\leq c_0 \varepsilon |t|^{p-1} + c_0 C_\varepsilon |t|^{\zeta-1}, \quad \forall t \in \mathbb{R}, \forall n \in \mathbb{Z}. \end{aligned} \quad (1.6)$$

Then,  $\mathcal{D}$  is the closed subspace of  $E$ ,  $I \in C^1(\mathcal{D}, \mathbb{R})$  and

$$\begin{aligned} \langle I'(u), v \rangle &= \sum_{n \in \mathbb{Z}} [a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta v(n) + b(n) |u(n)|^{p-2} u(n) v(n)] \\ &\quad - \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) v(n) \ln |u(n)|^r, \quad \forall u, v \in \mathcal{D}. \end{aligned} \quad (1.7)$$

Using Abel's partial summation formula (also known as Abel's transformation) in [16] and the definition of  $\Delta u(n)$ , we have

$$\sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta v(n) = - \sum_{n \in \mathbb{Z}} \Delta(a(n-1) \varphi_p(\Delta u(n-1))) v(n),$$

which implies that

$$\begin{aligned} \langle I'(u), v \rangle &= \sum_{n \in \mathbb{Z}} [-\Delta(a(n-1) \varphi_p(\Delta u(n-1))) v(n) + b(n) |u(n)|^{p-2} u(n) v(n)] \\ &\quad - \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) v(n) \ln |u(n)|^r, \quad \forall u, v \in \mathcal{D}. \end{aligned}$$

According to the above equations, we can derive that  $\langle I'(u), v \rangle = 0$  for any  $v \in \mathcal{D}$  if and only if

$$-\Delta(a(n-1) \varphi_p(\Delta u(n-1))) + b(n) |u(n)|^{p-2} u(n) = c(n) |u(n)|^{q-2} u(n) \ln |u(n)|^r.$$

Therefore, it is easy to see that the critical points of  $I$  in  $\mathcal{D}$  are solutions of equation (1.2). Furthermore, if  $u \in \mathcal{D}$  is a solution of equation (1.2) and  $u^\pm \neq 0$ , then  $u$  is a sign-changing

solution of equation (1.2), where

$$u^+(n) := \max\{u(n), 0\} \quad \text{and} \quad u^-(n) := \min\{u(n), 0\}.$$

To be precise, we obtain the following results.

**Theorem 1.1** *Assume that  $(C_1)$  and  $(C_2)$  hold. Then, problem (1.2) has a sign-changing solution  $u_0 \in \mathcal{M}$  such that  $I(u_0) = \inf_{\mathcal{M}} I := m_* > 0$  and  $u_0$  only changes the sign once, where*

$$\mathcal{M} = \{u \in \mathcal{D} : u^\pm \neq 0, \langle I'(u), u^+ \rangle = 0 \text{ and } \langle I'(u), u^- \rangle = 0\}.$$

**Theorem 1.2** *Assume that  $(C_1)$  and  $(C_2)$  hold. Then, problem (1.2) has a solution  $\bar{u} \in \mathcal{N}$  such that  $I(\bar{u}) = \inf_{\mathcal{N}} I := c_* > 0$ , where*

$$\mathcal{N} = \{u \in \mathcal{D} : u \neq 0 \text{ and } \langle I'(u), u \rangle = 0\}.$$

In addition,  $m_* \geq 2c_*$ .

## 2 Preliminaries

In this section, we provide some lemmas that play some important roles in the proofs of our results.

**Lemma 2.1** *Assume that  $(C_1)$  holds. Then,  $\mathcal{D}$  is continuously embedded into  $l^\kappa(\mathbb{Z}, \mathbb{R})$  for any  $p \leq \kappa \leq +\infty$ , that is, for all  $u \in \mathcal{D}$ ,*

$$\|u\|_{l^\kappa} \leq b_0^{-\frac{1}{p}} \|u\|. \quad (2.1)$$

Moreover,  $\mathcal{D}$  is compactly embedded in  $l^\kappa(\mathbb{Z}, \mathbb{R})$  for any  $p \leq \kappa \leq +\infty$ .

*Proof* For any  $u \in \mathcal{D}$ , when  $\kappa = p$ , there holds

$$\|u\| \geq \left( \sum_{n \in \mathbb{Z}} b(n) |u(n)|^p \right)^{\frac{1}{p}} \geq b_0^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{\frac{1}{p}} = b_0^{\frac{1}{p}} \|u\|_p.$$

When  $\kappa = +\infty$ , we can also obtain that

$$\|u\| \geq \left( \sum_{n \in \mathbb{Z}} b(n) |u(n)|^p \right)^{\frac{1}{p}} \geq b_0^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{\frac{1}{p}} \geq b_0^{\frac{1}{p}} \left( \sup_{n \in \mathbb{Z}} |u(n)|^p \right)^{\frac{1}{p}} = b_0^{\frac{1}{p}} \|u\|_{l^\infty}. \quad (2.2)$$

For any  $p < \kappa < +\infty$ , it follows from (2.2) that

$$\begin{aligned} \|u\|_{l^\kappa}^\kappa &= \sum_{n \in \mathbb{Z}} |u(n)|^\kappa = \sum_{n \in \mathbb{Z}} |u(n)|^{\kappa-p} |u(n)|^p \\ &\leq \|u\|_{l^\infty}^{\kappa-p} \sum_{n \in \mathbb{Z}} |u(n)|^p = \|u\|_{l^\infty}^{\kappa-p} \sum_{n \in \mathbb{Z}} \frac{1}{b(n)} b(n) |u(n)|^p \end{aligned}$$

$$\leq b_0^{-\frac{\kappa-p}{p}} \|u\|^{\kappa-p} \frac{1}{b_0} \|u\|^p = b_0^{-\frac{\kappa}{p}} \|u\|^\kappa.$$

Hence, (2.1) holds.

Next, we prove that the embeddings are also compact. Suppose that  $\{u_k\}$  is a bounded sequence in  $\mathcal{D}$ . Then, there is a subsequence of  $\{u_k\}$ , still denoted by  $\{u_k\}$ , such that  $u_k \rightharpoonup u$  weakly in  $\mathcal{D}$  for some point  $u \in \mathcal{D}$ . In particular,

$$\lim_{k \rightarrow +\infty} \sum_{n \in \mathbb{Z}} u_k \varphi = \sum_{n \in \mathbb{Z}} u \varphi,$$

where  $\varphi \in \mathcal{D}$  is defined by

$$\varphi(m) = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

for any fixed  $n$ . Thus, we have

$$\lim_{k \rightarrow +\infty} u_k(n) = u(n) \quad \text{for any fixed } n \in \mathbb{Z}. \quad (2.3)$$

We now prove  $u_k \rightarrow u$  in  $l^\kappa(\mathbb{Z}, \mathbb{R})$  for all  $p \leq \kappa \leq +\infty$ . When  $\kappa = p$ , since  $u \in \mathcal{D}$ , according to the boundedness of  $\{u_k\}$  and the definition of  $\|\cdot\|$ , there appears a positive constant  $\delta_0$  such that

$$\sum_{n \in \mathbb{Z}} b(n) |u_k(n) - u(n)|^p \leq \delta_0.$$

For any given positive constant  $\varepsilon_1$ , there is a  $n_0 \in \mathbb{Z}$  such that  $\frac{1}{b(n)} < \varepsilon_1$  as  $|n| > n_0$ . Therefore, we can obtain that

$$\sum_{|n| > n_0} |u_k(n) - u(n)|^p = \sum_{|n| > n_0} \frac{1}{b(n)} b(n) |u_k(n) - u(n)|^p \leq \varepsilon_1 \delta_0. \quad (2.4)$$

On the other hand, (2.3) implies that  $\lim_{k \rightarrow +\infty} \sum_{|n| \leq n_0} |u_k(n) - u(n)|^p = 0$  since  $\{n \in \mathbb{Z} : |n| \leq n_0\}$  is a finite set. Then, according to the arbitrariness of  $\varepsilon_1$  and (2.4), we have

$$\lim_{k \rightarrow +\infty} \sum_{n \in \mathbb{Z}} |u_k(n) - u(n)|^p = 0. \quad (2.5)$$

For  $\kappa = +\infty$ , according to the definition of  $\|\cdot\|_{l^\infty}$  and (2.5), as  $k \rightarrow +\infty$ , we have

$$\|u_k - u\|_{l^\infty}^p \leq \sum_{n \in \mathbb{Z}} |u_k(n) - u(n)|^p \rightarrow 0 \quad (2.6)$$

and for  $p < \kappa < +\infty$ , by (2.5) and (2.6), there exists

$$\begin{aligned} \|u_k - u\|_{l^\kappa}^\kappa &= \sum_{n \in \mathbb{Z}} |u_k(n) - u(n)|^{\kappa-p} |u_k(n) - u(n)|^p \\ &\leq \|u_k - u\|_{l^\infty}^{\kappa-p} \sum_{n \in \mathbb{Z}} |u_k(n) - u(n)|^p \rightarrow 0, \quad \text{as } k \rightarrow +\infty. \end{aligned} \quad (2.7)$$

Consequently, by (2.5), (2.6), and (2.7), we can derive that  $u_k \rightarrow u$  in  $l^k(\mathbb{Z}, \mathbb{R})$  for all  $p \leq \kappa \leq +\infty$ .  $\square$

**Proposition 2.1** For all  $\frac{p}{2} \in \mathbb{N}^*$  and  $u \in \mathcal{D}$ , there hold

$$\begin{aligned} I(u) &= I(u^+) + I(u^-) + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right|, \\ \langle I'(u), u^+ \rangle &= \langle I'(u^+), u^+ \rangle + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \end{aligned}$$

and

$$\begin{aligned} \langle I'(u), u^- \rangle &= \langle I'(u^-), u^- \rangle + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right|. \end{aligned}$$

*Proof* Let

$$\mathbb{Z}_1 := \{n \in \mathbb{Z} : u(n) \geq 0\} \quad \text{and} \quad \mathbb{Z}_2 := \{n \in \mathbb{Z} : u(n) < 0\}.$$

Note that  $\Delta u^+(n) \Delta u^-(n) = -u^+(n+1)u^-(n) - u^+(n)u^-(n+1) \geq 0$ . Then, according to the definition of  $\|\cdot\|$ , Appendix 1 below, and the binomial theorem, we have

$$\begin{aligned} \|u\|^p &= \sum_{n \in \mathbb{Z}} [a(n)|\Delta u^+(n) + \Delta u^-(n)|^p + b(n)|u^+(n) + u^-(n)|^p] \\ &= \sum_{n \in \mathbb{Z}} [a(n)|(\Delta u^+(n))^2 + 2\Delta u^+(n)\Delta u^-(n) + (\Delta u^-(n))^2|^{\frac{p}{2}} + b(n)|u^+(n) + u^-(n)|^p] \\ &= \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=0}^{\frac{p}{2}} C_{\frac{p}{2}}^i (\Delta u^+(n))^{p-2i} \sum_{j=0}^i C_i^j (2\Delta u^+(n)\Delta u^-(n))^{i-j} (\Delta u^-(n))^{2j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} b(n)|u^+(n) + u^-(n)|^p \end{aligned}$$

$$\begin{aligned}
&= \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=0}^{\frac{p}{2}} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}_1} b(n) |u^+(n)|^p + \sum_{n \in \mathbb{Z}_2} b(n) |u^-(n)|^p \\
&= \sum_{n \in \mathbb{Z}} [a(n) |\Delta u^+(n)|^p + b(n) |u^+(n)|^p] \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} [a(n) |\Delta u^-(n)|^p + b(n) |u^-(n)|^p] \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
&= \|u^+\|^p + \|u^-\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right|. \tag{2.8}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{n \in \mathbb{Z}} [a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta u^+(n) + b(n) |u(n)|^{p-2} u(n) u^+(n)] \\
&= \sum_{n \in \mathbb{Z}} [a(n) |\Delta u^+(n) + \Delta u^-(n)|^{p-2} (\Delta u^+(n) + \Delta u^-(n)) \Delta u^+(n) + b(n) | \\
&\quad \times u(n)|^{p-2} u(n) u^+(n)] \\
&= \sum_{n \in \mathbb{Z}} [a(n) |(\Delta u^+(n))^2 + 2\Delta u^+(n) \Delta u^-(n) + (\Delta u^-(n))^2|^{\frac{p-2}{2}} (\Delta u^+(n))^2] \\
&\quad + \sum_{n \in \mathbb{Z}_1} b(n) |u^+(n)|^p \\
&\quad + \sum_{n \in \mathbb{Z}} [a(n) |(\Delta u^+(n))^2 + 2\Delta u^+(n) \Delta u^-(n) + (\Delta u^-(n))^2|^{\frac{p-2}{2}} \Delta u^-(n) \Delta u^+(n)] \\
&= \sum_{n \in \mathbb{Z}} \left[ a(n) \left| \sum_{i=0}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-2-(i+j)} (\Delta u^-(n))^{i+j} \right| (\Delta u^+(n))^2 \right] \\
&\quad + \sum_{n \in \mathbb{Z}_1} b(n) |u^+(n)|^p \\
&\quad + \sum_{n \in \mathbb{Z}} \left[ a(n) \left| \sum_{i=0}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-2-(i+j)} (\Delta u^-(n))^{i+j} \right| \Delta u^-(n) \Delta u^+(n) \right]
\end{aligned}$$

$$\begin{aligned}
&= \|u^+\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \tag{2.9}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n \in \mathbb{Z}} [a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta u^-(n) + b(n) |u(n)|^{p-2} u(n) u^-(n)] \\
&= \sum_{n \in \mathbb{Z}} \left[ a(n) \left| \sum_{i=0}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-2-(i+j)} (\Delta u^-(n))^{i+j} \right| (\Delta u^-(n))^2 \right] \\
&\quad + \sum_{n \in \mathbb{Z}_2} b(n) |u^-(n)|^p \\
&\quad + \sum_{n \in \mathbb{Z}} \left[ a(n) \left| \sum_{i=0}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-2-(i+j)} (\Delta u^-(n))^{i+j} \right| \Delta u^+(n) \Delta u^-(n) \right] \\
&= \|u^-\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=0}^{\frac{p}{2}-2} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j+2)} (\Delta u^-(n))^{i+j+2} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-2} C_{\frac{p}{2}-1}^j 2^{\frac{p}{2}-1-j} (\Delta u^+(n))^{p-(\frac{p}{2}-1+j+2)} (\Delta u^-(n))^{\frac{p}{2}-1+j+2} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=0}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-1-(i+j)} (\Delta u^-(n))^{i+j+1} \right| \\
&= \|u^-\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right|. \tag{2.10}
\end{aligned}$$

By (1.4), (1.7), (2.8), (2.9), and (2.10), it is easy to see that the conclusions hold.  $\square$

Next, we establish an inequality associated to  $I(u)$ ,  $I(su^+ + tu^-)$ ,  $\langle I'(u), u^+ \rangle$ , and  $\langle I'(u), u^- \rangle$ .

