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# Blow-up solutions for a 4-dimensional semilinear elliptic system of Liouville type in some general cases

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## Abstract

This paper is devoted to the existence of singular limit solutions for a nonlinear elliptic system of Liouville type under Navier boundary conditions in a bounded open domain of  $\mathbb{R}^4$ . The concerned results are obtained employing the nonlinear domain decomposition method and a Pohozaev-type identity.

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**Keywords:** Liouville type system; Singular limit solution; Nonlinear domain decomposition method; Green's function; Pohozaev identity

## 1 Introduction and statement of the results

In recent decades, the nonlinear system has received a very significant attention in the field of mathematics and physics, since several phenomena in these areas are described by the nonlinear differential system, such as thermionic emissions, isothermal gas sphere, gas combustion and gauge theory [28]. The main objective of studying nonlinear initial boundary value problems involving partial differential equations is to designate whether solutions to a given equation develop a singularity. The blow-up problem can have an impact on the physical relevance and validity of the underlying model. Therefore, it is interesting to solve and characterize this type of problem.

In this paper, we investigate the existence of singular limit solutions for a four-dimensional semilinear elliptic system of Liouville type. More precisely, we consider the following elliptic system with Navier boundary conditions

$$\begin{cases} \Delta(a(u_1)\Delta u_1) = \rho^4 a(u_1) e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ \Delta(a(u_2)\Delta u_2) = \rho^4 a(u_2) e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ u_i = \Delta u_i = 0; \quad i = 1, 2 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a regular bounded open domain in  $\mathbb{R}^4$ ,  $\gamma, \xi$  and  $\rho$  are constants. We assume that  $\gamma, \xi \in (0, 1)$  such that  $\gamma + \xi > 1$ . If we take  $a(u) = e^{\lambda u}$  for the small parameter  $\lambda > 0$ .

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Then, problem (1) becomes as follows

$$\begin{cases} \Delta^2 u_1 + \mathcal{L}_\lambda(u_1) = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ \Delta^2 u_2 + \mathcal{L}_\lambda(u_2) = \rho^4 e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ u_i = \Delta u_i = 0; \quad i = 1, 2 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where

$$\mathcal{L}_\lambda(u) = \lambda(\Delta u)^2 + 2\lambda \nabla u \cdot \nabla(\Delta u) + \lambda^2 |\nabla u|^2 \Delta u. \quad (3)$$

The aim of this paper is to prove the existence of solutions  $(u_1, u_2)$  for the previous system. More precisely, we are interested in studying the existence of this solution with singular limits as the parameters  $\rho$  and  $\lambda$  tend to 0. We can use by deduction in all the following

$$\frac{1-\xi}{\gamma} \quad \text{and} \quad \frac{1-\gamma}{\xi} \in (0, 1). \quad (4)$$

Using the following transformation

$$\omega_1 = (\lambda \rho^4 e^{\gamma u_1})^\lambda \quad \text{and} \quad \omega_2 = (\lambda \rho^4 e^{\xi u_2})^\lambda.$$

Then,  $(\omega_1, \omega_2)$  satisfies the following system

$$\begin{cases} \Delta^2 \omega_1 = c_1 \omega_1^{\frac{\lambda+1}{\lambda}} \omega_2^{\frac{1-\gamma}{\xi\lambda}} & \text{in } \Omega, \\ \Delta^2 \omega_2 = c_2 \omega_2^{\frac{\lambda+1}{\lambda}} \omega_1^{\frac{1-\xi}{\gamma\lambda}} & \text{in } \Omega, \end{cases} \quad (5)$$

with  $\omega_1 = \omega_2 = (\lambda \rho^4)^\lambda$  on  $\partial\Omega$ , where  $c_1 = (\lambda \rho^4)^{\frac{\gamma-1}{\xi}}$  and  $c_2 = (\lambda \rho^4)^{\frac{\xi-1}{\gamma}}$ .

Note that the system (1) can be seen as a natural generalization of the following equation

$$\Delta^2 u = 6e^{4u} \quad \text{in } \mathbb{R}^4. \quad (6)$$

This type of equations appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature on four-dimensional Riemannian manifolds. For more details and background material we refer to [1, 13, 25]. The classification of solutions to the last equation has been studied by Lin, see [21]. More precisely, the author proved the following classification result.

**Theorem ([21])** *Let  $u$  be a solution of (6), satisfying the finite-mass condition*

$$\int_{\mathbb{R}^4} e^{4u} dx < \infty \quad (7)$$

*and  $|u(x)| = o(|x|^2)$  at  $\infty$ . Then there exists a point  $x^0 \in \mathbb{R}^4$  such that  $u$  is radially symmetric about  $x^0$  and*

$$u(x) = \ln \left( \frac{2\lambda}{1 + \lambda^2 |x - x^0|^2} \right).$$

This result is decisive for solving completely (6) under (7), because it reduces the problem to a simple ODE problem.

Wei in [29], has studied the behavior of solutions to the following nonlinear eigenvalue problem for the biharmonic operator  $\Delta^2$  in  $\mathbb{R}^4$ . More precisely, consider the following problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (8)$$

when  $f(u) = e^u$ . Before announcing the result of [29], we will introduce some notations. Let  $G(x, x')$  defined over  $\Omega \times \Omega$ , be the Green function associated with the bi-laplacian operator with Navier boundary conditions, which is the solution of

$$\begin{cases} \Delta_x^2 G(x, x') = 64\pi^2 \delta_{x=x'} & \text{in } \Omega, \\ G(x, x') = \Delta_x G(x, x') = 0 & \text{on } \partial\Omega, \end{cases}$$

and denote by  $H(x, x') = G(x, x') + 8 \ln |x - x'|$  its smooth part. Consider now the functional

$$E(x^1, \dots, x^m) = \sum_{j=1}^m H(x^j, x^j) + \sum_{j \neq l} G(x^j, x^l)$$

and denote by  $u^*$  the solution of

$$\begin{cases} \Delta^2 u^* = 64\pi^2 \sum_{j=1}^m \delta_{x^j} & \text{in } \Omega, \\ u^* = \Delta u^* = 0 & \text{on } \partial\Omega. \end{cases} \quad (9)$$

In [29], the author proved the following result.

**Theorem 1 ([29])** *Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^4$  and  $f$  a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \text{ tends to } 1, \quad \text{as } u \rightarrow +\infty. \quad (10)$$

For  $u_\lambda$  solution of (8), denote by  $\Sigma_\lambda = \lambda \int_\Omega f(u_\lambda) dx$ . Then there are only three possibilities:

- (i) The  $\{\Sigma_\lambda\}$  accumulate to 0. Then  $\|u_\lambda\|_{L^\infty(\Omega)} \rightarrow 0$  as  $\lambda \rightarrow 0$ .
- (ii) The  $\{\Sigma_\lambda\}$  accumulate to  $+\infty$ . Then  $u_\lambda \rightarrow +\infty$  as  $\lambda \rightarrow 0$ .
- (iii) The  $\{\Sigma_\lambda\}$  accumulate to  $64\pi^2 m$ , for some positive integer  $m$ . Then the limiting function  $u^* = \lim_{\lambda \rightarrow 0} u_\lambda$  has  $m$  blow-up points,  $\{x^1, \dots, x^m\}$ , where  $u_\lambda(x^i) \rightarrow +\infty$  as  $\lambda \rightarrow 0$ .

Moreover,  $(x^1, \dots, x^m)$  is a critical point of  $E$ .

In [8], Baraket et al. proved the inverse problem of the above result. More precisely, they considered the following problem

$$\Delta^2 u = \rho^4 e^u \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad (11)$$

where  $\Omega$  is a regular bounded domain in  $\mathbb{R}^4$  and  $\rho$  is a small parameter. More precisely, they have constructed a family of solutions  $(u_\rho)_\rho$  that converges to the function  $u^*$  as  $\rho$  tends to 0. Specifically, the authors proved the following result in [8].

**Theorem 2** [8] *Let  $\Omega$  be a smooth open subset of  $\mathbb{R}^4$  and  $x^1, \dots, x^m \in \Omega$  be given points. Assume that  $(x^1, \dots, x^m)$  is a nondegenerate critical point of  $E$ . Then, there exist  $\rho_0 > 0$  and  $(u_\rho)_{\rho \in (0, \rho_0)}$  a one parameter family of solutions of (11), such that*

$$\lim_{\rho \rightarrow 0} u_\rho = u^* \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega - \{x^1, \dots, x^m\}).$$

This result was extended in [4] for a general nonlinearity of type  $f(u) = e^u + e^{\gamma u}$  with  $\gamma \in (0, 1)$  instead of  $e^u$ . Moreover, the author proved in [3] a similar result for the following problem

$$\Delta^2 u + \mathcal{D}_\lambda(u) = \rho^4 e^u \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where  $\mathcal{D}_\lambda(u) := \lambda[(\Delta u)^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla(\Delta u)] + 2\lambda^2[\Delta u|\nabla u|^2 + \nabla u \cdot \nabla(|\nabla u|^2)] + \lambda^3|\nabla u|^4$ . Similar results were proved by other authors, see for instance [7, 17].

In dimension 2, we consider the analogous problem as follows

$$-\Delta u = \rho^2 e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{12}$$

where the parameter  $\rho$  tends to 0. The study of this equation goes back to 1853, when Liouville derived a representation formula for all solutions of (12) that are defined in  $\mathbb{R}^2$  [24]. It is well known that as the parameter  $\rho$  tends to 0, non-minimal solutions exist and they have singular limits. In [10], Baraket and Pacard proved the following result.

**Theorem 3** ([10]) *Let  $\Omega$  be a smooth open subset of  $\mathbb{R}^2$  and  $z^1, \dots, z^m \in \Omega$ . Assume that  $(z^1, \dots, z^m)$  is a nondegenerate critical point of the function*

$$F : (z^1, \dots, z^m) \in \mathbb{C}^m \longrightarrow \sum_j h(z^j, z^j) + \sum_{j \neq l} g(z^j, z^l),$$

*then there exist  $\rho_0 > 0$  and  $(u_\rho)_{\rho \in (0, \rho_0)}$  a one parameter family of solutions of (12) such that*

$$\lim_{\rho \rightarrow 0} u_\rho = u^* := \sum_{j=1}^m g(., z^j) \quad \text{in } C_{\text{loc}}^{2,\alpha}(\Omega - \{z^1, \dots, z^m\}).$$

Here  $g$  is the Green's function defined as the solution of

$$\begin{cases} -\Delta_z g(z, z') = 8\pi \delta_{z=z'} & \text{in } \Omega, \\ g(z, z') = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $h$  is its smooth part defined by  $h(z, z') = g(z, z') + 4\ln|z - z'|$ . Some generalizations can be found in [6, 12, 18]. Problems like (12) are considered by many researchers. For example, we cite [19].

When  $\lambda = 0$ , the system (1) has been studied in [2]. More precisely, the authors considered the following system

$$\begin{cases} \Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ \Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ u_i = \Delta u_i = 0, \quad i = 1, 2 & \text{on } \partial\Omega. \end{cases}$$

They proved the existence of singular limit solutions with blow-up on common points as  $\rho$  tends to 0, using the nonlinear domain decomposition method. Similar result is proved in [20]. In [9], Baraket et al. proved the existence of singular limit solutions for the above system in the case where the singular sets are disjoint.

Recently, in dimension two, Baraket et al. in [5] studied the existence of a singular limit solution for the following system in the case where the singular sets are not necessarily disjoint

$$\begin{cases} -\Delta u_1 - \lambda |\nabla u_1|^2 = \rho^2 e^{\gamma u_1 + (1-\gamma)u_2} & \text{in } \Omega, \\ -\Delta u_2 - \lambda |\nabla u_2|^2 = \rho^2 e^{\xi u_2 + (1-\xi)u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (13)$$

For  $\lambda = 0$  similar results are proved in [11]. Considerable effort has been devoted to the study of singular elliptic problems in recent years, as can be seen in, e.g., [2, 5, 11, 20, 23, 26, 28] and references therein.

In this paper, we will extend the result of [5] in dimension four. More precisely, we will show the existence of singular limit solutions of (1), which blow-up on common points when the parameter  $\rho$  and  $\lambda$  tend to 0, using the nonlinear domain decomposition method and the Pohozaev identity.

We will suppose in the following that  $\lambda$  satisfies

$$(A_1) \quad \text{If } 0 < \varepsilon < \lambda, \text{ then } \lambda^{1+\frac{\mu}{2}} \varepsilon^{-\mu} \rightarrow 0 \text{ as } \lambda \rightarrow 0, \text{ for any } \mu \in (1, 2).$$

$$(A_2) \quad \text{If } 0 < \varepsilon < \lambda, \text{ then } \lambda^{1+\frac{\delta}{2}} \varepsilon^{-\delta} \rightarrow 0 \text{ as } \lambda \rightarrow 0, \text{ for any } \delta \in \left(0, \min \left\{ \left( \frac{\gamma + \xi - 1}{\gamma} \right), \left( \frac{\gamma + \xi - 1}{\xi} \right) \right\} \right).$$

In order to facilitate the presentation of the main theorems, we consider the special case where we have only three singularities  $x^1, x^2$  and  $x^3$ . Specifically, our aim in this paper is to construct singular limits  $(u_1, u_2)$  such that  $u_1$  blow-up in  $x^1, x^2$  and  $u_2$  blow-up in  $x^2, x^3$ .

At first, we give a necessary condition about the position of the points  $x^1, x^2$ , and  $x^3$  thanks to Pohozaev identity. More precisely, we prove the following result.

**Theorem 4** *Let  $\Omega$  be a regular open subset of  $\mathbb{R}^4$  and  $x^1, x^2, x^3 \in \Omega$  be given disjoint points. Suppose that  $(u_1^{\rho, \lambda}, u_2^{\rho, \lambda})$  is a one parameter family of solutions of (1), such that*

$$\lim_{\rho, \lambda \rightarrow 0} u_1^{\rho, \lambda} = \frac{1}{\gamma} G(\cdot, x^1) + G(\cdot, x^2) = u_1^* \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^1, x^2\})$$

and

$$\lim_{\rho, \lambda \rightarrow 0} u_2^{\rho, \lambda} = \frac{1}{\xi} G(\cdot, x^3) + G(\cdot, x^2) = u_2^* \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^2, x^3\}).$$

Then  $(x^1, x^2, x^3)$  is a critical point of the functional

$$\begin{aligned} \mathcal{E}(x^1, x^2, x^3) &= \frac{1-\xi}{\gamma} H(x^1, x^1) + (2-\gamma-\xi) H(x^2, x^2) + \frac{1-\gamma}{\xi} H(x^3, x^3) \\ &\quad + \frac{(1-\gamma)(1-\xi)}{\gamma} G(x^1, x^3) + \frac{1-\xi}{\gamma} G(x^1, x^2) + \frac{1-\gamma}{\xi} G(x^2, x^3). \end{aligned}$$

A natural question that arises: can one find a solution that concentrates in a common point  $x^2$ . Before giving a partial answer of this question, we define an auxiliary function which is a cut-off function in  $C_0^\infty(\Omega)$  such that  $\varphi \equiv 1$  in  $B(x^1, r_0) \cup B(x^3, r_0)$  and  $\varphi \equiv 0$  in  $\Omega \setminus (B(x^1, r_0) \cup B(x^3, r_0))$ , where  $r_0 > 0$  and such that  $B(x^i, 2r_0) \subset \Omega$  for  $i = 1, 3$  and  $B(x^1, 2r_0) \cap B(x^3, 2r_0) = \emptyset$ .

**Theorem 5** Let  $\Omega$  be a regular open subset of  $\mathbb{R}^4$ ,  $\lambda > 0$  satisfying (A<sub>1</sub>)-(A<sub>2</sub>) and  $x^1, x^2, x^3 \in \Omega$  be given disjoint points. We Suppose that  $(x^1, x^2, x^3)$  is a nondegenerate critical point of the functional

$$\begin{aligned} \mathcal{E}(x^1, x^2, x^3) &= \frac{1-\xi}{\gamma} H(x^1, x^1) + (2-\gamma-\xi) H(x^2, x^2) + \frac{1-\gamma}{\xi} H(x^3, x^3) \\ &\quad + \frac{(1-\gamma)(1-\xi)}{\gamma} G(x^1, x^3) + \frac{1-\xi}{\gamma} G(x^1, x^2) + \frac{1-\gamma}{\xi} G(x^2, x^3). \end{aligned} \quad (14)$$

Then there exist  $\gamma_0$  and  $\xi_0$  in  $(0, 1)$  such that for all  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ , there exist  $\rho_0 > 0$ ,  $\lambda_0 > 0$  and  $(u_1^{\rho, \lambda}, u_2^{\rho, \lambda})_{\rho \leq \rho_0, \lambda \leq \lambda_0}$  a one parameter family of solutions of (1), such that

$$\begin{aligned} \lim_{\rho, \lambda \rightarrow 0} \varphi u_1^{\rho, \lambda} &= \frac{\varphi}{\gamma} G(\cdot, x^1) \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^1\}), \\ \lim_{\rho, \lambda \rightarrow 0} \varphi u_2^{\rho, \lambda} &= \frac{\varphi}{\xi} G(\cdot, x^3) \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^3\}) \end{aligned}$$

and

$$\begin{aligned} \lim_{\rho, \lambda \rightarrow 0} ((1-\xi) u_1^{\rho, \lambda} + (1-\gamma) u_2^{\rho, \lambda}) \\ = \frac{1-\xi}{\gamma} G(\cdot, x^1) + \frac{1-\gamma}{\xi} G(\cdot, x^3) + (2-\gamma-\xi) G(\cdot, x^2) \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^1, x^2, x^3\}). \end{aligned}$$

Unfortunately, we are not able to give the asymptotic behavior of  $u_1^{\rho, \lambda}$  and  $u_2^{\rho, \lambda}$  separately. But, under an additional assumption on the points set  $\{x^1, x^2, x^3\}$ , we give a positive answer.

**Theorem 6** Let  $\Omega$  be a regular open subset of  $\mathbb{R}^4$ ,  $\lambda > 0$  satisfying (A<sub>1</sub>)-(A<sub>2</sub>) and  $x^1, x^2, x^3 \in \Omega$  be given disjoint points. We Suppose that  $(x^1, x^2, x^3)$  is a nondegenerate critical point of the functional

$$\mathcal{E}(x^1, x^2, x^3) = \frac{1-\xi}{\gamma} H(x^1, x^1) + (2-\gamma-\xi) H(x^2, x^2) + \frac{1-\gamma}{\xi} H(x^3, x^3)$$

$$+ \frac{(1-\gamma)}{\xi} \frac{(1-\xi)}{\gamma} G(x^1, x^3) + \frac{1-\xi}{\gamma} G(x^1, x^2) + \frac{1-\gamma}{\xi} G(x^2, x^3) \quad (15)$$

such that

$$\frac{1}{\gamma} G(x^2, x^1) = \frac{1}{\xi} G(x^2, x^3) \quad \text{and} \quad \frac{1}{\gamma} \nabla G(\cdot, x^1)(x^2) = \frac{1}{\xi} \nabla G(\cdot, x^3)(x^2). \quad (16)$$

Then there exist  $\gamma_0$  and  $\xi_0$  in  $(0, 1)$  such that for all  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ , there exist  $\rho_0 > 0$ ,  $\lambda_0 > 0$ , and  $(u_1^{\rho, \lambda}, u_2^{\rho, \lambda})_{\rho \leq \rho_0, \lambda \leq \lambda_0}$  a one parameter family of solutions of (1) such that

$$\begin{aligned} \lim_{\rho, \lambda \rightarrow 0} u_1^{\rho, \lambda} &= \frac{1}{\gamma} G(\cdot, x^1) + G(\cdot, x^2) \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^1, x^2\}), \\ \lim_{\rho, \lambda \rightarrow 0} u_2^{\rho, \lambda} &= \frac{1}{\xi} G(\cdot, x^3) + G(\cdot, x^2) \quad \text{in } C_{\text{loc}}^{4,\alpha}(\Omega \setminus \{x^2, x^3\}). \end{aligned}$$

Let us now briefly outline the organization of the content of this paper: In Sect. 2, we give necessary conditions for the position of the blow-up points  $(x^1, x^2, x^3)$ , thanks to the Pohozaev identity and by using the techniques inspired by the work of Suzuki [27]. In Sect. 3, we prove Theorem 5, motivated by the technics of Baraket et al. [8]. Indeed, we discuss rotationally symmetric solutions of (1), we study the linearized operators around the radially symmetric solution. We recall some known results about the analysis of the BiLaplace operator in weighted spaces. Next, we study a nonlinear interior problem proving the existence of a family of solutions of (1) that are close to the rotationally symmetric solution. Then, we prove the existence of a family of solutions to (1) defined on  $\Omega$  with small balls removed. Finally, we show how elements of these families can be connected to produce solutions of (1) described in Theorem 6. In fact, we patch these pieces together via a nonlinear version of the Cauchy data matching.

*Remark* The conditions of Theorem 6 are certainly not valid on all domain of  $\mathbb{R}^4$ . It is thought that a certain symmetry of the domain must be imposed for the condition (16) to be verified.

## 2 Proof of Theorem 4

We first give the green identity for the bilaplacian operator:

$$\int_{\Omega} (\Delta^2 u) \cdot v - (\Delta^2 v) \cdot u = \int_{\partial\Omega} \left( \frac{\partial \Delta u}{\partial \nu} \cdot v - \Delta u \frac{\partial v}{\partial \nu} + \frac{\partial u}{\partial \nu} \cdot \Delta v - u \cdot \frac{\partial \Delta v}{\partial \nu} \right) d\sigma. \quad (17)$$

### 2.1 Behavior of solution around $x^2$

For  $i = 1, 2$ , let  $\mathcal{L}_\lambda(u_i)$  be defined by (3). We multiply the equation  $\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} - \mathcal{L}_\lambda(u_1)$  by  $\nabla(\gamma u_1 + (1-\gamma)u_2)$  and then integrating over  $B_2 = B(x^2, \eta)$  where  $\eta$  fixed small enough, we obtain a Pohozaev-type identity

$$\begin{aligned} \gamma \int_{B_2} (\Delta^2 u_1) \nabla u_1 + (1-\gamma) \int_{B_2} (\Delta^2 u_1) \nabla u_2 \\ = \rho^4 \int_{\partial B_2} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v d\sigma - \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 + (1-\gamma)u_2). \end{aligned} \quad (18)$$

Using the Green's formula, we obtain

$$\begin{aligned} \int_{B_2} (\Delta^2 u_1) \nabla u_1 &= - \int_{B_2} (\Delta u_1) \nabla (\Delta(u_1)) - \int_{B_2} \nabla (\nabla(\Delta u_1) \cdot \nabla u_1) \\ &\quad + \int_{\partial B_2} \nabla(\Delta u_1) \cdot v \nabla u_1 d\sigma + \int_{\partial B_2} \nabla u_1 \cdot v \nabla(\Delta u_1) d\sigma \\ &= -\frac{1}{2} \int_{\partial B_2} (\Delta u_1)^2 v d\sigma - \int_{\partial B_2} (\nabla(\Delta u_1) \cdot \nabla u_1) v d\sigma \\ &\quad + \int_{\partial B_2} \nabla(\Delta u_1) \cdot v \nabla u_1 d\sigma + \int_{\partial B_2} \nabla u_1 \cdot v \nabla(\Delta u_1) d\sigma. \end{aligned}$$

Similarly, we multiply the equation  $\Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1} - \mathcal{L}_\lambda(u_2)$  by  $\nabla(\xi u_2 + (1-\xi)u_1)$  and then integrating over  $B_2 = B(x^2, \eta)$ , we obtain a Pohozaev-type identity

$$\begin{aligned} \xi \int_{B_2} (\Delta^2 u_2) \nabla u_2 + (1-\xi) \int_{B_2} (\Delta^2 u_2) \nabla u_1 \\ = \rho^4 \int_{\partial B_2} (e^{\xi u_2 + (1-\xi)u_1} - 1) v d\sigma - \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1-\xi)u_1). \end{aligned} \quad (19)$$

Using the Green's formula, we obtain

$$\begin{aligned} \int_{B_2} (\Delta^2 u_2) \nabla u_2 &= - \int_{B_2} (\Delta u_2) \nabla (\Delta(u_2)) - \int_{B_2} \nabla (\nabla(\Delta u_2) \cdot \nabla u_2) \\ &\quad + \int_{\partial B_2} \nabla(\Delta u_2) \cdot v \nabla u_2 d\sigma + \int_{\partial B_2} \nabla u_2 \cdot v \nabla(\Delta u_2) d\sigma \\ &= -\frac{1}{2} \int_{\partial B_2} (\Delta u_2)^2 v d\sigma - \int_{\partial B_2} (\nabla(\Delta u_2) \cdot \nabla u_2) v d\sigma \\ &\quad + \int_{\partial B_2} \nabla(\Delta u_2) \cdot v \nabla u_2 d\sigma + \int_{\partial B_2} \nabla u_2 \cdot v \nabla(\Delta u_2) d\sigma. \end{aligned}$$

Using the identity

$$\begin{aligned} \int_{B_2} \Delta^2 u_2 \nabla u_1 + \int_{B_2} \Delta^2 u_1 \nabla u_2 \\ = - \int_{B_2} \nabla(\Delta u_2 \cdot \Delta u_1) + \int_{\partial B_2} \frac{\partial(\Delta u_2)}{\partial v} \nabla u_1 d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_1)}{\partial v} \nabla u_2 d\sigma \\ = - \int_{\partial B_2} (\Delta u_2 \cdot \Delta u_1) v d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_2)}{\partial v} \nabla u_1 d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_1)}{\partial v} \nabla u_2 d\sigma, \end{aligned}$$

then by combination of (18) and (19), we obtain

$$\begin{aligned} \gamma(1-\xi) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_1)^2 v - (\nabla(\Delta u_1) \cdot \nabla u_1) v + \nabla(\Delta u_1) \cdot v \nabla u_1 \right. \\ \left. + \nabla u_1 \cdot v \nabla(\Delta u_1) \right] d\sigma \\ + \xi(1-\gamma) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_2)^2 v - (\nabla(\Delta u_2) \cdot \nabla u_2) v \right. \\ \left. + \nabla(\Delta u_2) \cdot v \nabla u_2 + \nabla u_2 \cdot v \nabla(\Delta u_2) \right] d\sigma \end{aligned}$$

$$\begin{aligned}
& + \nabla(\Delta u_2) \cdot v \nabla u_2 + \nabla u_2 \cdot v \nabla(\Delta u_2) \Big] d\sigma \\
& + (1-\gamma)(1-\xi) \left[ - \int_{\partial B_2} (\Delta u_2 \cdot \Delta u_1) v \, d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_2)}{\partial v} \nabla u_1 \, d\sigma \right. \\
& \quad \left. + \int_{\partial B_2} \frac{\partial(\Delta u_1)}{\partial v} \nabla u_2 \, d\sigma \right] \\
& = \rho^4(1-\xi) \int_{\partial B_2} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v \, d\sigma + \rho^4(1-\gamma) \int_{\partial B_2} (e^{\xi u_2 + (1-\xi)u_1} - 1) v \, d\sigma \\
& \quad - (1-\xi) \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 + (1-\gamma)u_2) \\
& \quad - (1-\gamma) \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1-\xi)u_1). \tag{20}
\end{aligned}$$

In the desire to construct solutions of the system that blow-up in the point  $x^2$ , this means that if  $\rho$  and  $\lambda$  tend to zero,

$$u_1 \rightarrow u_1^*(x) = G(x, x^2) + \frac{1}{\gamma} G(x, x^1) \quad \text{and} \quad u_2 \rightarrow u_2^*(x) = G(x, x^2) + \frac{1}{\xi} G(x, x^3).$$

Since we have  $G(x, x^2) = -8 \ln|x - x^2| + H(x, x^2)$ , where  $H$  is a smooth function in  $\Omega$ , then

$$\begin{aligned}
u_1^*(x) &= G(x, x^2) + \frac{1}{\gamma} G(x, x^1) = -8 \ln|x - x^2| + H(x, x^2) + \frac{1}{\gamma} G(x, x^1) \\
&= -8 \ln|x - x^2| + R(x, x^2)
\end{aligned}$$

and

$$\begin{aligned}
u_2^*(x) &= G(x, x^2) + \frac{1}{\xi} G(x, x^3) = -8 \ln|x - x^2| + H(x, x^2) + \frac{1}{\xi} G(x, x^3) \\
&= -8 \ln|x - x^2| + K(x, x^2).
\end{aligned}$$

Thanks to the fact that the solutions of the system (1) are regular on  $\Omega \setminus \{x^1, x^2, x^3\}$  and by inserting the profile of the limits of the solutions in the identity (20), when  $\rho$  and  $\lambda$  tend to zero and  $\eta$  fixed small enough, we obtain

$$\begin{aligned}
& \lim_{\rho, \lambda \rightarrow 0} \left( \rho^4(1-\xi) \int_{\partial B_2} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v \, d\sigma + \rho^4(1-\gamma) \int_{\partial B_2} (e^{\xi u_2 + (1-\xi)u_1} - 1) v \, d\sigma \right. \\
& \quad \left. - (1-\xi) \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 + (1-\gamma)u_2) \right. \\
& \quad \left. - (1-\gamma) \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1-\xi)u_1) \right) = 0,
\end{aligned}$$

then

$$\begin{aligned}
& \gamma(1-\xi) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_1^*)^2 v - (\nabla(\Delta u_1^*) \cdot \nabla u_1^*) v + \nabla(\Delta u_1^*) \cdot v \nabla u_1^* \right. \\
& \quad \left. + \nabla u_1^* \cdot v \nabla(\Delta u_1^*) \right] d\sigma
\end{aligned}$$

$$\begin{aligned}
& + \xi(1-\gamma) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_2^*)^2 v - (\nabla(\Delta u_2^*) \cdot \nabla u_2^*) v \right. \\
& \quad \left. + \nabla(\Delta u_2^*) \cdot v \nabla u_2^* + \nabla u_2^* \cdot v \nabla(\Delta u_2^*) \right] d\sigma \\
& + (1-\gamma)(1-\xi) \left[ - \int_{\partial B_2} (\Delta u_2^* \cdot \Delta u_1^*) v d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_2^*)}{\partial v} \nabla u_1^* d\sigma \right. \\
& \quad \left. + \int_{\partial B_2} \frac{\partial(\Delta u_1^*)}{\partial v} \nabla u_2^* d\sigma \right] \\
& = 0. \tag{21}
\end{aligned}$$