**Lemma 2.2** *Assume that (C<sub>1</sub>) and (C<sub>2</sub>) hold. For all  $u \in \mathcal{D}$  and  $s, t \geq 0$ , there exists*

$$\begin{aligned}
I(u) &\geq I(su^+ + tu^-) + \frac{1-s^q}{q} \langle I'(u), u^+ \rangle + \frac{1-t^q}{q} \langle I'(u), u^- \rangle + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u^+\|^p \\
&\quad + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p
\end{aligned}$$

$$+ \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \Theta, \quad (2.11)$$

$$\text{where } \Theta = \frac{2s^p C_{\frac{p}{2}-1}^i C_i^j + s^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2s^{p-(i+j)} t^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \geq 0.$$

*Proof* It is easy to see that (2.11) holds for  $u = 0$ . Next, we let  $u \neq 0$ . According to Appendix 1 in [17], there holds

$$r(1 - \tau^q) + q\tau^q \ln \tau^r > 0, \quad \forall \tau \in (0, 1) \cup (1, +\infty). \quad (2.12)$$

For  $u \in \mathcal{D} \setminus \{0\}$  and all  $s, t \geq 0$ , we have

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} |su^+(n) + tu^-(n)|^q \ln |su^+(n) + tu^-(n)|^r \\ &= \sum_{n \in \mathbb{Z}_1} |su^+(n) + tu^-(n)|^q \ln |su^+(n) + tu^-(n)|^r \\ & \quad + \sum_{n \in \mathbb{Z}_2} |su^+(n) + tu^-(n)|^q \ln |su^+(n) + tu^-(n)|^r \\ &= \sum_{n \in \mathbb{Z}_1} |su^+(n)|^q \ln |su^+(n)|^r + \sum_{n \in \mathbb{Z}_2} |tu^-(n)|^q \ln |tu^-(n)|^r \\ &= \sum_{n \in \mathbb{Z}} [|su^+(n)|^q (\ln |u^+(n)|^r + \ln s^r) + |tu^-(n)|^q (\ln |u^-(n)|^r + \ln t^r)]. \end{aligned} \quad (2.13)$$

By virtue of Appendix 2 in [17], the function  $f(x) = \frac{1-a^x}{x}$  is strictly monotonically decreasing on  $(0, +\infty)$  for  $a > 0$  and  $a \neq 1$ . Then, by (1.4), (2.8), (2.13), (2.9), (2.10), (1.7), (2.12), and Appendix 2 below, we can derive the following inequality

$$\begin{aligned} & I(u) - I(su^+ + tu^-) \\ &= \frac{1}{p} (\|u\|^p - \|su^+ + tu^-\|^p) + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) (|u(n)|^q - |su^+(n) + tu^-(n)|^q) \\ & \quad - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) (|u(n)|^q \ln |u(n)|^r - |su^+(n) + tu^-(n)|^q \ln |su^+(n) + tu^-(n)|^r) \\ &= \frac{1-s^p}{p} \|u^+\|^p + \frac{1-t^p}{p} \|u^-\|^p + \frac{r(1-s^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^+(n)|^q \\ & \quad + \frac{r(1-t^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^-(n)|^q \\ & \quad - \frac{1-s^q}{q} \sum_{n \in \mathbb{Z}} c(n) |u^+(n)|^q \ln |u^+(n)|^r - \frac{1-t^q}{q} \sum_{n \in \mathbb{Z}} c(n) |u^-(n)|^q \ln |u^-(n)|^r \\ & \quad + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |su^+(n)|^q \ln s^r + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |tu^-(n)|^q \ln t^r \\ & \quad + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta s u^+(n))^{p-(i+j)} (\Delta t u^-(n))^{i+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta s u^+(n))^{\frac{p}{2}-j} (\Delta t u^-(n))^{\frac{p}{2}+j} \right| \\
& = \frac{1-s^q}{q} \left[ \langle I'(u), u^+ \rangle - \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta u^+(n) - \sum_{n \in \mathbb{Z}} b(n) |u^+(n)|^p \right] \\
& + \frac{1-t^q}{q} \left[ \langle I'(u), u^- \rangle - \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta u^-(n) - \sum_{n \in \mathbb{Z}} b(n) |u^-(n)|^p \right] \\
& + \frac{1-s^p}{p} \|u^+\|^p + \frac{1-t^p}{p} \|u^-\|^p + \frac{r(1-s^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^+(n)|^q \\
& + \frac{r(1-t^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^-(n)|^q \\
& + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |s u^+(n)|^q \ln s^r + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |t u^-(n)|^q \ln t^r \\
& + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta s u^+(n))^{p-(i+j)} (\Delta t u^-(n))^{i+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta s u^+(n))^{\frac{p}{2}-j} (\Delta t u^-(n))^{\frac{p}{2}+j} \right| \\
& = \frac{1-s^q}{q} \langle I'(u), u^+ \rangle + \frac{1-t^q}{q} \langle I'(u), u^- \rangle + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u^+\|^p \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p \\
& + \frac{r(1-s^q) + q s^q \ln s^r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^+(n)|^q + \frac{r(1-t^q) + q t^q \ln t^r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u^-(n)|^q \\
& + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right|
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& - \frac{1-s^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& - \frac{1-s^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& - \frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& - \frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& - \frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& + \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta su^+(n))^{p-(i+j)} (\Delta tu^-(n))^{i+j} \right| \\
& - \frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta su^+(n))^{\frac{p}{2}-j} (\Delta tu^-(n))^{\frac{p}{2}+j} \right| \\
& \geq \frac{1-s^q}{q} \langle I'(u), u^+ \rangle + \frac{1-t^q}{q} \langle I'(u), u^- \rangle + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u^+\|^p \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p \\
& - \frac{1-s^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& - \frac{1-s^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right|
\end{aligned}$$

$$\begin{aligned}
& -\frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& -\frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& -\frac{1-t^p}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& +\frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\
& +\frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& -\frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta su^+(n))^{p-(i+j)} (\Delta tu^-(n))^{i+j} \right| \\
& -\frac{1}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta su^+(n))^{\frac{p}{2}-j} (\Delta tu^-(n))^{\frac{p}{2}+j} \right| \\
& = \frac{1-s^q}{q} \langle I'(u), u^+ \rangle + \frac{1-t^q}{q} \langle I'(u), u^- \rangle + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u^+\|^p \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p \\
& + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \Theta' \\
& + \frac{2C_{\frac{p}{2}}^i - 2C_{\frac{p}{2}-1}^i - C_{\frac{p}{2}-1}^{i-1} - C_{\frac{p}{2}-1}^{i-1}}{2p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} 2^i (\Delta u^+(n))^{p-i} (\Delta u^-(n))^i \right| \\
& + \frac{C_{\frac{p}{2}}^i - C_{\frac{p}{2}-1}^i - C_{\frac{p}{2}-1}^{i-1}}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} (\Delta u^+(n))^{p-2i} (\Delta u^-(n))^{2i} \right| \\
& + \frac{2C_{\frac{p}{2}}^j - C_{\frac{p}{2}-1}^j - 2C_{\frac{p}{2}-1}^{j-1} - C_{\frac{p}{2}-1}^j}{2p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \Theta \\
& + \frac{2s^p C_{\frac{p}{2}-1}^i + s^p C_{\frac{p}{2}-1}^{i-1} + t^p C_{\frac{p}{2}-1}^{i-1} - 2s^{p-i} t^i C_{\frac{p}{2}}^i}{2p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} 2^i (\Delta u^+(n))^{p-i} (\Delta u^-(n))^i \right| \\
& + \frac{s^p C_{\frac{p}{2}-1}^i + t^p C_{\frac{p}{2}-1}^{i-1} - s^{p-2i} t^{2i} C_{\frac{p}{2}}^i}{p} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} (\Delta u^+(n))^{p-2i} (\Delta u^-(n))^{2i} \right|
\end{aligned}$$

$$\begin{aligned}
& + \frac{s^p C_{\frac{p}{2}-1}^j + 2t^p C_{\frac{p}{2}-1}^{j-1} + t^p C_{\frac{p}{2}-1}^j - 2s^{\frac{p}{2}-j} t^{\frac{p}{2}+j} C_{\frac{p}{2}}^j}{2p} \sum_{n \in \mathbb{Z}} a(n) \\
& \times \left| \sum_{j=0}^{\frac{p}{2}-1} 2^{\frac{p}{2}-j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\
& \geq \frac{1-s^q}{q} \langle I'(u), u^+ \rangle + \frac{1-t^q}{q} \langle I'(u), u^- \rangle + \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u^+\|^p \\
& + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p \\
& + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \Theta, \tag{2.14}
\end{aligned}$$

where  $\Theta' = \frac{2C_p^i C_i^j - 2C_{\frac{p}{2}-1}^i C_i^{j-1} - C_{\frac{p}{2}-1}^j C_{i-1}^{j-1} - 2C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} - C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j}{2p} = 0$  (use the combination number formula) and  $\Theta \geq 0$  (see Appendix 2). Hence, we obtain that (2.11) holds for all  $u \in \mathcal{D}$ ,  $s, t \geq 0$ .  $\square$

**Remark 2.1** Let  $s = t$  in (2.11). It is easy to see that  $\Theta = 0$ . Then, for all  $u \in \mathcal{D}$  and  $t \geq 0$ , there holds

$$I(u) \geq I(tu) + \frac{1-t^q}{q} \langle I'(u), u \rangle + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^+\|^p + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u^-\|^p.$$

**Corollary 2.3** Assume that (C<sub>1</sub>) and (C<sub>2</sub>) hold. For all  $u \in \mathcal{D}$  and  $t \geq 0$ , we have

$$I(u) \geq I(tu) + \frac{1-t^q}{q} \langle I'(u), u \rangle + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u\|^p. \tag{2.15}$$

*Proof* According to (1.4), (1.7), and (2.12), there exists

$$\begin{aligned}
& I(u) - I(tu) \\
& = \frac{1}{p} (\|u\|^p - \|tu\|^p) + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) (|u(n)|^q - |tu(n)|^q) \\
& - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) (|u(n)|^q \ln |u(n)|^r - |tu(n)|^q \ln |tu(n)|^r) \\
& = \frac{1-t^p}{p} \|u\|^p + \frac{r(1-t^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q - \frac{1-t^q}{q} \sum_{n \in \mathbb{Z}} c(n) |u|^q \ln |u|^r \\
& + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |tu|^q \ln t^r \\
& = \frac{1-t^q}{q} (\langle I'(u), u \rangle - \|u\|^p) + \frac{1-t^p}{p} \|u\|^p + \frac{r(1-t^q)}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \\
& + \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |tu|^q \ln t^r
\end{aligned}$$

$$\begin{aligned}
&= \frac{1-t^q}{q} \langle I'(u), u \rangle + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u\|^p + \frac{r(1-t^q) + qt^q \ln t^r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \\
&\geq \frac{1-t^q}{q} \langle I'(u), u \rangle + \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u\|^p.
\end{aligned} \tag{2.16}$$

Hence, (2.15) holds for all  $u \in \mathcal{D}$  and  $t \geq 0$ .  $\square$

Note that  $1 < p < q$ ,  $\Theta \geq 0$  and the function  $f(x) = \frac{1-a^x}{x}$  is strictly monotonically decreasing on  $(0, +\infty)$  for  $a > 0$  and  $a \neq 1$ . Then, in combination with Lemma 2.2, we have the following corollary.

**Corollary 2.4** Assume that  $(C_1)$  and  $(C_2)$  hold. For any  $u \in \mathcal{M}$ , we can obtain that  $I(u) = \max_{s,t \geq 0} I(su^+ + tu^-)$ .

In combination with Corollary 2.3 or Remark 2.1, we have the following corollary.

**Corollary 2.5** Assume that  $(C_1)$  and  $(C_2)$  hold. For any  $u \in \mathcal{N}$ , there holds  $I(u) = \max_{t \geq 0} I(tu)$ .

**Lemma 2.6** Assume that  $(C_1)$  and  $(C_2)$  hold. For any  $u \in \mathcal{D}$  with  $u \neq 0$ , there exists a unique positive constant  $t_0$  such that  $t_0 u \in \mathcal{N}$ .

*Proof* First, we prove the existence of  $t_0$ . For any  $u \in \mathcal{D}$  with  $u \neq 0$ , let  $u \in \mathcal{N}$  be fixed and define a function  $g(t) = \langle I'(tu), tu \rangle$  on  $(0, +\infty)$ . On the one hand, by (1.6) and Lemma 2.1, there exist two positive constants  $\varepsilon_2 < \frac{b_0}{c_0}$  and  $C_{\varepsilon_2}$  such that

$$\begin{aligned}
g(t) &= t^p \|u\|^p - \sum_{n \in \mathbb{Z}} c(n) |tu(n)|^q \ln |tu(n)|^r \\
&\geq t^p \|u\|^p - \sum_{n \in \mathbb{Z}} c_0 \varepsilon_2 |tu(n)|^p - \sum_{n \in \mathbb{Z}} c_0 C_{\varepsilon_2} |tu(n)|^\zeta \\
&\geq t^p \|u\|^p - b_0^{-1} c_0 \varepsilon_2 t^p \|u\|^p - t^\zeta \sum_{n \in \mathbb{Z}} c_0 C_{\varepsilon_2} |u(n)|^\zeta.
\end{aligned} \tag{2.17}$$

Then, according to  $\zeta > q$  and  $q > p > 1$ , we have that  $g(t) > 0$  for all sufficiently small  $t > 0$ .

On the other hand, noting that  $c(n) > 0$  for all  $n \in \mathbb{Z}$ , by  $(C_2)$  and (1.6), there exists

$$\begin{aligned}
g(t) &= t^p \|u\|^p - t^q \ln t^r \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q - t^q \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r \\
&\leq t^p \|u\|^p - t^q \ln t^r \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q + t^q \sum_{n \in \mathbb{Z}} c_0 (\varepsilon |u(n)|^p + C_\varepsilon |u(n)|^\zeta).
\end{aligned} \tag{2.18}$$

Then, by  $1 < p < q$ ,  $r \geq 1$  and (2.18), it is easy to see  $g(t) < 0$  for all large  $t$ . Hence, it follows from the continuity of  $g(t)$  that there exists a  $t_0 \in (0, +\infty)$  such that  $g(t_0) = 0$ , which implies that there exists a positive constant  $t_0$  such that  $t_0 u \in \mathcal{N}$ .

Next, we prove the uniqueness of  $t_0$ . Proofing by contradiction, we assume that there exist  $u \in \mathcal{D}$  and two positive numbers  $t_1 \neq t_2$  such that  $t_1 u \in \mathcal{N}$  and  $t_2 u \in \mathcal{N}$ . Note that the function  $f(x) = \frac{1-a^x}{x}$  is strictly monotonically decreasing on  $(0, +\infty)$  for  $a > 0$  and  $a \neq 1$ .

Taking  $u$  as  $t_1 u$  and  $t$  as  $\frac{t_2}{t_1}$  in Corollary 2.3, there holds

$$I(t_1 u) \geq I(t_2 u) + t_1^p \left( \frac{1 - (\frac{t_2}{t_1})^p}{p} - \frac{1 - (\frac{t_2}{t_1})^q}{q} \right) \|u\|^p > I(t_2 u). \quad (2.19)$$

On the other hand, taking  $u$  as  $t_2 u$  and  $t$  as  $\frac{t_1}{t_2}$  in Corollary 2.3, there also holds

$$I(t_2 u) \geq I(t_1 u) + t_2^p \left( \frac{1 - (\frac{t_1}{t_2})^p}{p} - \frac{1 - (\frac{t_1}{t_2})^q}{q} \right) \|u\|^p > I(t_1 u). \quad (2.20)$$

Hence, (2.19) contradicts (2.20). Hence,  $t_1 = t_2$ , that is, there exists a unique positive constant  $t_0$  such that  $t_0 u \in \mathcal{N}$ .  $\square$

**Lemma 2.7** *Assume that (C<sub>1</sub>) and (C<sub>2</sub>) hold. For any  $u \in \mathcal{D}$  with  $u^\pm \neq 0$ , there exists a unique pair of positive constants  $(s_0, t_0)$  such that  $s_0 u^+ + t_0 u^- \in \mathcal{M}$ .*

*Proof* First, we prove the existence of  $(s_0, t_0)$ . For any  $u \in \mathcal{D}$  with  $u^\pm \neq 0$ , according to (2.9) and (2.10), we have

$$\begin{aligned} h_1(s, t) &:= \langle I'(su^+ + tu^-), su^+ \rangle \\ &= s^p \|u^+\|^p - \sum_{n \in \mathbb{Z}} c(n) |su^+(n)|^q \ln |su^+(n)|^r \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} s^{p-(i+j)} t^{i+j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} s^{p-(i+j)} t^{i+j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} h_2(s, t) &:= \langle I'(su^+ + tu^-), tu^- \rangle \\ &= t^p \|u^-\|^p - \sum_{n \in \mathbb{Z}} c(n) |tu^-(n)|^q \ln |tu^-(n)|^r \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} s^{p-(i+j)} t^{i+j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} s^{\frac{p}{2}-j} t^{\frac{p}{2}+j} (\Delta u^+(n))^{\frac{p}{2}-j} (\Delta u^-(n))^{\frac{p}{2}+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} s^{p-(i+j)} t^{i+j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right|. \end{aligned} \quad (2.22)$$

It follows from (2.17) and (2.18) that  $h_1(s, s) > 0$  and  $h_2(s, s) > 0$  for  $s > 0$  sufficiently small and  $h_1(t, t) < 0$  and  $h_2(t, t) < 0$  for  $t > 0$  large enough. Thus, there are two constants  $0 < \theta_1 <$

$\theta_2$  such that

$$h_1(\theta_1, \theta_1) > 0, \quad h_2(\theta_1, \theta_1) > 0, \quad h_1(\theta_2, \theta_2) < 0, \quad h_2(\theta_2, \theta_2) < 0. \quad (2.23)$$

For all  $s, t \in [\theta_1, \theta_2]$ , according to (2.21), (2.22), and (2.23), there exists

$$\begin{aligned} h_1(\theta_1, t) &\geq \theta_1^p \|u^+\|^p - \sum_{n \in \mathbb{Z}} c(n) |\theta_1 u^+(n)|^q \ln |\theta_1 u^+(n)|^r \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} \theta_1^p (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} \theta_1^p (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| \\ &= h_1(\theta_1, \theta_1) \\ &> 0 \end{aligned} \quad (2.24)$$

and similarly, we can obtain that

$$\begin{aligned} h_1(\theta_2, t) &\leq h_1(\theta_2, \theta_2) < 0, \quad h_2(s, \theta_1) \geq h_2(\theta_1, \theta_1) > 0 \quad \text{and} \\ h_2(s, \theta_2) &\leq h_2(\theta_2, \theta_2) < 0. \end{aligned} \quad (2.25)$$

Therefore, by virtue of (2.24), (2.25), and the Pincaré–Miranda Theorem [12], there appears a point  $(s_0, t_0)$  with  $\theta_1 < s_0, t_0 < \theta_2$  such that  $h_1(s_0, t_0) = h_2(s_0, t_0) = 0$ , that is, there exist a pair of positive constants  $(s_0, t_0)$  such that  $s_0 u^+ + t_0 u^- \in \mathcal{M}$ .

Next, we prove the uniqueness of  $(s_0, t_0)$ . Proofing by contradiction, we suppose that there are two unequal pairs of positive constants  $(s_1, t_1)$  and  $(s_2, t_2)$  such that  $s_1 u^+ + t_1 u^- \in \mathcal{M}$  and  $s_2 u^+ + t_2 u^- \in \mathcal{M}$ . Note that the function  $f(x) = \frac{1-a^x}{x}$  is strictly monotonically decreasing on  $(0, +\infty)$  for  $a > 0$  and  $a \neq 1$ . Hence, taking  $u, s$ , and  $t$  as  $s_1 u^+ + t_1 u^-$ ,  $\frac{s_2}{s_1}$ , and  $\frac{t_2}{t_1}$  in Lemma 2.2, respectively, and noting that  $p < q$ , then we have

$$\begin{aligned} I(s_1 u^+ + t_1 u^-) &\geq I(s_2 u^+ + t_2 u^-) + s_1^p \left( \frac{1 - (\frac{s_2}{s_1})^p}{p} - \frac{1 - (\frac{s_2}{s_1})^q}{q} \right) \|u^+\|^p \\ &\quad + t_1^p \left( \frac{1 - (\frac{t_2}{t_1})^p}{p} - \frac{1 - (\frac{t_2}{t_1})^q}{q} \right) \|u^-\|^p \\ &\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u^+(n))^{p-(i+j)} (\Delta u^-(n))^{i+j} \right| s_1^{p-i+j} t_1^{i+j} \Theta'' \\ &> I(s_2 u^+ + t_2 u^-), \end{aligned} \quad (2.26)$$

where  $\Theta'' = \frac{2(\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^i C_i^j + (\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2(\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + (\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2(\frac{s_2}{s_1})^{p-(i+j)} (\frac{t_2}{t_1})^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \geq 0$  (see Appendix 3). Also, taking  $u, s$ , and  $t$  as  $s_2 u^+ + t_2 u^-$ ,  $\frac{s_1}{s_2}$ , and  $\frac{t_1}{t_2}$ , respectively, we have

$$I(s_2 u^+ + t_2 u^-) > I(s_1 u^+ + t_1 u^-). \quad (2.27)$$

As a consequence, there is a contradiction between (2.26) and (2.27). Hence,  $(s_1, t_1) = (s_2, t_2)$  that implies that there is a unique pair of positive constants  $(s_0, t_0)$  such that  $s_0 u^+ + t_0 u^- \in \mathcal{M}$ .  $\square$

**Lemma 2.8** *Assume that  $(C_1)$  and  $(C_2)$  hold. Then,*

$$\inf_{u \in \mathcal{N}} I(u) =: c_* = \inf_{u \in \mathcal{D}, u \neq 0} \max_{t \geq 0} I(tu) \quad (2.28)$$

and

$$\inf_{u \in \mathcal{M}} I(u) =: m_* = \inf_{u \in \mathcal{D}, u^\pm \neq 0} \max_{s, t \geq 0} I(su^+ + tu^-). \quad (2.29)$$

*Proof* On the one hand, according to Corollary 2.4 and the definition of  $\mathcal{M}$ , there holds

$$\inf_{u \in \mathcal{M}} I(u) = \inf_{u \in \mathcal{M}} \max_{s, t \geq 0} I(su^+ + tu^-) \geq \inf_{u \in \mathcal{D}, u^\pm \neq 0} \max_{s, t \geq 0} I(su^+ + tu^-).$$

On the other hand, for any  $u \in \mathcal{D}$  with  $u^\pm \neq 0$ , by virtue of Lemma 2.7 there appear two positive constants  $s_0, t_0$  such that  $s_0 u^+ + t_0 u^- \in \mathcal{M}$ . Then, we have

$$\max_{s, t \geq 0} I(su^+ + tu^-) \geq I(s_0 u^+ + t_0 u^-) \geq \inf_{u \in \mathcal{M}} I(u),$$

which implies that

$$\inf_{u \in \mathcal{D}, u^\pm \neq 0} \max_{s, t \geq 0} I(su^+ + tu^-) \geq \inf_{u \in \mathcal{D}, u^\pm \neq 0} I(s_0 u^+ + t_0 u^-) \geq \inf_{u \in \mathcal{M}} I(u).$$

Hence, it is easy to see that the conclusion (2.29) holds. Similarly, it follows from Corollary 2.5, the definition of  $\mathcal{N}$ , and Lemma 2.6 that (2.28) also holds.  $\square$

**Lemma 2.9** *Assume that  $(C_1)$  and  $(C_2)$  hold. Then,  $m_* > 0$  and  $c_* > 0$  can be achieved.*

*Proof* For any  $u \in \mathcal{M}$ , there holds  $\langle I'(u), u \rangle = 0$ . For  $\varepsilon_3 = \frac{b_0}{pc_0} > 0$ , by (1.6), (1.7), and Lemma 2.1, there is a positive constant  $C_{\varepsilon_3}$  such that

$$\|u\|^p = \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r \leq c_0 \varepsilon_3 \|u\|_{l^p}^p + c_0 C_{\varepsilon_3} \|u\|_{l^{\zeta}}^{\zeta} \leq \frac{1}{p} \|u\|^p + c_0 C_{\varepsilon_3} b_0^{-\frac{\zeta}{p}} \|u\|^{\zeta}.$$

Since  $1 < p < q < \zeta$ , then  $\|u\| \geq \rho := (\frac{(p-1)b_0^{\frac{\zeta}{p}}}{pc_0 C_{\varepsilon_3}})^{\zeta-p}$  for any  $u \in \mathcal{M}$ .