We set

$$\begin{aligned}
I_{\text{lhs}} = & \gamma(1-\xi) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_1^*)^2 v - (\nabla(\Delta u_1^*) \cdot \nabla u_1^*) v + \nabla(\Delta u_1^*) \cdot v \nabla u_1^* \right. \\
& \quad \left. + \nabla u_1^* \cdot v \nabla(\Delta u_1^*) \right] d\sigma \\
& + \xi(1-\gamma) \int_{\partial B_2} \left[ \frac{-1}{2} (\Delta u_2^*)^2 v - (\nabla(\Delta u_2^*) \cdot \nabla u_2^*) v \right. \\
& \quad \left. + \nabla(\Delta u_2^*) \cdot v \nabla u_2^* + \nabla u_2^* \cdot v \nabla(\Delta u_2^*) \right] d\sigma \\
& + (1-\gamma)(1-\xi) \left[ - \int_{\partial B_2} (\Delta u_2^* \cdot \Delta u_1^*) v d\sigma + \int_{\partial B_2} \frac{\partial(\Delta u_2^*)}{\partial v} \nabla u_1^* d\sigma \right. \\
& \quad \left. + \int_{\partial B_2} \frac{\partial(\Delta u_1^*)}{\partial v} \nabla u_2^* d\sigma \right],
\end{aligned}$$

by computation, we prove that

$$\begin{aligned}
I_{\text{lhs}} = & -\frac{8}{\eta} \gamma(1-\xi) \int_{\partial B_2} \nabla \Delta R(x, x^2) d\sigma - \frac{8}{\eta} \xi(1-\gamma) \int_{\partial B_2} \nabla \Delta K(x, x^2) d\sigma \\
& + \frac{8}{\eta} (1-\xi)(1-\gamma) \int_{\partial B_2} ((\nabla \Delta K(x, x^2) + \nabla \Delta R(x, x^2)) \cdot v) v d\sigma \\
& + \frac{16}{\eta^2} \left[ (1-\xi) \int_{\partial B_2} \Delta R(x, x^2) v d\sigma + (1-\gamma) \int_{\partial B_2} \Delta K(x, x^2) v d\sigma \right] \\
& + \frac{32}{\eta^3} \left[ (1-\xi) \int_{\partial B_2} \nabla R(x, x^2) d\sigma + (1-\gamma) \int_{\partial B_2} \nabla K(x, x^2) d\sigma \right] + O(\eta). \tag{22}
\end{aligned}$$

Then we have

$$\begin{aligned}
& -8\eta\gamma(1-\xi) \int_{\partial B_2} \nabla \Delta R(x, x^2) d\sigma - 8\eta\xi(1-\gamma) \int_{\partial B_2} \nabla \Delta K(x, x^2) d\sigma \\
& + 8\eta(1-\xi)(1-\gamma) \int_{\partial B_2} ((\nabla \Delta K(x, x^2) + \nabla \Delta R(x, x^2)) \cdot v) v d\sigma \\
& + 16 \left[ (1-\xi) \int_{\partial B_2} \Delta R(x, x^2) v d\sigma + (1-\gamma) \int_{\partial B_2} \Delta K(x, x^2) v d\sigma \right] \\
& + \frac{32}{\eta} \left[ (1-\xi) \int_{\partial B_2} \nabla R(x, x^2) d\sigma + (1-\gamma) \int_{\partial B_2} \nabla K(x, x^2) d\sigma \right] = O(\eta^3), \tag{23}
\end{aligned}$$

writing  $\nabla R(x, x^2) = \nabla R(x^2, x^2) + O(\eta)$  and  $\nabla K(x, x^2) = \nabla K(x^2, x^2) + O(\eta)$ , we obtain

$$(1 - \xi)\nabla R(x^2, x^2) + (1 - \gamma)\nabla K(x^2, x^2) = O(\eta^3),$$

which means that  $x^2$  is a critical point of the functional

$$\mathcal{E}_2 : x \mapsto \frac{1 - \xi}{\gamma(2 - \gamma - \xi)}G(x, x^1) + H(x, x^2) + \frac{1 - \gamma}{\xi(2 - \gamma - \xi)}G(x, x^3). \quad (24)$$

## 2.2 Behavior of solution around $x^1$ and $x^3$

We multiply the equation  $\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} - \mathcal{L}_\lambda(u_1)$  by  $\nabla(\gamma u_1 + (1-\gamma)u_2)$  and then integrating over

$B_1 = B(x^1, \eta)$ , we obtain a Pohozaev-type identity

$$\begin{aligned} & \gamma \int_{B_1} (\Delta^2 u_1) \nabla u_1 + (1 - \gamma) \int_{B_1} (\Delta^2 u_1) \nabla u_2 \\ &= \rho^4 \int_{\partial B_1} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v \, d\sigma - \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 + (1 - \gamma)u_2). \end{aligned} \quad (25)$$

Using the Green's formula, we obtain

$$\begin{aligned} & \int_{B_1} (\Delta^2 u_1) \nabla u_1 \\ &= - \int_{B_1} (\Delta u_1) \nabla(\Delta(u_1)) - \int_{B_1} \nabla(\nabla(\Delta u_1) \cdot \nabla u_1) \\ & \quad + \int_{\partial B_1} \nabla(\Delta u_1) \cdot v \nabla u_1 \, d\sigma + \int_{\partial B_1} \nabla u_1 \cdot v \nabla(\Delta u_1) \, d\sigma \\ &= -\frac{1}{2} \int_{\partial B_1} (\Delta u_1)^2 v \, d\sigma - \int_{\partial B_1} (\nabla(\Delta u_1) \cdot \nabla u_1) v \, d\sigma \\ & \quad + \int_{\partial B_1} \nabla(\Delta u_1) \cdot v \nabla u_1 \, d\sigma + \int_{\partial B_1} \nabla u_1 \cdot v \nabla(\Delta u_1) \, d\sigma. \end{aligned}$$

Similarly, we multiply the equation  $\Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1} - \mathcal{L}_\lambda(u_2)$  by  $\nabla(\xi u_2 + (1 - \xi)u_1)$  and then integrating over  $B_1 = B(x^1, \eta)$ , we obtain a Pohozaev-type identity

$$\begin{aligned} & \xi \int_{B_1} (\Delta^2 u_2) \nabla u_2 + (1 - \xi) \int_{B_1} (\Delta^2 u_2) \nabla u_1 \\ &= \rho^4 \int_{\partial B_1} (e^{\xi u_2 + (1-\xi)u_1} - 1) v \, d\sigma - \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1 - \xi)u_1). \end{aligned} \quad (26)$$

Using the Green's formula, we obtain

$$\begin{aligned} & \int_{B_1} (\Delta^2 u_2) \nabla u_2 \\ &= - \int_{B_1} (\Delta u_2) \nabla(\Delta(u_2)) - \int_{B_1} \nabla(\nabla(\Delta u_2) \cdot \nabla u_2) \\ & \quad + \int_{\partial B_1} \nabla(\Delta u_2) \cdot v \nabla u_2 \, d\sigma + \int_{\partial B_1} \nabla u_2 \cdot v \nabla(\Delta u_2) \, d\sigma \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \int_{\partial B_1} (\Delta u_2)^2 v \, d\sigma - \int_{\partial B_1} (\nabla(\Delta u_2) \cdot \nabla u_2) v \, d\sigma \\
&\quad + \int_{\partial B_1} \nabla(\Delta u_2) \cdot v \nabla u_2 \, d\sigma + \int_{\partial B_1} \nabla u_2 \cdot v \nabla(\Delta u_2) \, d\sigma.
\end{aligned}$$

Using the identity

$$\begin{aligned}
&\int_{B_1} \Delta^2 u_2 \nabla u_1 + \int_{B_1} \Delta^2 u_1 \nabla u_2 \\
&= - \int_{B_1} \nabla(\Delta u_2 \cdot \Delta u_1) + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial \nu} \nabla u_1 \, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial \nu} \nabla u_2 \, d\sigma \\
&= - \int_{\partial B_1} (\Delta u_2 \cdot \Delta u_1) v \, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial \nu} \nabla u_1 \, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial \nu} \nabla u_2 \, d\sigma,
\end{aligned}$$

then combining (25) and (26), we obtain

$$\begin{aligned}
&\gamma(1-\xi) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_1)^2 v - (\nabla(\Delta u_1) \cdot \nabla u_1) v + \nabla(\Delta u_1) \cdot v \nabla u_1 + \nabla u_1 \cdot v \nabla(\Delta u_1) \right] d\sigma \\
&\quad + \xi(1-\gamma) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_2)^2 v - (\nabla(\Delta u_2) \cdot \nabla u_2) v \right. \\
&\quad \left. + \nabla(\Delta u_2) \cdot v \nabla u_2 + \nabla u_2 \cdot v \nabla(\Delta u_2) \right] d\sigma \\
&\quad + (1-\gamma)(1-\xi) \left[ - \int_{\partial B_1} (\Delta u_2 \cdot \Delta u_1) v \, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial \nu} \nabla u_1 \, d\sigma \right. \\
&\quad \left. + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial \nu} \nabla u_2 \, d\sigma \right] \\
&= \rho^4(1-\xi) \int_{\partial B_1} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v \, d\sigma + \rho^4(1-\gamma) \int_{\partial B_1} (e^{\xi u_2 + (1-\xi)u_1} - 1) v \, d\sigma \\
&\quad - (1-\xi) \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 \\
&\quad + (1-\gamma)u_2) - (1-\gamma) \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1-\xi)u_1). \tag{27}
\end{aligned}$$

In the desire to construct solutions of the system that blow-up at the point  $x^1$ , this means that if  $\rho$  and  $\lambda$  tend to zero,

$$u_1 \rightarrow u_1^*(x) = G(x, x^2) + \frac{1}{\gamma} G(x, x^1) \quad \text{and} \quad u_2 \rightarrow u_2^*(x) = G(x, x^2) + \frac{1}{\xi} G(x, x^3).$$

Since we have  $G(x, x^1) = -8 \ln|x - x^1| + H(x, x^1)$ , where  $H$  is a smooth function in  $\Omega$ , then

$$\begin{aligned}
u_1^*(x) &= G(x, x^2) + \frac{1}{\gamma} G(x, x^1) = -\frac{8}{\gamma} \ln|x - x^1| + \frac{1}{\gamma} H(x, x^1) + G(x, x^2) \\
&= -\frac{8}{\gamma} \ln|x - x^1| + S(x, x^1)
\end{aligned}$$

and

$$u_2^*(x) = G(x, x^2) + \frac{1}{\xi} G(x, x^3) = T(x, x^2, x^3).$$

Thanks to the fact that the solutions of the system (1) are regular on  $\Omega \setminus \{x^1, x^2, x^3\}$  and by inserting the profile of the limits of the solutions in the identity (20), when  $\rho$  and  $\lambda$  tend to zero and  $\eta$  fixed small enough, we obtain

$$\begin{aligned} & \lim_{\rho, \lambda \rightarrow 0} \left( \rho^4 (1 - \xi) \int_{\partial B_2} (e^{\gamma u_1 + (1-\gamma)u_2} - 1) v d\sigma + \rho^4 (1 - \gamma) \int_{\partial B_2} (e^{\xi u_2 + (1-\xi)u_1} - 1) v d\sigma \right. \\ & \quad \left. - (1 - \xi) \int_{B_2} \mathcal{L}_\lambda(u_1) \nabla(\gamma u_1 + (1 - \gamma)u_2) \right. \\ & \quad \left. - (1 - \gamma) \int_{B_2} \mathcal{L}_\lambda(u_2) \nabla(\xi u_2 + (1 - \xi)u_1) \right) = 0, \end{aligned}$$

then

$$\begin{aligned} & \gamma(1 - \xi) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_1^*)^2 v - (\nabla(\Delta u_1^*) \cdot \nabla u_1^*) v + \nabla(\Delta u_1^*) \cdot v \nabla u_1^* + \nabla u_1^* \cdot v \nabla(\Delta u_1^*) \right] d\sigma \\ & + \xi(1 - \gamma) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_2^*)^2 v - (\nabla(\Delta u_2^*) \cdot \nabla u_2^*) v \right. \\ & \quad \left. + \nabla(\Delta u_2^*) \cdot v \nabla u_2^* + \nabla u_2^* \cdot v \nabla(\Delta u_2^*) \right] d\sigma \\ & + (1 - \gamma)(1 - \xi) \left[ - \int_{\partial B_1} (\Delta u_2^* \cdot \Delta u_1^*) v d\sigma \right. \\ & \quad \left. + \int_{\partial B_1} \frac{\partial(\Delta u_2^*)}{\partial v} \nabla u_1^* d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1^*)}{\partial v} \nabla u_2^* d\sigma \right] \\ & = 0. \end{aligned} \tag{28}$$

We set

$$\begin{aligned} I_{\text{lhs}} = & \gamma(1 - \xi) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_1^*)^2 v - (\nabla(\Delta u_1^*) \cdot \nabla u_1^*) v \right. \\ & \quad \left. + \nabla(\Delta u_1^*) \cdot v \nabla u_1^* + \nabla u_1^* \cdot v \nabla(\Delta u_1^*) \right] d\sigma \\ & + \xi(1 - \gamma) \int_{\partial B_1} \left[ \frac{-1}{2} (\Delta u_2^*)^2 v - (\nabla(\Delta u_2^*) \cdot \nabla u_2^*) v \right. \\ & \quad \left. + \nabla(\Delta u_2^*) \cdot v \nabla u_2^* + \nabla u_2^* \cdot v \nabla(\Delta u_2^*) \right] d\sigma \\ & + (1 - \gamma)(1 - \xi) \left[ - \int_{\partial B_1} (\Delta u_2^* \cdot \Delta u_1^*) v d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_2^*)}{\partial v} \nabla u_1^* d\sigma \right. \\ & \quad \left. + \int_{\partial B_1} \frac{\partial(\Delta u_1^*)}{\partial v} \nabla u_2^* d\sigma \right], \end{aligned}$$

by computation, we prove that

$$\begin{aligned}
I_{\text{lhs}} = & -\frac{8}{\eta}(1-\xi)\int_{\partial B_1} \nabla \Delta S(x, x^1) d\sigma + \frac{8}{\gamma\eta}(1-\gamma)(1-\xi)\int_{\partial B_1} \left(\frac{\partial \Delta T(x, x^2, x^3)}{\partial v}\right)_v d\sigma \\
& + \frac{16}{\eta^2} \left[ (1-\xi)\int_{\partial B_1} \Delta S(x, x^1) v d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma} \int_{\partial B_1} \Delta T(x, x^2, x^3) v d\sigma \right] \\
& + \frac{32}{\eta^3} \left[ (1-\xi)\int_{\partial B_1} \nabla S(x, x^1) d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma} \int_{\partial B_1} \nabla T(x, x^2, x^3) d\sigma \right] \\
& + O(\eta).
\end{aligned} \tag{29}$$

Then we have

$$\begin{aligned}
& -8\eta(1-\xi)\int_{\partial B_1} \nabla \Delta S(x, x^1) d\sigma + \frac{8\eta}{\gamma}(1-\gamma)(1-\xi)\int_{\partial B_1} \left(\frac{\partial \Delta T(x, x^2, x^3)}{\partial v}\right)_v d\sigma \\
& + 16 \left[ (1-\xi)\int_{\partial B_1} \Delta S(x, x^1) v d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma} \int_{\partial B_1} \Delta T(x, x^2, x^3) v d\sigma \right] \\
& + \frac{32}{\eta} \left[ (1-\xi)\int_{\partial B_1} \nabla S(x, x^1) d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma} \int_{\partial B_1} \nabla T(x, x^2, x^3) d\sigma \right] \\
& = O(\eta^3),
\end{aligned} \tag{30}$$

writing  $\nabla S(x, x^1) = \nabla S(x^1, x^1) + O(\eta)$  and  $\nabla T(x, x^2, x^3) = \nabla T(x^1, x^2, x^3) + O(\eta)$ , we obtain

$$(1-\xi)\nabla S(x^1, x^1) + \frac{(1-\gamma)(1-\xi)}{\gamma} \nabla T(x^1, x^2, x^3) = O(\eta^3),$$

which means that  $x^1$  is a critical point of the functional

$$\mathcal{E}_1 : x \mapsto H(., x^1) + G(., x^2) + \frac{1-\gamma}{\xi} G(., x^3). \tag{31}$$

In  $B_3 = B(x^3, \eta)$ , we proceed similarly as in  $B_1 = B(x^1, \eta)$  and, taking into account the changes, we obtain that  $x^3$  is a critical point of the functional

$$\mathcal{E}_3 : x \mapsto H(., x^3) + G(., x^2) + \frac{1-\xi}{\gamma} G(., x^1). \tag{32}$$

Finally, by combination of (24), (31), and (32), we conclude that the point  $(x^1, x^2, x^3)$  is a critical point of the functional  $\mathcal{E}$  defined by

$$\begin{aligned}
\mathcal{E}(x^1, x^2, x^3) = & \frac{1-\xi}{\gamma} H(x^1, x^1) + (2-\gamma-\xi) H(x^2, x^2) + \frac{1-\gamma}{\xi} H(x^3, x^3) \\
& + \frac{(1-\xi)}{\gamma} G(x^1, x^2) + \frac{(1-\xi)(1-\gamma)}{\gamma\xi} G(x^1, x^3) + \frac{(1-\gamma)}{\xi} G(x^3, x^2).
\end{aligned}$$

### 3 Proof of Theorem 5

#### 3.1 Construction of the approximate solution

We denote by  $\varepsilon$  the smallest positive parameter satisfying

$$\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Let

$$u_\varepsilon(x) := 4\ln(1+\varepsilon^2) - 4\ln(\varepsilon^2 + |x|^2), \quad (33)$$

which is a solution of

$$\Delta^2 u = \rho^4 e^u \quad \text{in } \mathbb{R}^4. \quad (34)$$

Hence for all  $\tau > 0$  the function

$$u_{\varepsilon,\tau}(x) := 4\ln(1+\varepsilon^2) + 4\ln\tau - 4\ln(\varepsilon^2 + |\tau x|^2) \quad (35)$$

is also solution to (34).

##### 3.1.1 A linearized operator

First we introduce some definitions and notations:

**Definition 1** Given  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ ,  $\mu \in \mathbb{R}$ , and  $|x| = r$ , we define the Hölder weighted space  $C_\mu^{k,\alpha}(\mathbb{R}^4)$  as the space of functions  $w \in C_{\text{loc}}^{k,\alpha}(\mathbb{R}^4)$  for which the following norm

$$\|u\|_{C_\mu^{k,\alpha}(\mathbb{R}^4)} = \|u\|_{C^{k,\alpha}(\bar{B}_1(0))} + \sup_{r \geq 1} ((1+r^2)^{-\frac{\mu}{2}} \|u(r \cdot)\|_{C^{k,\alpha}(\bar{B}_1(0)-B_{\frac{1}{2}}(0))})$$

is finite. Similarly, for given  $\bar{r} \geq 1$ , let  $C_\mu^{k,\alpha}(B_{\bar{r}}(0))$  be the space of functions in  $C^{k,\alpha}(B_{\bar{r}}(0))$  for which the following norm

$$\|u\|_{C_\mu^{k,\alpha}(B_{\bar{r}}(0))} = \|u\|_{C^{k,\alpha}(B_1(0))} + \sup_{1 \leq r \leq \bar{r}} (r^{-\mu} \|u(r \cdot)\|_{C^{k,\alpha}(\bar{B}_1(0)-B_{\frac{1}{2}}(0))})$$

is finite. Finally, set  $B_r^*(x^i) = B_r(x^i) - \{x^i\}$ , let  $C_\mu^{k,\alpha}(\bar{B}_1^*(0))$  be the space of functions in  $C_{\text{loc}}^{k,\alpha}(\bar{B}_1^*(0))$  for which the following norm

$$\|u\|_{C_\mu^{k,\alpha}(\bar{B}_1^*(0))} = \sup_{r \leq \frac{1}{2}} (r^{-\mu} \|u(r \cdot)\|_{C^{k,\alpha}(\bar{B}_2(0)-B_1(0))})$$

is finite.

We define the linear elliptic operator  $\mathbb{L}$  by

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+r^2)^4},$$

which is the linearized operator of  $\Delta^2 u - \rho^4 e^u = 0$  about the radial symmetric solution  $u_{\varepsilon=1, \tau=1}$  defined by (35). When  $k \geq 2$ , we let  $[\mathcal{C}_\mu^{k,\alpha}(\bar{\Omega})]_0$  to be the subspace of functions  $w \in \mathcal{C}_\mu^{k,\alpha}(\bar{\Omega})$  satisfying  $\Delta w = w = 0$  on  $\partial\Omega$ .

For all  $\varepsilon, \lambda, \tau_i > 0$ ,  $i = 1, 2, 3$  and  $\gamma, \xi \in (0, 1)$ , we define

$$r_{\varepsilon, \lambda} := \max\left(\varepsilon^{\frac{1}{2}}, \lambda^{\frac{1}{2}}, \varepsilon^{\frac{\gamma+\xi-1}{\gamma}}, \varepsilon^{\frac{\gamma+\xi-1}{\xi}}\right) \quad \text{and} \quad R_{\varepsilon, \lambda}^i := \tau_i \frac{r_{\varepsilon, \lambda}}{\varepsilon}. \quad (36)$$

**Proposition 1** [8] All bounded solutions of  $\mathbb{L}w = 0$  on  $\mathbb{R}^4$  are linear combination of

$$\phi_0(x) = 4 \frac{1 - |x|^2}{1 + |x|^2} \quad \text{and} \quad \phi_i(x) = \frac{8x_i}{1 + |x|^2} \quad \text{for } i = 1, \dots, 4.$$

Moreover, for  $\mu > 1$ ,  $\mu \notin \mathbb{Z}$ , the operator  $\mathbb{L} : \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \rightarrow \mathcal{C}_{\mu-4}^{0,\alpha}(\mathbb{R}^4)$  is surjective.

In the following, we denote by  $\mathcal{G}_\mu$  to be a right inverse of  $\mathbb{L}$ . Similarly, using the fact that any bounded bi-harmonic solution on  $\mathbb{R}^4$  is constant, we claim

**Proposition 2** Let  $\delta > 0$ ,  $\delta \notin \mathbb{Z}$  then  $\Delta^2$  is surjective from  $\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$  to  $\mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4)$ .

We denote by  $\mathcal{K}_\delta : \mathcal{C}_{\delta-4}^{0,\alpha}(\mathbb{R}^4) \rightarrow \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$  a right inverse of  $\Delta^2$  for  $\delta > 0$ ,  $\delta \notin \mathbb{Z}$ .

Finally, we consider punctured domains. Given  $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3$  three distinct points in  $\Omega$ , we define  $\tilde{\mathbf{x}} := (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$  and  $\bar{\Omega}^*(\tilde{\mathbf{x}}) := \bar{\Omega} - \{\tilde{x}^1, \tilde{x}^2, \tilde{x}^3\}$ . Let  $r_0 > 0$  be small such that  $\bar{B}_{r_0}(\tilde{x}^i)$  are disjoint and included in  $\Omega$ . For all  $r \in (0, r_0)$ , we define

$$\bar{\Omega}_r(\tilde{\mathbf{x}}) := \bar{\Omega} - \bigcup_{i=1}^3 B_r(\tilde{x}^i).$$

**Definition 2** Let  $k \in \mathbb{R}, \alpha \in (0, 1)$  and  $\nu \in \mathbb{R}$ , we introduce the Hölder weighted space  $\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$  as the space of functions  $w \in \mathcal{C}_{\text{loc}}^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$  such that

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} := \|w\|_{\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}_{\frac{r_0}{2}}(\tilde{\mathbf{x}}))} + \sum_{i=1}^3 \sup_{0 < r \leq \frac{r_0}{2}} (r^{-\nu} \|w(\tilde{x}^i + r \cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2(0) - B_1(0))})$$

is finite.

Furthermore, for  $k \geq 2$ , let  $[\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))]_0$  to be the set of  $w \in \mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$  satisfying  $\Delta w = w = 0$  on  $\partial\Omega$ .

We recall the following result.

**Proposition 3** [17] Let  $\nu < 0$ ,  $\nu \notin \mathbb{Z}$ , then  $\Delta^2$  is surjective from  $[\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))]_0$  to  $\mathcal{C}_{\nu-4}^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$ .

We denote by  $\tilde{\mathcal{G}}_\nu : \mathcal{C}_{\nu-4}^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}})) \rightarrow [\mathcal{C}_\nu^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))]_0$  a right inverse of  $\Delta^2$  for  $\nu < 0$ ,  $\nu \notin \mathbb{Z}$ .

### 3.1.2 Ansatz and first estimates

For all  $\sigma \geq 1$ , we denote by  $\xi_{\mu,\sigma} : \mathcal{C}_\mu^{0,\alpha}(\bar{B}_\sigma(0)) \rightarrow \mathcal{C}_\mu^{0,\alpha}(\mathbb{R}^4)$  the extension operator defined by

$$\begin{cases} \xi_{\mu,\sigma}(f)(x) \equiv f(x) & \text{for } |x| \leq \sigma, \\ \xi_{\mu,\sigma}(f)(x) = \chi\left(\frac{|x|}{\sigma}\right)f\left(\sigma \frac{x}{|x|}\right) & \text{for } |x| \geq \sigma. \end{cases} \quad (37)$$

Here  $\chi$  is a cut-off function over  $\mathbb{R}_+$ , which is equal to 1 for  $t \leq 1$  and equal to 0 for  $t \geq 2$ . It is easy to check that there exists a constant  $c = \bar{c}(\mu) > 0$ , independent of  $\sigma$  such that

$$\|\xi_{\mu,\sigma}(w)\|_{C_\mu^{0,\alpha}(\mathbb{R}^4)} \leq \bar{c}\|w\|_{C_\mu^{0,\alpha}(\bar{B}_\sigma(0))}. \quad (38)$$

Here, we are interested to study the system

$$\begin{cases} \Delta^2 u_1 + \mathcal{L}_\lambda(u_1) = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2}, \\ \Delta^2 u_2 + \mathcal{L}_\lambda(u_2) = \rho^4 e^{\xi u_2 + (1-\xi)u_1}, \end{cases} \quad (39)$$

where

$$\mathcal{L}_\lambda(u_i) = \lambda(\Delta u_i)^2 + \lambda \nabla u_i \cdot \nabla(\Delta u_i) + \lambda^2 |\nabla u_i|^2 \Delta u_i, \quad \text{for } i = 1, 2. \quad (40)$$

Using the following transformations

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_1}x\right) + \frac{8}{\gamma} \ln \varepsilon - \frac{4}{\gamma} \ln\left(\frac{\tau_1(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^1), \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_1}x\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^1), \end{cases} \quad (41)$$

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_2}x\right) + 8 \ln \varepsilon - 4 \ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^2), \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_2}x\right) + 8 \ln \varepsilon - 4 \ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^2) \end{cases} \quad (42)$$

and

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_3}x\right) \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_3}x\right) + \frac{8}{\xi} \ln \varepsilon - \frac{4}{\xi} \ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right) \end{cases} \quad \begin{array}{l} \text{in } B_{r_{\varepsilon,\lambda}}(x^3), \\ \text{in } B_{r_{\varepsilon,\lambda}}(x^3). \end{array} \quad (43)$$

Thus, the previous systems can be written as

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24C_{1,\varepsilon}^{\frac{4\gamma+\xi-1}{\gamma}} \varepsilon^{\frac{8\gamma+\xi-1}{\gamma}} e^{\xi v_2 + (1-\xi)v_1} \end{cases} \quad \begin{array}{l} \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \\ \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \end{array} \quad (44)$$

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} \end{cases} \quad \begin{array}{l} \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2), \\ \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2), \end{array} \quad (45)$$

and

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24C_{3,\varepsilon}^{\frac{4\gamma+\xi-1}{\xi}} \varepsilon^{\frac{8\gamma+\xi-1}{\xi}} e^{\gamma v_1 + (1-\gamma)v_2} \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} \end{cases} \quad \begin{array}{l} \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \\ \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \end{array} \quad (46)$$

where  $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$  for  $i = 1, 3$ . Here  $\tau_i > 0$  is a constant which will be fixed later.

We denote by  $\bar{u} = u_{\varepsilon=1, \tau_i=1}$ , we look for a solution of (44) of the form

$$\begin{cases} v_1(x) = \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln \gamma}{\gamma} + h_1^1(x), \\ v_2(x) = \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x). \end{cases} \quad (47)$$

Using the fact that  $e^{\bar{u}(x-x^1)} = \frac{16}{(1+|x-x^1|^2)^4}$ , we see that this amounts to solve the system

$$\begin{cases} \mathbb{L}h_1^1 = \frac{384}{\gamma(1+r^2)^4} (e^\gamma h_1^1 + (1-\gamma)h_2^1 - \gamma h_1^1 - 1) - \mathcal{L}_\lambda(\frac{1}{\gamma}\bar{u}(x-x^1) - \frac{1-\gamma}{\gamma}G(\frac{\varepsilon x}{\tau_1}, x^2) \\ \quad - \frac{1-\gamma}{\gamma\xi}G(\frac{\varepsilon x}{\tau_1}, x^3) - \frac{\ln\gamma}{\gamma} + h_1^1(x)), \\ \Delta^2 h_2^1 = \frac{24C_{1,\varepsilon} \frac{4}{\gamma} \frac{\gamma+\xi-1}{\gamma} \frac{1-\xi}{\gamma} \varepsilon^{\frac{\gamma+\xi-1}{\gamma}} e^{\frac{\gamma+\xi-1}{\gamma}} G(\frac{\varepsilon x}{\tau_1}, x^2) + \frac{\gamma+\xi-1}{\gamma\xi} G(\frac{\varepsilon x}{\tau_1}, x^3) + \xi h_2^1 + (1-\xi)h_1^1}{\frac{1-\xi}{\gamma} (1+r^2)^4 \frac{1-\xi}{\gamma}} \\ \quad - \mathcal{L}_\lambda(\frac{1}{\xi}G(\frac{\varepsilon x}{\tau_1}, x^3) + G(\frac{\varepsilon x}{\tau_1}, x^2) + h_2^1(x)), \end{cases} \quad (48)$$

We denote by

$$\mathbb{L}h_1^1 = \mathcal{R}_1(h_1^1, h_2^1) \quad \text{and} \quad \Delta^2 h_2^1 = \mathcal{R}_2(h_1^1, h_2^1).$$

Fix  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\})$ . To find a solution of (48), it is enough to find a fixed point  $(h_1^1, h_2^1)$  in a small ball of  $C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)$  solutions of

$$\begin{cases} h_1^1 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon, \lambda}^1} \circ \mathcal{R}_1(h_1^1, h_2^1) = \mathcal{N}_1(h_1^1, h_2^1), \\ h_2^1 = \mathcal{K}_\delta \circ \xi_{\delta, R_{\varepsilon, \lambda}^1} \circ \mathcal{R}_2(h_1^1, h_2^1) = \mathcal{M}_1(h_1^1, h_2^1). \end{cases} \quad (49)$$

Here  $\xi_{\mu, R_{\varepsilon, \lambda}^1}$  is defined in (37),  $\mathcal{G}_\mu$  and  $\mathcal{K}_\delta$  are defined after Propositions 1, 2, respectively. Then we have the following result.

**Lemma 1** Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$ ,  $\bar{c}_\kappa > 0$  and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\})$ . We have

$$\begin{aligned} \|\mathcal{N}_1(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon, \lambda}^2, & \|\mathcal{M}_1(0, 0)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\mathcal{N}_1(h_1^1, h_2^1) - \mathcal{N}_1(k_1^1, k_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \end{aligned}$$

and

$$\|\mathcal{M}_1(h_1^1, h_2^1) - \mathcal{M}_1(k_1^1, k_2^1)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|(h_1^1, h_2^1) - (k_1^1, k_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)},$$

provided  $(h_1^1, h_2^1), (k_1^1, k_2^1) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(h_1^1, h_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2, \quad \|(k_1^1, k_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (50)$$

*Proof* Using the fact that  $|\nabla^i G(x, y)| \leq c|x-y|^{-i}$  for  $i \geq 1$ , we get

$$\begin{aligned} &\sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\mathcal{R}_1(0, 0)| \\ &\leq \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln\gamma}{\gamma} \right) \right| \\ &\leq \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \left( \Delta \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right) \right)^2 \right| \end{aligned}$$

$$\begin{aligned}
& + 2\lambda \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\mu} \left| \nabla \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right) \right| \\
& \times \left| \nabla \left( \Delta \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right) \right) \right| \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\mu} \left| \nabla \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right) \right|^2 \\
& \times \left| \Delta \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right) \right| \\
& \leq c_\kappa \lambda (1 + \varepsilon + \varepsilon^4 R_{\varepsilon,\lambda}^{4-\mu} + \varepsilon^2 R_{\varepsilon,\lambda}^{2-\mu} + \varepsilon^3 R_{\varepsilon,\lambda}^{3-\mu}) + c_\kappa \lambda^2 (1 + \varepsilon^2 R_{\varepsilon,\lambda}^{3-\mu} + \varepsilon R_{\varepsilon,\lambda}^{2-\mu} + \varepsilon^3 R_{\varepsilon,\lambda}^{4-\mu}) \\
& \leq c_\kappa \lambda (1 + \varepsilon + \varepsilon^\mu r_{\varepsilon,\lambda}^{4-\mu}) + c_\kappa \lambda^2 (1 + \varepsilon^{\mu-1} r_{\varepsilon,\lambda}^{4-\mu}).
\end{aligned}$$

Making use of Proposition 1 together with (38), for  $\mu \in (1, 2)$ , we get that there exists  $c_\kappa > 0$  such that

$$\|\mathcal{N}_1(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (51)$$

For the second estimate, we have

$$\begin{aligned}
& \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} |\mathcal{R}_2(0, 0)| \\
& \leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda}^1} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} r^{4-\delta} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} e^{\frac{\gamma+\xi-1}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + \frac{\gamma+\xi-1}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right)} \\
& \quad + \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \mathcal{L}_\lambda \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \\
& \leq c_\kappa C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r \leq R_{\varepsilon,\lambda}^1} S(r) + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left| \Delta \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \right|^2 \\
& \quad + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \right| \\
& \quad \times \left| \nabla \left( \Delta \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \right) \right| \\
& \quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \right|^2 \\
& \quad \times \left| \Delta \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right) \right|,
\end{aligned}$$

where  $S(r) = \frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}$ . Then, using the fact that  $|\nabla^i G(x, y)| \leq c|x-y|^{-i}$  for  $i \geq 1$ , we get

$$\sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} |\mathcal{R}_2(0, 0)| \leq c_\kappa C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r \leq R_{\varepsilon,\lambda}^1} S(r) + c_\kappa \lambda \varepsilon^4 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} + c_\kappa \lambda^2 \varepsilon^4 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta}.$$

If  $4 - \delta - 8\frac{1-\xi}{\gamma} \leq 0$ , then  $S$  is bounded on  $\mathbb{R}_+$ . If  $4 - \delta - 8\frac{1-\xi}{\gamma} > 0$ ,  $\sup_{[0, \frac{r_{\varepsilon, \lambda}}{\varepsilon}]} S(r) = S(\frac{r_{\varepsilon, \lambda}}{\varepsilon})$ , then we get

$$\sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} |\mathcal{R}_2(0, 0)| \leq c_\kappa \max \left\{ \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}, \varepsilon^{\delta+4} r_{\varepsilon, \lambda}^{4-\delta-8\frac{1-\xi}{\gamma}} \right\} + c_\kappa \lambda \varepsilon^\delta r_{\varepsilon, \lambda}^{4-\delta} + c_\kappa \lambda^2 \varepsilon^\delta r_{\varepsilon, \lambda}^{4-\delta} \leq c_\kappa r_{\varepsilon, \lambda}^2.$$

Using the same argument as above, we get  $\|\mathcal{M}_1(0, 0)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2$ .

Recall the following conditions.

(A<sub>1</sub>) If  $0 < \varepsilon < \lambda$ , then  $\lambda^{1+\frac{\mu}{2}} \varepsilon^{-\mu} \rightarrow 0$  as  $\lambda \rightarrow 0$ , for any  $\mu \in (1, 2)$ .

(A<sub>2</sub>) If  $0 < \varepsilon < \lambda$ , then  $\lambda^{1+\frac{\delta}{2}} \varepsilon^{-\delta} \rightarrow 0$  as  $\lambda \rightarrow 0$ , for any  $\delta \in \left(0, \min \left\{ \left( \frac{\gamma+\xi-1}{\gamma} \right), \left( \frac{\gamma+\xi-1}{\xi} \right) \right\} \right)$ .

To derive the third estimate, using the fact that for all functions in  $C_\mu^{k,\alpha}(\mathbb{R}^4)$  bounded by a constant times  $(1+r^2)^{\mu/2}$  have their  $l$ -th partial derivatives that are bounded by  $(1+r^2)^{(\mu-l)/2}$ , for  $l = 1, \dots, k+\alpha, \dots$  (a.e.  $|\nabla^l w| \leq c_\kappa r^{\mu-l} \|w\|_{C_\mu^{k,\alpha}(\mathbb{R}^4)}$ ,  $(1+r^2)^{(\mu-l)/2} \sim r^{\mu-l}$  for  $r$  very large) and the fact that  $|\nabla^i G(x, y)| \leq c|x-y|^{-i}$  for  $i \geq 1$ . Then for  $(h_1^1, h_2^1), (k_1^1, k_2^1)$  verifying (50), we have

$$\begin{aligned} & \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\mathcal{R}_1(h_1^1, h_2^1) - \mathcal{R}_1(k_1^1, k_2^1)| \\ & \leq \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} |(e^{\gamma h_1^1 + (1-\gamma)h_2^1} - \gamma h_1^1 - 1) - (e^{\gamma k_1^1 + (1-\gamma)k_2^1} - \gamma k_1^1 - 1)| \\ & \quad + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\gamma \xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) - \frac{\ln \gamma}{\gamma} + h_1^1(x) \right) \right. \\ & \quad \left. - \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) - \frac{1-\gamma}{\gamma \xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) - \frac{\ln \gamma}{\gamma} + k_1^1(x) \right) \right| \\ & \leq c_\kappa \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} [\gamma^2 ((h_1^1)^2 - (k_1^1)^2) + (1-\gamma) |h_2^1 - k_2^1|] \\ & \quad + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta(h_1^1 - k_1^1)| \\ & \quad \times \left( \frac{2}{\gamma} |\Delta \bar{u}| + 2 \frac{1-\gamma}{\gamma} \left| \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right| + 2 \frac{1-\gamma}{\gamma \xi} \left| \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) \right| + |\Delta h_1^1| + |\Delta k_1^1| \right) \\ & \quad + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \nabla(h_1^1 - k_1^1) \cdot \nabla \left( \frac{2}{\gamma} \Delta \bar{u} + 2 \frac{1-\gamma}{\gamma} \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right) \right. \\ & \quad \left. + 2 \frac{1-\gamma}{\gamma \xi} \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) + \Delta h_1^1 + \Delta k_1^1 \right) \\ & \quad + \nabla(\Delta(h_1^1 - k_1^1)) \cdot \nabla \left( \frac{2}{\gamma} \bar{u} - 2 \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right) \end{aligned}$$

$$\begin{aligned}
& -2 \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_1^1(x) + k_1^1(x) \Big| \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta(h_1^1 - k_1^1)| \left[ \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \right. \\
& \left. \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_1^1(x) \right) \right|^2 \\
& + \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + k_1^1(x) \right) \right|^2 \Big] \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta \bar{u}| + 2 \frac{1-\gamma}{\gamma} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right| + 2 \frac{1-\gamma}{\gamma \xi} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right| + |\Delta h_1^1| \right. \\
& \left. + |\Delta k_1^1| \right) \left[ \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \right. \\
& \left. \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_1^1(x) \right) \right|^2 \\
& \left. - \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + k_1^1(x) \right) \right|^2 \right] \\
& \leq c_\kappa (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}}) \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} \\
& + c_\kappa \lambda (1 + \varepsilon R_{\varepsilon, \lambda} + \varepsilon^2 R_{\varepsilon, \lambda}^2 + \varepsilon^3 R_{\varepsilon, \lambda}^3 \\
& + R_{\varepsilon, \lambda}^\mu (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}})) \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} \\
& + c_\kappa \lambda^2 (1 + \varepsilon R_{\varepsilon, \lambda} + \varepsilon^2 R_{\varepsilon, \lambda}^2 + \varepsilon^3 R_{\varepsilon, \lambda}^3 + R_{\varepsilon, \lambda}^\mu (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}})) \\
& + \varepsilon R_{\varepsilon, \lambda}^{\mu+1} (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}}) + \varepsilon^2 R_{\varepsilon, \lambda}^{\mu+2} (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}}) \\
& + \varepsilon^2 R_{\varepsilon, \lambda}^{2\mu} (\|h_1^1\|_{C_\mu^{4,\alpha}} + \|k_1^1\|_{C_\mu^{4,\alpha}})^2 \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} \\
& \leq c_\kappa r_{\varepsilon, \lambda}^2 \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} \\
& + c_\kappa \lambda (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu}) \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} \\
& + c_\kappa \lambda^2 (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu} + \varepsilon^{-2\mu} r_{\varepsilon, \lambda}^{2(2+\mu)}) \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}}.
\end{aligned}$$

Using the following estimates

$$c_\kappa \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu} \leq \begin{cases} c_\kappa \varepsilon^{1-\frac{\mu}{2}} & \text{for } \varepsilon > \lambda, \\ c_\kappa \lambda^{1+\frac{\mu}{2}} \varepsilon^{-\mu} & \text{for } \lambda > \varepsilon, \end{cases}$$

together with condition (A<sub>1</sub>), yield  $c_\kappa \lambda (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu}) \leq c_\kappa r_{\varepsilon, \lambda}^2$  and  $c_\kappa \lambda^2 (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu} + \varepsilon^{-2\mu} r_{\varepsilon, \lambda}^{2(2+\mu)}) \leq c_\kappa r_{\varepsilon, \lambda}^2$ . Making use of Proposition 1 together with (38) and using the condition (A<sub>1</sub>) for  $\mu \in (1, 2)$ , we get that there exists  $\bar{c}_\kappa > 0$  such that

$$\begin{aligned}
& \|\mathcal{N}_1(h_1^1, h_2^1) - \mathcal{N}_1(k_1^1, k_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
& \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{52}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} |\mathcal{R}_2(h_1^1, h_2^1) - \mathcal{R}_2(k_1^1, k_2^1)| \\
& \leq \sup_{r \leq R_{\varepsilon,\lambda}^1} 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \gamma^{-\frac{1-\xi}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} \\
& \quad \times r^{4-\delta} e^{\frac{\gamma+\xi-1}{\gamma} G(\frac{\varepsilon x}{\tau_1}, x^2) + \frac{\gamma+\xi-1}{\gamma\xi} G(\frac{\varepsilon x}{\tau_1}, x^3)} \\
& \quad \times |e^{\xi h_2^1 + (1-\xi)h_1^1} - e^{\xi k_2^1 + (1-\xi)k_1^1}| \\
& \quad + \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left| \mathcal{L}_\lambda \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) \right) \right. \\
& \quad \left. - \mathcal{L}_\lambda \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + k_2^1(x) \right) \right| \\
& \leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda}^1} 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \gamma^{-\frac{1-\xi}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} \\
& \quad \times r^{4-\delta} [\xi |h_2^1 - k_2^1| + (1-\xi) |h_1^1 - k_1^1|] \\
& \quad + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left[ |\Delta(h_2^1 - k_2^1)| \left( 2 \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right| \right. \right. \\
& \quad \left. \left. + \frac{2}{\xi} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right| + |\Delta h_2^1| + |\Delta k_2^1| \right) \\
& \quad + \left| \nabla(h_2^1 - k_2^1) \cdot \nabla \left( 2 \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + \frac{2}{\xi} \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + \Delta h_2^1 + \Delta k_2^1 \right) \right. \\
& \quad \left. + \nabla(\Delta(h_2^1 - k_2^1)) \right. \\
& \quad \left. \times \nabla \left( 2 G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + \frac{2}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + h_2^1 + k_2^1 \right) \right] + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} |\Delta(h_2^1 - k_2^1)| \\
& \quad \times \left[ \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) \right) \right|^2 \right. \\
& \quad \left. + \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + k_2^1(x) \right) \right|^2 \right] \\
& \quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\delta} \left( 2 \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right| + \frac{2}{\xi} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right| + |\Delta h_2^1| + |\Delta k_2^1| \right) \\
& \quad \times \left[ \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) \right) \right|^2 \right. \\
& \quad \left. - \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + k_2^1(x) \right) \right|^2 \right] \\
& \leq c_\kappa \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} \sup_{r \leq R_{\varepsilon,\lambda}^1} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} r^{4-\delta} [\xi r^\delta \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} + (1-\xi) r^\mu \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}}] \\
& \quad + c_\kappa \lambda (\varepsilon R_{\varepsilon,\lambda} + \varepsilon^2 R_{\varepsilon,\lambda}^2 + \varepsilon^3 R_{\varepsilon,\lambda}^3 + R_{\varepsilon,\lambda}^\delta (\|h_2^1\|_{C_\delta^{4,\alpha}} + \|k_2^1\|_{C_\delta^{4,\alpha}})) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}}
\end{aligned}$$

$$\begin{aligned}
& + c_\kappa \lambda^2 (\varepsilon^2 R_{\varepsilon, \lambda}^2 + \varepsilon^3 R_{\varepsilon, \lambda}^3 \\
& + \varepsilon R_{\varepsilon, \lambda}^{\delta+1} (\|h_2^1\|_{C_\delta^{4,\alpha}} + \|k_2^1\|_{C_\delta^{4,\alpha}}) + \varepsilon^2 R_{\varepsilon, \lambda}^{\delta+2} (\|h_2^1\|_{C_\delta^{4,\alpha}} + \|k_2^1\|_{C_\delta^{4,\alpha}}) \\
& + R_{\varepsilon, \lambda}^{2\delta} (\|h_2^1\|_{C_\delta^{4,\alpha}} + \|k_2^1\|_{C_\delta^{4,\alpha}})^2) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} \\
& \leq c_\kappa r_{\varepsilon, \lambda}^2 \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} + c_\kappa r_{\varepsilon, \lambda}^2 \|h_1^1 - k_1^1\|_{C_\mu^{4,\alpha}} + c_\kappa \lambda (r_{\varepsilon, \lambda} + \varepsilon^{-\delta} r_{\varepsilon, \lambda}^{\delta+2}) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}} \\
& + c_\kappa \lambda^2 (r_{\varepsilon, \lambda}^2 + \varepsilon^{-\delta} r_{\varepsilon, \lambda}^{\delta+3} + \varepsilon^{-2\delta} r_{\varepsilon, \lambda}^{2(\delta+2)}) \|h_2^1 - k_2^1\|_{C_\delta^{4,\alpha}}.
\end{aligned}$$

Using the following estimates

$$c_\kappa \varepsilon^{-\delta} r_{\varepsilon, \lambda}^{2+\delta} \leq \begin{cases} c_\kappa \varepsilon^{1-\frac{\delta}{2}} & \text{for } \varepsilon > \lambda, \\ c_\kappa \lambda^{1+\frac{\delta}{2}} \varepsilon^{-\delta} & \text{for } \lambda > \varepsilon, \end{cases}$$

together with condition (A<sub>2</sub>), yield  $c_\kappa \lambda (r_{\varepsilon, \lambda} + \varepsilon^{-\delta} r_{\varepsilon, \lambda}^{\delta+2}) \leq c_\kappa r_{\varepsilon, \lambda}^2$  and  $c_\kappa \lambda^2 (r_{\varepsilon, \lambda}^2 + \varepsilon^{-\delta} r_{\varepsilon, \lambda}^{\delta+3} + \varepsilon^{-2\delta} r_{\varepsilon, \lambda}^{2(\delta+2)}) \leq c_\kappa r_{\varepsilon, \lambda}^2$ . Making use of Proposition 1 together with (38) and using the condition (A<sub>2</sub>) for  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , we get that there exists  $\bar{c}_\kappa > 0$  such that

$$\|\mathcal{M}_1(h_1^1, h_2^1) - \mathcal{M}_1(k_1^1, k_2^1)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|(h_1^1, h_2^1) - (k_1^1, k_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)}. \quad (53)$$

□

Reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa r_{\varepsilon, \lambda}^2 < \frac{1}{2}$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ . There exists also  $\gamma_0 \in (0, 1)$  such that  $\bar{c}_\kappa (1 - \gamma) \leq \frac{1}{2}$  for all  $\gamma \in (\gamma_0, 1)$ . Therefore (52) and (53) are enough to show that

$$(h_1^1, h_2^1) \mapsto (\mathcal{N}_1(h_1^1, h_2^1), \mathcal{M}_1(h_1^1, h_2^1))$$

is a contraction from the ball

$$\{(h_1^1, h_2^1) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4) : \|(h_1^1, h_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2\}$$

into itself. Then, applying a contraction mapping argument, we obtain the following proposition.

**Proposition 4** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , and for all  $\tau_1$  in some fixed compact subset of  $[\tau_1^-, \tau_1^+] \subset (0, \infty)$  there exists a unique  $(h_1^1, h_2^1) := (h_{1,\varepsilon, \tau_1}, h_{2,\varepsilon, \tau_1})$  solution of (49) such that

$$\|(h_1^1, h_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Hence (47) solves (44) in  $B_{R_{\varepsilon, \lambda}^1}(x^1)$ .

In  $B_{R_\varepsilon^3}(x^3)$ , following the same arguments as the first case by reversing the roles of the functions  $v_1$  and  $v_2$  and by respecting the changes of the coefficients, we can prove that there exists  $(h_1^3, h_2^3) \in C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$  such that

$$\|(h_1^3, h_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Furthermore,  $(h_1^3, h_2^3)$  solves the equations

$$\begin{cases} \Delta^2 h_1^3 = \frac{24C_{3,\varepsilon}}{\xi^{1-\gamma}(1+r^2)^4} e^{\frac{4\gamma+\xi-1}{\xi}} G(\frac{\varepsilon x}{\tau_3}, x^2) + \frac{\gamma+\xi-1}{\gamma\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) + \gamma h_1^3 + (1-\gamma) h_2^3 \\ \quad - \mathcal{L}_\lambda(\frac{1}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) + G(\frac{\varepsilon x}{\tau_3}, x^2) + h_1^3(x)), \\ \mathbb{L} h_2^3 = \frac{384}{\xi(1+r^2)^4} [e^{\xi h_2^3 + (1-\xi)h_1^3} - \xi h_2^3 - 1] - \mathcal{L}_\lambda(\frac{1}{\xi} \bar{u}(x - x^3) - \frac{1-\xi}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^2) \\ \quad - \frac{1-\xi}{\gamma\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) - \frac{\ln \xi}{\xi} + h_2^3(x)). \end{cases} \quad (54)$$

Then we have the following proposition.

**Proposition 5** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\xi_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\xi \in (\xi_0, 1)$  and for all  $\tau_3$  in some fixed compact subset of  $[\tau_3^-, \tau_3^+] \subset (0, \infty)$ , there exists a unique  $(h_1^3, h_2^3) := (h_{1,\varepsilon,\tau_3}, h_{2,\varepsilon,\tau_3})$  solution of (49) such that

$$\|(h_1^3, h_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} G(\frac{\varepsilon x}{\tau_3}, x^1) + G(\frac{\varepsilon x}{\tau_3}, x^2) + h_1^3(x), \\ v_2(x) := \frac{1}{\xi} \bar{u}(x - x^3) - \frac{1-\xi}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^2) - \frac{1-\xi}{\gamma\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) - \frac{\ln \xi}{\xi} + h_2^3(x) \end{cases}$$

solves (46) in  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ .

In  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ , we look for a solution of (45) of the form

$$\begin{cases} v_1(x) = \bar{u}(x - x^2) + h_1^2(x), \\ v_2(x) = \bar{u}(x - x^2) + h_2^2(x). \end{cases} \quad (55)$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L} h_1^2 = \frac{384}{(1+r^2)^4} [e^{\gamma h_1^2 + (1-\gamma)h_2^2} - h_1^2 - 1] - \mathcal{L}_\lambda(\bar{u}(x - x^2) + h_1^2(x)), \\ \mathbb{L} h_2^2 = \frac{384}{(1+r^2)^4} [e^{\xi h_2^2 + (1-\xi)h_1^2} - h_2^2 - 1] - \mathcal{L}_\lambda(\bar{u}(x - x^2) + h_2^2(x)). \end{cases} \quad (56)$$

We denote by

$$\mathbb{L} h_1^2 = \mathcal{R}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L} h_2^2 = \mathcal{R}_4(h_1^2, h_2^2).$$

To find a solution of (56), it is enough to find a fixed point  $(h_1^2, h_2^2)$  in a small ball of  $C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$ , solutions of

$$\begin{cases} h_1^2 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon,\lambda}^2} \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2), \\ h_2^2 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon,\lambda}^2} \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{cases} \quad (57)$$

Then, we have the following result.

**Lemma 2** Let  $\mu \in (1, 2)$ ,  $\gamma_0$  and  $\xi_0 \in (0, 1)$ . Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\bar{c}_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . We have

$$\begin{aligned} \|\mathcal{N}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \quad \|\mathcal{M}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \gamma) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ \|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq \bar{c}_\kappa (1 - \xi) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}, \end{aligned}$$

provided  $(h_1^2, h_2^2), (k_1^2, k_2^2)$  in  $C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(h_1^2, h_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (58)$$

*Proof* We have

$$\begin{aligned} \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{R}_3(0, 0)| &\leq \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{L}_\lambda(\bar{u}(x - x^2))| \\ &\leq \lambda \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} (|\Delta \bar{u}(x - x^2)|^2 + |\nabla \bar{u}(x - x^2) \nabla (\Delta \bar{u}(x - x^2))|) \\ &\quad + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\nabla \bar{u}(x - x^2)|^2 |\Delta \bar{u}(x - x^2)|. \end{aligned}$$

Making use of Proposition 1 together with (38), for  $\mu \in (1, 2)$ , we get that there exists  $c_\kappa > 0$  such that

$$\|\mathcal{N}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2. \quad (59)$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2. \quad (60)$$

To derive the third estimate, using the fact that for all functions in  $C_\mu^{k,\alpha}(\mathbb{R}^4)$  bounded by a constant times  $(1 + r^2)^{\mu/2}$  have their  $l$ -th partial derivatives that are bounded by  $(1 + r^2)^{(\mu-l)/2}$ , for  $l = 1, \dots, k + \alpha, \dots$  (a.e.  $|\nabla^l w| \leq c_\kappa r^{\mu-l} \|w\|_{C_\mu^{k,\alpha}(\mathbb{R}^4)}$ ,  $(1 + r^2)^{(\mu-l)/2} \sim r^{\mu-l}$  for  $r$  very large) and the fact that  $|\nabla^i G(x, y)| \leq c|x - y|^{-i}$  for  $i \geq 1$ . Then for  $(h_1^2, h_2^2), (k_1^2, k_2^2)$  verifying (58), we get

$$\begin{aligned} \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{R}_3(h_1^2, h_2^2) - \mathcal{R}_3(k_1^2, k_2^2)| \\ \leq \sup_{r \leq R_{\varepsilon, \lambda}^2} \frac{384r^{4-\mu}}{(1 + r^2)^4} |(e^{\gamma h_1^2 + (1-\gamma)h_2^2} - h_1^2) - (e^{\gamma k_1^2 + (1-\gamma)k_2^2} - k_1^2)| \\ + \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{L}_\lambda(\bar{u}(x - x^2) + h_1^2(x)) - \mathcal{L}_\lambda(\bar{u}(x - x^2) + k_1^2(x))| \end{aligned}$$

$$\begin{aligned}
&\leq c \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} |(\gamma-1)(h_1^2-k_1^2)+(1-\gamma)(h_2^2-k_2^2)| + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(h_1^2-k_1^2)| \\
&\quad \times (2|\Delta \bar{u}| + |\Delta h_1^2| + |\Delta k_1^2|) + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\nabla(h_1^2-k_1^2) \cdot \nabla(2\Delta \bar{u} + \Delta h_1^2 + \Delta k_1^2) \\
&\quad + \nabla(\Delta(h_1^2-k_1^2)) \cdot \nabla(2\bar{u} + h_1^2 + k_1^2)| \\
&\quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(h_1^2-k_1^2)| [|\nabla(\bar{u}(x-x^2) + h_1^2(x))|^2 \\
&\quad + |\nabla(\bar{u}(x-x^2) + k_1^2(x))|^2] \\
&\quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta \bar{u}| + 2 |\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}}| + |\Delta h_1^2| + |\Delta k_1^2| \right) \\
&\quad \times [|\nabla(\bar{u}(x-x^2) + h_1^2(x))|^2 - |\nabla(\bar{u}(x-x^2) + k_1^2(x))|^2] \\
&\leq c_\kappa (1-\gamma) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}} + c_\kappa \lambda (1+R_{\varepsilon,\lambda}^\mu (\|h_1^2\|_{C_\mu^{4,\alpha}} \\
&\quad + \|k_1^2\|_{C_\mu^{4,\alpha}})) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}} \\
&\quad + c_\kappa \lambda^2 (1+R_{\varepsilon,\lambda}^\mu (\|h_1^2\|_{C_\mu^{4,\alpha}} + \|k_1^2\|_{C_\mu^{4,\alpha}}) + R_{\varepsilon,\lambda}^{2\mu} (\|h_1^2\|_{C_\mu^{4,\alpha}} + \|k_1^2\|_{C_\mu^{4,\alpha}})^2) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}} \\
&\leq c_\kappa (1-\gamma) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}} + c_\kappa \lambda (1+\varepsilon^{-\mu} r_{\varepsilon,\lambda}^{2+\mu}) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}} \\
&\quad + c_\kappa \lambda^2 (1+\varepsilon^{-\mu} r_{\varepsilon,\lambda}^{2+\mu} + \varepsilon^{-2\mu} r_{\varepsilon,\lambda}^{2(2+\mu)}) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}}.
\end{aligned}$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get that there exists  $\bar{c}_\kappa > 0$  such that

$$\begin{aligned}
&\|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
&\leq \bar{c}_\kappa (1-\gamma + r_{\varepsilon,\lambda}^2) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{61}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
&\leq \bar{c}_\kappa (1-\xi) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\xi + r_{\varepsilon,\lambda}^2) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{62}
\end{aligned}$$

□

Then there exist  $\gamma_0$  and  $\xi_0 \in (0, 1)$ , reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa (1-\gamma + r_{\varepsilon,\lambda}^2) \leq 1/2$  and  $\bar{c}_\kappa (1-\xi + r_{\varepsilon,\lambda}^2) \leq 1/2$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . Therefore (61) and (62) are enough to show that

$$(h_1^2, h_2^2) \mapsto (\mathcal{N}_2(h_1^2, h_2^2), \mathcal{M}_2(h_1^2, h_2^2))$$

is a contraction from the ball

$$\{(h_1^2, h_2^2) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4) : \|(h_1^2, h_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

**Proposition 6** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$ ,  $\gamma_0 \in (0, 1)$  and  $\xi_0 \in (0, 1)$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$  and  $c_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$  and for all  $\tau_2$  in some fixed compact subset of  $[\tau_2^-, \tau_2^+] \subset (0, \infty)$ , there exists a unique  $(h_1^2, h_2^2)$  ( $:= (h_{1,\varepsilon,\tau_2}, h_{2,\varepsilon,\tau_2})$ ) solution of (57) such that

$$\|(h_1^2, h_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence (55) solves (45) in  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ .

### 3.1.3 Bi-harmonic extensions

Next, we will study the properties of interior and exterior bi-harmonic extensions. Given  $(\varphi, \psi), (\tilde{\varphi}, \tilde{\psi}) \in \mathcal{C}^{4,\alpha}(S^3) \times \mathcal{C}^{2,\alpha}(S^3)$ , we define respectively  $H^{\text{int}} = H^{\text{int}}(\varphi, \psi; \cdot) = H_{\varphi, \psi}^{\text{int}}$  and  $H^{\text{ext}} = H^{\text{ext}}(\tilde{\varphi}, \tilde{\psi}; \cdot) = H_{\tilde{\varphi}, \tilde{\psi}}^{\text{ext}}$  to be the solution of

$$\begin{cases} \Delta^2 H^{\text{int}} = 0 & \text{in } B_1(0), \\ H^{\text{int}} = \varphi & \text{on } \partial B_1(0), \\ \Delta H^{\text{int}} = \psi & \text{on } \partial B_1(0), \end{cases} \quad (63)$$

and

$$\begin{cases} \Delta^2 H^{\text{ext}} = 0 & \text{in } \mathbb{R}^4 - B_1(0), \\ H^{\text{ext}} = \tilde{\varphi} & \text{on } \partial B_1(0), \\ \Delta H^{\text{ext}} = \tilde{\psi} & \text{on } \partial B_1(0), \end{cases} \quad (64)$$

which decays at infinity. We will also use

**Definition 3** Given  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$  and  $v \in \mathbb{R}$ , we define the space  $\mathcal{C}_v^{k,\alpha}(\mathbb{R}^4 - B_1(0))$  as the space of functions  $w \in \mathcal{C}_{\text{loc}}^{k,\alpha}(\mathbb{R}^4 - B_1(0))$  for which the following norm

$$\|w\|_{\mathcal{C}_v^{k,\alpha}(\mathbb{R}^4 - B_1(0))} = \sup_{r \geq 1} (r^{-v} \|w(r \cdot)\|_{\mathcal{C}_v^{k,\alpha}(\bar{B}_2(0) - B_1(0))})$$

is finite.

We denote by  $e_1, \dots, e_4$  the coordinate functions on  $S^3$ .

**Lemma 3** [4] Assume that

$$\int_{S^3} (8\varphi - \psi) d\nu_{S^3} = 0 \quad \text{and} \quad \int_{S^3} (12\varphi - \psi) e_\ell d\nu_{S^3} = 0 \quad \text{for } \ell = 1, \dots, 4. \quad (65)$$

Then there exists  $c > 0$  such that

$$\|H_{\varphi, \psi}^{\text{int}}\|_{\mathcal{C}_2^{4,\alpha}(\bar{B}_1^*(0))} \leq c(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}).$$

Similarly, there exists  $c > 0$  such that if

$$\int_{S^3} \tilde{\psi} d\nu_{S^3} = 0, \quad (66)$$

then

$$\|H_{\tilde{\varphi}, \tilde{\psi}}^{\text{ext}}\|_{C_{-1}^{4,\alpha}(\mathbb{R}^4 - B_1(0))} \leq c(\|\tilde{\varphi}\|_{C^{4,\alpha}(S^3)} + \|\tilde{\psi}\|_{C^{2,\alpha}(S^3)}).$$

If  $F \subset L^2(S^3)$  is a subspace  $S^3$ , we denote  $F^\perp$  to be the subspace of  $F$ , which are  $L^2(S^3)$ -orthogonal to the functions  $1, e_1, \dots, e_4$ . We will need the following result.

**Lemma 4** [4] *The mapping*

$$\begin{aligned} \mathcal{P} : C^{4,\alpha}(S^3)^\perp \times C^{2,\alpha}(S^3)^\perp &\longrightarrow C^{3,\alpha}(S^3)^\perp \times C^{1,\alpha}(S^3)^\perp, \\ (\varphi, \psi) &\longmapsto (\partial_r(H_{\varphi, \psi}^{\text{int}} - H_{\varphi, \psi}^{\text{ext}}), \partial_r(\Delta H_{\varphi, \psi}^{\text{int}} - \Delta H_{\varphi, \psi}^{\text{ext}})) \end{aligned}$$

is an isomorphism.

### 3.2 The nonlinear interior problem

Here, we are looking for a solution of the following systems as in the above subsection, we only add the interior harmonic extension and the perturbation term  $\nu_i^j$  for  $i, j = 1, 2$ .

$$\begin{cases} \Delta^2 \nu_1 + \mathcal{L}_\lambda(\nu_1) = 24e^{\gamma \nu_1 + (1-\gamma)\nu_2} & \text{in } B_{R_{\varepsilon, \lambda}^1}(x^1), \\ \Delta^2 \nu_2 + \mathcal{L}_\lambda(\nu_2) = 24C_{1,\varepsilon}^{\frac{4\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} e^{\xi \nu_2 + (1-\xi)\nu_1} & \text{in } B_{R_{\varepsilon, \lambda}^1}(x^1), \end{cases} \quad (67)$$

$$\begin{cases} \Delta^2 \nu_1 + \mathcal{L}_\lambda(\nu_1) = 24e^{\gamma \nu_1 + (1-\gamma)\nu_2} & \text{in } B_{R_{\varepsilon, \lambda}^2}(x^2), \\ \Delta^2 \nu_2 + \mathcal{L}_\lambda(\nu_2) = 24e^{\xi \nu_2 + (1-\xi)\nu_1} & \text{in } B_{R_{\varepsilon, \lambda}^2}(x^2) \end{cases} \quad (68)$$

and

$$\begin{cases} \Delta^2 \nu_1 + \mathcal{L}_\lambda(\nu_1) = 24C_{3,\varepsilon}^{\frac{4\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}} e^{\gamma \nu_1 + (1-\gamma)\nu_2} & \text{in } B_{R_{\varepsilon, \lambda}^3}(x^3), \\ \Delta^2 \nu_2 + \mathcal{L}_\lambda(\nu_2) = 24e^{\xi \nu_2 + (1-\xi)\nu_1} & \text{in } B_{R_{\varepsilon, \lambda}^3}(x^3), \end{cases} \quad (69)$$

where  $\mathcal{L}_\lambda$  is defined by (40) and  $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$  for  $i = 1, 3$ . Here  $\tau_i > 0$  is a constant, which will be fixed later.

Given  $\varphi^i := (\varphi_1^i, \varphi_2^i) \in (C^{4,\alpha}(S^3))^2$  and  $\psi^i := (\psi_1^i, \psi_2^i) \in (C^{2,\alpha}(S^3))^2$  such that  $(\varphi_1^i, \psi_1^i)$  and  $(\varphi_2^i, \psi_2^i)$  are satisfying (65). We denote by  $\bar{u} = u_{\varepsilon=1, \tau_i=1}$ , we write for  $x \in B_{R_{\varepsilon, \lambda}^1}(x^1)$  the following system

$$\begin{cases} \nu_1(x) = \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln \gamma}{\gamma} + h_1^1(x) \\ \quad + H^{\text{int}}(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_{\varepsilon, \lambda}^1}) + v_1^1(x), \\ \nu_2(x) = \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) + H^{\text{int}}(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_{\varepsilon, \lambda}^1}) + v_2^1(x). \end{cases}$$

Using the fact that  $H^{\text{int}}$  is bi-harmonic and that  $e^{\bar{u}(x-x^1)} = \frac{16}{(1+|x-x^1|^2)^4}$ , we see that this amounts to solve the system

$$\left\{ \begin{array}{l} \mathbb{L}v_1^1 = \frac{384}{\gamma(1+r^2)^4} (e^{\gamma(h_1^1 + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + v_1^1) + (1-\gamma)(h_2^1 + H_{\varphi_2^1, \psi_2^1}^{\text{int}} + v_2^1)} - \gamma v_1^1 - 1) - \mathcal{L}_\lambda(\frac{1}{\gamma}\bar{u}(x-x^1) \\ \quad - \frac{1-\gamma}{\gamma}G(\frac{\varepsilon x}{\tau_1}, x^2) - \frac{1-\gamma}{\gamma\xi}G(\frac{\varepsilon x}{\tau_1}, x^3) - \frac{\ln\gamma}{\gamma} \\ \quad + H_{\varphi_1^1, \psi_1^1}^{\text{int}}(\frac{x-x^1}{R_{\varepsilon, \lambda}}) + h_1^1(x) + v_1^1(x)) - \Delta^2 h_1^1, \\ \Delta^2 v_2^1 = \frac{24C_{1,\varepsilon}}{\frac{1-\xi}{\varepsilon}(1+\varepsilon^2)^4} \frac{1-\xi}{\varepsilon} e^{\frac{\gamma+\xi-1}{\varepsilon}} \\ \quad \times e^{(\frac{\varepsilon x}{\tau_1}, x^2) + \frac{\gamma+\xi-1}{\gamma\xi}G(\frac{\varepsilon x}{\tau_1}, x^3) + \xi(h_2^1 + H_{\varphi_2^1, \psi_2^1}^{\text{int}} + v_2^1) + (1-\xi)(h_1^1 + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + v_1^1)} \\ \quad - \mathcal{L}_\lambda(\frac{1}{\xi}G(\frac{\varepsilon x}{\tau_1}, x^3) + G(\frac{\varepsilon x}{\tau_1}, x^2) + H_{\varphi_2^1, \psi_2^1}^{\text{int}}(\frac{x-x^1}{R_{\varepsilon, \lambda}}) + h_2^1(x) + v_2^1(x)) - \Delta^2 h_2^1, \end{array} \right. \quad (70)$$

We denote by

$$\mathbb{L}v_1^1 = \mathcal{R}_1(v_1^1, v_2^1) \quad \text{and} \quad \Delta^2 v_2^1 = \mathcal{R}_2(v_1^1, v_2^1).$$

Fix  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ . To find a solution of (48), it is enough to find a fixed point  $(v_1^1, v_2^1)$  in a small ball of  $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$  solutions of

$$\left\{ \begin{array}{l} v_1^1 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon, \lambda}^1} \circ \mathcal{R}_1(v_1^1, v_2^1) = \mathcal{N}_1(v_1^1, v_2^1), \\ v_2^1 = \mathcal{K}_\delta \circ \xi_{\delta, R_{\varepsilon, \lambda}^1} \circ \mathcal{R}_2(v_1^1, v_2^1) = \mathcal{M}_1(v_1^1, v_2^1). \end{array} \right. \quad (71)$$

Here  $\xi_{\mu, R_{\varepsilon, \lambda}^1}$  is defined in (37),  $\mathcal{G}_\mu$  and  $\mathcal{K}_\delta$  are defined after Propositions 1, 2, respectively.

Given  $\kappa > 0$  (whose value will be fixed later), we further assume that the functions  $\varphi_j^1$  and  $\psi_j^1$  satisfy

$$\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2, \quad \text{for } j = 1, 2. \quad (72)$$

Then we have the following result.

**Lemma 5** Let  $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{4,\alpha}(S^3))^2$  and  $\psi^1 := (\psi_1^1, \psi_2^1) \in (\mathcal{C}^{2,\alpha}(S^3))^2$  such that  $(\varphi_1^1, \psi_1^1)$  and  $(\varphi_2^1, \psi_2^1)$  are satisfying (65) and (72). Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$ ,  $\bar{c}_\kappa > 0$  and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ . We have

$$\|\mathcal{N}_1(0, 0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2, \quad \|\mathcal{M}_1(0, 0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2,$$

$$\|\mathcal{N}_1(v_1^1, v_2^1) - \mathcal{N}_1(t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|v_1^1 - t_1^1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|v_2^1 - t_2^1\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}_1(v_1^1, v_2^1) - \mathcal{M}_1(t_1^1, t_2^1)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)},$$

provided  $(v_1^1, v_2^1), (t_1^1, t_2^1) \in \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2, \quad \|(t_1^1, t_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (73)$$

*Proof* The proof of the first and the second estimates follows from the asymptotic behavior of  $H^{\text{int}}$  together with the assumption on the norms of  $\varphi_j^1$  and  $\psi_j^1$  given by (72) and it follows from the estimate of  $H^{\text{int}}$ , given by Lemma 3, that

$$\left\| H_{\varphi_j^1, \psi_j^1}^{\text{int}} \left( \frac{r}{R_{\varepsilon, \lambda}^1} \cdot \right) \right\|_{C^{4,\alpha}(\bar{B}_2(0)-B_1(0))} \leq Cr^2(R_{\varepsilon, \lambda}^1)^{-2} (\|\varphi_j^1\|_{C^{4,\alpha}(S^3)} + \|\psi_j^1\|_{C^{2,\alpha}(S^3)}),$$

for all  $r \leq \frac{R_{\varepsilon, \lambda}^1}{2}$ . Then by (72), we get

$$\left\| H_{\varphi_j^1, \psi_j^1}^{\text{int}} \left( \frac{r}{R_{\varepsilon, \lambda}^1} \cdot \right) \right\|_{C^{4,\alpha}(\bar{B}_2(0)-B_1(0))} \leq c_\kappa \varepsilon^2 r^2. \quad (74)$$

On the other hand,

$$\begin{aligned} & \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\mathcal{R}_1(0, 0)| \\ & \leq \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} \left| e^{\gamma(H_{\varphi_1^1, \psi_1^1}^{\text{int}} + h_1^1) + (1-\gamma)(H_{\varphi_2^1, \psi_2^1}^{\text{int}} + h_2^1)} - 1 \right| + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left( \left| \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x-x^1) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln \gamma}{\gamma} \right. \right. \\ & \quad \left. \left. + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x-x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) \right) \right| + |\Delta^2 h_1^1| \right) \\ & \leq \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} (\gamma r^2 \|H_{\varphi_1^1, \psi_1^1}^{\text{int}}\|_{C_2^{4,\alpha}} + \gamma r^\mu \|h_1^1\|_{C_\mu^{4,\alpha}} \\ & \quad + (1-\gamma)r^2 \|H_{\varphi_2^1, \psi_2^1}^{\text{int}}\|_{C_2^{4,\alpha}} + (1-\gamma)r^\mu \|h_2^1\|_{C_\mu^{4,\alpha}}) \\ & \quad + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \left( \Delta \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x-x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) \right) \right)^2 \right| \\ & \quad + 2\lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \nabla \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x-x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) \right) \right| \\ & \quad \times \left| \nabla \left( \Delta \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right. \right. \right. \\ & \quad \left. \left. \left. + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x-x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) \right) \right) \right| \\ & \quad + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \nabla \left( \frac{1}{\gamma} \bar{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right. \right. \\ & \quad \left. \left. + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x-x^1}{R_{\varepsilon, \lambda}^1} \right) \right) + h_1^1(x) \right|^2 \end{aligned}$$

$$\begin{aligned}
& \times \left| \Delta \left( \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right. \right. \\
& \quad \left. \left. + H_{\varphi_1^1, \psi_1^1}^{\text{int}}\left(\frac{x-x^1}{R_{\varepsilon, \lambda}^1}\right) + h_1^1(x) \right) \right| + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta^2 h_1^1| \\
& \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda (1 + \varepsilon^4 R_{\varepsilon, \lambda}^{4-\mu} + \varepsilon^2 R_{\varepsilon, \lambda}^{2-\mu} + \varepsilon^3 R_{\varepsilon, \lambda}^{3-\mu} + \varepsilon^6 R_{\varepsilon, \lambda}^{5-\mu} + \varepsilon \|h_1^1\|_{C_\mu^{4,\alpha}} \\
& \quad + \varepsilon^2 R_{\varepsilon, \lambda}^2 \|h_1^1\|_{C_\mu^{4,\alpha}} + \varepsilon^3 R_{\varepsilon, \lambda}^3 \|h_1^1\|_{C_\mu^{4,\alpha}} \\
& \quad + R_{\varepsilon, \lambda}^\mu \|h_1^1\|_{C_\mu^{4,\alpha}}^2) + c_\kappa \lambda^2 (1 + \varepsilon R_{\varepsilon, \lambda}^{2-\mu} + \varepsilon^2 R_{\varepsilon, \lambda}^{3-\mu} + \varepsilon^3 R_{\varepsilon, \lambda}^{4-\mu} + \varepsilon^5 R_{\varepsilon, \lambda}^{5-\mu} \\
& \quad + R_{\varepsilon, \lambda} \|h_1^1\|_{C_\mu^{4,\alpha}} + \varepsilon^2 R_{\varepsilon, \lambda}^3 \|h_1^1\|_{C_\mu^{4,\alpha}} \\
& \quad + \varepsilon R_{\varepsilon, \lambda}^2 \|h_1^1\|_{C_\mu^{4,\alpha}} + R_{\varepsilon, \lambda}^{\mu+1} \|h_1^1\|_{C_\mu^{4,\alpha}}^2) \\
& \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda (1 + \varepsilon r_{\varepsilon, \lambda}^2 + \varepsilon^\mu r_{\varepsilon, \lambda}^{2-\mu} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{\mu+4}) \\
& \quad + c_\kappa \lambda^2 (1 + \varepsilon^{\mu-1} r_{\varepsilon, \lambda}^{2-\mu} + \varepsilon^\mu r_{\varepsilon, \lambda}^{5-\mu} + \varepsilon^{-1} r_{\varepsilon, \lambda}^3 + \varepsilon^{-\mu-1} r_{\varepsilon, \lambda}^{\mu+5}).
\end{aligned}$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get that there exists  $c_\kappa > 0$  such that

$$\|\mathcal{N}_1(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2. \quad (75)$$

For the second estimate, we have

$$\begin{aligned}
& \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} |\mathcal{R}_2(0, 0)| \\
& \leq c_\kappa \sup_{r \leq R_{\varepsilon, \lambda}^1} C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} r^{4-\delta} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} \\
& \quad \times e^{\frac{\gamma+\xi-1}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + \frac{\gamma+\xi-1}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right)} e^{\xi(H_{\varphi_2^1, \psi_2^1}^{\text{int}} + h_2^1) + (1-\xi)(H_{\varphi_1^1, \psi_1^1}^{\text{int}} + h_1^1)} \\
& \quad + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} \left( \left| \mathcal{L}_\lambda \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \right. \\
& \quad \left. \left. \left. + H_{\varphi_2^1, \psi_2^1}^{\text{int}}\left(\frac{x-x^1}{R_{\varepsilon, \lambda}^1}\right) + h_2^1(x) \right) \right| + |\Delta^2 h_2^1| \right) \\
& \leq c_\kappa \sup_{r \leq R_{\varepsilon, \lambda}^1} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} r^{4-\delta} \left( \frac{16}{(1+r^2)^4} \right)^{\frac{1-\xi}{\gamma}} (\xi r^2 \|H_{\varphi_2^1, \psi_2^1}^{\text{int}}\|_{C_2^{4,\alpha}} + r^\mu \|h_2^1\|_{C_\mu^{4,\alpha}} \\
& \quad + (1-\xi)r^2 \|H_{\varphi_1^1, \psi_1^1}^{\text{int}}\|_{C_2^{4,\alpha}} \\
& \quad + (1-\xi)r^\mu \|h_1^1\|_{C_\mu^{4,\alpha}} + 1) + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} \left| \Delta \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \\
& \quad \left. \left. + H_{\varphi_2^1, \psi_2^1}^{\text{int}}\left(\frac{x-x^1}{R_{\varepsilon, \lambda}^1}\right) + h_2^1(x) \right) \right|^2 \\
& \quad + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} \left| \nabla \left( \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + H_{\varphi_2^1, \psi_2^1}^{\text{int}}\left(\frac{x-x^1}{R_{\varepsilon, \lambda}^1}\right) + h_2^1(x) \right) \right|
\end{aligned}$$

$$\begin{aligned}
& \times \left| \nabla \left( \Delta \left( \frac{1}{\xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) + G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) + H_{\varphi_2^1, \psi_2^1}^{\text{int}} \left( \frac{x - x^1}{R_{\varepsilon, \lambda}^1} \right) + h_2^1(x) \right) \right) \right| \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\delta} \left| \nabla \left( \frac{1}{\xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) \right. \right. \\
& \quad \left. \left. + G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) + H_{\varphi_2^1, \psi_2^1}^{\text{int}} \left( \frac{x - x^1}{R_{\varepsilon, \lambda}^1} \right) + h_2^1(x) \right) \right|^2 \left| \Delta \left( \frac{1}{\xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) \right. \right. \\
& \quad \left. \left. + G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) + H_{\varphi_2^1, \psi_2^1}^{\text{int}} \left( \frac{x - x^1}{R_{\varepsilon, \lambda}^1} \right) + h_2^1(x) \right) \right| \\
& + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta^2 h_2^1|.
\end{aligned}$$

With the same argument as above, but using the condition  $(A_2)$ , we get  $\|\mathcal{M}_1(0, 0)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2$ .

To derive the third estimate, for  $(v_1^1, v_2^1), (t_1^1, t_2^1)$  verifying (73), we have

$$\begin{aligned}
& \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\mathcal{R}_1(v_1^1, v_2^1) - \mathcal{R}_1(t_1^1, t_2^1)| \\
& \leq \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} \left| \left( e^{\gamma(h_1^1 + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + v_1^1) + (1-\gamma)(h_2^1 + H_{\varphi_2^1, \psi_2^1}^{\text{int}} + v_2^1)} - \gamma v_1^1 \right) \right. \\
& \quad \left. - \left( e^{\gamma(h_1^1 + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + t_1^1) + (1-\gamma)(h_2^1 + H_{\varphi_2^1, \psi_2^1}^{\text{int}} + t_2^1)} - \gamma t_1^1 \right) \right| \\
& + \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right. \right. \\
& \quad \left. \left. - \frac{1-\gamma}{\gamma \xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) - \frac{\ln \gamma}{\gamma} + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x - x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) + v_1^1(x) \right) \right. \\
& \quad \left. - \mathcal{L}_\lambda \left( \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right. \right. \\
& \quad \left. \left. - \frac{1-\gamma}{\gamma \xi} G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) - \frac{\ln \gamma}{\gamma} + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \left( \frac{x - x^1}{R_{\varepsilon, \lambda}^1} \right) + h_1^1(x) + t_1^1(x) \right) \right| \\
& \leq c_\kappa \sup_{r \leq R_{\varepsilon, \lambda}^1} \frac{384r^{4-\mu}}{(1+r^2)^4} \frac{1}{\gamma} [\gamma^2 ((v_1^1)^2 - (t_1^1)^2) + (1-\gamma)|v_2^1 - t_2^1|] + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta(v_1^1 - t_1^1)| \\
& \quad \times \left( \frac{2}{\gamma} |\Delta \bar{u}| + 2 \frac{1-\gamma}{\gamma} \left| \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right| + 2 \frac{1-\gamma}{\gamma \xi} \left| \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) \right| + 2 |\Delta H_{\varphi_1^1, \psi_1^1}^{\text{int}}| \right. \\
& \quad \left. + 2 |\Delta h_1^1| + |\Delta v_1^1| + |\Delta t_1^1| \right) \\
& + 2 \lambda \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left| \nabla(v_1^1 - t_1^1) \cdot \nabla \left( \frac{2}{\gamma} \Delta \bar{u} + 2 \frac{1-\gamma}{\gamma} \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right) \right. \\
& \quad \left. + 2 \frac{1-\gamma}{\gamma \xi} \Delta G \left( \frac{\varepsilon x}{\tau_1}, x^3 \right) + 2 \Delta H_{\varphi_1^1, \psi_1^1}^{\text{int}} + 2 \Delta h_1^1 \right. \\
& \quad \left. + \Delta v_1^1 + \Delta t_1^1 \right) + \nabla(\Delta(v_1^1 - t_1^1)) \cdot \nabla \left( \frac{2}{\gamma} \bar{u} - 2 \frac{1-\gamma}{\gamma} G \left( \frac{\varepsilon x}{\tau_1}, x^2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -2 \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + 2H_{\varphi_1^1, \psi_1^1}^{\text{int}} + 2h_1^1 \\
& + v_1^1 + t_1^1 \Big| + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} |\Delta(v_1^1 - t_1^1)| \Big[ \Big| \nabla \left( \frac{1}{\gamma} \bar{u} \right. \\
& \left. - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} \right. \\
& \left. + h_1^1 + v_1^1 \right)^2 + \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \\
& \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + h_1^1 + t_1^1 \right)^2 \right] \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^1} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta \bar{u}| + 2 \frac{1-\gamma}{\gamma} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right| + 2 \frac{1-\gamma}{\gamma \xi} \left| \Delta G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \right| \right. \\
& \left. + 2 |\Delta H_{\varphi_1^1, \psi_1^1}^{\text{int}}| + 2 |h_1^1| \right. \\
& \left. + |\Delta v_1^1| + |\Delta t_1^1| \right) \Big[ \Big| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) \right. \right. \\
& \left. \left. - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + h_1^1 + v_1^1 \right)^2 \right. \\
& \left. - \left| \nabla \left( \frac{1}{\gamma} \bar{u} - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma \xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + H_{\varphi_1^1, \psi_1^1}^{\text{int}} + h_1^1 + t_1^1 \right) \right|^2 \right] \\
& \leq c_\kappa r_{\varepsilon, \lambda}^2 \|v_1^1 - t_1^1\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|v_2^1 - t_2^1\|_{C_\delta^{4,\alpha}} + c_\kappa \lambda (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{\mu+2}) \|v_1^1 - t_1^1\|_{C_\mu^{4,\alpha}} \\
& + c_\kappa \lambda^2 (1 + r_{\varepsilon, \lambda} + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{\mu+2} + \varepsilon^{-2\mu} r_{\varepsilon, \lambda}^{2\mu+4}) \|v_1^1 - t_1^1\|_{C_\mu^{4,\alpha}}.
\end{aligned}$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get that there exists  $\bar{c}_\kappa > 0$  such that

$$\begin{aligned}
& \|\mathcal{N}_1(v_1^1, v_2^1) - \mathcal{N}_1(t_1^1, t_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
& \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|v_1^1 - t_1^1\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|v_2^1 - t_2^1\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{76}
\end{aligned}$$

Similarly, we get the estimate for

$$\|\mathcal{M}_1(v_1^1, v_2^1) - \mathcal{M}_1(t_1^1, t_2^1)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq \bar{c}_\kappa r_{\varepsilon, \lambda}^2 \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)}. \tag{77}$$

□

Reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa r_{\varepsilon, \lambda}^2 < \frac{1}{2}$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ . There exists also  $\gamma_0 \in (0, 1)$  such that  $\bar{c}_\kappa (1-\gamma) \leq \frac{1}{2}$  for all  $\gamma \in (\gamma_0, 1)$ . Therefore (76) and (77) are enough to show that

$$(v_1^1, v_2^1) \mapsto (\mathcal{N}_1(v_1^1, v_2^1), \mathcal{M}_1(v_1^1, v_2^1))$$

is a contraction from the ball

$$\{(v_1^1, v_2^1) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4) : \|(v_1^1, v_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

**Proposition 7** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , for all  $\tau_1$  in some fixed compact subset of  $[\tau_1^-, \tau_1^+] \subset (0, \infty)$  and for  $\varphi_j^1$  and  $\psi_j^1$  satisfying (65) and (72), there exists a unique  $(v_1^1, v_2^1) := (v_{1,\varepsilon, \tau_1, \varphi_1^1, \psi_1^1}, v_{2,\varepsilon, \tau_1, \varphi_2^1, \psi_2^1})$  solution of (71) such that

$$\|(v_1^1, v_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G(\frac{\varepsilon x}{\tau_1}, x^2) - \frac{1-\gamma}{\gamma\xi} G(\frac{\varepsilon x}{\tau_1}, x^3) \\ \quad - \frac{\ln \gamma}{\gamma} + h_1^1(x) + H^{\text{int}}(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_{\varepsilon, \lambda}^1}) + v_1^1(x), \\ v_2(x) := \frac{1}{\xi} G(\frac{\varepsilon x}{\tau_1}, x^3) + G(\frac{\varepsilon x}{\tau_1}, x^2) + h_2^1(x) + H^{\text{int}}(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_{\varepsilon, \lambda}^1}) + v_2^1(x) \end{cases}$$

solves (67) in  $B_{R_{\varepsilon, \lambda}^1}(x^1)$ .

In  $B_{R_\varepsilon^3}(x^3)$ , following the same arguments as the first case by reversing the roles of the functions  $v_1$  and  $v_2$  and by respecting the changes of the coefficients we can prove that there exists  $(v_1^3, v_2^3) \in C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$  such that

$$\|(v_1^3, v_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Furthermore,  $(v_1^3, v_2^3)$  solves the equations

$$\begin{cases} \Delta^2 v_1^3 = \frac{24C_{3,\varepsilon} \frac{4\gamma+\xi-1}{\xi} 16 \frac{1-\gamma}{\xi} \varepsilon \frac{8\gamma+\xi-1}{\xi}}{\xi \frac{1-\gamma}{\xi} (1+r^2)^4} \\ \quad \times e^{\frac{\gamma+\xi-1}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^2) + \frac{\gamma+\xi-1}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) + \gamma(h_1^3 + H^{\text{int}}_{\varphi_1^3, \psi_1^3} + v_1^3) + (1-\gamma)(h_2^3 + H^{\text{int}}_{\varphi_2^3, \psi_2^3} + v_2^3)} \\ \quad - \mathcal{L}_\lambda(\frac{1}{\gamma} G(\frac{\varepsilon x}{\tau_3}, x^1) + G(\frac{\varepsilon x}{\tau_3}, x^2) \\ \quad + H^{\text{int}}(\varphi_1^3, \psi_1^3; \frac{x-x^3}{R_{\varepsilon, \lambda}^3}) + h_1^3(x) + v_1^3(x)) - \Delta^2 h_1^3, \\ \mathbb{L} v_2^3 = \frac{384}{\xi(1+r^2)^4} [e^{\xi(h_2^3 + H^{\text{int}}_{\varphi_2^3, \psi_2^3} + v_2^3) + (1-\xi)(h_1^3 + H^{\text{int}}_{\varphi_1^3, \psi_1^3} + v_1^3)} \\ \quad - \xi v_2^3 - 1] - \mathcal{L}_\lambda(\frac{1}{\xi} \bar{u}(x - x^3) \\ \quad - \frac{1-\xi}{\xi} G(\frac{\varepsilon x}{\tau_3}, x^2) - \frac{1-\xi}{\gamma\xi} G(\frac{\varepsilon x}{\tau_3}, x^1) - \frac{\ln \xi}{\xi} \\ \quad + H^{\text{int}}(\varphi_2^3, \psi_2^3; \frac{x-x^3}{R_{\varepsilon, \lambda}^3}) + h_2^3(x) + v_2^3(x)) - \Delta^2 h_2^3. \end{cases} \quad (78)$$

Given  $\kappa > 0$  (whose value will be fixed later), we further assume that the functions  $\varphi_j^3$  and  $\psi_j^3$  satisfy

$$\|\varphi_j^3\|_{C^{4,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|\psi_j^3\|_{C^{2,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2, \quad \text{for } j = 1, 2. \quad (79)$$

Then we have the following proposition.