Let  $\{u_k\} \subset \mathcal{M}$  be such that  $I(u_k) \rightarrow m_*$ . By (1.4) and (1.7), there holds

$$\begin{aligned} m_* + o(1) &= I(u_k) - \frac{1}{q} \langle I'(u_k), u_k \rangle \\ &= \left( \frac{1}{p} - \frac{1}{q} \right) \|u_k\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q \geq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_k\|^p. \end{aligned}$$

This shows that the sequence  $\{u_k\}$  is bounded in  $\mathcal{D}$ , that is, there exists a  $M_1 > 0$  such that  $\|u_k\| \leq M_1$ . Thus, there appears a  $u_0 \in \mathcal{D}$  such that  $u_k^\pm \rightharpoonup u_0^\pm$  in  $\mathcal{D}$ . Then, according to

Lemma 2.1, we can obtain that  $u_k^\pm \rightarrow u_0^\pm$  in  $l^\kappa(\mathbb{Z}, \mathbb{R})$  for  $\kappa \in [p, +\infty]$  and  $u_k^\pm(n) \rightarrow u_0^\pm(n)$  for all  $n \in \mathbb{Z}$ .

Since  $\{u_k\} \subset \mathcal{M}$ , there exists  $\langle I'(u_k), u_k^\pm \rangle = 0$  and then by Proposition 2.1, we have

$$\begin{aligned} & \|u_k^+\|^p - \sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q \ln |u_k^+(n)|^r \\ &= - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} & \|u_k^-\|^p - \sum_{n \in \mathbb{Z}} c(n) |u_k^-(n)|^q \ln |u_k^-(n)|^r \\ &= - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u_k^+(n))^{\frac{p}{2}-j} (\Delta u_k^-(n))^{\frac{p}{2}+j} \right| \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right|. \end{aligned} \quad (2.31)$$

It follows from (1.6), (2.30), Lemma 2.1, and the boundedness of  $\{u_k\}$  that there exists  $\varepsilon_4 \in (0, \frac{b_0 \rho p}{c_0 M_1^p})$  and a positive constant  $C_{\varepsilon_4}$  such that

$$\begin{aligned} & \rho^p \leq \|u_k^+\|^p \\ &= \sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q \ln |u_k^+(n)|^r \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ &\leq \sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q \ln |u_k^+(n)|^r \\ &\leq c_0 \varepsilon_4 \|u_k^+\|_{l^\kappa}^p + c_0 C_{\varepsilon_4} \|u_k^+\|_{l^\kappa}^\zeta \\ &\leq c_0 \varepsilon_4 b_0^{-1} \|u_k^+\|^p + c_0 C_{\varepsilon_4} \|u_k^+\|_{l^\kappa}^\zeta \\ &\leq c_0 \varepsilon_4 b_0^{-1} M_1^p + c_0 C_{\varepsilon_4} \|u_k^+\|_{l^\kappa}^\zeta, \end{aligned}$$

which implies that  $\|u_k^+\|_{l^\zeta}^\zeta \geq \frac{\rho^p - c_0 \varepsilon_4 b_0^{-1} M_1^p}{c_0 C_{\varepsilon_4}} > 0$ . Similarly, by (1.6), (2.31), Lemma 2.1, and the boundedness of  $\{u_k\}$ , there exists  $\varepsilon_5 \in (0, \frac{b_0 \rho^p}{c_0 M_1^p})$  and a positive constant  $C_{\varepsilon_5}$  such that  $\|u_k^-\|_{l^\zeta}^\zeta \geq \frac{\rho^p - c_0 \varepsilon_5 b_0^{-1} M_1^p}{c_0 C_{\varepsilon_5}} > 0$ . Then, let  $\varepsilon' := \max\{\varepsilon_4, \varepsilon_5\}$  and  $C_{\varepsilon'} := \max\{C_{\varepsilon_4}, C_{\varepsilon_5}\}$ , we have that  $\|u_k^\pm\|_{l^\zeta}^\zeta \geq \frac{\rho^p - c_0 \varepsilon' b_0^{-1} M_1^p}{c_0 C_{\varepsilon'}} > 0$ . For any  $p \leq \kappa \leq +\infty$ , by virtue of the compactness of the embedding  $\mathcal{D} \hookrightarrow l^\kappa(\mathbb{Z}, \mathbb{R})$  and

$$\|u_0^\pm\|_{l^\zeta}^\zeta \geq \frac{\rho^p - c_0 \varepsilon' b_0^{-1} M_1^p}{c_0 C_{\varepsilon'}} > 0,$$

which implies that  $u_0^\pm \neq 0$ . Note that  $\Delta u^\pm(n) = u^\pm(n+1) - u^\pm(n)$ . By the fact that  $u_k^\pm(n) \rightarrow u_0^\pm(n)$  for all  $n \in \mathbb{Z}$ , we can derive that

$$|\Delta u_k^\pm(n) - \Delta u_0^\pm(n)| = |u_k^\pm(n+1) - u_0^\pm(n+1) - (u_k^\pm(n) - u_0^\pm(n))| \rightarrow 0,$$

which implies that  $\Delta u_k^\pm(n) \rightarrow \Delta u_0^\pm(n)$  for all  $n \in \mathbb{Z}$ . Note that

$$|u_k^\pm(n)| \leq \|u_k\|_{l^\infty} \leq b_0^{-\frac{1}{p}} \|u\| \leq b_0^{-\frac{1}{p}} M_1, \quad \text{for all } n \in \mathbb{Z}. \quad (2.32)$$

Also, by (1.6), for any given  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that

$$|t|^q |\ln t^r| \leq \varepsilon |t|^p + C_\varepsilon |t|^\zeta, \quad \forall t \in \mathbb{R}. \quad (2.33)$$

Then, we can obtain that  $|u_k^+(n)|^q |\ln |u_k^+(n)|^r| \leq \varepsilon |u_k^+(n)|^p + C_\varepsilon |u_k^+(n)|^\zeta \leq \varepsilon b_0^{-1} M_1^p + C_\varepsilon b_0^{-\frac{\zeta}{p}} M_1^\zeta$ , which implies that

$$\sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q |\ln |u_k^+(n)|^r| \leq (\varepsilon b_0^{-1} M_1^p + C_\varepsilon b_0^{-\frac{\zeta}{p}} M_1^\zeta) \sum_{n \in \mathbb{Z}} c(n).$$

Note that  $\sum_{n \in \mathbb{Z}} c(n) < \infty$  (by  $(C_2)$ ). Thus, it follows from (2.30), the weak lower semicontinuity of norm, Fatou's Lemma, and the Lebesgue dominated convergence theorem that

$$\begin{aligned} & \langle I'(u_0), u_0^+ \rangle \\ & \leq \liminf_{k \rightarrow \infty} \|u_k^+\|^p + \liminf_{k \rightarrow \infty} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ & \quad + \liminf_{k \rightarrow \infty} \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \\ & \quad - \liminf_{k \rightarrow \infty} \sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q |\ln |u_k^+(n)|^r| \\ & \leq \liminf_{k \rightarrow \infty} \left[ \|u_k^+\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \right. \\ & \quad \left. + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_k^+(n))^{p-(i+j)} (\Delta u_k^-(n))^{i+j} \right| \right] \end{aligned}$$

$$\begin{aligned} & - \sum_{n \in \mathbb{Z}} c(n) |u_k^+(n)|^q \ln |u_k^+(n)|^r \Big] \\ & = \liminf_{k \rightarrow \infty} \langle I'(u_k), u_k^+ \rangle = 0, \end{aligned}$$

which implies that

$$\langle I'(u_0), u_0^+ \rangle \leq 0. \quad (2.34)$$

Similarly, by (2.31), the weak lower semicontinuity of norm, Fatou's Lemma, and the Lebesgue dominated convergence theorem, there exists

$$\langle I'(u_0), u_0^- \rangle \leq 0 \quad \text{and then} \quad \langle I'(u_0), u_0 \rangle \leq 0. \quad (2.35)$$

According to Lemma 2.7, there are two positive constants  $s_3, t_3$  such that

$$s_3 u_0^+ + t_3 u_0^- \in \mathcal{M} \quad \text{and} \quad I(s_3 u_0^+ + t_3 u_0^-) \geq m_*. \quad (2.36)$$

By (1.4), (1.7), the weak lower semicontinuity of norm, (C<sub>2</sub>), (2.32), Lemma 2.2, (2.34), (2.35), and (2.36), there exists

$$\begin{aligned} m_* &= \lim_{k \rightarrow \infty} \left[ I(u_k) - \frac{1}{q} \langle I'(u_k), u_k \rangle \right] \\ &\geq \liminf_{k \rightarrow \infty} \left[ \left( \frac{1}{p} - \frac{1}{q} \right) \|u_k\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q \right] \\ &\geq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_0\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_0(n)|^q \\ &= I(u_0) - \frac{1}{q} \langle I'(u_0), u_0 \rangle \\ &\geq I(s_3 u_0^+ + t_3 u_0^-) + \frac{1-s_3^q}{q} \langle I'(u_0), u_0^+ \rangle + \frac{1-t_3^q}{q} \langle I'(u_0), u_0^- \rangle - \frac{1}{q} \langle I'(u_0), u_0 \rangle \\ &\geq m_* - \frac{s_3^q}{q} \langle I'(u_0), u_0^+ \rangle - \frac{t_3^q}{q} \langle I'(u_0), u_0^- \rangle \\ &\geq m_*. \end{aligned} \quad (2.37)$$

Moreover, in combination (2.37) with (2.35), we can obtain that

$$m_* - \frac{s_3^q}{q} \langle I'(u_0), u_0^+ \rangle \leq \frac{t_3^q}{q} \langle I'(u_0), u_0^- \rangle + m_* \leq m_*,$$

which implies that  $\langle I'(u_0), u_0^+ \rangle \geq 0$ . Similarly, we can also obtain that  $\langle I'(u_0), u_0^- \rangle \geq 0$ . Then, by (2.34) and (2.35), we have  $\langle I'(u_0), u_0^\pm \rangle = 0$  and then  $\langle I'(u_0), u_0 \rangle = 0$ . Furthermore, according to (2.37), we can obtain that  $I(u_0) = m_*$  and  $u_0 \in \mathcal{M}$ . Note that  $u_0^+ \neq 0$ . If we let  $s_3 = 0$  and  $t_3 = 0$  in (2.11), then we have

$$m_* = I(u_0) \geq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_0^+\|^p + \left( \frac{1}{p} - \frac{1}{q} \right) \|u_0^-\|^p > 0.$$

Through arguments similar to the above, we can also conclude that  $c_* > 0$  can be achieved.  $\square$

**Lemma 2.10** *Assume that  $(C_1)$  and  $(C_2)$  hold. If  $u_0 \in \mathcal{M}$  and  $I(u_0) = m_*$ , then  $u_0$  is a critical point of  $I$ .*

*Proof* Arguing by contradiction. If we suppose that  $I'(u_0) \neq 0$  for all  $u_0 \in \mathcal{D}$ , then there are two positive constants  $\delta$  and  $\vartheta$  such that

$$\|I'(u)\| \geq \vartheta, \quad \forall \|u - u_0\| \leq 3\delta.$$

Since  $u_0 \in \mathcal{M}$ , we have  $\langle I'(u_0), u_0^\pm \rangle = 0$ , and by Lemma 2.2, for all  $s, t \geq 0$ , there exists

$$\begin{aligned} I(su_0^+ + tu_0^-) &\leq I(u_0) - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \Theta \\ &\quad - \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u_0^+\|^p - \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u_0^-\|^p \\ &= m_* - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^{i-1} 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \Theta \\ &\quad - \left( \frac{1-s^p}{p} - \frac{1-s^q}{q} \right) \|u_0^+\|^p - \left( \frac{1-t^p}{p} - \frac{1-t^q}{q} \right) \|u_0^-\|^p. \end{aligned} \tag{2.38}$$

Let  $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ . It follows from (2.38) that

$$\mathcal{Y} := \max_{(s,t) \in \partial D} I(su_0^+ + tu_0^-) < m_*. \tag{2.39}$$

For  $\epsilon := \min\{\frac{m_* - \mathcal{Y}}{3}, \frac{\vartheta \delta}{8}\}$  and  $S_\delta := B(u_0, \delta)$ , by [22], we can obtain a deformation  $\eta \in \mathcal{C}([0, 1] \times \mathcal{D}, \mathcal{D})$  such that

- (i)  $\eta(1, u) = u$  if  $|I(u) - m_*| > 2\epsilon$ ;
- (ii)  $\eta(1, I^{m_* + \epsilon} \cap S_\delta) \subset I^{m_* - \epsilon}$ , where  $I^c := \{u \in \mathcal{D} : I(u) \leq c\}$ ;
- (iii)  $I(\eta(1, u)) \leq I(u)$ ,  $\forall u \in \mathcal{D}$ ;
- (iv)  $\eta(1, u)$  is a homeomorphism of  $\mathcal{D}$ .

By virtue of (2.38), (iii), and for all  $s, t \geq 0$ , which makes  $|s-1|^2 + |t-1|^2 \geq \delta^2/\|u_0\|^2$  hold, there exists

$$I(\eta(1, su_0^+ + tu_0^-)) \leq I(su_0^+ + tu_0^-) < I(u_0) = m_*. \tag{2.40}$$

Also, by Corollary 2.4, for all  $s, t \geq 0$ , we have that  $I(su_0^+ + tu_0^-) \leq I(u_0) = m_*$ . Then, by (ii), we have

$$I(\eta(1, su_0^+ + tu_0^-)) \leq m_* - \epsilon, \quad \text{for } s, t \geq 0, |s-1|^2 + |t-1|^2 < \delta^2/\|u_0\|^2. \tag{2.41}$$

By virtue of (2.39), (2.40), and (2.41), we have

$$\max_{(s,t) \in \bar{D}} I(\eta(1, su_0^+ + tu_0^-)) < m_*. \tag{2.42}$$

Define  $k(s, t) = su_0^+ + tu_0^-$ . Next, we prove that  $\eta(1, k(D)) \cap \mathcal{M} \neq \emptyset$ . Set  $\gamma(s, t) := \eta(1, k(s, t))$ ,

$$\begin{aligned}\phi_1(s, t) &:= (\langle I'(k(s, t)), u_0^+ \rangle, \langle I'(k(s, t)), u_0^- \rangle) \\ &:= (y_1(s, t), y_2(s, t))\end{aligned}$$

and

$$\phi_2(s, t) := \left( \frac{1}{s} \langle I'(\gamma(s, t)), (\gamma(s, t))^+ \rangle, \frac{1}{t} \langle I'(\gamma(s, t)), (\gamma(s, t))^- \rangle \right).$$

Note that  $\phi_1(s, t)$  and  $\phi_2(s, t)$  are two-dimensional vectors. According to (2.21) and (2.22), it is obvious that  $y_1(s, t) = \frac{1}{s} h_1(s, t)$  and  $y_2(s, t) = \frac{1}{t} h_2(s, t)$ . Hence,  $\phi_1(s, t)$  is a  $C^1$  function of  $s, t$  and we have

$$\begin{aligned}&\left. \frac{\partial y_1(s, t)}{\partial s} \right|_{(1,1)} \\ &= (p-1) \|u_0^+\|^p - \sum_{n \in \mathbb{Z}} (q-1)c(n) |u_0^+(n)|^q \ln |u_0^+(n)|^r - \sum_{n \in \mathbb{Z}} rc(n) |u_0^+(n)|^q \\ &\quad + (p-i-j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\ &\quad + (p-i-j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right|, \\ &\left. \frac{\partial y_1(s, t)}{\partial t} \right|_{(1,1)} \\ &= (i+j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\ &\quad + (i+j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right|.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\left. \frac{\partial y_2(s, t)}{\partial s} \right|_{(1,1)} \\ &= (p-i-j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\ &\quad + \left( \frac{p}{2} - j \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u_0^+(n))^{\frac{p}{2}-j} (\Delta u_0^-(n))^{\frac{p}{2}+j} \right| \\ &\quad + (p-i-j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right|,\end{aligned}$$

$$\begin{aligned}
& \left. \frac{\partial y_2(s, t)}{\partial t} \right|_{(1,1)} \\
&= (p-1) \|u_0^-\|^p - \sum_{n \in \mathbb{Z}} (q-1)c(n)|u_0^-(n)|^q \ln |u_0^-(n)|^r - \sum_{n \in \mathbb{Z}} rc(n)|u_0^-(n)|^q \\
&\quad + (i+j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&\quad + \left( \frac{p}{2} + j - 1 \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u_0^+(n))^{\frac{p}{2}-j} (\Delta u_0^-(n))^{\frac{p}{2}+j} \right| \\
&\quad + (i+j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right|.
\end{aligned}$$

Let

$$M = \begin{bmatrix} \frac{\partial y_1(s, t)}{\partial s}|_{(1,1)} & \frac{\partial y_2(s, t)}{\partial s}|_{(1,1)} \\ \frac{\partial y_1(s, t)}{\partial t}|_{(1,1)} & \frac{\partial y_2(s, t)}{\partial t}|_{(1,1)} \end{bmatrix}.$$

By (2.30) and (2.31), we have

$$\begin{aligned}
& \left. \frac{\partial y_1(s, t)}{\partial s} \right|_{(1,1)} \\
&= (p-1) \|u_0^+\|^p - \sum_{n \in \mathbb{Z}} rc(n)|u_0^+(n)|^q \\
&\quad + (p-i-j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&\quad + (p-i-j-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_i^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&\quad - (q-1) \|u_0^+\|^p - (q-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&\quad - (q-1) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_i^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&= (p-q) \left[ \|u_0^+\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \right. \\
&\quad \left. + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_i^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \right] \\
&\quad - \sum_{n \in \mathbb{Z}} rc(n)|u_0^+(n)|^q
\end{aligned}$$

$$\begin{aligned}
& + (-i-j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
& + (-i-j) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
& = -\frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} + C^+
\end{aligned}$$

and

$$\frac{\partial y_2(s, t)}{\partial t} \Big|_{(1,1)} = -\frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)} + C^-,$$

where

$$\begin{aligned}
C^+ &= (p-q) \left[ \|u_0^+\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}-1}^i C_i^j 2^{i-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \right. \\
&\quad \left. + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \right] \\
&\quad - \sum_{n \in \mathbb{Z}} r c(n) |u_0^+(n)|^q
\end{aligned}$$

and

$$\begin{aligned}
C^- &= (p-q) \left[ \|u_0^-\|^p + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=1}^i C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{i-1} 2^{i-j} (\Delta u_0^-(n))^{p-(i+j)} (\Delta u_0^+(n))^{i+j} \right| \right. \\
&\quad \left. + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^{j-1} 2^{\frac{p}{2}-j} (\Delta u_0^+(n))^{\frac{p}{2}-j} (\Delta u_0^-(n))^{\frac{p}{2}+j} \right| \right] \\
&\quad + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}} \sum_{j=0}^{i-1} C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j 2^{i-1-j} (\Delta u_0^+(n))^{p-(i+j)} (\Delta u_0^-(n))^{i+j} \right| \\
&\quad - \sum_{n \in \mathbb{Z}} r c(n) |u_0^-(n)|^q.
\end{aligned}$$

It follows from  $1 < p < q$  and  $u_0^\pm \neq 0$  that  $C^+ < 0$ ,  $C^- < 0$ ,  $\frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} \geq 0$ , and  $\frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)} \geq 0$ . Then, we have

$$\begin{aligned}
\det M &= \frac{\partial y_1(s, t)}{\partial s} \Big|_{(1,1)} \times \frac{\partial y_2(s, t)}{\partial t} \Big|_{(1,1)} - \frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} \times \frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)} \\
&= \left( -\frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} + C^+ \right) \times \left( -\frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)} + C^- \right) \\
&\quad - \frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} \times \frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)}
\end{aligned}$$

$$= -C^- \cdot \frac{\partial y_1(s, t)}{\partial t} \Big|_{(1,1)} - C^+ \cdot \frac{\partial y_2(s, t)}{\partial s} \Big|_{(1,1)} + C^+ C^- \\ > 0,$$

which implies that  $\det M \neq 0$ . According to the topological degree theory [1], we can obtain that  $\deg(\phi_1, D, (0, 0)) = 1$ . By virtue of (2.39) and (i), there exists  $\gamma = k$  on  $\partial D$ . As a consequence, it follows from the homotopy invariance of the Brouwer degree that

$$\deg(\phi_2, D, (0, 0)) = \deg(\phi_1, D, (0, 0)) = 1,$$

which implies that  $\phi_2(s_4, t_4) = 0$  for some  $(s_4, t_4) \in D$  and so  $\eta(1, k(s_4, t_4)) = \gamma(s_4, t_4) \in \mathcal{M}$ . Then, by the definition of  $\mathcal{M}$ , we know that  $I(\eta(1, k(s_4, t_4))) \geq m^*$ . This contradicts (2.42). Hence,  $I'(u_0) = 0$ , that is,  $u_0$  is a critical point of  $I$ .  $\square$

### 3 The existence of sign-changing solutions

In this section, we will prove the existence of sign-changing solutions that only change sign once.

*Proof of Theorem 1.1* First, it follows from Lemma 2.9 and Lemma 2.10 that problem (1.2) has a sign-changing solution  $u_0 \in \mathcal{M}$  such that

$$I(u_0) = m_* \quad \text{and} \quad I'(u_0) = 0. \quad (3.1)$$

Next, we prove that  $u_0$  only changes sign once. Denote  $u_0 = u_1 + u_2 + u_3$ , where

$$\begin{aligned} u_1 &\geq 0, & u_2 &\leq 0, & V_1 \cap V_2 &= \emptyset, \\ u_1|_{\mathbb{Z} \setminus (V_1 \cup V_2)} &= u_2|_{\mathbb{Z} \setminus (V_1 \cup V_2)} = u_3|_{V_1 \cup V_2} = 0, \\ V_1 &:= \{n \in \mathbb{Z} : u_1(n) > 0\}, & V_2 &:= \{n \in \mathbb{Z} : u_2(n) < 0\} \end{aligned} \quad (3.2)$$

and  $V_1 = \{n_1, n_1 + 1, \dots, n_1 + m_1\}$ ,  $V_2 = \{n_2, n_2 + 1, \dots, n_2 + m_2\}$ , where the value of  $n_1$  or  $n_2$  may be  $-\infty$  and the value of  $n_1 + m_1$  or  $n_2 + m_2$  may be  $+\infty$ .

Setting  $w = u_1 + u_2$ , it is easy to see that  $w^+ = u_1$ ,  $w^- = u_2$ , and  $w^\pm \neq 0$ . According to Lemma 2.7, there is a unique pair of positive constants  $s_4, t_4$  such that  $s_4 w^+ + t_4 w^- \in \mathcal{M}$ . By virtue of  $I'(u_0) = 0$ , we can derive that  $\langle I'(u_0), w^\pm \rangle = 0$ . Then, by (1.7), we can obtain that

$$\begin{aligned} \langle I'(u_0), w^+ \rangle &= \sum_{n \in \mathbb{Z}} [a(n)|\Delta w(n) + \Delta u_3(n)|^{p-2} (\Delta w(n) + \Delta u_3(n)) \Delta w^+(n)] \\ &\quad + \sum_{n \in \mathbb{Z}} [b(n)|w(n) + u_3(n)|^{p-2} (w(n) + u_3(n)) w^+(n)] \\ &\quad - \sum_{n \in \mathbb{Z}} c(n)|w(n) + u_3(n)|^{q-2} (w(n) + u_3(n)) w^+(n) \ln |w(n) + u_3(n)|^r \\ &= \langle I'(w), w^+ \rangle + \sum_{n \in V_1} a(n)|\Delta w(n) + \Delta u_3(n)|^{p-2} \Delta u_3(n) \Delta w^+(n) \\ &\quad + \sum_{n \in \mathbb{Z} \setminus V_1} a(n)|\Delta w(n) + \Delta u_3(n)|^{p-2} \Delta u_3(n) \Delta w^+(n) \end{aligned}$$

$$\begin{aligned}
& + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^i (\Delta w(n))^{p-2-i} (\Delta u_3(n))^i \right| \Delta w(n) \Delta w^+(n) \\
& = \langle I'(w), w^+ \rangle - a(n_1 - 1) |w^+(n_1) - u_3(n_1 - 1)|^{p-2} w^+(n_1) u_3(n_1 - 1) \\
& \quad - a(n_1 + m_1) |-w^+(n_1 + m_1) + u_3(n_1 + m_1 + 1)|^{p-2} \\
& \quad \times w^+(n_1 + m_1) u_3(n_1 + m_1 + 1) \\
& + \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^i (\Delta w(n))^{p-2-i} (\Delta u_3(n))^i \right| \Delta w(n) \Delta w^+(n). \tag{3.3}
\end{aligned}$$

Note that

$$\begin{aligned}
\Delta w(n) \Delta w^+(n) &= \Delta w^+(n) \Delta w^+(n) + \Delta w^-(n) \Delta w^+(n) \\
&= (\Delta w^+(n))^2 + \Delta w^-(n) \Delta w^+(n) \\
&= (\Delta w^+(n))^2 - w^-(n+1) w^+(n) - w^-(n) w^+(n+1) \geq 0.
\end{aligned}$$

According to (3.3), one has

$$\begin{aligned}
\langle I'(w), w^+ \rangle &= a(n_1 - 1) |w^+(n_1) - u_3(n_1 - 1)|^{p-2} w^+(n_1) u_3(n_1 - 1) \\
&\quad + a(n_1 + m_1) |-w^+(n_1 + m_1) + u_3(n_1 + m_1 + 1)|^{p-2} \\
&\quad \times w^+(n_1 + m_1) u_3(n_1 + m_1 + 1) \\
&\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^i (\Delta w(n))^{p-2-i} (\Delta u_3(n))^i \right| \Delta w(n) \Delta w^+(n) \\
&\leq 0. \tag{3.4}
\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}
\langle I'(w), w^- \rangle &= a(n_2 - 1) |w^-(n_2) - u_3(n_2 - 1)|^{p-2} w^-(n_2) u_3(n_2 - 1) \\
&\quad + a(n_2 + m_2) |-w^-(n_2 + m_2) + u_3(n_2 + m_2 + 1)|^{p-2} \\
&\quad \times w^-(n_2 + m_2) u_3(n_2 + m_2 + 1) \\
&\quad - \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} C_{\frac{p}{2}-1}^i (\Delta w(n))^{p-2-i} (\Delta u_3(n))^i \right| \Delta w(n) \Delta w^-(n) \\
&\leq 0. \tag{3.5}
\end{aligned}$$