**Proposition 8** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$ , and  $\xi_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\xi \in (\xi_0, 1)$ , for all  $\tau_3$  in

some fixed compact subset of  $[\tau_3^-, \tau_3^+] \subset (0, \infty)$  and for  $\varphi_j^3$  and  $\psi_j^3$  satisfying (65) and (79), there exists a unique  $(v_1^3, v_2^3) := (\nu_{1,\varepsilon,\tau_3,\varphi_1^3,\psi_1^3}, \nu_{2,\varepsilon,\tau_3,\varphi_2^3,\psi_2^3})$  solution of (71) such that

$$\|(v_1^3, v_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) + H^{\text{int}}(\varphi_1^3, \psi_1^3; \frac{x-x^3}{R_{\varepsilon,\lambda}^3}) + v_1^3(x), \\ v_2(x) := \frac{1}{\xi} \bar{u}(x - x^3) - \frac{1-\xi}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) \\ \quad - \frac{\ln \xi}{\xi} + h_2^3(x) + H^{\text{int}}(\varphi_2^3, \psi_2^3; \frac{x-x^3}{R_{\varepsilon,\lambda}^3}) + v_2^3(x) \end{cases}$$

solves (69) in  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ .

In  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ , we look for a solution of (45) of the form

$$\begin{cases} v_1(x) = \bar{u}(x - x^2) + H^{\text{int}}(\varphi_1^2, \psi_1^2; \frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_1^2(x) + v_1^2(x), \\ v_2(x) = \bar{u}(x - x^2) + H^{\text{int}}(\varphi_2^2, \psi_2^2; \frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_2^2(x) + v_2^2(x). \end{cases}$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L}v_1^2 = \frac{384}{(1+r^2)^4} [e^{\gamma(h_1^2+H^{\text{int}}_{\varphi_1^2,\psi_1^2}+v_1^2)+(1-\gamma)(h_2^2+H^{\text{int}}_{\varphi_2^2,\psi_2^2}+v_2^2)} - v_1^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x - x^2) + H^{\text{int}}_{\varphi_1^2,\psi_1^2}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_1^2(x) + v_1^2(x)) - \Delta^2 h_1^2, \\ \mathbb{L}v_2^2 = \frac{384}{(1+r^2)^4} [e^{\xi(h_2^2+H^{\text{int}}_{\varphi_2^2,\psi_2^2}+v_2^2)+(1-\xi)(h_1^2+H^{\text{int}}_{\varphi_1^2,\psi_1^2}+v_1^2)} - v_2^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x - x^2) + H^{\text{int}}_{\varphi_2^2,\psi_2^2}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_2^2(x) + v_2^2(x)) - \Delta^2 h_2^2. \end{cases} \quad (80)$$

We denote by

$$\mathbb{L}v_1^2 = \mathcal{R}_3(v_1^2, v_2^2) \quad \text{and} \quad \mathbb{L}v_2^2 = \mathcal{R}_4(v_1^2, v_2^2).$$

To find a solution of (80), it is enough to find a fixed point  $(v_1^2, v_2^2)$  in a small ball of  $C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$ , solutions of

$$\begin{cases} v_1^2 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon,\lambda}^2} \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(v_1^2, v_2^2), \\ v_2^2 = \mathcal{G}_\mu \circ \xi_{\mu, R_{\varepsilon,\lambda}^2} \circ \mathcal{R}_4(v_1^2, v_2^2) = \mathcal{M}_2(v_1^2, v_2^2). \end{cases} \quad (81)$$

Given  $\kappa > 0$  (whose value will be fixed later), we further assume that the functions  $\varphi_j^2$  and  $\psi_j^2$  satisfy

$$\|\varphi_j^2\|_{C^{4,\alpha}(S^3)} \leq \kappa r_{\varepsilon,\lambda}^2 \quad \text{and} \quad \|\psi_j^2\|_{C^{2,\alpha}(S^3)} \leq \kappa r_{\varepsilon,\lambda}^2, \quad \text{for } j = 1, 2. \quad (82)$$

Then, we have the following result.

**Lemma 6** Let  $\mu \in (1, 2)$ ,  $\gamma_0$  and  $\xi_0 \in (0, 1)$ . Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\bar{c}_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . We have

$$\begin{aligned} \|\mathcal{N}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \quad \|\mathcal{M}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\mathcal{N}_2(v_1^2, v_2^2) - \mathcal{N}_2(t_1^2, t_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \gamma) \|v_2^2 - t_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ \|\mathcal{M}_2(v_1^2, v_2^2) - \mathcal{M}_2(t_1^2, t_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq \bar{c}_\kappa (1 - \xi) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \|v_2^2 - t_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}, \end{aligned}$$

provided  $(v_1^2, v_2^2), (t_1^2, t_2^2)$  in  $C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(v_1^2, v_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|(t_1^2, t_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (83)$$

*Proof* The proof of the first and the second estimates follows from the asymptotic behavior of  $H^{\text{int}}$  together with the assumption on the norms of  $\varphi_j^2$  and  $\psi_j^2$  given by (82) and it follows from the estimate of  $H^{\text{int}}$ , given by Lemma 3, that

$$\left\| H_{\varphi_j^2, \psi_j^2}^{\text{int}} \left( \frac{r}{R_{\varepsilon, \lambda}^2} \cdot \right) \right\|_{C^{4,\alpha}(\bar{B}_2(0) - B_1(0))} \leq Cr^2(R_{\varepsilon, \lambda}^2)^{-2} (\|\varphi_j^2\|_{C^{4,\alpha}(S^3)} + \|\psi_j^2\|_{C^{2,\alpha}(S^3)}),$$

for all  $r \leq \frac{R_{\varepsilon, \lambda}^2}{2}$ . Then by (82), we get

$$\left\| H_{\varphi_j^2, \psi_j^2}^{\text{int}} \left( \frac{r}{R_{\varepsilon, \lambda}^2} \cdot \right) \right\|_{C^{4,\alpha}(\bar{B}_2(0) - B_1(0))} \leq c_\kappa \varepsilon^2 r^2.$$

On the other hand,

$$\begin{aligned} &\sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{R}_3(0, 0)| \\ &\leq \sup_{r \leq R_{\varepsilon, \lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}}) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}})} - 1 \right| \\ &\quad + \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} (|\mathcal{L}_\lambda(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2)| + |\Delta^2 h_1^2|) \\ &\leq c_\kappa \sup_{r \leq R_{\varepsilon, \lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} (\gamma r^2 \|H_{\varphi_1^2, \psi_1^2}^{\text{int}}\|_{C_2^{4,\alpha}} + \gamma \|h_1^2\|_{C_\mu^{4,\alpha}} \\ &\quad + (1-\gamma)r^2 \|H_{\varphi_2^2, \psi_2^2}^{\text{int}}\|_{C_2^{4,\alpha}} + (1-\gamma)\|h_2^2\|_{C_\mu^{4,\alpha}}) \\ &\quad + \lambda \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} \left[ |\Delta \bar{u} + \Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}} + \Delta h_1^2|^2 \right] \\ &\quad + 2 |\nabla(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2) \nabla(\Delta(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2))| \\ &\quad + \lambda^2 \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\nabla(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2)|^2 |\Delta(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2)| + \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\Delta^2 h_1^2| \end{aligned}$$

$$\begin{aligned} &\leq c_\kappa r_{\varepsilon,\lambda}^2 + c_\kappa \lambda (1 + \varepsilon^\mu r_{\varepsilon,\lambda}^{2-\mu} + r_{\varepsilon,\lambda}^2 + \varepsilon^{-\mu} r_{\varepsilon,\lambda}^{\mu+4}) + c_\kappa \lambda^2 (1 + \varepsilon^\mu r_{\varepsilon,\lambda}^{2-\mu} + \varepsilon^{\mu+2} r_{\varepsilon,\lambda}^{6-\mu} \\ &\quad + \varepsilon^{\mu+1} r_{\varepsilon,\lambda}^{4-\mu} + \varepsilon^{-\mu} r_{\varepsilon,\lambda}^{\mu+4} + r_{\varepsilon,\lambda}^4 + \varepsilon^{-2\mu} r_{\varepsilon,\lambda}^{2\mu+6}). \end{aligned}$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get

$$\|\mathcal{N}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (84)$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (85)$$

To derive the third estimate, using the asymptotic behavior of  $H^{\text{int}}$  given by the estimate (74) and for  $l = 1, \dots, 4 + \alpha$ ,  $|\nabla^l w| \leq c_\kappa r^{\mu-l} \|w\|_{C_\mu^{k,\alpha}(\mathbb{R}^4)}$ . Then for  $(v_1^2, v_2^2), (t_1^2, t_2^2)$  verifying (58), we have

$$\begin{aligned} &\sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\mathcal{R}_3(v_1^2, v_2^2) - \mathcal{R}_3(t_1^2, t_2^2)| \\ &\leq \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| \left( e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + v_1^2) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}} + v_2^2)} - v_1^2 \right) \right. \\ &\quad \left. - \left( e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + t_1^2) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}} + t_2^2)} - t_1^2 \right) \right| \\ &\quad + \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\mathcal{L}_\lambda(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2) - \mathcal{L}_\lambda(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2)| \\ &\leq c \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} |(\gamma-1)(v_1^2 - t_1^2) + (1-\gamma)(v_2^2 - t_2^2)| + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(v_1^2 - t_1^2)| \\ &\quad \times (2|\Delta \bar{u}| + 2|\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}}| + 2|\Delta h_1^2| + |\Delta v_1^2| + |\Delta t_1^2|) \\ &\quad + 2\lambda \sup_{r \leq R_{\varepsilon,\lambda}^1} r^{4-\mu} |\nabla(v_1^2 - t_1^2) \cdot \nabla(2\Delta \bar{u} + 2\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}} + 2\Delta h_1^2 \\ &\quad + \Delta v_1^2 + \Delta t_1^2) + \nabla(\Delta(v_1^2 - t_1^2)) \cdot \nabla(2\bar{u} + 2H_{\varphi_1^2, \psi_1^2}^{\text{int}} + 2h_1^2 + v_1^2 + t_1^2)| \\ &\quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(v_1^2 - t_1^2)| \\ &\quad \times [|\nabla(\bar{u}(x-x^2) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2)|^2 + |\nabla(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2)|^2] \\ &\quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta \bar{u}| \right. \\ &\quad \left. + 2|\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}}| + 2|\Delta h_1^2| + |\Delta v_1^2| + |\Delta t_1^2| \right) [|\nabla(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2)|^2 \\ &\quad - |\nabla(\bar{u} + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2)|^2] \\ &\leq c_\kappa (1-\gamma) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}} + c_\kappa (1-\gamma) \|v_2^2 - t_2^2\|_{C_\mu^{4,\alpha}} \\ &\quad + c_\kappa \lambda (1 + r_{\varepsilon,\lambda}^2 + \varepsilon^{-\mu} r_{\varepsilon,\lambda}^{2+\mu}) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}} \end{aligned}$$

$$+ c_\kappa \lambda^2 (1 + r_{\varepsilon, \lambda}^2 + \varepsilon^{-\mu} r_{\varepsilon, \lambda}^{2+\mu} + \varepsilon^{-2\mu} r_{\varepsilon, \lambda}^{2(2+\mu)}) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}}.$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get that there exists  $\bar{c}_\kappa > 0$  such that

$$\begin{aligned} & \|N_2(v_1^2, v_2^2) - N_2(t_1^2, t_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ & \leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \gamma) \|v_2^2 - t_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned} \quad (86)$$

Similarly, we get

$$\begin{aligned} & \|M_2(v_1^2, v_2^2) - M_2(t_1^2, t_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ & \leq \bar{c}_\kappa (1 - \xi) \|v_1^2 - t_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \|v_2^2 - t_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned} \quad (87)$$

□

Then there exist  $\gamma_0$  and  $\xi_0 \in (0, 1)$ , reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \leq 1/2$  and  $\bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \leq 1/2$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . Therefore (61) and (62) are enough to show that

$$(v_1^2, v_2^2) \mapsto (N_2(v_1^2, v_2^2), M_2(v_1^2, v_2^2))$$

is a contraction from the ball

$$\{(v_1^2, v_2^2) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4) : \|(v_1^2, v_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

**Proposition 9** *Given  $\kappa > 0$ ,  $\mu \in (1, 2)$ ,  $\gamma_0 \in (0, 1)$  and  $\xi_0 \in (0, 1)$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$  and  $c_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ , for all  $\tau_2$  in some fixed compact subset of  $[\tau_2^-, \tau_2^+] \subset (0, \infty)$  and for  $\varphi_j^2$  and  $\psi_j^2$  satisfying (65) and (82), there exists a unique  $(v_1^2, v_2^2) := (v_{1,\varepsilon, \tau_2, \varphi_1^2, \psi_1^2}, v_{2,\varepsilon, \tau_2, \varphi_2^2, \psi_2^2})$  solution of (81) such that*

$$\|(v_1^2, v_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \bar{u}(x - x^2) + h_1^2(x) + H^{\text{int}}(\varphi_1^2, \psi_1^2; \frac{x-x^2}{R_{\varepsilon, \lambda}^2}) + v_1^2(x), \\ v_2(x) := \bar{u}(x - x^2) + h_2^2(x) + H^{\text{int}}(\varphi_2^2, \psi_2^2; \frac{x-x^2}{R_{\varepsilon, \lambda}^2}) + v_2^2(x), \end{cases}$$

solves (68) in  $B_{R_{\varepsilon, \lambda}^2}(x^2)$ .

Remark also that the functions  $(v_1^i, v_2^i) := (v_{1,\varepsilon, \tau_i, \varphi_1^i, \psi_1^i}^i, v_{2,\varepsilon, \tau_i, \varphi_2^i, \psi_2^i}^i)$ , for  $i \in \{1, 2, 3\}$ , depend continuously on the parameter  $\tau_i$ .

### 3.3 The nonlinear exterior problem

Given  $\tilde{\mathbf{x}} := (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \Omega^3$  close to  $\mathbf{x} := (x^1, x^2, x^3) \in \mathbb{R}^3$ ,  $\eta := (\eta_1, \eta_2, \eta_3) \in \mathbb{R}$  close to 0,  $\tilde{\varphi}_1 := (\tilde{\varphi}_1^1, \tilde{\varphi}_1^2, \tilde{\varphi}_1^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$ ,  $\tilde{\varphi}_2 := (\tilde{\varphi}_2^1, \tilde{\varphi}_2^2, \tilde{\varphi}_2^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$ ,  $\tilde{\psi}_1 := (\tilde{\psi}_1^1, \tilde{\psi}_1^2, \tilde{\psi}_1^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$  and  $\tilde{\psi}_2 := (\tilde{\psi}_2^1, \tilde{\psi}_2^2, \tilde{\psi}_2^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$  satisfying (66). Let  $\tilde{\mathbf{w}}_1$  and  $\tilde{\mathbf{w}}_2$  be defined by

$$\begin{aligned}\tilde{\mathbf{w}}_1(x) &:= \frac{1 + \eta_1}{\gamma} G(x, \tilde{x}^1) + (1 + \eta_2) G(x, \tilde{x}^2) \\ &\quad + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_1^i, \tilde{\psi}_1^i; \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}}\right) \quad \text{and} \\ \tilde{\mathbf{w}}_2(x) &:= \frac{1 + \eta_3}{\xi} G(x, \tilde{x}^3) + (1 + \eta_2) G(x, \tilde{x}^2) \\ &\quad + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_2^i, \tilde{\psi}_2^i; \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}}\right).\end{aligned}\tag{88}$$

Here  $\chi_{r_0}$  is a cut-off function identically equal to 1 in  $B_{\frac{r_0}{2}}(0)$  and identically equal to 0 outside  $B_{r_0}(0)$ . We would like to find a solution of the system

$$\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2} - \mathcal{L}_\lambda(u_1) \quad \text{and} \quad \Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1} - \mathcal{L}_\lambda(u_2)\tag{89}$$

in the domain  $\bar{\Omega}_{r_{\varepsilon, \lambda}}(\tilde{\mathbf{x}})$ , which is a perturbation of  $\tilde{\mathbf{w}}_k$ ,  $k = 1, 2$ , with

$$\mathcal{L}_\lambda(u_i) = \lambda(\Delta u_i)^2 + \lambda \nabla u_i \cdot \nabla(\Delta u_i) + \lambda^2 |\nabla u_i|^2 \Delta u_i, \quad \text{for } i = 1, 2.$$

Writing  $v_k = \tilde{\mathbf{w}}_k + \tilde{v}_k$ , this amounts to solve in  $\bar{\Omega}_{r_{\varepsilon, \lambda}}(\tilde{\mathbf{x}})$

$$\begin{cases} \Delta^2 \tilde{v}_1 = \rho^4 e^{\gamma(\tilde{\mathbf{w}}_1 + \tilde{v}_1) + (1-\gamma)(\tilde{\mathbf{w}}_2 + \tilde{v}_2)} - \mathcal{L}_\lambda(\tilde{\mathbf{w}}_1 + \tilde{v}_1) - \Delta^2 \tilde{\mathbf{w}}_1, \\ \Delta^2 \tilde{v}_2 = \rho^4 e^{\xi(\tilde{\mathbf{w}}_2 + \tilde{v}_2) + (1-\xi)(\tilde{\mathbf{w}}_1 + \tilde{v}_1)} - \mathcal{L}_\lambda(\tilde{\mathbf{w}}_2 + \tilde{v}_2) - \Delta^2 \tilde{\mathbf{w}}_2. \end{cases}\tag{90}$$

For all  $\sigma \in (0, \frac{r_0}{2})$  and all  $\tilde{\mathbf{x}} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \Omega^3$  such that  $\|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \frac{r_0}{2}$ , where  $\mathbf{x} = (x^1, x^2, x^3)$ , we denote by  $\tilde{\xi}_{\sigma, \tilde{\mathbf{x}}} : \mathcal{C}_v^{0,\alpha}(\bar{\Omega}_\sigma(\tilde{\mathbf{x}})) \rightarrow \mathcal{C}_v^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))$  the extension operator defined by

$$\begin{cases} \tilde{\xi}_{\sigma, \tilde{\mathbf{x}}}(f) \equiv f & \text{in } \bar{\Omega}_\sigma(\tilde{\mathbf{x}}), \\ \tilde{\xi}_{\sigma, \tilde{\mathbf{x}}}(f)(\tilde{x}^j + x) = \tilde{\chi}\left(\frac{|x|}{\sigma}\right) f(\tilde{x}^j + \sigma \frac{x}{|x|}) & \text{in } B_\sigma(\tilde{x}^j) - B_{\frac{\sigma}{2}}(\tilde{x}^j) \forall 1 \leq j \leq 3, \\ \tilde{\xi}_{\sigma, \tilde{\mathbf{x}}}(f) \equiv 0 & \text{in } B_{\frac{\sigma}{2}}(\tilde{x}^1) \cup B_{\frac{\sigma}{2}}(\tilde{x}^2) \cup B_{\frac{\sigma}{2}}(\tilde{x}^3). \end{cases}$$

Here  $\tilde{\chi}$  is a cut-off function over  $\mathbb{R}_+$ , which is equal to 1 for  $t \geq 1$  and equal to 0 for  $t \leq \frac{1}{2}$ . Obviously, there exists a constant  $\bar{c} = \bar{c}(\nu) > 0$  only depending on  $\nu$  such that

$$\|\tilde{\xi}_{\sigma, \tilde{\mathbf{x}}}(w)\|_{\mathcal{C}_v^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \leq \bar{c} \|w\|_{\mathcal{C}_v^{0,\alpha}(\bar{\Omega}_\sigma(\tilde{\mathbf{x}}))}.\tag{91}$$

We fix  $\nu \in (-1, 0)$ , to solve (90), it is enough to find  $(\tilde{v}_1, \tilde{v}_2) \in (\mathcal{C}_v^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}})))^2$  solution of

$$\tilde{v}_1 = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_{\varepsilon, \lambda}, \tilde{\mathbf{x}}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \quad \text{and} \quad \tilde{v}_2 = \tilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_{\varepsilon, \lambda}, \tilde{\mathbf{x}}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2),\tag{92}$$

where

$$\begin{cases} \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) := \rho^4 e^{\gamma(\tilde{w}_1 + \tilde{v}_1) + (1-\gamma)(\tilde{w}_2 + \tilde{v}_2)} - \mathcal{L}_\lambda(\tilde{w}_1 + \tilde{v}_1) - \Delta^2 \tilde{w}_1, \\ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2) := \rho^4 e^{\xi(\tilde{w}_2 + \tilde{v}_2) + (1-\xi)(\tilde{w}_1 + \tilde{v}_1)} - \mathcal{L}_\lambda(\tilde{w}_2 + \tilde{v}_2) - \Delta^2 \tilde{w}_2. \end{cases}$$

We denote by

$$\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2) := \tilde{\mathcal{K}}_v \circ \tilde{\xi}_{r_{\varepsilon, \lambda}, \tilde{x}} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2) \quad \text{and} \quad \tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) := \tilde{\mathcal{K}}_v \circ \tilde{\xi}_{r_{\varepsilon, \lambda}, \tilde{x}} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2).$$

Given  $\kappa > 0$  (whose value will be fixed later), we further assume that for  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$  the functions  $\tilde{\varphi}_j^i, \tilde{\psi}_j^i$ , the parameters  $\eta_i$  and the point  $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$  satisfy

$$\|\tilde{\varphi}_j^i\|_{C^{4,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2, \quad \|\tilde{\psi}_j^i\|_{C^{2,\alpha}(S^3)} \leq \kappa r_{\varepsilon, \lambda}^2, \quad (93)$$

$$|\eta_i| \leq \kappa r_{\varepsilon, \lambda}^2, \quad |\tilde{x}^i - x^i| \leq \kappa r_{\varepsilon, \lambda}. \quad (94)$$

Then the following result holds.

**Lemma 7** *Under the above assumptions, there exists a constant  $c_\kappa > 0$  such that*

$$\begin{aligned} \|\tilde{\mathcal{N}}(0, 0)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\tilde{\mathcal{M}}(0, 0)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}'_1, \tilde{v}'_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} &\leq c_\kappa r_{\varepsilon, \lambda}^2 \|(\tilde{v}_1, \tilde{v}_2) - (\tilde{v}'_1, \tilde{v}'_2)\|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2} \end{aligned}$$

and

$$\|\tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{M}}(\tilde{v}'_1, \tilde{v}'_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} \leq c_\kappa r_{\varepsilon, \lambda}^2 \|(\tilde{v}_1, \tilde{v}_2) - (\tilde{v}'_1, \tilde{v}'_2)\|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2},$$

provided  $(\tilde{v}_1, \tilde{v}_2, \tilde{v}'_1, \tilde{v}'_2) \in (C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^4$  satisfy

$$\|(\tilde{v}_1, \tilde{v}_2)\|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2} \leq 2c_\kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|(\tilde{v}'_1, \tilde{v}'_2)\|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (95)$$

*Proof* As for the interior problem, the proof of the two first estimates follows from the asymptotic behavior of  $H^{\text{ext}}$  together with the assumption on the norm of boundary data  $\tilde{\varphi}_j^i$  and  $\tilde{\psi}_j^i$ , given by (93). Indeed, let  $c_\kappa$  be a constant depending only on  $\kappa$ , by Lemma 3,

$$\left| H^{\text{ext}} \left( \tilde{\varphi}_j^i, \tilde{\psi}_j^i; \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right| \leq c_\kappa r_{\varepsilon, \lambda}^3 r^{-1}. \quad (96)$$

On the other hand,

$$\tilde{S}_1(0, 0) = \rho^4 e^{\gamma \tilde{w}_1 + (1-\gamma) \tilde{w}_2} - \mathcal{L}_\lambda(\tilde{w}_1) - \Delta^2 \tilde{w}_1 \quad \text{and}$$

$$\tilde{S}_2(0, 0) = \rho^4 e^{\xi \tilde{w}_2 + (1-\xi) \tilde{w}_1} - \mathcal{L}_\lambda(\tilde{w}_2) - \Delta^2 \tilde{w}_2.$$

We will estimate  $\tilde{S}_1(0, 0)$  in different subregions of  $\bar{\Omega}^*(\tilde{x})$ .

- In  $B_{\frac{r_0}{2}}(\tilde{x}^1) - B_{r_{\varepsilon,\lambda}}(\tilde{x}^1)$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 1$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_1 = 0$ , so that  $|\tilde{S}_1(0, 0)| = \rho^4 e^{\gamma \tilde{w}_1 + (1-\gamma)\tilde{w}_2} - \mathcal{L}_{\lambda}(\tilde{\mathbf{w}}_1)$ . Then

$$\begin{aligned}
& |\tilde{S}_1(0, 0)| \\
& \leq c_{\kappa} \varepsilon^4 |x - \tilde{x}^1|^{-8(1+\eta_1)} \\
& \quad + \lambda \left| \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right) \right)^2 \right| \\
& \quad + \lambda \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\
& \quad \left. \left. + H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right) \cdot \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\
& \quad \left. \left. + H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right) \right) \right| \\
& \quad + \lambda^2 \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right) \right|^2 \\
& \quad \times \left| \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right) \right| \\
& \leq c_{\kappa} \varepsilon^4 r^{-8(1+\eta_1)} + c_{\kappa} \lambda \left| \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-2} + (1+\eta_2) |x - \tilde{x}^2|^{-2} + r_{\varepsilon,\lambda}^3 r^{-3} \right)^2 \right| \\
& \quad + c_{\kappa} \lambda \left| \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-1} + (1+\eta_2) |x - \tilde{x}^2|^{-1} + r_{\varepsilon,\lambda}^3 r^{-2} \right) \right. \\
& \quad \times \left. \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-3} + (1+\eta_2) |x - \tilde{x}^2|^{-3} + r_{\varepsilon,\lambda}^3 r^{-4} \right) \right| \\
& \quad + c_{\kappa} \lambda^2 \left| \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-1} + (1+\eta_2) |x - \tilde{x}^2|^{-1} + r_{\varepsilon,\lambda}^3 r^{-2} \right|^2 \\
& \quad \times \left| \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-2} + (1+\eta_2) |x - \tilde{x}^2|^{-2} + r_{\varepsilon,\lambda}^3 r^{-3} \right|.
\end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_1$  small enough, we get

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{\frac{r_0}{2}}(\tilde{x}^1))} \leq \sup_{r_{\varepsilon,\lambda} \leq r \leq \frac{r_0}{2}} r^{4-\nu} |\tilde{S}_1(0, 0)| \leq c_{\kappa} \varepsilon^4 r_{\varepsilon,\lambda}^{-4} + c_{\kappa} \lambda + c_{\kappa} \lambda r_{\varepsilon,\lambda}^3.$$

- In  $B_{r_0}(\tilde{x}^1) - B_{\frac{r_0}{2}}(\tilde{x}^1)$ , using the estimate (96), we have

$$\begin{aligned}
& |\tilde{S}_1(0, 0)| \\
& \leq c_{\kappa} \varepsilon^4 r^{-8(1+\eta_1)} + \left| [\Delta^2, \chi_{r_0}(x - \tilde{x}^1)] H_{\tilde{\varphi}_1^1, \tilde{\psi}_1^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \right| \\
& \quad + \lambda \left| \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}} \right) \right) \right)^2 \right| \\
& \quad + \lambda \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}} \right) \right) \right|
\end{aligned}$$

$$\begin{aligned}
& \times \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right) \\
& + \lambda^2 \left| \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right. \\
& \times \left. \left| \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right| \right. \\
& \leq c_\kappa (\varepsilon^4 r^{-8(1+\eta_1)} + r^{-1} r_{\varepsilon, \lambda}^3) + c_\kappa \lambda \left| \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-2} + (1+\eta_2) |x - \tilde{x}^2|^{-2} + r_{\varepsilon, \lambda}^3 r^{-3} \right)^2 \right|^2 \\
& + c_\kappa \lambda \left| \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-1} + (1+\eta_2) |x - \tilde{x}^2|^{-1} + r_{\varepsilon, \lambda}^3 r^{-2} \right) \left( \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-3} \right. \right. \\
& \left. \left. + (1+\eta_2) |x - \tilde{x}^2|^{-3} + r_{\varepsilon, \lambda}^3 r^{-4} \right) \right| \\
& + c_\kappa \lambda^2 \left| \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-1} + (1+\eta_2) |x - \tilde{x}^2|^{-1} + r_{\varepsilon, \lambda}^3 r^{-2} \right|^2 \\
& \times \left| \frac{1+\eta_1}{\gamma} |x - \tilde{x}^1|^{-2} + (1+\eta_2) |x - \tilde{x}^2|^{-2} + r_{\varepsilon, \lambda}^3 r^{-3} \right|,
\end{aligned}$$

where

$$\begin{aligned}
[\Delta^2, \chi_{r_0}]w &= w \Delta^2 \chi_{r_0} + 2\Delta w \Delta \chi_{r_0} + 4\nabla(\Delta w) \cdot \nabla \chi_{r_0} + 4\nabla w \cdot \nabla(\Delta \chi_{r_0}) \\
&+ 4 \sum_{i,j=1}^4 \frac{\partial^2 \chi_{r_0}}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}.
\end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_1$  small enough, we get

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0/2}(\tilde{x}^1) - B_{\frac{r_0}{2}}(\tilde{x}^1))} \leq \sup_{\frac{r_0}{2} \leq r \leq r_0} r^{4-\nu} |\tilde{S}_1(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

• In  $B_{r_0/2}(\tilde{x}^2) - B_{r_{\varepsilon, \lambda}}(\tilde{x}^2)$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 1$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_1 = 0$ , so that  $\tilde{S}_1(0, 0) = \rho^4 e^{\gamma \tilde{\mathbf{w}}_1 + (1-\gamma) \tilde{\mathbf{w}}_2} - \mathcal{L}_\lambda(\tilde{\mathbf{w}}_1)$ . Then

$$\begin{aligned}
& |\tilde{S}_1(0, 0)| \\
& \leq c_\kappa \varepsilon^4 |x - \tilde{x}^2|^{-8(1+\eta_2)} \\
& + \lambda \left| \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\
& + \lambda \left| \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right) \right. \\
& \cdot \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\
& \left. \left. + H_{\tilde{\varphi}_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right) \\
& + \lambda^2 \left| \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right|
\end{aligned}$$

$$\times \left| \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\varphi_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x-\tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right|.$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_2$  small enough, we get

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0}(\tilde{x}^2))} \leq \sup_{r_{\varepsilon, \lambda} \leq r \leq r_0/2} r^{4-\nu} |\tilde{S}_1(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

- In  $B_{r_0}(\tilde{x}^2) - B_{r_0/2}(\tilde{x}^2)$ , using the estimate (96), there holds

$$\begin{aligned} & |\tilde{S}_1(0, 0)| \\ & \leq c_\kappa \varepsilon^4 r^{-8(1+\eta_2)} + \left| [\Delta^2, \chi_{r_0}(x - \tilde{x}^2)] H_{\varphi_1^2, \tilde{\psi}_1^2}^{\text{ext}} \left( \frac{x-\tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right| \\ & \quad + \lambda \left| \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\varphi_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\ & \quad + \lambda \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\varphi_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right| \\ & \quad \times \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\varphi_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right| \\ & \quad + \lambda^2 \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\varphi_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|^2 \\ & \quad \times \left| \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\varphi_1^i, \tilde{\psi}_1^i}^{\text{ext}} \left( \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|. \end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_2$  small enough, we get

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0}(\tilde{x}^2))} \leq \sup_{r_0/2 \leq r \leq r_0} r^{4-\nu} |\tilde{S}_1(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

Similarly, for  $\nu \in (-1, 0)$  and  $\eta_3$  small enough, we can prove the same result for  $\tilde{x}^3$ .

- In  $\Omega - (B_{r_0}(\tilde{x}^1) \cup B_{r_0}(\tilde{x}^2) \cup B_{r_0}(\tilde{x}^3))$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_1 = 0$ . Then

$$\begin{aligned} & |\tilde{S}_1(0, 0)| \\ & \leq c_\kappa \varepsilon^4 + \lambda \left| \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right) \right)^2 \right| \\ & \quad + \lambda \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right) \right| \\ & \quad \times \nabla \left( \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right) \right) \\ & \quad + \lambda^2 \left| \nabla \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right) \right|^2 \\ & \quad \times \left| \Delta \left( \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) \right) \right|. \end{aligned}$$

So for  $\nu \in (-1, 0)$ , we have

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(\bar{\Omega} - \bigcup_{i=1}^3 B_{r_0}(\tilde{x}^i))} \leq \sup_{r \geq r_0} r^{4-\nu} |\tilde{S}_1(0, 0)| \leq c_\kappa \varepsilon^4 + c_\kappa \lambda.$$

We conclude that

$$\|\tilde{S}_1(0, 0)\|_{C_{\nu-4}^{0,\alpha}(\bar{\Omega}_{r_0}(\tilde{x}))} \leq c_\kappa r_{\varepsilon, \lambda}^2. \quad (97)$$

Now, we are interested in the second equation of the previous system.

• In  $B_{\frac{r_0}{2}}(\tilde{x}^1) - B_{r_{\varepsilon, \lambda}}(\tilde{x}^1)$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 1$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_1 = 0$ , so that  $|\tilde{S}_2(0, 0)| = \rho^4 e^{\xi \tilde{w}_2 + (1-\xi) \tilde{w}_1} - \mathcal{L}_\lambda(\tilde{\mathbf{w}}_2)$ . Then

$$\begin{aligned} & |\tilde{S}_2(0, 0)| \\ & \leq c_\kappa \varepsilon^4 |x - \tilde{x}^1|^{-8 \frac{(1-\xi)(1+\eta_1)}{\gamma}} \\ & \quad + \lambda \left| \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^1, \tilde{\psi}_2^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\ & \quad + \lambda \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^1, \tilde{\psi}_2^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right) \right| \\ & \quad \times \nabla \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\ & \quad \left. \left. + H_{\tilde{\varphi}_2^1, \tilde{\psi}_2^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right) \right) \Big| + \lambda^2 \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\ & \quad \left. \left. + H_{\tilde{\varphi}_2^1, \tilde{\psi}_2^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right) \right|^2 \\ & \quad \times \left| \Delta \left( \frac{1+\eta_3}{\gamma} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^1, \tilde{\psi}_2^1}^{\text{ext}} \left( \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right) \right|. \end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_1$  small enough, we get

$$\|\tilde{S}_2(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{\frac{r_0}{2}}(\tilde{x}^1))} \leq \sup_{r_{\varepsilon, \lambda} \leq r \leq \frac{r_0}{2}} r^{4-\nu} |\tilde{S}_2(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

• In  $B_{r_0}(\tilde{x}^1) - B_{\frac{r_0}{2}}(\tilde{x}^1)$ , using the estimate (96), we have

$$\begin{aligned} & |\tilde{S}_2(0, 0)| \\ & \leq c_\kappa \varepsilon^4 r^{-8 \frac{(1-\xi)(1+\eta_1)}{\gamma}} + \left| [\Delta^2, \chi_{r_0}(x - \tilde{x}^1)] H^{\text{ext}} \left( \tilde{\varphi}_2^1, \tilde{\psi}_2^1; \frac{x - \tilde{x}^1}{r_{\varepsilon, \lambda}} \right) \right| \\ & \quad + \lambda \left| \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\ & \quad + \lambda \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right| \\ & \quad \times \nabla \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right) \Big| \end{aligned}$$

$$\begin{aligned}
& + \lambda^2 \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|^2 \\
& \times \left| \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|,
\end{aligned}$$

where

$$\begin{aligned}
[\Delta^2, \chi_{r_0}]w &= w \Delta^2 \chi_{r_0} + 2\Delta w \Delta \chi_{r_0} + 4\nabla(\Delta w) \cdot \nabla \chi_{r_0} + 4\nabla w \cdot \nabla(\Delta \chi_{r_0}) \\
&+ 4 \sum_{i,j=1}^4 \frac{\partial^2 \chi_{r_0}}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}.
\end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_1$  small enough, we get

$$\|\tilde{S}_2(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0}(\tilde{x}^1) - B_{\frac{r_0}{2}}(\tilde{x}^1))} \leq \sup_{\frac{r_0}{2} \leq r \leq r_0} r^{4-\nu} |\tilde{S}_2(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

• In  $B_{r_0/2}(\tilde{x}^2) - B_{r_{\varepsilon, \lambda}}(\tilde{x}^2)$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 1$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_2 = 0$ , so that  $\tilde{S}_2(0, 0) = \rho^4 e^{\xi \tilde{\mathbf{w}}_2 + (1-\xi) \tilde{\mathbf{w}}_1}$ . Then

$$\begin{aligned}
& |\tilde{S}_2(0, 0)| \\
& \leq c_\kappa \varepsilon^4 |x - \tilde{x}^2|^{-8(1+\eta_2)} \\
& + \lambda \left| \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^2, \tilde{\psi}_2^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\
& + \lambda \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^2, \tilde{\psi}_2^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right| \\
& \times \nabla \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) \right. \right. \\
& \quad \left. \left. + H_{\tilde{\varphi}_2^2, \tilde{\psi}_2^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right) \\
& + \lambda^2 \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^2, \tilde{\psi}_2^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right|^2 \\
& \times \left| \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + H_{\tilde{\varphi}_2^2, \tilde{\psi}_2^2}^{\text{ext}} \left( \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right) \right|.
\end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_2$  small enough, we get

$$\|\tilde{S}_2(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0}(\tilde{x}^2))} \leq \sup_{r_{\varepsilon, \lambda} \leq r \leq r_0/2} r^{4-\nu} |\tilde{S}_2(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

• In  $B_{r_0}(\tilde{x}^2) - B_{r_0/2}(\tilde{x}^2)$ , using the estimate (96), there holds

$$\begin{aligned}
& |\tilde{S}_2(0, 0)| \\
& \leq c_\kappa \varepsilon^4 r^{-8(1+\eta_2)} + \left| [\Delta^2, \chi_{r_0}(x - \tilde{x}^2)] H^{\text{ext}} \left( \tilde{\varphi}_2^2, \tilde{\psi}_2^2; \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \right|
\end{aligned}$$

$$\begin{aligned}
& + \lambda \left| \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2)G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right)^2 \right| \\
& + \lambda \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2)G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right| \\
& \times \nabla \left( \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2)G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right) \\
& + \lambda^2 \left| \nabla \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2)G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|^2 \\
& \times \left| \Delta \left( \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2)G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H_{\tilde{\varphi}_2^i, \tilde{\psi}_2^i}^{\text{ext}} \left( \frac{x - \tilde{x}^i}{r_{\varepsilon, \lambda}} \right) \right) \right|.
\end{aligned}$$

Hence, for  $\nu \in (-1, 0)$  and  $\eta_2$  small enough, we get

$$\|\tilde{S}_2(0, 0)\|_{C_{\nu-4}^{0,\alpha}(B_{r_0}(\tilde{x}^2))} \leq \sup_{r_0/2 \leq r \leq r_0} r^{4-\nu} |\tilde{S}_2(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2 + c_\kappa \lambda + c_\kappa \lambda r_{\varepsilon, \lambda}^3.$$

Similarly, for  $\nu \in (-1, 0)$  and  $\eta_3$  small enough, we can prove the same result for  $\tilde{x}^3$ .

- In  $\Omega - (B_{r_0}(\tilde{x}^1) \cup B_{r_0}(\tilde{x}^2) \cup B_{r_0}(\tilde{x}^3))$ , we have  $\chi_{r_0}(x - \tilde{x}^1) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^2) = 0$ ,  $\chi_{r_0}(x - \tilde{x}^3) = 0$  and  $\Delta^2 \tilde{\mathbf{w}}_1 = 0$ . So for  $\nu \in (-1, 0)$ , we have

$$\|\tilde{S}_2(0, 0)\|_{C_{\nu-4}^{0,\alpha}(\bar{\Omega} - \bigcup_{i=1}^3 B_{r_0}(\tilde{x}^i))} \leq \sup_{r \geq r_0} r^{4-\nu} |\tilde{S}_2(0, 0)| \leq c_\kappa r_{\varepsilon, \lambda}^2.$$

Making use of Proposition 3 together with (91), we conclude that

$$\|\tilde{\mathcal{N}}(0, 0)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \leq c_\kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|\tilde{\mathcal{M}}(0, 0)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))} \leq c_\kappa r_{\varepsilon, \lambda}^2. \quad (98)$$

For the proof of the third estimate, let  $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}'_1$  and  $\tilde{\nu}'_2 \in C_v^{4,\alpha}(\bar{\Omega}^*)$  satisfy (95), we have

$$\begin{aligned}
& |\tilde{S}_1(\tilde{\nu}_1, \tilde{\nu}_2) - \tilde{S}_1(\tilde{\nu}'_1, \tilde{\nu}'_2)| \\
& \leq c_\kappa \varepsilon^4 e^{\gamma \tilde{\mathbf{w}}_1 + (1-\gamma) \tilde{\mathbf{w}}_2} |e^{\gamma \tilde{\nu}_1 + (1-\gamma) \tilde{\nu}_2} - e^{\gamma \tilde{\nu}'_1 + (1-\gamma) \tilde{\nu}'_2}| + |\mathcal{L}_\lambda(\tilde{\mathbf{w}}_1 + \tilde{\nu}_1) - \mathcal{L}_\lambda(\tilde{\mathbf{w}}_1 + \tilde{\nu}'_1)| \\
& \leq c_\kappa \varepsilon^4 (\gamma |\tilde{\nu}_1 - \tilde{\nu}'_1| + (1-\gamma) |\tilde{\nu}_2 - \tilde{\nu}'_2|) + \lambda |\Delta(\tilde{\nu}_1 - \tilde{\nu}'_1)| (2|\Delta \tilde{\mathbf{w}}_1| + |\Delta \tilde{\nu}_1| + |\Delta \tilde{\nu}'_1|) \\
& \quad + \lambda |\nabla(\tilde{\nu}_1 - \tilde{\nu}'_1) \cdot \nabla(2\Delta \tilde{\mathbf{w}}_1 + \Delta \tilde{\nu}_1 + \Delta \tilde{\nu}'_1) + \nabla(\Delta(\tilde{\nu}_1 - \tilde{\nu}'_1) \cdot \nabla(2\Delta \tilde{\mathbf{w}}_1 + \tilde{\nu}_1 + \tilde{\nu}'_1))| \\
& \quad + \lambda^2 |\Delta(\tilde{\nu}_1 - \tilde{\nu}'_1)| [|\nabla(\tilde{\mathbf{w}}_1 + \tilde{\nu}_1)|^2 + |\nabla(\tilde{\mathbf{w}}_1 + \tilde{\nu}'_1)|^2] \\
& \quad + \lambda^2 (|\Delta \tilde{\mathbf{w}}_1| + |\Delta \tilde{\nu}_1| + |\Delta \tilde{\nu}'_1|) [|\nabla(\tilde{\mathbf{w}}_1 + \tilde{\nu}_1)|^2 - |\nabla(\tilde{\mathbf{w}}_1 + \tilde{\nu}'_1)|^2].
\end{aligned}$$

So, for  $\eta_i, i = 1, 2, 3$ , small enough and using the fact that for all  $w \in C_v^{4,\alpha}(\bar{\Omega}^*)$ , there exists a constant  $c > 0$  such that  $|\nabla^l w| \leq c_\kappa r^{\nu-l} \|w\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}$ , we get

$$\begin{aligned}
& \sup_{x \in \bar{\Omega}^*} r^{4-\nu} |\tilde{S}_1(\tilde{\nu}_1, \tilde{\nu}_2) - \tilde{S}_1(\tilde{\nu}'_1, \tilde{\nu}'_2)| \\
& \leq c_\kappa \varepsilon^4 \sup_{x \in \bar{\Omega}^*} r^{4-\nu} (\gamma r^\nu \|\tilde{\nu}_1 - \tilde{\nu}'_1\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{x}}))}
\end{aligned}$$

$$+ (1 - \gamma)r^v \|\tilde{v}_2 - \tilde{v}'_2\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} + c_\kappa \lambda \|\tilde{v}_1 - \tilde{v}'_1\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))}.$$

Using the estimate (91), there exists  $c_\kappa$  (depending on  $\kappa$ ) such that

$$\begin{aligned} & \|\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}'_1, \tilde{v}'_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} \\ & \leq c_\kappa r_{\varepsilon,\lambda}^2 (\|\tilde{v}_1 - \tilde{v}'_1\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} + \|\tilde{v}_2 - \tilde{v}'_2\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))}). \end{aligned} \quad (99)$$

Similarly, we can use the same arguments to prove,

$$\begin{aligned} & \|\tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{M}}(\tilde{v}'_1, \tilde{v}'_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} \\ & \leq c_\kappa r_{\varepsilon,\lambda}^2 (\|\tilde{v}_1 - \tilde{v}'_1\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} + \|\tilde{v}_2 - \tilde{v}'_2\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))}). \end{aligned} \quad (100)$$

□

Reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa > 0$ , if necessary, we can assume that  $c_\kappa r_{\varepsilon,\lambda}^2 \leq \frac{1}{2}$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ . Then, (99) and (100) are enough to show that

$$(\tilde{v}_1, \tilde{v}_2) \mapsto (\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2), \tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2))$$

is a contraction from the ball

$$\{(\tilde{v}_1, \tilde{v}_2) \in (C_v^{4,\alpha}(\mathbb{R}^4))^2 : \|(\tilde{v}_1, \tilde{v}_2)\|_{(C_v^{4,\alpha}(\mathbb{R}^4))^2} \leq 2c_\kappa r_{\varepsilon,\lambda}^2\},$$

into itself. Hence there exists a unique fixed point  $(\tilde{v}_1, \tilde{v}_2)$  in this set, which is a solution of (92). Applying a fixed point theorem for contraction mappings, we conclude that

**Proposition 10** *Given  $\kappa > 0$ , there exists  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$  (depending on  $\kappa$ ) such that for any  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\eta_i$  and  $\tilde{x}^i$  satisfying (94) and functions  $\tilde{\varphi}_j^i$  and  $\tilde{\psi}_j^i$  satisfying (66) and (93), there exists a unique  $(\tilde{v}_1, \tilde{v}_2)$  ( $:= (\tilde{v}_{1,\varepsilon, \eta_1, \eta_2, \tilde{x}, \tilde{\varphi}_1^i, \tilde{\psi}_1^i}, \tilde{v}_{2,\varepsilon, \eta_2, \eta_3, \tilde{x}, \tilde{\varphi}_2^i, \tilde{\psi}_2^i})$ ) solution of (92) so that for  $v_k$  ( $k = 1, 2$ ) defined by*

$$\begin{aligned} v_1(x) &:= \frac{1 + \eta_1}{\gamma} G(x, \tilde{x}^1) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_1^i, \tilde{\psi}_1^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}\right) + \tilde{v}_1(x), \\ v_2(x) &:= \frac{1 + \eta_3}{\xi} G(x, \tilde{x}^3) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_2^i, \tilde{\psi}_2^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}\right) + \tilde{v}_2(x) \end{aligned}$$

solves (89) in  $\bar{\Omega}_{r_{\varepsilon,\lambda}}(\tilde{x})$ . In addition, we have

$$\|(\tilde{v}_1, \tilde{v}_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

### 3.4 The nonlinear Cauchy-data matching

We will summarize the results of the previous sections. Using the previous notations, assume that  $\tilde{x} := (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \Omega^3$  are given close to  $x := (x^1, x^2, x^3)$ . Assume also that

$$\tau := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3,$$

are given (the values of  $\tau_l^-$  and  $\tau_l^+$ , for  $l = 1, 2, 3$  will be fixed later). First, we consider some set of boundary data  $\varphi^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(S^3))^2$  and  $\psi^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ . Let  $\varepsilon \in (0, \varepsilon_\kappa)$  and according to the result of Proposition 7, 8, and 9, we can find,  $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$  a solution of (39) in  $B_{r_{\varepsilon,\lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$ , which can be decomposed as

$$u_{\text{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \tilde{x}^1) - \frac{1-\gamma}{\gamma} G(x, \tilde{x}^2) - \frac{1-\gamma}{\gamma\xi} G(x, \tilde{x}^3) - \frac{\ln\gamma}{\gamma} + H_{\varphi_1^1, \psi_1^1}^{\text{int}}\left(\frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) \\ \quad + h_1^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) + v_1^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \tilde{x}^2) + H_{\varphi_1^2, \psi_1^2}^{\text{int}}\left(\frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) + h_1^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) + v_1^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^2), \\ \frac{1}{\gamma} G(x, \tilde{x}^1) + G(x, \tilde{x}^2) + H_{\varphi_1^3, \psi_1^3}^{\text{int}}\left(\frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}}\right) + h_1^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) + v_1^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^3) \end{cases}$$

and

$$u_{\text{int},2}(x) := \begin{cases} \frac{1}{\xi} G(x, \tilde{x}^3) + G(x, \tilde{x}^2) + H_{\varphi_2^1, \psi_2^1}^{\text{int}}\left(\frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) + h_2^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) + v_2^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \tilde{x}^2) + H_{\varphi_2^2, \psi_2^2}^{\text{int}}\left(\frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) + h_2^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) + v_2^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^2), \\ \frac{1}{\xi} u_{\varepsilon,\tau_3}(x - \tilde{x}^3) - \frac{1-\xi}{\xi} G(x, \tilde{x}^2) - \frac{1-\xi}{\gamma\xi} G(x, \tilde{x}^1) - \frac{\ln\xi}{\xi} + H_{\varphi_2^3, \psi_2^3}^{\text{int}}\left(\frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}}\right) \\ \quad + h_2^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) + v_2^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^3), \end{cases}$$

where for  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ ,  $R_{\varepsilon,\lambda}^i = \tau_i \frac{r_{\varepsilon,\lambda}}{\varepsilon}$  and the functions  $h_j^i$  satisfy

$$\begin{aligned} \| (h_1^1, h_2^1) \|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} &\leq 2c_\kappa r_{\varepsilon,\lambda}^2, & \| (h_1^2, h_2^2) \|_{(\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))^2} &\leq 2c_\kappa r_{\varepsilon,\lambda}^2 \quad \text{and} \\ \| (h_1^3, h_2^3) \|_{(\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4))} &\leq 2c_\kappa r_{\varepsilon,\lambda}^2. \end{aligned}$$

Similarly, given some boundary data  $\tilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$ ,  $\tilde{\psi}_j^i \in C^{2,\alpha}(S^3)$  satisfying (66),  $(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$  satisfying (94), provided  $\varepsilon \in (0, \varepsilon_\kappa)$ , by Proposition 10, we find a solution  $u_{\text{ext}} := (u_{\text{ext},1}, u_{\text{ext},2})$  of (39) in  $\bar{\Omega} \setminus (B_{r_{\varepsilon,\lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^3))$ , which can be decomposed as

$$\begin{cases} u_{\text{ext},1}(x) := \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}(\tilde{\varphi}_1^i, \tilde{\psi}_1^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}) + \tilde{v}_1(x), \\ u_{\text{ext},2}(x) := \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}(\tilde{\varphi}_2^i, \tilde{\psi}_2^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}) + \tilde{v}_2(x), \end{cases}$$

with  $\tilde{v}_1, \tilde{v}_2 \in C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))$  satisfying

$$\| (\tilde{v}_1, \tilde{v}_2) \|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

It remains to determine the parameters and the boundary data in such a way that the function equal to  $u_{\text{int}}$  in  $B_{r_{\varepsilon,\lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$  and equal to  $u_{\text{ext}}$  in  $\bar{\Omega}_{r_{\varepsilon,\lambda}}(\tilde{x})$  is a smooth

function. This amounts to find the boundary data and the parameters so that, for each  $j = 1, 2$

$$\begin{aligned} u_{\text{int},j} &= u_{\text{ext},j}, & \partial_r u_{\text{int},j} &= \partial_r u_{\text{ext},j}, \\ \Delta u_{\text{int},j} &= \Delta u_{\text{ext},j} & \text{and} & \quad \partial_r \Delta u_{\text{int},j} = \partial_r \Delta u_{\text{ext},j} \end{aligned} \tag{101}$$

on  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^1)$ ,  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^2)$  and  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$ .

Suppose that (101) is verified, this provides that for each  $\varepsilon$  small enough  $u_{\varepsilon,\lambda} \in C^{4,\alpha}$  (which is obtained by patching together the functions  $u_{\text{int}}$  and the function  $u_{\text{ext}}$ ), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof, since  $\varepsilon$  and  $\lambda$  tend to 0, the sequence of solutions we have obtained satisfies the required singular limit behavior.

Before we proceed, the following remarks are due. First it is convenient to observe that the function  $u_{\varepsilon,\tau_i}$  can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -4 \ln \tau_i - 8 \ln |x| + \mathcal{O}\left(\frac{\varepsilon^2 \tau_i^{-2}}{|x|^2}\right) \quad \text{on } \partial B_{r_{\varepsilon,\lambda}}(0). \tag{102}$$

- On  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^1)$ , we have

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(x) &= -\frac{4}{\gamma} \ln \tau_1 + \frac{8\eta_1}{\gamma} \ln |x - \tilde{x}^1| - \frac{1-\gamma}{\gamma \xi} G(x, \tilde{x}^3) - \frac{\ln \gamma}{\gamma} \\ &\quad + h_1^1\left(R_{\varepsilon,\lambda}^1 \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) + H^{\text{int}}\left(\varphi_1^1, \psi_1^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) - H^{\text{ext}}\left(\tilde{\varphi}_1^1, \tilde{\psi}_1^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) \\ &\quad - \frac{1+\eta_1}{\gamma} H(x, \tilde{x}^1) - \left(1 + \eta_2 + \frac{1-\gamma}{\gamma}\right) G(x, \tilde{x}^2) + \mathcal{O}\left(\frac{\varepsilon^2 \tau_1^{-2}}{|x - \tilde{x}^1|^2}\right) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \tag{103}$$

Next, even though all functions are defined on  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^1)$  in (101), it will be more convenient to solve on  $S^3$  the following set of equations

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, & \partial_r(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, \\ \Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0 \quad \text{and} \quad \partial_r \Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) = 0. \end{aligned} \tag{104}$$

Since the boundary data are chosen to satisfy (65) or (66). We decompose

$$\begin{aligned} \varphi_1^1 &= \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_{1,\perp}^1, & \psi_1^1 &= 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_{1,\perp}^1, \\ \tilde{\varphi}_1^1 &= \tilde{\varphi}_{1,0}^1 + \tilde{\varphi}_{1,1}^1 + \tilde{\varphi}_{1,\perp}^1 \quad \text{and} \quad \tilde{\psi}_1^1 = \tilde{\psi}_{1,1}^1 + \tilde{\psi}_{1,\perp}^1, \end{aligned}$$

where  $\varphi_{1,0}^1, \tilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_{1,1}^1, \tilde{\varphi}_{1,1}^1, \tilde{\psi}_{1,1}^1$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi_{1,\perp}^1, \tilde{\varphi}_{1,\perp}^1, \psi_{1,\perp}^1, \tilde{\psi}_{1,\perp}^1$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

Using (103), we have for  $x \in S^3$

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} x) &= -\frac{4}{\gamma} \ln \tau_1 + \frac{8\eta_1}{\gamma} \ln(r_{\varepsilon,\lambda} |x|) - \frac{1}{\gamma} \left( H(\tilde{x}^1, \tilde{x}^1) + G(\tilde{x}^1, \tilde{x}^2) \right) \\ &\quad + h_1^1\left(R_{\varepsilon,\lambda}^1 \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) + H^{\text{int}}\left(\varphi_1^1, \psi_1^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) - H^{\text{ext}}\left(\tilde{\varphi}_1^1, \tilde{\psi}_1^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) \\ &\quad - \frac{1+\eta_1}{\gamma} H(x, \tilde{x}^1) - \left(1 + \eta_2 + \frac{1-\gamma}{\gamma}\right) G(x, \tilde{x}^2) + \mathcal{O}\left(\frac{\varepsilon^2 \tau_1^{-2}}{|x - \tilde{x}^1|^2}\right) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned}$$

$$\begin{aligned}
& + \frac{1-\gamma}{\xi} G(\tilde{x}^1, \tilde{x}^3) \Big) + H^{\text{int}}(\varphi_1^1, \psi_1^1; x) - H^{\text{ext}}(\tilde{\varphi}_1^1, \tilde{\psi}_1^1; x) \\
& - \frac{\ln \gamma}{\gamma} - \frac{\eta_1}{\gamma} H(\tilde{x}^1, \tilde{x}^1) - \eta_2 G(\tilde{x}^1, \tilde{x}^2) + \mathcal{O}(r_{\varepsilon, \lambda}^2).
\end{aligned}$$

Then, the projection of the equations (104) over  $\mathbb{E}_0$  yields

$$\begin{cases}
-4 \ln \tau_1 + 8\eta_1 \ln r_\varepsilon - \ln \gamma + \gamma \varphi_{1,0}^1 - \gamma \tilde{\varphi}_{1,0}^1 - \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
8\eta_1 + 2\gamma \varphi_{1,0}^1 + 2\gamma \tilde{\varphi}_{1,0}^1 + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
16\eta_1 + 8\gamma \varphi_{1,0}^1 + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
-32\eta_1 + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0,
\end{cases} \quad (105)$$

where

$$\mathcal{E}_1(\cdot, \tilde{\mathbf{x}}) := H(\cdot, \tilde{x}^1) + G(\cdot, \tilde{x}^2) + \frac{1-\gamma}{\xi} G(\cdot, \tilde{x}^3).$$

The system (105) can be simply written as

$$\begin{aligned}
\eta_1 &= \mathcal{O}(r_{\varepsilon, \lambda}^2), \quad \varphi_{1,0}^1 = \mathcal{O}(r_{\varepsilon, \lambda}^2), \\
\tilde{\varphi}_{1,0}^1 &= \mathcal{O}(r_{\varepsilon, \lambda}^2) \quad \text{and} \quad \frac{1}{\ln r_{\varepsilon, \lambda}} [4 \ln \tau_1 + \ln \gamma + \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}})] = \mathcal{O}(r_{\varepsilon, \lambda}^2).
\end{aligned}$$

We are now in a position to define  $\tau_1^-$  and  $\tau_1^+$ . In fact, according to the above analysis, as  $\varepsilon$  and  $\lambda$  tend to 0, we expect  $\tilde{x}^i$  to converge to  $x^i$  for  $i \in \{1, 2, 3\}$  and  $\tau_1$  to converge to  $\tau_1^*$ , satisfying

$$4 \ln \tau_1^* = -\ln \gamma - \mathcal{E}_1(x^1, \mathbf{x}).$$

Hence, it is enough to choose  $\tau_1^-$  and  $\tau_1^+$  in such a way that

$$4 \ln(\tau_1^-) < -\ln \gamma - \mathcal{E}_1(x^1, \mathbf{x}) < 4 \ln(\tau_1^+).$$

Consider now the projection of (104) over  $\mathbb{E}_1$ . Given a smooth function  $f$  defined in  $\Omega$ , we identify its gradient  $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$  with the element of  $\mathbb{E}_1$

$$\bar{\nabla} f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{cases}
\varphi_{1,1}^1 - \tilde{\varphi}_{1,1}^1 - \bar{\nabla} \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
3\varphi_{1,1}^1 + 3\tilde{\varphi}_{1,1}^1 + \frac{1}{2}\tilde{\psi}_{1,1}^1 - \bar{\nabla} \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
15\varphi_{1,1}^1 - 3\tilde{\varphi}_{1,1}^1 - \tilde{\psi}_{1,1}^1 + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0, \\
15\varphi_{1,1}^1 + 15\tilde{\varphi}_{1,1}^1 + \frac{18}{4}\tilde{\psi}_{1,1}^1 + \mathcal{O}(r_{\varepsilon, \lambda}^2) = 0,
\end{cases} \quad (106)$$

which can be simplified as follows

$$\begin{aligned}\varphi_{1,1}^1 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \tilde{\varphi}_{1,1}^1 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\psi}_{1,1}^1 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \bar{\nabla}\mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}) &= \mathcal{O}(r_{\varepsilon,\lambda}^2).\end{aligned}\tag{107}$$

Finally, we consider the projection onto  $L^2(S^3)^\perp$ . This yields the system

$$\begin{cases} \varphi_1^{1,\perp} - \tilde{\varphi}_1^{1,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi_1^{1,\perp}, \psi_1^{1,\perp}}^{\text{int}} - H_{\tilde{\varphi}_1^{1,\perp}, \tilde{\psi}_1^{1,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi_1^{1,\perp} - \tilde{\psi}_1^{1,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r\Delta(H_{\varphi_1^{1,\perp}, \psi_1^{1,\perp}}^{\text{int}} - H_{\tilde{\varphi}_1^{1,\perp}, \tilde{\psi}_1^{1,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases}\tag{108}$$

Thanks to the result of Lemma 4, this last system can be rewritten as

$$\begin{aligned}\varphi_1^{1,\perp} &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \tilde{\varphi}_1^{1,\perp} &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \psi_1^{1,\perp} &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \text{and} \quad \tilde{\psi}_1^{1,\perp} &= \mathcal{O}(r_{\varepsilon,\lambda}^2).\end{aligned}$$

If we define the parameter  $t_1 \in \mathbb{R}$  by

$$t_1 = \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_1 + \ln \gamma + \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}})],$$

then the systems found by projecting (104) gather in this equality

$$\begin{aligned}T_{1,\varepsilon}^1 &= (t_1, \eta_1, \varphi_{1,0}^1, \tilde{\varphi}_{1,0}^1, \varphi_{1,1}^1, \tilde{\varphi}_{1,1}^1, \tilde{\psi}_{1,1}^1, \bar{\nabla}\mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}), \varphi_1^{1,\perp}, \tilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \tilde{\psi}_1^{1,\perp}) \\ &= \mathcal{O}(r_{\varepsilon,\lambda}^2),\end{aligned}\tag{109}$$

where, as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

- On  $\partial B_{r_\varepsilon}(\tilde{x}^1)$ , we have

$$\begin{aligned}(u_{\text{int},2} - u_{\text{ext},2})(x) &= -\frac{\eta_3}{\xi} G(x, \tilde{x}^3) + G(x, \tilde{x}^2) + h_2^1 \left( R_{\varepsilon,\lambda}^1 \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) + H^{\text{int}} \left( \varphi_2^1, \psi_2^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) \\ &\quad - (1 + \eta_2) G(x, \tilde{x}^2) - H^{\text{ext}} \left( \tilde{\varphi}_2^1, \tilde{\psi}_2^1; \frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}} \right) + \mathcal{O}(r_{\varepsilon,\lambda}^2).\end{aligned}\tag{110}$$

In the same manner as above, we will solve on  $S^3$  the following system

$$\begin{aligned}(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, & \partial_r(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, \\ \Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0 \quad \text{and} \quad \partial_r\Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0.\end{aligned}\tag{111}$$

We decompose

$$\varphi_2^1 = \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, \quad \psi_2^1 = 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp},$$

$$\tilde{\varphi}_2^1 = \varphi_{2,0}^1 + \tilde{\varphi}_{2,1}^1 + \tilde{\varphi}_2^{1,\perp} \quad \text{and} \quad \tilde{\psi}_2^1 = \tilde{\psi}_{2,1}^1 + \tilde{\psi}_2^{1,\perp},$$

with  $\varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1 \in \mathbb{E}_0$ ,  $\varphi_{2,1}^1, \tilde{\varphi}_{2,1}^1, \tilde{\psi}_{2,1}^1 \in \mathbb{E}_1$  and  $\varphi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}$  belong to  $(L^2(S^3))^\perp$ .