On the basis of (1.4), (1.7), (2.11), (3.1), (3.2), (3.4), and (3.5), using the same processing method as (2.7), we have

$$\begin{aligned}
m_* &= I(u_0) - \frac{1}{q} \langle I'(u_0), u_0 \rangle \\
&= I(w) + I(u_3) + \left( \frac{1}{p} - \frac{1}{q} \right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{i=1}^{\frac{p}{2}-1} \sum_{j=0}^i C_{\frac{p}{2}}^i C_i^j 2^{i-j} (\Delta w(n))^{p-(i+j)} (\Delta u_3(n))^{i+j} \right|
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{q}\langle I'(w), w \rangle - \frac{1}{q}\langle I'(u_3), u_3 \rangle \\
& + \left(\frac{1}{p} - \frac{1}{q}\right) \sum_{n \in \mathbb{Z}} a(n) \left| \sum_{j=0}^{\frac{p}{2}-1} C_{\frac{p}{2}}^j 2^{\frac{p}{2}-j} (\Delta w(n))^{\frac{p}{2}-j} (\Delta u_3(n))^{\frac{p}{2}+j} \right| \\
& \geq I(s_0 w^+ + t_0 w^-) + \frac{1-s_0^q}{q} \langle I'(w), w^+ \rangle + \frac{1-t_0^q}{q} \langle I'(w), w^- \rangle \\
& + I(u_3) - \frac{1}{q} \langle I'(w), w \rangle - \frac{1}{q} \langle I'(u_3), u_3 \rangle \\
& = I(s_0 w^+ + t_0 w^-) - \frac{s_0^q}{q} \langle I'(w), w^+ \rangle - \frac{t_0^q}{q} \langle I'(w), w^- \rangle + I(u_3) - \frac{1}{q} \langle I'(u_3), u_3 \rangle \\
& = I(s_0 w^+ + t_0 w^-) - \frac{s_0^q}{q} \langle I'(w), w^+ \rangle - \frac{t_0^q}{q} \langle I'(w), w^- \rangle \\
& + \left(\frac{1}{p} - \frac{1}{q}\right) \|u_3\| + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_3(n)|^q \\
& \geq m_* + \left(\frac{1}{p} - \frac{1}{q}\right) \|u_3\|,
\end{aligned}$$

which implies that  $u_3 = 0$ . Thus,  $u_0$  only changes sign once.  $\square$

#### 4 The existence of ground-state solutions

In this section, we will prove the existence of Nehari-type ground-state solutions for (1.2) and provide the relationship between the sign-changing ground-state energy and the ground-state energy. We mainly use the method in [2, 4] to prove that the functional  $I$  satisfies the Cerami condition at any level  $d \in (0, \infty)$ , and then use the method in [5] to prove that the functional  $I$  has a mountain-pass geometry. To prove the above conclusions, we need the following lemmas.

**Lemma 4.1** ([18]) *Let  $X$  be a real Banach space. For some constants  $\alpha, \beta, \rho > 0$ , and  $e \in X$  with  $\|e\|_X > \rho$ , there exists a functional  $I \in C^1(X, \mathbb{R})$  satisfying the following mountain-pass geometry:*

$$\max\{I(0), I(e)\} \leq \alpha < \beta \leq \inf_{\|u\|_X=\rho} I(u).$$

Set  $d_0 = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$ , where  $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$ . Then, there exists a Cerami sequence  $\{u_k\} \subset X$  of  $I$  at level  $d_0$ , where a sequence  $\{u_k\}$  is called a Cerami sequence at a level  $d_0$  if it satisfies

$$I(u_k) \rightarrow d_0 \quad \text{and} \quad \|I'(u_k)\| (1 + \|u_k\|) \rightarrow 0. \quad (4.1)$$

**Remark 4.1** It is easy to obtain that  $d_0 \geq \beta > 0$  (for example, see the proof of Theorem 1.15 in [22]).

**Lemma 4.2** *The Cerami sequence  $\{u_k\} \subset \mathcal{D}$  at any level  $d_0 \in (0, +\infty)$  has at least one convergent subsequence in  $\mathcal{D}$ .*

*Proof* Since  $\{u_k\}$  is a Cerami sequence at the level  $d_0$ , then (4.1) holds. We claim that  $\{u_k\}$  is bounded in  $\mathcal{D}$ . Arguing by contradiction, we suppose that  $\{u_k\}$  is not bounded in  $\mathcal{D}$ , that is, there appears a subsequence, still denoted by  $\{u_k\}$ , such that  $\|u_k\| \rightarrow +\infty$  as  $k \rightarrow +\infty$ .

Let  $w_k = \frac{u_k}{\|u_k\|}$ . Then, there exists a subsequence, still denoted by  $\{w_k\}$ , and a function  $w \in \mathcal{D}$  such that

$$\begin{cases} w_k \rightharpoonup w & \text{in } \mathcal{D}, \\ w_k(n) \rightarrow w(n) & \text{for each } n \in \mathbb{Z}, \\ w_k \rightarrow w & \text{in } l^\kappa(\mathbb{Z}, \mathbb{R}), \kappa \in [p, +\infty]. \end{cases} \quad (4.2)$$

Then, we will prove the claim by discussing the following two cases.

*Case 1:*  $w = 0$ .

Set  $t_k \in [0, 1]$  such that  $I(t_k u_k) = \max_{t \in [0, 1]} I(t u_k)$ . For any given constants  $\tau > 0$  and  $N > 0$ , it follows from the unboundedness of  $\{u_k\}$  that

$$\|u_k\| > (p\tau + 1)^{\frac{1}{p}}, \quad \text{for large enough } k \geq N.$$

Set  $\bar{w}_k = (p\tau + 1)^{\frac{1}{p}} w_k$ . By virtue of  $(C_2)$ , the boundedness of  $\{w_k\}$ , (2.1), (1.6), and the Lebesgue dominate convergence theorem, we have

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \sum_{n \in \mathbb{Z}} c(n) |\bar{w}_k(n)|^q \ln |\bar{w}_k(n)|^r \\ &= \lim_{k \rightarrow +\infty} \sum_{n \in \mathbb{Z}} c(n) (p\tau + 1)^{\frac{q}{p}} |w_k(n)|^q \ln ((p\tau + 1)^{\frac{r}{p}} |w_k(n)|^r) \\ &= \lim_{k \rightarrow +\infty} (p\tau + 1)^{\frac{q}{p}} \left( \sum_{n \in \mathbb{Z}} c(n) |w_k(n)|^q \ln (p\tau + 1)^{\frac{r}{p}} + \sum_{n \in \mathbb{Z}} c(n) |w_k(n)|^q \ln |w_k(n)|^r \right) \\ &= (p\tau + 1)^{\frac{q}{p}} \left( \sum_{n \in \mathbb{Z}} c(n) |w(n)|^q \ln (p\tau + 1)^{\frac{r}{p}} + \sum_{n \in \mathbb{Z}} c(n) |w(n)|^q \ln |w(n)|^r \right) \\ &= 0. \end{aligned}$$

Then, for  $k$  large enough, we can obtain that

$$I(t_k u_k) \geq I\left(\frac{(p\tau + 1)^{\frac{1}{p}}}{\|u_k\|} u_k\right) = I(\bar{w}_k) \geq \frac{1}{p} \|\bar{w}_k\|^p - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |\bar{w}_k(n)|^q \ln |\bar{w}_k(n)|^r \geq \tau.$$

According to the arbitrariness of  $\tau$ , we can obtain that

$$\lim_{k \rightarrow +\infty} I(t_k u_k) = +\infty. \quad (4.3)$$

If  $t_k = 1$ , substituting it into (4.3) can obtain  $\lim_{k \rightarrow +\infty} I(u_k) = +\infty$ , which contradicts (4.1). Then, it follows from  $I(0) = 0$  that  $t_k \in (0, 1)$ . Thus,  $\frac{d}{dt} I(t u_k) |_{t=t_k} = 0$ . Therefore, according

to the definition of  $\{u_k\}$ , we can obtain that

$$\begin{aligned} I(t_k u_k) &= I(t_k u_k) - \frac{1}{q} \langle I'(t_k u_k), t_k u_k \rangle = \left( \frac{1}{p} - \frac{1}{q} \right) \|t_k u_k\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |t_k u_k(n)|^q \\ &\leq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_k\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q = I(u_k) - \frac{1}{q} \langle I'(u_k), u_k \rangle \\ &\leq d + o(1), \end{aligned}$$

which is contrary to (4.3). Hence, the assumption is not valid, that is,  $\{u_k\}$  is bounded in  $\mathcal{D}$ .

*Case 2:  $w \neq 0$ .*

Let  $V' = \{n \in \mathbb{Z}; w \neq 0\}$ . Then,  $|u_k(n)| \rightarrow +\infty$  as  $k \rightarrow +\infty$  for each  $n \in V'$ . According to the fact that  $\|u_k\| \rightarrow +\infty$ , as  $k \rightarrow +\infty$ , and  $I(u_k) \leq c_*$ , there holds  $\frac{I(u_k)}{\|u_k\|^p} \rightarrow 0$ , as  $k \rightarrow +\infty$ , that is

$$\frac{1}{p} + \frac{r \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q}{q^2 \|u_k\|^p} - \frac{\sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q \ln |u_k(n)|^r}{q \|u_k\|^p} = o_k(1),$$

which together with the definition of  $\mathcal{D}$ ,  $(C_1)$ , and  $(C_2)$  implies that  $G(u_k) := \frac{r \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q}{q^2 \|u_k\|^p} - \frac{\sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^q \ln |u_k(n)|^r}{q \|u_k\|^p}$  is bounded, that is, there is a positive constant  $M_2$  such that

$$|G(u_k)| \leq M_2, \quad \forall k \in \mathbb{N}. \quad (4.4)$$

We set

$$G(u_k) = \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| \leq M_3} (\cdot) + \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| > M_3} (\cdot) + \sum_{n \in V'} (\cdot) := I + II + III, \quad (4.5)$$

where  $M_3 = e^{\frac{1}{q}} > 0$  and  $(\cdot) = \frac{rc(n)|u_k(n)|^q}{q^2 \|u_k\|^p} - \frac{c(n)|u_k(n)|^q \ln |u_k(n)|^r}{q \|u_k\|^p}$ . For  $I$  in (4.5), according to (1.6), Lemma 2.1, (2.33), and  $\sum_{n \in \mathbb{Z}} c(n) < +\infty$ , there are two positive constants  $\varepsilon_6$  and  $C_{\varepsilon_6}$  such that

$$\begin{aligned} I &= \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| \leq M_3} \left( \frac{rc(n)|u_k(n)|^q}{q^2 \|u_k\|^p} - \frac{c(n)|u_k(n)|^q \ln |u_k(n)|^r}{q \|u_k\|^p} \right) \\ &\leq \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| \leq M_3} \left( \frac{rc(n)|u_k(n)|^q}{q^2 \|u_k\|^p} + \frac{c(n)\varepsilon_6|u_k(n)|^p + c(n)C_{\varepsilon_6}|u_k(n)|^\zeta}{q \|u_k\|^p} \right) \\ &\leq \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| \leq M_3} \left( \frac{rc(n)M_3}{q^2 \|u_k\|^p} + \frac{c(n)\varepsilon_6 M_3 + c(n)C_{\varepsilon_6} M_3}{q \|u_k\|^p} \right) \\ &\leq \left( \frac{rM_3 \sum_{n \in \mathbb{Z}} c(n)}{q^2 \|u_k\|^p} + \frac{(\varepsilon_6 + C_{\varepsilon_6})M_3 \sum_{n \in \mathbb{Z}} c(n)}{q \|u_k\|^p} \right) \\ &\rightarrow 0, \quad \text{as } k \rightarrow +\infty. \end{aligned} \quad (4.6)$$

For  $II$  in (4.5), we have

$$\begin{aligned} II &= \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| > M_3} \left( \frac{rc(n)|u_k(n)|^q}{q^2\|u_k\|^p} - \frac{c(n)|u_k(n)|^q \ln |u_k(n)|^r}{q\|u_k\|^p} \right) \\ &= \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| > M_3} \frac{c(n)|u_k(n)|^q}{q^2\|u_k\|^p} (r - q \ln |u_k(n)|^r) \\ &\leq \sum_{n \in \mathbb{Z} \setminus V'; |u_k(n)| > M_3} \frac{c(n)|w_k(n)|^p|u_k(n)|^{q-p}}{q^2} (r - q \ln M_3^r) \rightarrow -\infty, \quad \text{as } k \rightarrow +\infty. \end{aligned} \quad (4.7)$$

Note that  $|u_k(n)| \rightarrow +\infty$  as  $k \rightarrow +\infty$  for each  $n \in V'$ . Then, similar to the argument of  $II$ , we also have

$$III = \sum_{n \in V'} \left( \frac{rc(n)|u_k(n)|^q}{q^2\|u_k\|^p} - \frac{c(n)|u_k(n)|^q \ln |u_k(n)|^r}{q\|u_k\|^p} \right) \rightarrow -\infty, \quad \text{as } k \rightarrow +\infty. \quad (4.8)$$

Thus,  $\lim_{k \rightarrow \infty} G(u_k) = -\infty$ , which contradicts (4.4). Therefore, we deduce that  $\{u_k\}$  is bounded in  $\mathcal{D}$ .