Projecting the set of equations (111) over  $\mathbb{E}_0$ , we get

$$\begin{cases} \varphi_{2,0}^1 - \tilde{\varphi}_{2,0}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 2\varphi_{2,0}^1 + 2\tilde{\varphi}_{2,0}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 8\varphi_{2,0}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (112)$$

From the  $L^2$ -projection of (111) over  $\mathbb{E}_1$ , we obtain the system of equations

$$\begin{cases} \varphi_{2,1}^1 - \tilde{\varphi}_{2,1}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 3\varphi_{2,1}^1 + 3\tilde{\varphi}_{2,1}^1 + \frac{1}{2}\tilde{\psi}_{2,1}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{2,1}^1 - 3\tilde{\varphi}_{2,1}^1 - \tilde{\psi}_{2,1}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{2,1}^1 + 15\tilde{\varphi}_{2,1}^1 + \frac{18}{4}\tilde{\psi}_{2,1}^1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (113)$$

Finally, we consider the  $L^2$ -projection onto  $(L^2(S^3))^\perp$ . This yields the system

$$\begin{cases} \varphi_2^{1,\perp} - \tilde{\varphi}_2^{1,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi_2^{1,\perp}, \psi_2^{1,\perp}}^{\text{int}} - H_{\tilde{\varphi}_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi_2^{1,\perp} - \tilde{\psi}_2^{1,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r \Delta(H_{\varphi_2^{1,\perp}, \psi_2^{1,\perp}}^{\text{int}} - H_{\tilde{\varphi}_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (114)$$

Using again Lemma 4, the above system can be rewritten as

$$\varphi_2^{1,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}_2^{1,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_2^{1,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \tilde{\psi}_2^{1,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

Then the systems found by projecting (111) gather in this equality

$$T_{2,\varepsilon}^1 = (\varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1, \varphi_{2,1}^1, \tilde{\varphi}_{2,1}^1, \tilde{\psi}_{2,1}^1, \varphi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2). \quad (115)$$

- On  $\partial B_{r_\varepsilon}(\tilde{x}^2)$ , we have

$$\begin{aligned} & (1-\xi)(u_{\text{int},1} - u_{\text{ext},1})(x) + (1-\gamma)(u_{\text{int},2} - u_{\text{ext},2})(x) \\ &= -4(2-\gamma-\xi) \ln \tau_2 + 8(2-\gamma-\xi)\eta_2 \ln|x - \tilde{x}^2| \\ &+ (1-\xi)h_1^2\left(R_{\varepsilon,\lambda}^2 \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) + (1-\gamma)h_2^2\left(R_{\varepsilon,\lambda}^2 \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) \\ &+ (1-\xi)H^{\text{int}}\left(\varphi_1^2, \psi_1^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) + (1-\gamma)H^{\text{int}}\left(\varphi_2^2, \psi_2^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) \\ &- (1-\xi)H^{\text{ext}}\left(\tilde{\varphi}_1^2, \tilde{\psi}_1^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) - (1-\gamma)H^{\text{ext}}\left(\tilde{\varphi}_2^2, \tilde{\psi}_2^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) \\ &- \left[ (2-\gamma-\xi)H(x, \tilde{x}^2) + \frac{1-\xi}{\gamma}G(x, \tilde{x}^1) + \frac{1-\gamma}{\xi}G(x, \tilde{x}^3) \right] \end{aligned}$$

$$+ \mathcal{O}\left(\frac{\varepsilon^2 \tau_2^{-2}}{|x - \tilde{x}^2|^2}\right) + \mathcal{O}(r_{\varepsilon, \lambda}^2). \quad (116)$$

We denote by

$$\begin{aligned} \varphi^2 &= (1 - \xi)\varphi_1^2 + (1 - \gamma)\varphi_2^2, & \psi^2 &= (1 - \xi)\psi_1^2 + (1 - \gamma)\psi_2^2, \\ \tilde{\varphi}^2 &= (1 - \xi)\tilde{\varphi}_1^2 + (1 - \gamma)\tilde{\varphi}_2^2, & \tilde{\psi}^2 &= (1 - \xi)\tilde{\psi}_1^2 + (1 - \gamma)\tilde{\psi}_2^2 \\ h^2 &= (1 - \xi)h_1^2 + (1 - \gamma)h_2^2, \end{aligned}$$

Then we have

$$\begin{aligned} &(1 - \xi)(u_{\text{int},1} - u_{\text{ext},1})(x) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2})(x) \\ &= -4(2 - \gamma - \xi)\ln \tau_2 + 8(2 - \gamma - \xi)\eta_2 \ln |x - \tilde{x}^2| + h^2 \left( R_{\varepsilon, \lambda}^2 \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}} \right) \\ &\quad + H^{\text{int}}\left(\varphi^2, \psi^2; \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}}\right) - H^{\text{ext}}\left(\tilde{\varphi}^2, \tilde{\psi}^2; \frac{x - \tilde{x}^2}{r_{\varepsilon, \lambda}}\right) \\ &\quad - \left[ (2 - \gamma - \xi)H(x, \tilde{x}^2) + \frac{1 - \xi}{\gamma}G(x, \tilde{x}^1) + \frac{1 - \gamma}{\xi}G(x, \tilde{x}^3) \right] \\ &\quad + \mathcal{O}\left(\frac{\varepsilon^2 \tau_2^{-2}}{|x - \tilde{x}^2|^2}\right) + \mathcal{O}(r_{\varepsilon, \lambda}^2). \end{aligned} \quad (117)$$

Next, even though all functions are defined on  $\partial B_{r_{\varepsilon, \lambda}}(\tilde{x}^2)$  in (101), it will be more convenient to solve on  $S^3$ , the following set of equations

$$\begin{aligned} &((1 - \xi)(u_{\text{int},1} - u_{\text{ext},1}) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2}))(\tilde{x}^2 + r_{\varepsilon, \lambda} \cdot) = 0, \\ &\partial_r((1 - \xi)(u_{\text{int},1} - u_{\text{ext},1}) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2}))(\tilde{x}^2 + r_{\varepsilon, \lambda} \cdot) = 0, \\ &\Delta((1 - \xi)(u_{\text{int},1} - u_{\text{ext},1}) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2}))(\tilde{x}^2 + r_{\varepsilon, \lambda} \cdot) = 0, \\ &\partial_r \Delta((1 - \xi)(u_{\text{int},1} - u_{\text{ext},1}) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2}))(\tilde{x}^2 + r_{\varepsilon, \lambda} \cdot) = 0. \end{aligned} \quad (118)$$

Since the boundary data are chosen to satisfy (65) or (66). We decompose

$$\begin{aligned} \varphi^2 &= \varphi_0^2 + \varphi_1^2 + \varphi^{2,\perp}, & \psi^2 &= 8\varphi_0^2 + 12\varphi_1^2 + \psi^{2,\perp}, \\ \tilde{\varphi}^2 &= \tilde{\varphi}_0^2 + \tilde{\varphi}_1^2 + \tilde{\varphi}^{2,\perp} \quad \text{and} \quad \tilde{\psi}^2 = \tilde{\psi}_1^2 + \tilde{\psi}^{2,\perp}, \end{aligned}$$

where  $\varphi_0^2, \tilde{\varphi}_0^2 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_1^2, \tilde{\varphi}_1^2, \tilde{\psi}_1^2$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi^{2,\perp}, \tilde{\varphi}^{2,\perp}, \psi^{2,\perp}, \tilde{\psi}^{2,\perp}$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

We insist that, for  $x \in S^3$ , both equations (117) involve the same relation of the parameter  $\tau_2$  and the appropriate energy  $\mathcal{E}_2$ . Then we have

$$\begin{aligned} &(1 - \xi)(u_{\text{int},1} - u_{\text{ext},1})(x) + (1 - \gamma)(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^2 + r_{\varepsilon, \lambda} x) \\ &= -4(2 - \gamma - \xi)\ln \tau_2 + 8(2 - \gamma - \xi)\eta_2 \ln r_{\varepsilon, \lambda} |x| + H^{\text{int}}(\varphi^2, \psi^2; x) - H^{\text{ext}}(\tilde{\varphi}^2, \tilde{\psi}^2; x) \\ &\quad - \left[ (2 - \gamma - \xi)H(\tilde{x}^2, \tilde{x}^2) + \frac{1 - \xi}{\gamma}G(\tilde{x}^2, \tilde{x}^1) + \frac{1 - \gamma}{\xi}G(\tilde{x}^2, \tilde{x}^3) \right] + \mathcal{O}(r_{\varepsilon, \lambda}^2). \end{aligned} \quad (119)$$

Projecting the set of equations (118) over  $\mathbb{E}_0$ , we get

$$\begin{cases} -4(2-\gamma-\xi)\ln\tau_2 + 8(2-\gamma-\xi)\eta_2 \ln r_{\varepsilon,\lambda} + \varphi_0^2 - \tilde{\varphi}_0^2 - \mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) \\ = 0, \\ 8(2-\gamma-\xi)\eta_2 + 2\varphi_0^2 + 2\tilde{\varphi}_0^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 16(2-\gamma-\xi)\eta_2 + 8\varphi_0^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ -32(2-\gamma-\xi)\eta_2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \end{cases} \quad (120)$$

where

$$\mathcal{E}_2(\cdot, \tilde{\mathbf{x}}) := (2-\gamma-\xi)H(\cdot, \tilde{x}^2) + \frac{1-\xi}{\gamma}G(\cdot, \tilde{x}^1) + \frac{1-\gamma}{\xi}G(\cdot, \tilde{x}^3).$$

The system (120) can be simply written as

$$\begin{aligned} \eta_2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \varphi_0^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\varphi}_0^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \frac{1}{\ln r_{\varepsilon,\lambda}} \left[ 4\ln\tau_2 + \frac{\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}})}{2-\gamma-\xi} \right] = \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned}$$

We are now in a position to define  $\tau_2^-$  and  $\tau_2^+$ . In fact, according to the above analysis, as  $\varepsilon$  and  $\lambda$  tend to 0, we expect  $\tilde{x}^i$  to converge to  $x^i$  for  $i \in \{1, 2, 3\}$  and  $\tau_2$  to converge to  $\tau_2^*$ , satisfying

$$4\ln\tau_2^* = -\frac{\mathcal{E}_2(x^2, \mathbf{x})}{2-\gamma-\xi}.$$

Hence it is enough to choose  $\tau_2^-$  and  $\tau_2^+$  in such a way that

$$4\ln(\tau_2^-) < -\frac{\mathcal{E}_2(x^2, \mathbf{x})}{2-\gamma-\xi} < 4\ln(\tau_2^+).$$

Consider now the projection of (118) over  $\mathbb{E}_1$ . Given a smooth function  $f$  defined in  $\Omega$ , we identify its gradient  $\nabla f = (\partial_{x_1}f, \dots, \partial_{x_4}f)$  with the element of  $\mathbb{E}_1$

$$\bar{\nabla}f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{cases} \varphi_1^2 - \tilde{\varphi}_1^2 - \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 3\varphi_1^2 + 3\tilde{\varphi}_1^2 + \frac{1}{2}\tilde{\psi}_1^2 - \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_1^2 - 3\tilde{\varphi}_1^2 - \tilde{\psi}_1^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_1^2 + 15\tilde{\varphi}_1^2 + \frac{18}{4}\tilde{\psi}_1^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (121)$$

Which can be simplified as follows

$$\varphi_1^2 = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}_1^2 = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\psi}_1^2 = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) = \mathcal{O}(r_{\varepsilon,\lambda}^2). \quad (122)$$

Finally, we consider the projection onto  $L^2(S^3)^\perp$ . This yields the system

$$\begin{cases} \varphi^{2,\perp} - \tilde{\varphi}^{2,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi^{2,\perp},\psi^{2,\perp}}^{\text{int}} - H_{\tilde{\varphi}^{2,\perp},\tilde{\psi}^{2,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi^{2,\perp} - \tilde{\psi}^{2,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r\Delta(H_{\varphi^{2,\perp},\psi^{2,\perp}}^{\text{int}} - H_{\tilde{\varphi}^{2,\perp},\tilde{\psi}^{2,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (123)$$

Thanks to the result of Lemma 4, this last system can be rewritten as

$$\varphi^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \tilde{\psi}^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

If we define the parameter  $t_2 \in \mathbb{R}$  by

$$t_2 = \frac{1}{\ln r_{\varepsilon,\lambda}} \left[ 4 \ln \tau_2 + \frac{\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}})}{2 - \gamma - \xi} \right],$$

then the systems found by projecting (118) gather in this equality

$$T_{c,\varepsilon}^2 = (t_2, \eta_2, \varphi_0^2, \tilde{\varphi}_0^2, \varphi_1^2, \tilde{\varphi}_1^2, \tilde{\psi}_1^2, \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}), \varphi^{2,\perp}, \tilde{\varphi}^{2,\perp}, \psi^{2,\perp}, \tilde{\psi}^{2,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad (124)$$

where, as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

- On  $\partial B_{r_\varepsilon}(\tilde{x}^3)$ , we have

$$\begin{aligned} & (u_{\text{int},1} - u_{\text{ext},1})(x) \\ &= -\frac{\eta_1}{\gamma} G(x, \tilde{x}^1) - \eta_2 G(x, \tilde{x}^2) + h_1^3 \left( R_{\varepsilon,\lambda}^3 \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) + H^{\text{int}} \left( \varphi_1^3, \psi_1^3; \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) \\ & \quad - H^{\text{ext}} \left( \tilde{\varphi}_1^3, \tilde{\psi}_1^3; \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (125)$$

In the same manner as above, we will solve on  $S^3$  the following system

$$\begin{aligned} & (u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0, \quad \partial_r(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0, \\ & \Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0 \quad \text{and} \quad \partial_r\Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0. \end{aligned} \quad (126)$$

We decompose

$$\begin{aligned} \varphi_1^3 &= \varphi_{1,0}^3 + \varphi_{1,1}^3 + \varphi_1^{3,\perp}, \quad \psi_1^3 = 8\varphi_{1,0}^3 + 12\varphi_{1,1}^3 + \psi_1^{3,\perp}, \\ \tilde{\varphi}_1^3 &= \tilde{\varphi}_{1,0}^3 + \tilde{\varphi}_{1,1}^3 + \tilde{\varphi}_1^{3,\perp} \quad \text{and} \quad \tilde{\psi}_1^3 = \tilde{\psi}_{1,1}^3 + \tilde{\psi}_1^{3,\perp}, \end{aligned}$$

with  $\varphi_{1,0}^3, \tilde{\varphi}_{1,0}^3 \in \mathbb{E}_0$ ,  $\varphi_{1,1}^3, \tilde{\varphi}_{1,1}^3 \in \mathbb{E}_1$  and  $\varphi_1^{3,\perp}, \tilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}$  belong to  $(L^2(S^3))^\perp$ .

Projecting the set of equations (126) over  $\mathbb{E}_0$ , we get

$$\begin{cases} \varphi_{1,0}^3 - \tilde{\varphi}_{1,0}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 2\varphi_{1,0}^3 + 2\tilde{\varphi}_{1,0}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 8\varphi_{1,0}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (127)$$

From the  $L^2$ -projection of (126) over  $\mathbb{E}_1$ , we obtain the system of equations

$$\begin{cases} \varphi_{1,1}^3 - \tilde{\varphi}_{1,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 3\varphi_{1,1}^3 + 3\tilde{\varphi}_{1,1}^3 + \frac{1}{2}\tilde{\psi}_{1,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{1,1}^3 - 3\tilde{\varphi}_{1,1}^3 - \tilde{\psi}_{1,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{1,1}^3 + 15\tilde{\varphi}_{1,1}^3 + \frac{18}{4}\tilde{\psi}_{1,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (128)$$

Finally, we consider the  $L^2$ -projection onto  $(L^2(S^3))^\perp$ . This yields the system

$$\begin{cases} \varphi_1^{3,\perp} - \tilde{\varphi}_1^{3,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi_1^{3,\perp}, \psi_1^{3,\perp}}^{\text{int}} - H_{\tilde{\varphi}_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi_1^{3,\perp} - \tilde{\psi}_1^{3,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r \Delta(H_{\varphi_1^{3,\perp}, \psi_1^{3,\perp}}^{\text{int}} - H_{\tilde{\varphi}_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (129)$$

Using again Lemma 4, the above system can be rewritten as

$$\varphi_1^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}_1^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_1^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \tilde{\psi}_1^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

Then the systems found by projecting (126) gather in this equality

$$T_{1,\varepsilon}^3 = (\varphi_{1,0}^3, \tilde{\varphi}_{1,0}^3, \varphi_{1,1}^3, \tilde{\varphi}_{1,1}^3, \tilde{\psi}_{1,1}^3, \varphi_1^{3,\perp}, \tilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2). \quad (130)$$

• On  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$ , we have

$$\begin{aligned} & (u_{\text{int},2} - u_{\text{ext},2})(x) \\ &= -\frac{4}{\xi} \ln \tau_3 + \frac{8\eta_3}{\xi} \ln |x - \tilde{x}^3| - \frac{1-\xi}{\gamma\xi} G(x, \tilde{x}^1) - \frac{\ln \xi}{\xi} \\ &+ h_2^3 \left( R_{\varepsilon,\lambda}^3 \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) + H^{\text{int}} \left( \varphi_2^3, \psi_2^3; \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) - H^{\text{ext}} \left( \tilde{\varphi}_2^3, \tilde{\psi}_2^3; \frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}} \right) \\ &- \frac{1+\eta_3}{\xi} H(x, \tilde{x}^3) - \left( 1 + \eta_2 - \frac{1-\xi}{\xi} \right) G(x, \tilde{x}^2) + \mathcal{O} \left( \frac{\varepsilon^2 \tau_3^{-2}}{|x - \tilde{x}^3|^2} \right) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (131)$$

Next, even though all functions are defined on  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$  in (101), it will be more convenient to solve on  $S^3$  the following set of equations

$$\begin{aligned} & (u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0, \quad \partial_r(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0, \\ & \Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0 \quad \text{and} \quad \partial_r \Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^3 + r_{\varepsilon,\lambda} \cdot) = 0. \end{aligned} \quad (132)$$

Since the boundary data are chosen to satisfy (65) or (66). We decompose

$$\begin{aligned}\varphi_2^3 &= \varphi_{2,0}^3 + \varphi_{2,1}^3 + \varphi_2^{3,\perp}, & \psi_2^3 &= 8\varphi_{2,0}^3 + 12\varphi_{2,1}^3 + \psi_2^{3,\perp}, \\ \tilde{\varphi}_2^3 &= \tilde{\varphi}_{2,0}^3 + \tilde{\varphi}_{2,1}^3 + \tilde{\varphi}_2^{3,\perp} & \text{and} & \tilde{\psi}_2^3 = \tilde{\psi}_{2,1}^3 + \tilde{\psi}_2^{3,\perp},\end{aligned}$$

where  $\varphi_{2,0}^3, \tilde{\varphi}_{2,0}^3 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_{2,1}^3, \tilde{\varphi}_{2,1}^3, \tilde{\psi}_{2,1}^3$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi_2^{3,\perp}, \tilde{\varphi}_2^{3,\perp}, \psi_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

Using (131), we have for  $x \in S^3$

$$\begin{aligned}(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^3 + r_{\varepsilon,\lambda}x) &= -\frac{4}{\xi} \ln \tau_3 + \frac{8\eta_3}{\xi} \ln(r_{\varepsilon,\lambda}|x|) - \frac{1}{\xi} \left( H(\tilde{x}^3, \tilde{x}^3) + G(\tilde{x}^3, \tilde{x}^2) \right. \\ &\quad \left. + \frac{1-\xi}{\gamma} G(\tilde{x}^3, \tilde{x}^1) \right) + H^{\text{int}}(\varphi_2^3, \psi_2^3; x) - H^{\text{ext}}(\tilde{\varphi}_2^3, \tilde{\psi}_2^3; x) \\ &\quad - \frac{\ln \xi}{\xi} - \frac{\eta_3}{\xi} H(\tilde{x}^3, \tilde{x}^3) - \eta_2 G(\tilde{x}^3, \tilde{x}^2) + \mathcal{O}(r_{\varepsilon,\lambda}^2).\end{aligned}$$

Then, the projection of the set equations (132) over  $\mathbb{E}_0$  yields

$$\begin{cases} -4 \ln \tau_3 + 8\eta_3 \ln r_{\varepsilon} - \ln \xi + \xi \varphi_{2,0}^3 - \xi \tilde{\varphi}_{2,0}^3 - \mathcal{E}_3(\tilde{x}^3, \tilde{x}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 8\eta_3 + 2\xi \varphi_{2,0}^3 + 2\xi \tilde{\varphi}_{2,0}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 16\eta_3 + 8\xi \varphi_{2,0}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ -32\eta_3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \end{cases} \quad (133)$$

where

$$\mathcal{E}_3(\cdot, \tilde{x}) := H(\cdot, \tilde{x}^3) + G(\cdot, \tilde{x}^2) + \frac{1-\xi}{\gamma} G(\cdot, \tilde{x}^1).$$

The system (133) can be simply written as

$$\begin{aligned}\eta_3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \varphi_{2,0}^3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\varphi}_{2,0}^3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_3 + \ln \xi + \mathcal{E}_3(\tilde{x}^3, \tilde{x})] = \mathcal{O}(r_{\varepsilon,\lambda}^2).\end{aligned}$$

We are now in a position to define  $\tau_3^-$  and  $\tau_3^+$ . In fact, according to the above analysis, as  $\varepsilon$  and  $\lambda$  tend to 0, we expect  $\tilde{x}^i$  to converge to  $x^i$  for  $i \in \{1, 2, 3\}$  and  $\tau_3$  to converge to  $\tau_3^*$ , satisfying

$$4 \ln \tau_3^* = -\ln \xi - \mathcal{E}_3(x^3, \mathbf{x}).$$

Hence it is enough to choose  $\tau_3^-$  and  $\tau_3^+$  in such a way that

$$4 \ln(\tau_3^-) < -\ln \xi - \mathcal{E}_3(x^3, \mathbf{x}) < 4 \ln(\tau_3^+).$$

Consider now the projection of (132) over  $\mathbb{E}_1$ . Given a smooth function  $f$  defined in  $\Omega$ , we identify its gradient  $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$  with the element of  $\mathbb{E}_1$

$$\bar{\nabla}f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{cases} \varphi_{2,1}^3 - \tilde{\varphi}_{2,1}^3 - \bar{\nabla}\mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 3\varphi_{2,1}^3 + 3\tilde{\varphi}_{2,1}^3 + \frac{1}{2}\tilde{\psi}_{2,1}^3 - \bar{\nabla}\mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{2,1}^3 - 3\tilde{\varphi}_{2,1}^3 - \tilde{\psi}_{2,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{2,1}^3 + 15\tilde{\varphi}_{2,1}^3 + \frac{18}{4}\tilde{\psi}_{2,1}^3 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \end{cases} \quad (134)$$

which can be simplified as follows

$$\begin{aligned} \varphi_{2,1}^3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \tilde{\varphi}_{2,1}^3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\psi}_{2,1}^3 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \bar{\nabla}\mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}}) &= \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (135)$$

Finally, we consider the projection onto  $L^2(S^3)^\perp$ . This yields the system

$$\begin{cases} \varphi_2^{3,\perp} - \tilde{\varphi}_2^{3,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi_2^{3,\perp}, \psi_2^{3,\perp}}^{\text{int}} - H_{\tilde{\varphi}_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi_2^{3,\perp} - \tilde{\psi}_2^{3,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r\Delta(H_{\varphi_2^{3,\perp}, \psi_2^{3,\perp}}^{\text{int}} - H_{\tilde{\varphi}_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (136)$$

Thanks to the result of Lemma 4, this last system can be rewritten as

$$\varphi_2^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}_2^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_2^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \tilde{\psi}_2^{3,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

If we define the parameter  $t_3 \in \mathbb{R}$  by

$$t_3 = \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_3 + \ln \xi + \mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}})],$$

then the systems found by projecting (132) gather in this equality

$$T_{2,\varepsilon}^3 = (t_3, \eta_3, \varphi_{2,0}^3, \tilde{\varphi}_{2,0}^3, \varphi_{2,1}^3, \tilde{\varphi}_{2,1}^3, \tilde{\psi}_{2,1}^3, \bar{\nabla}\mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}}), \varphi_2^{3,\perp}, \tilde{\varphi}_2^{3,\perp}, \psi_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad (137)$$

where, as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but are bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

We recall that  $\mathbf{d} = r_{\varepsilon,\lambda}(\tilde{\mathbf{x}} - \mathbf{x})$ , in addition the previous systems can be written as for  $i = 1, 2, 3$ :

$$(\mathbf{d}, t_i, \eta_i, \varphi^i, \tilde{\varphi}^i, \psi^i, \tilde{\psi}^i, \bar{\nabla}\mathcal{E}_i) = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

Combining (109), (115), (124), (130), and (137), we have

$$T_{i,\varepsilon} = (T_{i,\varepsilon}^1, T_{i,\varepsilon}^2, T_{i,\varepsilon}^3) = (\mathcal{O}(r_{\varepsilon,\lambda}^2), \mathcal{O}(r_{\varepsilon,\lambda}^2), \mathcal{O}(r_{\varepsilon,\lambda}^2)), \quad \text{for } i = 1, 2. \quad (138)$$

Then the nonlinear mapping, which appears on the right-hand side of (138), is continuous and compact. In addition, reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, this nonlinear mapping sends the ball of radius  $\kappa r_{\varepsilon,\lambda}^2$  (for the natural product norm) into itself, provided  $\kappa$  is fixed large enough. Applying Schauder's fixed point theorem in the ball of radius  $\kappa r_{\varepsilon,\lambda}^2$  in the product space where the entries live, we obtain the existence of a solution of equation (138).