As a consequence, both of the above cases indicate that the assumption is not valid, that is,  $\{u_k\}$  is bounded in  $\mathcal{D}$ . Then, there exists a subsequence, still denoted by  $\{u_k\}$ , and a function  $u \in \mathcal{D}$  such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \mathcal{D}, \\ u_k(n) \rightarrow u(n) & \text{for each } n \in \mathbb{Z}, \\ u_k \rightarrow u & \text{in } l^\kappa(\mathbb{Z}, \mathbb{R}), \kappa \in [p, +\infty]. \end{cases} \quad (4.9)$$

Note that  $\{u_k\}$  is a Cerami sequence. Then, there holds

$$\lim_{k \rightarrow +\infty} \langle I'(u_k), u_k - u \rangle = 0. \quad (4.10)$$

Moreover, by (4.9), we have

$$\lim_{k \rightarrow +\infty} \langle I'(u), u_k - u \rangle = 0. \quad (4.11)$$

Note that  $\{u_k\}$  is bounded in  $E$ . On the basis of (1.6), (2.1), (4.9), and  $\sum_{n \in \mathbb{Z}} c(n) < +\infty$  there exist two positive constants  $\varepsilon$  and  $C_\varepsilon$  such that

$$\begin{aligned} &\sum_{n \in \mathbb{Z}} c(n)|u(n)|^{q-2}u(n)(u_k(n) - u(n)) \ln |u(n)|^r \\ &\leq \sum_{n \in \mathbb{Z}} c(n)(u_k(n) - u(n))(\varepsilon|u(n)|^{p-1} + C_\varepsilon|u(n)|^{\zeta-1}) \\ &\rightarrow 0 \quad \text{as } k \rightarrow +\infty. \end{aligned} \quad (4.12)$$

Similarly, it follows from the boundedness of  $\{\|u_k\|\}$ , (1.6), (2.1), (4.9), and  $\sum_{n \in \mathbb{Z}} c(n) < +\infty$  that

$$\lim_{k \rightarrow +\infty} \sum_{n \in \mathbb{Z}} c(n)|u_k(n)|^{q-2}u_k(n)(u_k(n) - u(n)) \ln |u_k(n)|^r = 0. \quad (4.13)$$

Then, using the Hölder inequality

$$d_1 d_2 + d_3 d_4 \leq (d_1^p + d_3^p)^{\frac{1}{p}} (d_2^{p^*} + d_4^{p^*})^{\frac{1}{p^*}},$$

where  $d_1, d_2, d_3, d_4$  are nonnegative constants and  $p^* = \frac{p}{p-1}$ ,  $p > 1$ , by virtue of (1.7) and (1.3), we have

$$\begin{aligned} & \langle I'(u_k), u_k - u \rangle - \langle I'(u), u_k - u \rangle \\ &= \sum_{n \in \mathbb{Z}} a(n) |\Delta u_k(n)|^{p-2} \Delta u_k(n) \Delta(u_k(n) - u(n)) \\ &\quad + \sum_{n \in \mathbb{Z}} b(n) |u_k(n)|^{p-2} u_k(n) (u_k(n) - u(n)) \\ &\quad - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta(u_k(n) - u(n)) \\ &\quad - \sum_{n \in \mathbb{Z}} b(n) |u(n)|^{p-2} u(n) (u_k(n) - u(n)) \\ &\quad + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\ &= \|u_k\|^p + \|u\|^p - \sum_{n \in \mathbb{Z}} a(n) |\Delta u_k(n)|^{p-2} \Delta u_k(n) \Delta u(n) - \sum_{n \in \mathbb{Z}} b(n) |u_k(n)|^{p-2} u_k(n) u(n) \\ &\quad - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\ &\quad - \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-2} \Delta u(n) \Delta u_k(n) \\ &\quad - \sum_{n \in \mathbb{Z}} b(n) |u(n)|^{p-2} u(n) u_k(n) + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\ &\geq \|u_k\|^p + \|u\|^p - \sum_{n \in \mathbb{Z}} a(n) |\Delta u_k(n)|^{p-1} |\Delta u(n)| - \sum_{n \in \mathbb{Z}} b(n) |u_k(n)|^{p-1} |u(n)| \\ &\quad - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r - \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^{p-1} |\Delta u_k(n)| \\ &\quad - \sum_{n \in \mathbb{Z}} b(n) |u(n)|^{p-1} |u_k(n)| + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\ &\geq \|u_k\|^p + \|u\|^p - \left( \sum_{n \in \mathbb{Z}} (a(n)^{\frac{1}{p}} |\Delta u(n)|)^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} (a(n)^{1-\frac{1}{p}} |\Delta u_k(n)|^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\quad - \left( \sum_{n \in \mathbb{Z}} (b(n)^{\frac{1}{p}} |u(n)|)^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} (b(n)^{1-\frac{1}{p}} |u_k(n)|^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\quad - \left( \sum_{n \in \mathbb{Z}} (a(n)^{\frac{1}{p}} |\Delta u_k(n)|)^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} (a(n)^{1-\frac{1}{p}} |\Delta u(n)|^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \end{aligned}$$

$$\begin{aligned}
& - \left( \sum_{n \in \mathbb{Z}} (b(n)^{\frac{1}{p}} |u_k(n)|)^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} (b(n)^{1-\frac{1}{p}} |u(n)|^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\
& - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\
& + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\
= & \|u_k\|^p + \|u\|^p - \left( \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} a(n) |\Delta u_k(n)|^p \right)^{\frac{p-1}{p}} \\
& - \left( \sum_{n \in \mathbb{Z}} b(n) |u(n)|^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} b(n) |u_k(n)|^p \right)^{\frac{p-1}{p}} \\
& - \left( \sum_{n \in \mathbb{Z}} a(n) |\Delta u_k(n)|^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^p \right)^{\frac{p-1}{p}} \\
& - \left( \sum_{n \in \mathbb{Z}} b(n) |u_k(n)|^p \right)^{\frac{1}{p}} \left( \sum_{n \in \mathbb{Z}} b(n) |u(n)|^p \right)^{\frac{p-1}{p}} \\
& - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\
& + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\
\geq & \|u_k\|^p + \|u\|^p - \left( \sum_{n \in \mathbb{Z}} [a(n) |\Delta u(n)|^p + b(n) |u(n)|^p] \right)^{\frac{1}{p}} \\
& \times \left( \sum_{n \in \mathbb{Z}} [a(n) |\Delta u_k(n)|^p + b(n) |u_k(n)|^p] \right)^{\frac{p-1}{p}} \\
& - \left( \sum_{n \in \mathbb{Z}} [a(n) |\Delta u_k(n)|^p + b(n) |u_k(n)|^p] \right)^{\frac{p}{p}} \\
& \times \left( \sum_{n \in \mathbb{Z}} [a(n) |\Delta u(n)|^p + b(n) |u(n)|^p] \right)^{\frac{p-1}{p}} \\
& - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\
& + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\
= & \|u_k\|^p + \|u\|^p - \|u\| \|u_k\|^{p-1} - \|u_k\| \|u\|^{p-1} \\
& - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\
& + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r \\
= & (\|u_k\|^{p-1} - \|u\|^{p-1})(\|u_k\| - \|u\|)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{n \in \mathbb{Z}} c(n) |u_k(n)|^{q-2} u_k(n) (u_k(n) - u(n)) \ln |u_k(n)|^r \\
& + \sum_{n \in \mathbb{Z}} c(n) |u(n)|^{q-2} u(n) (u_k(n) - u(n)) \ln |u(n)|^r.
\end{aligned} \tag{4.14}$$

According to (4.10), (4.11), (4.12), (4.13), and (4.14), we have  $\|u_k\| \rightarrow \|u\|$  as  $k \rightarrow +\infty$ . By the uniform convexity of  $\mathcal{D}$  (similar to the argument of the Appendix A.1 in [24]), the fact that  $u_k \rightarrow u$  in  $\mathcal{D}$  and the Kadec–Klee property, we can obtain that  $u_k \rightarrow u$  in  $\mathcal{D}$ . Thus,  $I$  satisfies the Cerami condition.  $\square$

Next, we prove that the functional  $I$  defined by (1.4) has a mountain-pass geometry.

**Lemma 4.3** (i) *There are two positive constants  $\rho$  and  $\delta'$  such that  $I(u) \geq \delta'$  for all  $u \in \mathcal{D}$  with  $\|u\| = \rho$ .*

(ii) *There is  $\varphi_j \in \mathcal{D} \setminus \{0\}$  such that  $I(t\varphi_j) \rightarrow -\infty$  as  $t \rightarrow +\infty$ .*

*Proof* For (i), it follows from (1.4), (1.6), and Lemma 2.1 that there exists  $\varepsilon_7 \in (0, \frac{qb_0}{pc_0})$  and  $C_{\varepsilon_7} > 0$  such that

$$\begin{aligned}
I(u) & \geq \frac{1}{p} \|u\|^p - \frac{1}{q} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r \\
& \geq \frac{1}{p} \|u\|^p - \frac{1}{q} c_0 \varepsilon_7 \|u\|_p^p - \frac{1}{q} c_0 C_{\varepsilon_7} \|u\|_L^\zeta \\
& \geq \left( \frac{1}{p} - \frac{1}{q} c_0 \varepsilon_7 b_0^{-1} \right) \|u\|^p - \frac{1}{q} c_0 C_{\varepsilon_7} b_0^{-\frac{\zeta}{p}} \|u\|^\zeta.
\end{aligned}$$

Choose  $\rho > 0$  sufficiently small. There appears a constants  $\beta_0 = \frac{\rho^p q - \rho^p p c_0 (\varepsilon_7 b_0^{-1} + C_{\varepsilon_7} b_0^{-\frac{\zeta}{p}})}{pq} > 0$  such that  $I(u) \geq \beta_0$  for all  $u \in \mathcal{D}$  with  $\|u\| = \rho$ .

By the definition of  $c_*$ , for any  $j > 0$ , we can choose a  $\varphi_j \in \mathcal{N} \subset \mathcal{D} \setminus \{0\}$  such that

$$c_* \leq I(\varphi_j) \leq c_* + \frac{1}{j}. \tag{4.15}$$

Then, for any  $t > 0$ , there holds

$$\begin{aligned}
I(t\varphi_j) & = \frac{t^p}{p} \|\varphi_j\|^p + \frac{rt^q}{q^2} \sum_{n \in \mathbb{Z}} c(n) |\varphi_j(n)|^q - \frac{t^q \ln t^r}{q} \sum_{n \in \mathbb{Z}} c(n) |\varphi_j(n)|^q \\
& - \frac{t^q}{q} \sum_{n \in \mathbb{Z}} c(n) |\varphi_j(n)|^q \ln |\varphi_j(n)|^r \\
& \leq \frac{t^p}{p} \|\varphi_j\|^p + \frac{rt^q}{q^2} \sum_{n \in \mathbb{Z}} c(n) |\varphi_j(n)|^q - \frac{t^q \ln t^r}{q} \sum_{n \in \mathbb{Z}} c(n) |\varphi_j(n)|^q \\
& + \frac{t^q}{q} \sum_{n \in \mathbb{Z}} c_0 (\varepsilon |\varphi_j(n)|^p + C_\varepsilon |\varphi_j(n)|^\zeta),
\end{aligned}$$

which implies that  $I(t\varphi_j) \rightarrow -\infty$  as  $t \rightarrow +\infty$ , and hence, (ii) holds.  $\square$

*Proof of Theorem 1.2* Lemma 4.1 and Lemma 4.3 imply that  $I$  has a Cerami sequence  $\{u_{kj}\}$  at the level  $d_j$ , that is,

$$I(u_{kj}) \rightarrow d_j \quad \text{and} \quad \|I'(u_{kj})\|(1 + \|u_{kj}\|) \rightarrow 0, \quad \text{as } kj \rightarrow +\infty.$$

By virtue of Remark 4.1 and the definition of  $d_j$ , it is easy to see that  $d_j \in [\beta_0, \max_{0 \leq t \leq 1} I(t\varphi_j)]$ . Furthermore, noting that  $\varphi_j \in \mathcal{N}$ , according to Corollary 2.5, we obtain that  $I(\varphi_j) = \max_{t \geq 0} I(t\varphi_j)$ , and hence,  $d_j \in [\beta_0, I(\varphi_j)]$ , which together with (4.15) implies that  $d_j \in [\beta_0, c_* + \frac{1}{j}]$ . Thus, we can choose a subsequence  $\{u_{k_j,j}\}$ , denoted by  $\{u_j\}$ , such that

$$I(u_j) \rightarrow d_* \quad \text{and} \quad \|I'(u_j)\|(1 + \|u_j\|) \rightarrow 0, \quad \text{as } j \rightarrow +\infty, \quad (4.16)$$

for some  $d_* \in [\frac{\beta_0}{2}, c_*]$ . Equation (4.16) and Lemma 4.2 imply that  $\{u_j\}$  has a convergent subsequence, still denoted by  $\{u_j\}$ , such that  $u_j \rightarrow \bar{u}$  as  $j \rightarrow +\infty$ . By the continuity of  $I$  and  $I'$ , we obtain that  $I(\bar{u}) = d_*$  and  $I'(\bar{u}) = 0$ , which together with the fact that  $d_* \geq \frac{\beta_0}{2} > 0$  implies that  $\bar{u} \in \mathcal{N}$  is a nontrivial solution of (1.2) and obviously,  $I(\bar{u}) \geq c_* = \inf_{u \in \mathcal{N}} I(u)$ . Moreover, according to (4.16), (1.4), (1.7), and the weak lower semicontinuity of norm, there exists

$$\begin{aligned} c_* \geq d_* &= \lim_{j \rightarrow +\infty} \left[ I(u_j) - \frac{1}{p} \langle I'(u_j), u_j \rangle \right] = \lim_{j \rightarrow +\infty} \left[ \left( \frac{1}{p} - \frac{1}{q} \right) \|u_j\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |u_j(n)|^q \right] \\ &\geq \left( \frac{1}{p} - \frac{1}{q} \right) \|\bar{u}\|^p + \frac{r}{q^2} \sum_{n \in \mathbb{Z}} c(n) |\bar{u}(n)|^q = I(\bar{u}) - \frac{1}{p} \langle I'(\bar{u}), \bar{u} \rangle = I(\bar{u}), \end{aligned}$$

which implies that  $I(\bar{u}) \leq c_*$ . Thus,  $I(\bar{u}) = c_* = \inf_{u \in \mathcal{N}} I(u) > 0$ .

Finally, we prove that  $m_* \geq 2c_*$ . In fact, it follows from Corollary 2.4, Proposition 2.1, Lemma 2.6, and Lemma 2.8 that there are two positive constants  $s'$  and  $t'$  such that  $s'u_0^+ \in \mathcal{N}$  and  $t'u_0^- \in \mathcal{N}$ , then one has

$$\begin{aligned} m_* &= I(u_0) = \max_{s,t \geq 0} I(su_0^+ + tu_0^-) \\ &\geq \max_{s,t \geq 0} [I(su_0^+) + I(tu_0^-)] \\ &= \max_{s \geq 0} I(su_0^+) + \max_{t \geq 0} I(tu_0^-) \\ &\geq I(s'u_0^+) + I(t'u_0^-) \\ &\geq I(\bar{u}) + I(\bar{u}) = 2c_*. \end{aligned}$$

The proof of Theorem 1.2 is completed.  $\square$

## Appendices

**Appendix 1** There exists  $u \in E$  such that  $\sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r = -\infty$ .

*Proof* Set

$$u(n) = \begin{cases} \frac{1}{|n| \ln |n|}, & |n| \geq p+2, \\ 0, & |n| \leq p+1, \end{cases}$$

where  $|n|$  represents the absolute value of  $n$ . Let

$$a(n), b(n) = \begin{cases} |n|^{p-1}, & |n| \geq p-1, \\ 1, & |n| \leq p-2, \end{cases} \quad \text{and} \quad c(n) = \begin{cases} |n|^{q-1}, & |n| \geq q-1, \\ 1, & |n| \leq q-2. \end{cases}$$

Using the method for discriminating the convergence of improper integrals, we can obtain the following results

$$\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^{\theta}} = \begin{cases} < +\infty, \text{ that is, the series convergence,} & \theta > 1, \\ = +\infty, \text{ that is, the series divergence,} & 0 \leq \theta \leq 1. \end{cases}$$

According to the definitions of  $\Delta$ ,  $u(n)$  and  $a(n)$ , using the  $C_p$  inequality, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} a(n) |\Delta u(n)|^p &= |p+1|^{p-1} \left( \frac{1}{|p+2| \ln |p+2|} \right)^p + \sum_{|n| \geq p+2} |n|^{p-1} \left| \frac{1}{|n+1| \ln |n+1|} - \frac{1}{|n| \ln |n|} \right|^p \\ &\leq \frac{1}{|p+2| (\ln |p+2|)^p} + \sum_{|n| \geq p+2} |n|^{p-1} \left( \frac{1}{|n+1| \ln |n+1|} + \frac{1}{|n| \ln |n|} \right)^p \\ &\leq \frac{1}{|p+2| (\ln |p+2|)^p} + \sum_{|n| \geq p+2} |n|^{p-1} 2^{p-1} \left[ \left( \frac{1}{|n+1| \ln |n+1|} \right)^p + \left( \frac{1}{|n| \ln |n|} \right)^p \right] \\ &\leq \frac{1}{|p+2| (\ln |p+2|)^p} + 2^{p-1} \sum_{|n| \geq p+2} \left[ \frac{1}{|n+1| (\ln |n+1|)^p} + \frac{1}{|n| (\ln |n|)^p} \right] \\ &< +\infty. \end{aligned}$$

Then, according to the definition of  $u(n)$  and  $b(n)$ , there holds

$$\sum_{n \in \mathbb{Z}} b(n) |u(n)|^p = \sum_{|n| \geq p+2} \frac{|n|^{p-1}}{(|n| \ln |n|)^p} = \sum_{|n| \geq p+2} \frac{1}{|n| (\ln |n|)^p} < +\infty.$$

Therefore, it is easy to see that  $u \in E$ .

Now, we prove that  $\sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r = -\infty$  if  $1 < q \leq 2$ . Note that if  $|n| \geq p+2$ , then

$$|u(n)|^q \ln |u(n)|^r = \frac{1}{(|n| \ln |n|)^q} \times \ln \frac{1}{(|n| \ln |n|)^r} = - \left( \frac{r}{|n|^q (\ln |n|)^{q-1}} + \frac{r \ln(\ln |n|)}{|n|^q (\ln |n|)^q} \right).$$

Thus, there holds

$$\begin{aligned} \sum_{n \in \mathbb{Z}} c(n) |u(n)|^q \ln |u(n)|^r &= \sum_{|n| \geq p+2} |n|^{q-1} \times \left[ - \left( \frac{r}{|n|^q (\ln |n|)^{q-1}} + \frac{r \ln(\ln |n|)}{|n|^q (\ln |n|)^q} \right) \right] \\ &= - \left( \sum_{|n| \geq p+2} \frac{r}{|n| (\ln |n|)^{q-1}} + \sum_{|n| \geq p+2} \frac{r \ln(\ln |n|)}{|n| (\ln |n|)^q} \right) \\ &:= -(I' + II'). \end{aligned}$$

As  $|n| \geq p + 2$ , we have  $\ln \ln |n| > 0$  and then  $II' > 0$ . On the other hand, it is easy to see that  $I' = \infty$  since  $1 < q \leq 2$ . Thus, we complete the proof.  $\square$

**Appendix 2** For all  $s, t \geq 0$  and  $0 \leq i, j \leq \frac{p}{2}$ , there exist

$$\begin{aligned} & \frac{2s^p C_{\frac{p}{2}-1}^i C_i^j + s^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2s^{p-(i+j)} t^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \geq 0, \\ & \frac{s^p C_{\frac{p}{2}-1}^i + t^p C_{\frac{p}{2}-1}^{i-1} - s^{p-2i} t^{2i} C_{\frac{p}{2}}^i}{p} \geq 0, \\ & \frac{2s^p C_{\frac{p}{2}-1}^i + s^p C_{\frac{p}{2}-1}^{i-1} + t^p C_{\frac{p}{2}-1}^{i-1} - 2s^{p-i} t^i C_{\frac{p}{2}}^i}{2p} \geq 0 \quad \text{and} \\ & \frac{s^p C_{\frac{p}{2}-1}^j + 2t^p C_{\frac{p}{2}-1}^{j-1} + t^p C_{\frac{p}{2}-1}^j - 2s^{\frac{p}{2}-j} t^{\frac{p}{2}+j} C_{\frac{p}{2}}^j}{2p} \geq 0. \end{aligned}$$

*Proof* Using the combination formula  $C_{p-1}^j = \frac{p-j}{p} C_p^j$ ,  $C_{p-1}^{j-1} = \frac{j}{p} C_p^j$  and deformation of the Young inequality  $x^\lambda y^{1-\lambda} \leq \lambda x + (1-\lambda)y$ , ( $0 < \lambda < 1$ ), there exists

$$\begin{aligned} & \frac{2s^p C_{\frac{p}{2}-1}^i C_i^j + s^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + t^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2s^{p-(i+j)} t^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \\ &= \frac{2s^p \left( \frac{\frac{p}{2}-i}{\frac{p}{2}} C_{\frac{p}{2}}^i \right) C_i^j + s^p \left( \frac{i}{\frac{p}{2}} C_{\frac{p}{2}}^i \right) \left( \frac{i-j}{i} C_i^j \right) + 2t^p \left( \frac{i}{\frac{p}{2}} C_{\frac{p}{2}}^i \right) \left( \frac{j}{i} C_i^j \right) + t^p \left( \frac{i}{\frac{p}{2}} C_{\frac{p}{2}}^i \right) \left( \frac{i-j}{i} C_i^j \right) - 2s^{p-(i+j)} t^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \\ &= \frac{C_{\frac{p}{2}}^i C_i^j [2(p-2i)s^p + 2(i-j)s^p + 4jt^p + 2(i-j)t^p - 2ps^{p-(i+j)} t^{i+j}]}{2p^2} \\ &= \frac{C_{\frac{p}{2}}^i C_i^j \left[ \frac{p-(i+j)}{p} s^p + \frac{i+j}{p} t^p - s^{p-(i+j)} t^{i+j} \right]}{p} \\ &\geq \frac{C_{\frac{p}{2}}^i C_i^j}{p} \left[ \left( s^p \right)^{\frac{p-(i+j)}{p}} \left( t^p \right)^{\frac{i+j}{p}} - s^{p-(i+j)} t^{i+j} \right] = 0. \end{aligned}$$

Similarly, we can obtain that  $\frac{2s^p C_{\frac{p}{2}-1}^j + s^p C_{\frac{p}{2}-1}^{j-1} + t^p C_{\frac{p}{2}-1}^{j-1} - 2s^{p-i} t^i C_{\frac{p}{2}}^j}{2p} \geq 0$ ,  $\frac{s^p C_{\frac{p}{2}-1}^j + t^p C_{\frac{p}{2}-1}^{j-1} - s^{p-2i} t^{2i} C_{\frac{p}{2}}^j}{p} \geq 0$  and  $\frac{s^p C_{\frac{p}{2}-1}^j + 2t^p C_{\frac{p}{2}-1}^{j-1} + t^p C_{\frac{p}{2}-1}^j - 2s^{\frac{p}{2}-j} t^{\frac{p}{2}+j} C_{\frac{p}{2}}^j}{2p} \geq 0$ . Thus, we can see that the conclusions hold.  $\square$

**Appendix 3** For all  $\frac{s_2}{s_1}, \frac{t_2}{t_1} \geq 0$  and  $0 \leq i, j \leq \frac{p}{2}$ , there exists

$$\frac{2(\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^i C_i^j + (\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2(\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + (\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2(\frac{s_2}{s_1})^{p-(i+j)} (\frac{t_2}{t_1})^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \geq 0.$$

*Proof* Using the method of Appendix 2, we have

$$\begin{aligned}
 & \frac{2(\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^i C_i^j + (\frac{s_2}{s_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j + 2(\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^{j-1} + (\frac{t_2}{t_1})^p C_{\frac{p}{2}-1}^{i-1} C_{i-1}^j - 2(\frac{s_2}{s_1})^{p-(i+j)} (\frac{t_2}{t_1})^{i+j} C_{\frac{p}{2}}^i C_i^j}{2p} \\
 &= \frac{C_{\frac{p}{2}}^i C_i^j [2(p-2i)(\frac{s_2}{s_1})^p + 2(i-j)(\frac{s_2}{s_1})^p + 4j(\frac{t_2}{t_1})^p + 2(i-j)(\frac{t_2}{t_1})^p - 2p(\frac{s_2}{s_1})^{p-(i+j)} (\frac{t_2}{t_1})^{i+j}]}{2p^2} \\
 &= \frac{C_{\frac{p}{2}}^i C_i^j}{p} \left[ \frac{p-(i+j)}{p} \left( \frac{s_2}{s_1} \right)^p + \frac{i+j}{p} \left( \frac{t_2}{t_1} \right)^p - \left( \frac{s_2}{s_1} \right)^{p-(i+j)} \left( \frac{t_2}{t_1} \right)^{i+j} \right] \\
 &\geq \frac{C_{\frac{p}{2}}^i C_i^j}{p} \left[ \left( \left( \frac{s_2}{s_1} \right)^p \right)^{\frac{p-(i+j)}{p}} \left( \left( \frac{t_2}{t_1} \right)^p \right)^{\frac{i+j}{p}} - \left( \frac{s_2}{s_1} \right)^{p-(i+j)} \left( \frac{t_2}{t_1} \right)^{i+j} \right] = 0.
 \end{aligned}$$

Thus, we complete the proof.  $\square$

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#### Data Availability

No datasets were generated or analysed during the current study.

## Declarations

#### Competing interests

The authors declare no competing interests.

#### Author contributions

Ou and Zhang contributed equally to this work. All authors reviewed the manuscript.

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