This completes the proof of Theorem 5.  $\square$

## 4 Proof of Theorem 6

### 4.1 Construction of the approximate solution

#### 4.1.1 Ansatz and first estimates

We define another ansatz for solving the following system.

$$\begin{cases} \Delta^2 u_1 + \mathcal{L}_\lambda(u_1) = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2}, \\ \Delta^2 u_2 + \mathcal{L}_\lambda(u_2) = \rho^4 e^{\xi u_2 + (1-\xi)u_1}, \end{cases} \quad (139)$$

where

$$\mathcal{L}_\lambda(u_i) = \lambda(\Delta u_i)^2 + 2\lambda \nabla u_i \cdot \nabla(\Delta u_i) + \lambda^2 |\nabla u_i|^2 \Delta u_i, \quad \text{for } i = 1, 2.$$

Using the following transformations

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_1}x\right) + \frac{8}{\gamma} \ln \varepsilon - \frac{4}{\gamma} \ln\left(\frac{\tau_1(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^1), \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_1}x\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^1), \end{cases} \quad (140)$$

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_2}x\right) + 8 \ln \varepsilon - 4 \ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^2), \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_2}x\right) + 8 \ln \varepsilon - 4 \ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^2) \end{cases} \quad (141)$$

and

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_3}x\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^3) \\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_3}x\right) + \frac{8}{\xi} \ln \varepsilon - \frac{4}{\xi} \ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_{\varepsilon,\lambda}}(x^3). \end{cases} \quad (142)$$

So the previous systems can be written as

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \end{cases} \quad (143)$$

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2) \end{cases} \quad (144)$$

and

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24C_{3,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}} e^{\nu v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \end{cases} \quad (145)$$

where  $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$  for  $i = 1, 3$ . Here  $\tau_i > 0$  is a constant, which will be fixed later.

In  $B_{R_{\varepsilon,\lambda}^1}(x^1)$  and  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ , we reproduce exactly the same as in the proof of Theorem 4, so we have the following propositions.

**Proposition 11** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$ , and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , and for all  $\tau_1$  in some fixed compact subset of  $[\tau_1^-, \tau_1^+] \subset (0, \infty)$  there exists a unique  $(h_1^1, h_2^1) := (h_{1,\varepsilon,\tau_1}, h_{2,\varepsilon,\tau_1})$  solution of (49) such that

$$\|(h_1^1, h_2^1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln \gamma}{\gamma} + H^{\text{int}}(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_{\varepsilon,\lambda}^1}) + h_1^1(x), \\ v_2(x) := \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + H^{\text{int}}(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_{\varepsilon,\lambda}^1}) + h_2^1(x) \end{cases}$$

solves (143) in  $B_{R_{\varepsilon,\lambda}^1}(x^1)$ .

**Proposition 12** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\xi_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\xi \in (\xi_0, 1)$  and for all  $\tau_3$  in some fixed compact subset of  $[\tau_3^-, \tau_3^+] \subset (0, \infty)$ , there exists a unique  $(h_1^3, h_2^3) := (h_{1,\varepsilon,\tau_3}, h_{2,\varepsilon,\tau_3})$  solution of (49) such that

$$\|(h_1^3, h_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x), \\ v_2(x) := \frac{1}{\xi} \bar{u}(x - x^3) - \frac{1-\xi}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{\ln \xi}{\xi} + h_2^3(x) \end{cases}$$

solves (145) in  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ .

In  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ , we look for a solution of (144) of the form

$$\begin{cases} v_1(x) = \bar{u}(x - x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x), \\ v_2(x) = \bar{u}(x - x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_2^2(x). \end{cases}$$

This amounts to solve the equations

$$\begin{cases} \mathbb{L}h_1^2 = \frac{384}{(1+r^2)^4} [e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}(\frac{1}{\gamma}G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi}G(\frac{\varepsilon x}{\tau_2}, x^3)) + \gamma h_1^2 + (1-\gamma)h_2^2} - h_1^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)}G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1-\gamma}{\xi(2-\gamma-\xi)}G(\frac{\varepsilon x}{\tau_2}, x^3) + h_1^2(x)), \\ \mathbb{L}h_2^2 = \frac{384}{(1+r^2)^4} [e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}(-\frac{1}{\gamma}G(\frac{\varepsilon x}{\tau_2}, x^1) + \frac{1}{\xi}G(\frac{\varepsilon x}{\tau_2}, x^3)) + \xi h_2^2 + (1-\xi)h_1^2} - h_2^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x-x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)}G(\frac{\varepsilon x}{\tau_2}, x^1) + \frac{1-\xi}{\xi(2-\gamma-\xi)}G(\frac{\varepsilon x}{\tau_2}, x^3) + h_2^2(x)). \end{cases} \quad (146)$$

We denote by

$$\mathbb{L}h_1^2 = \mathcal{R}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}h_2^2 = \mathcal{R}_4(h_1^2, h_2^2).$$

To find a solution of (146), it is enough to find a fixed point  $(h_1^2, h_2^2)$  in a small ball of  $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ , solutions of

$$\begin{cases} h_1^2 = \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{R}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2), \\ h_2^2 = \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{R}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{cases} \quad (147)$$

Then, we have the following result.

**Lemma 8** Let  $\mu \in (1, 2)$ ,  $\gamma_0$  and  $\xi_0 \in (0, 1)$ . Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\bar{c}_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . We have

$$\begin{aligned} \|\mathcal{N}_2(0, 0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon, \lambda}^2, \quad \|\mathcal{M}_2(0, 0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon, \lambda}^2, \\ \|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \|h_1^2 - k_1^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \gamma) \|h_2^2 - k_2^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \\ \|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq \bar{c}_\kappa (1 - \xi) \|h_1^2 - k_1^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \|h_2^2 - k_2^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}, \end{aligned}$$

provided  $(h_1^2, h_2^2)$ ,  $(k_1^2, k_2^2)$  in  $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2 \quad \text{and} \quad \|(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2. \quad (148)$$

*Proof* Using the assumption (16), we get

$$\begin{aligned} &\sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} |\mathcal{R}_3(0, 0)| \\ &\leq \sup_{r \leq R_{\varepsilon, \lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}(\frac{1}{\gamma}G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi}G(\frac{\varepsilon x}{\tau_2}, x^3))} - 1 \right| \\ &\quad + \sup_{r \leq R_{\varepsilon, \lambda}^2} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\ &\quad \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq c \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}r^2\varepsilon^2}{(1+r^2)^4} + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \left( \Delta \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\
&\quad + \frac{1-\gamma}{\xi(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \Big|^2 \\
&\quad + \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right) \right. \\
&\quad \times \nabla \left( \Delta \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right) \right) \Big| \\
&\quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\
&\quad - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \Big|^2 \\
&\quad \times \left| \Delta \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right) \right| \\
&\leq c_\kappa r_{\varepsilon,\lambda}^2 + c_\kappa \lambda (1 + \varepsilon^\mu r_{\varepsilon,\lambda}^{2-\mu}) + c_\kappa \lambda^2 (1 + \varepsilon^{\mu-1} r_{\varepsilon,\lambda}^{2-\mu}).
\end{aligned}$$

Making use of Proposition 1 together with (38), for  $\mu \in (1, 2)$ , we get that there exists  $c_\kappa$  such that

$$\|\mathcal{N}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (149)$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0, 0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (150)$$

To derive the third estimate, for  $(h_1^2, h_2^2), (k_1^2, k_2^2)$  verifying (148), we have

$$\begin{aligned}
&\sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\mathcal{R}_3(h_1^2, h_2^2) - \mathcal{R}_3(k_1^2, k_2^2)| \\
&\leq \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} (\frac{1}{\gamma} G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi} G(\frac{\varepsilon x}{\tau_2}, x^3))} e^{\gamma h_1^2 + (1-\gamma)h_2^2} - h_1^2 \right. \\
&\quad - \left. e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} (\frac{1}{\gamma} G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi} G(\frac{\varepsilon x}{\tau_2}, x^3))} e^{\gamma k_1^2 + (1-\gamma)k_2^2} + k_1^2 \right| \\
&\quad + \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\
&\quad - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x) \Big| \\
&\quad - \mathcal{L}_\lambda \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + k_1^2(x) \right) \Big| \\
&\leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} |(\gamma-1)(h_1^2 - k_1^2) + (1-\gamma)(h_2^2 - k_2^2)| + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \nabla(h_1^2 - k_1^2) \right. \\
&\quad \times \nabla \left( 2\Delta \bar{u} + 2\frac{1-\gamma}{\gamma(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right)
\end{aligned}$$

$$\begin{aligned}
& -2 \frac{1-\gamma}{\xi(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + \Delta h_1^2 + \Delta k_1^2 \\
& + \nabla(\Delta(h_1^2 - k_1^2)) \cdot \nabla \left( 2\bar{u}(x-x^2) + 2 \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \\
& - 2 \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \\
& \left. + h_1^2(x) + k_1^2(x) \right) \\
& + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(h_1^2 - k_1^2)| \left[ \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \right. \\
& - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x) \left. \right|^2 + \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\
& - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + k_1^2(x) \left. \right|^2 \left. \right] + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta \bar{u}| \right. \\
& \left. + 2 \frac{1-\gamma}{\gamma(2-\gamma-\xi)} \left| \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right| + 2 \frac{1-\gamma}{\xi(2-\gamma-\xi)} \left| \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right| + |\Delta h_1^2| + |\Delta k_1^2| \right) \\
& \times \left[ \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \right. \\
& - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x) \left. \right|^2 \\
& - \left| \nabla \left( \bar{u}(x-x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\
& - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + k_1^2(x) \left. \right|^2 \right]
\end{aligned}$$

Proceeding as in the proof of the Theorem 5, we obtain

$$\begin{aligned}
& \|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
& \leq \bar{c}_\kappa (1-\gamma + r_{\varepsilon,\lambda}^2) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\gamma) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{151}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\
& \leq \bar{c}_\kappa (1-\xi) \|h_1^2 - k_1^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1-\xi + r_{\varepsilon,\lambda}^2) \|h_2^2 - k_2^2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{152}
\end{aligned}$$

□

Then there exist  $\gamma_0$  and  $\xi_0 \in (0, 1)$ , reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa (1-\gamma + r_{\varepsilon,\lambda}^2) \leq 1/2$  and  $\bar{c}_\kappa (1-\xi + r_{\varepsilon,\lambda}^2) \leq 1/2$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . Therefore, (151) and (152) are enough to show that

$$(h_1^2, h_2^2) \mapsto (\mathcal{N}_2(h_1^2, h_2^2), \mathcal{M}_2(h_1^2, h_2^2))$$

is a contraction from the ball

$$\{(h_1^2, h_2^2) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4) : \| (h_1^2, h_2^2) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

**Proposition 13** *Given  $\kappa > 0$ ,  $\mu \in (1, 2)$ ,  $\gamma_0 \in (0, 1)$ , and  $\xi_0 \in (0, 1)$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$  and  $c_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , and  $\xi \in (\xi_0, 1)$  and for all  $\tau_2$  in some fixed compact subset of  $[\tau_2^-, \tau_2^+] \subset (0, \infty)$ , there exists a unique  $(h_1^2, h_2^2) := (h_{1,\varepsilon,\tau_2}, h_{2,\varepsilon,\tau_2})$  solution of (157) such that*

$$\| (h_1^2, h_2^2) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \bar{u}(x - x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_1^2(x), \\ v_2(x) := \bar{u}(x - x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + h_2^2(x) \end{cases}$$

solves (144) in  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ .

#### 4.2 The nonlinear interior problem

Here, we look for a solution of the following systems as in the above subsection, we only add the interior harmonic extension and the perturbation term  $v_i^j$  for  $i, j = 1, 2$ .

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24C_{1,\varepsilon}^{\frac{4\gamma+\xi-1}{\gamma}} \varepsilon^{\frac{8\gamma+\xi-1}{\gamma}} e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^1}(x^1), \end{cases} \quad (153)$$

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^2}(x^2), \end{cases} \quad (154)$$

and

$$\begin{cases} \Delta^2 v_1 + \mathcal{L}_\lambda(v_1) = 24C_{3,\varepsilon}^{\frac{4\gamma+\xi-1}{\xi}} \varepsilon^{\frac{8\gamma+\xi-1}{\xi}} e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \\ \Delta^2 v_2 + \mathcal{L}_\lambda(v_2) = 24e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R_{\varepsilon,\lambda}^3}(x^3), \end{cases} \quad (155)$$

where  $\mathcal{L}_\lambda(u_i) = \lambda(\Delta u_i)^2 + 2\lambda \nabla u_i \cdot \nabla(\Delta u_i) + \lambda^2 |\nabla u_i|^2 \Delta u_i$ , for  $i = 1, 2$  and  $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$  for  $i = 1, 3$ . Here  $\tau_i > 0$  is a constant, which will be fixed later.

Given  $\varphi^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(\mathbb{S}^3))^2$  and  $\psi^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^3))^2$  such that  $(\varphi_1^i, \psi_1^i)$  and  $(\varphi_2^i, \psi_2^i)$  are satisfying (65). We denote by  $\bar{u} = u_{\varepsilon=1, \tau_i=1}$ . In  $B_{R_{\varepsilon,\lambda}^1}(x^1)$  and  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ , we reproduce exactly the same as in the proof of Theorem 5, so we have the following propositions.

**Proposition 14** *Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\gamma_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , for all  $\tau_1$  in*

some fixed compact subset of  $[\tau_1^-, \tau_1^+] \subset (0, \infty)$  and for  $\varphi_j^1$  and  $\psi_j^1$  satisfying (65) and (72), there exists a unique  $(v_1^1, v_2^1) := (\nu_{1,\varepsilon,\tau_1,\varphi_1^1,\psi_1^1}, \nu_{2,\varepsilon,\tau_1,\varphi_2^1,\psi_2^1})$  solution of (49) such that

$$\| (v_1^1, v_2^1) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} \bar{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) - \frac{\ln \gamma}{\gamma} + h_1^1(x) \\ \quad + H^{\text{int}}(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_{\varepsilon,\lambda}^1}) + v_1^1(x), \\ v_2(x) := \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) + H^{\text{int}}(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_{\varepsilon,\lambda}^1}) + v_2^1(x) \end{cases}$$

solves (153) in  $B_{R_{\varepsilon,\lambda}^1}(x^1)$ .

**Proposition 15** Given  $\kappa > 0$ ,  $\mu \in (1, 2)$  and  $\delta \in (0, \min\{\frac{\gamma+\xi-1}{\gamma}, \frac{\gamma+\xi-1}{\xi}\})$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\xi_0 \in (0, 1)$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\xi \in (\xi_0, 1)$ , for all  $\tau_3$  in some fixed compact subset of  $[\tau_3^-, \tau_3^+] \subset (0, \infty)$  and for  $\varphi_j^3$  and  $\psi_j^3$  satisfying (65) and (79), there exists a unique  $(v_1^3, v_2^3) := (\nu_{1,\varepsilon,\tau_3,\varphi_1^3,\psi_1^3}, \nu_{2,\varepsilon,\tau_3,\varphi_2^3,\psi_2^3})$  solution of (49) such that

$$\| (v_1^3, v_2^3) \|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \frac{1}{\gamma} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) + H^{\text{int}}(\varphi_1^3, \psi_1^3; \frac{x-x^3}{R_{\varepsilon,\lambda}^3}) + v_1^3(x), \\ v_2(x) := \frac{1}{\xi} \bar{u}(x - x^3) - \frac{1-\xi}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) - \frac{\ln \xi}{\xi} + h_2^3(x) \\ \quad + H^{\text{int}}(\varphi_2^3, \psi_2^3; \frac{x-x^3}{R_{\varepsilon,\lambda}^3}) + v_2^3(x) \end{cases}$$

solves (155) in  $B_{R_{\varepsilon,\lambda}^3}(x^3)$ .

In  $B_{R_{\varepsilon,\lambda}^2}(x^2)$ , we look for a solution of (144) of the form

$$\begin{cases} v_1(x) = \bar{u}(x - x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H^{\text{int}}_{\varphi_1^2, \psi_1^2}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) \\ \quad + h_1^2(x) + v_1^2(x), \\ v_2(x) = \bar{u}(x - x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H^{\text{int}}_{\varphi_2^2, \psi_2^2}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) \\ \quad + h_2^2(x) + v_2^2(x). \end{cases}$$

This amounts to solve the equations

$$\left\{ \begin{array}{l} \mathbb{L}v_1^2 = \frac{384}{(1+r^2)^4} [e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}(\frac{1}{\gamma}G(\frac{\varepsilon x}{r_2},x^1)-\frac{1}{\xi}G(\frac{\varepsilon x}{r_2},x^3))+\gamma(h_1^2+H_{\varphi_1^2,\psi_1^2}^{\text{int}}+v_1^2)+(1-\gamma)(h_2^2+H_{\varphi_2^2,\psi_2^2}^{\text{int}}+v_2^2)} \\ \quad - v_1^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x-x^2)) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)}G(\frac{\varepsilon x}{r_2},x^1) - \frac{1-\gamma}{\xi(2-\gamma-\xi)}G(\frac{\varepsilon x}{r_2},x^3) \\ \quad + H_{\varphi_1^2,\psi_1^2}^{\text{int}}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_1^2(x) + v_1^2(x)) - \Delta^2 h_1^2, \\ \mathbb{L}v_2^2 = \frac{384}{(1+r^2)^4} [e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}(-\frac{1}{\gamma}G(\frac{\varepsilon x}{r_2},x^1)+\frac{1}{\xi}G(\frac{\varepsilon x}{r_2},x^3))+\xi(h_2^2+H_{\varphi_2^2,\psi_2^2}^{\text{int}}+v_2^2)+(1-\xi)(h_1^2+H_{\varphi_1^2,\psi_1^2}^{\text{int}}+v_1^2)} \\ \quad - v_2^2 - 1] \\ \quad - \mathcal{L}_\lambda(\bar{u}(x-x^2)) - \frac{1-\xi}{\gamma(2-\gamma-\xi)}G(\frac{\varepsilon x}{r_2},x^1) + \frac{1-\xi}{\xi(2-\gamma-\xi)}G(\frac{\varepsilon x}{r_2},x^3) \\ \quad + H_{\varphi_2^2,\psi_2^2}^{\text{int}}(\frac{x-x^2}{R_{\varepsilon,\lambda}^2}) + h_2^2(x) + v_2^2(x)) - \Delta^2 h_2^2. \end{array} \right. \quad (156)$$

We denote by

$$\mathbb{L}v_1^2 = \mathcal{R}_3(v_1^2, v_2^2) \quad \text{and} \quad \mathbb{L}v_2^2 = \mathcal{R}_4(v_1^2, v_2^2).$$

To find a solution of (156), it is enough to find a fixed point  $(v_1^2, v_2^2)$  in a small ball of  $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ , solutions of

$$\left\{ \begin{array}{l} v_1^2 = \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{R}_3(v_1^2, v_2^2) = \mathcal{N}_2(v_1^2, v_2^2), \\ v_2^2 = \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{R}_4(v_1^2, v_2^2) = \mathcal{M}_2(v_1^2, v_2^2). \end{array} \right. \quad (157)$$

Then, we have the following result.

**Lemma 9** Let  $\mu \in (1, 2)$ ,  $\gamma_0$  and  $\xi_0 \in (0, 1)$ . Given  $\kappa > 0$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ ,  $c_\kappa > 0$  and  $\bar{c}_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , and  $\xi \in (\xi_0, 1)$ . We have

$$\begin{aligned} \|\mathcal{N}_2(0, 0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq c_\kappa r_{\varepsilon,\lambda}^2, \quad \|\mathcal{M}_2(0, 0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2, \\ \|\mathcal{N}_2(v_1^2, v_2^2) - \mathcal{N}_2(t_1^2, t_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon,\lambda}^2) \|v_1^2 - t_1^2\|_{\mathcal{C}_\mu^{2,\alpha}(\mathbb{R}^2)} + \bar{c}_\kappa (1 - \gamma) \|v_2^2 - t_2^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \\ \|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} &\leq \bar{c}_\kappa (1 - \xi) \|v_1^2 - t_1^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon,\lambda}^2) \|v_2^2 - t_2^2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}, \end{aligned}$$

provided  $(v_1^2, v_2^2)$ ,  $(t_1^2, t_2^2)$  in  $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$  satisfying

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2 \quad \text{and} \quad \|(t_1^2, t_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon,\lambda}^2. \quad (158)$$

*Proof* The proof of the first and the second estimates follows from the asymptotic behavior of  $H^{\text{int}}$  together with the assumption on the norms of  $\varphi_j^2$  and  $\psi_j^2$  given by (82), and it follows from the estimate of  $H^{\text{int}}$ , given by Lemma 3, that

$$\left\| H_{\varphi_j^2, \psi_j^2}^{\text{int}} \left( \frac{r}{R_{\varepsilon,\lambda}^2} \cdot \right) \right\|_{\mathcal{C}^{4,\alpha}(\bar{B}_2(0) - B_1(0))} \leq C r^2 (R_{\varepsilon,\lambda}^2)^{-2} (\|\varphi_j^2\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^2\|_{\mathcal{C}^{2,\alpha}(S^3)}),$$

for all  $r \leq \frac{R_{\varepsilon,\lambda}^2}{2}$ . Then by (82), we get

$$\left\| H_{\varphi_j^2, \psi_j^2}^{\text{int}} \left( \frac{r}{R_{\varepsilon,\lambda}^2} \cdot \right) \right\|_{C^{4,\alpha}(\bar{B}_2(0)-B_1(0))} \leq c_\kappa \varepsilon^2 r^2.$$

On the other hand, using (16), we obtain

$$\begin{aligned} & \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\mathcal{R}_3(0,0)| \\ & \leq \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} (\frac{1}{\gamma} G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi} G(\frac{\varepsilon x}{\tau_2}, x^3))} e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}}) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}})} - 1 \right| \\ & \quad + \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left( \left| \mathcal{L}_\lambda \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) \right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 \right| + |\Delta^2 h_1^2| \right) \\ & \leq \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} (\gamma r^2 \|H_{\varphi_1^2, \psi_1^2}^{\text{int}}\|_{C_2^{4,\alpha}} + \gamma r^\mu \|h_1^2\|_{C_\mu^{4,\alpha}} \\ & \quad + (1-\gamma)r^2 \|H_{\varphi_2^2, \psi_2^2}^{\text{int}}\|_{C_2^{4,\alpha}} + (1-\gamma)r^\mu \|h_2^2\|_{C_\mu^{4,\alpha}}) \\ & \quad + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left[ \left| \left( \Delta \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} \Delta G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} \Delta G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + \Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}} + \Delta h_1^2 \right|^2 \right| \\ & \quad + 2 \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 \right) \right. \\ & \quad \times \nabla \left( \Delta \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 \right) \right) \left| \right] \\ & \quad + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 \right) \right|^2 \\ & \quad \times \left| \Delta \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^1 \right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G \left( \frac{\varepsilon x}{\tau_2}, x^3 \right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 \right) \right| \\ & \quad + \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta^2 h_1^2|. \end{aligned}$$

Making use of Proposition 1 together with (38) and using the condition ( $A_1$ ) for  $\mu \in (1, 2)$ , we get that there exists  $c_\kappa$  such that

$$\|\mathcal{N}_2(0,0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (159)$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0,0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_{\varepsilon,\lambda}^2. \quad (160)$$

To derive the third estimate, for  $(v_1^2, v_2^2), (t_1^2, t_2^2)$  verifying (158), we have

$$\begin{aligned} & \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\mathcal{R}_3(h_1^2, h_2^2) - \mathcal{R}_3(k_1^2, k_2^2)| \\ & \leq \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} \left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} (\frac{1}{\gamma}G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi}G(\frac{\varepsilon x}{\tau_2}, x^3))} e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + v_1^2) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}} + v_2^2)} - v_1^2 \right. \\ & \quad \left. - e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)} (\frac{1}{\gamma}G(\frac{\varepsilon x}{\tau_2}, x^1) - \frac{1}{\xi}G(\frac{\varepsilon x}{\tau_2}, x^3))} e^{\gamma(h_1^2 + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + t_1^2) + (1-\gamma)(h_2^2 + H_{\varphi_2^2, \psi_2^2}^{\text{int}} + t_2^2)} + t_1^2 \right| \\ & \quad + \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \mathcal{L}_\lambda \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \\ & \quad \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2 \right) \right. \\ & \quad \left. - \mathcal{L}_\lambda \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2 \right) \right| \\ & \leq c_\kappa \sup_{r \leq R_{\varepsilon,\lambda}^2} \frac{384r^{4-\mu}}{(1+r^2)^4} |(\gamma-1)(v_1^2 - t_1^2) + (1-\gamma)(v_2^2 - t_2^2)| + \lambda \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left| \nabla(v_1^2 - t_1^2) \right. \\ & \quad \times \nabla \left( 2\Delta\bar{u} + 2\frac{1-\gamma}{\gamma(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - 2\frac{1-\gamma}{\xi(2-\gamma-\xi)} \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right. \\ & \quad \left. + 2\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}} + 2\Delta h_1^2 + \Delta v_1^2 + \Delta t_1^2 \right) \\ & \quad + \nabla(\Delta(v_1^2 - t_1^2)) \cdot \nabla \left( 2\bar{u} + 2\frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \\ & \quad \left. - 2\frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + 2H_{\varphi_1^2, \psi_1^2}^{\text{int}} + 2h_1^2 \right. \\ & \quad \left. + v_1^2 + t_1^2 \right) \left| + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} |\Delta(v_1^2 - t_1^2)| \right| \left[ \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2 \right) \right|^2 + \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2 \right) \right|^2 \right] + \lambda^2 \sup_{r \leq R_{\varepsilon,\lambda}^2} r^{4-\mu} \left( \frac{2}{\gamma} |\Delta\bar{u}| + 2\frac{1-\gamma}{\gamma(2-\gamma-\xi)} |\Delta G\left(\frac{\varepsilon x}{\tau_2}, x^1\right)| \right) \\ & \quad + 2\frac{1-\gamma}{\xi(2-\gamma-\xi)} \left| \Delta G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right| \\ & \quad + 2|\Delta H_{\varphi_1^2, \psi_1^2}^{\text{int}}| + 2|\Delta h_1^2| + |\Delta v_1^2| + |\Delta t_1^2| \end{aligned}$$

$$\begin{aligned} & \times \left[ \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \right. \right. \right. \\ & \quad \left. \left. \left. + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + v_1^2 \right) \right|^2 - \left| \nabla \left( \bar{u} + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) + H_{\varphi_1^2, \psi_1^2}^{\text{int}} + h_1^2 + t_1^2 \right) \right|^2 \right] \end{aligned}$$

Proceeding as in the proof of the Theorem 5, we obtain

$$\begin{aligned} & \| \mathcal{N}_2(v_1^2, v_2^2) - \mathcal{N}_2(t_1^2, t_2^2) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ & \leq \bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \| v_1^2 - t_1^2 \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \gamma) \| v_2^2 - t_2^2 \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned} \quad (161)$$

Similarly, we get

$$\begin{aligned} & \| \mathcal{M}_2(v_1^2, v_2^2) - \mathcal{M}_2(t_1^2, t_2^2) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \\ & \leq \bar{c}_\kappa (1 - \xi) \| v_1^2 - t_1^2 \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} + \bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \| v_2^2 - t_2^2 \|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned} \quad (162)$$

□

Then there exist  $\gamma_0$  and  $\xi_0 \in (0, 1)$ , reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$ , if necessary, we can assume that  $\bar{c}_\kappa (1 - \gamma + r_{\varepsilon, \lambda}^2) \leq 1/2$  and  $\bar{c}_\kappa (1 - \xi + r_{\varepsilon, \lambda}^2) \leq 1/2$  for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$  and  $\xi \in (\xi_0, 1)$ . Therefore (161) and (162) are enough to show that

$$(v_1^2, v_2^2) \mapsto (\mathcal{N}_2(v_1^2, v_2^2), \mathcal{M}_2(v_1^2, v_2^2))$$

is a contraction from the ball

$$\{(v_1^2, v_2^2) \in C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4) : \|(v_1^2, v_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

**Proposition 16** *Given  $\kappa > 0$ ,  $\mu \in (1, 2)$ ,  $\gamma_0 \in (0, 1)$ , and  $\xi_0 \in (0, 1)$ , there exist  $\varepsilon_\kappa > 0$ ,  $\lambda_\kappa > 0$ , and  $c_\kappa > 0$  such that for all  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\gamma \in (\gamma_0, 1)$ , and  $\xi \in (\xi_0, 1)$ , for all  $\tau_2$  in some fixed compact subset of  $[\tau_2^-, \tau_2^+] \subset (0, \infty)$  and for  $\varphi_j^2$  and  $\psi_j^2$  satisfying (65) and (82), there exists a unique  $(v_1^2, v_2^2)$  ( $:= (v_{1,\varepsilon, \tau_2, \varphi_1^2, \psi_1^2}, v_{2,\varepsilon, \tau_2, \varphi_2^2, \psi_2^2})$ ) solution of (147) such that*

$$\|(v_1^2, v_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

Hence

$$\begin{cases} v_1(x) := \bar{u}(x - x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \\ \quad + h_1^2(x) + H^{\text{int}}(\varphi_1^2, \psi_1^2; \frac{x-x^2}{R_{\varepsilon, \lambda}^2}) + v_1^2(x), \\ v_2(x) := \bar{u}(x - x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \\ \quad + h_2^2(x) + H^{\text{int}}(\varphi_2^2, \psi_2^2; \frac{x-x^2}{R_{\varepsilon, \lambda}^2}) + h_2^2(x), \end{cases}$$

solves (154) in  $B_{R_{\varepsilon, \lambda}^2}(x^2)$ .

Note also that the functions  $(v_1^i, v_2^i)(:= (\nu_{1,\varepsilon,\tau_i,\varphi_1^i,\psi_1^i}^i, \nu_{2,\varepsilon,\tau_i,\varphi_2^i,\psi_2^i}^i))$ , for  $i \in \{1, 2, 3\}$ , depend continuously on the parameter  $\tau_i$ .

### 4.3 The nonlinear exterior problem

Using the same arguments as in the proof of Theorem 5, exterior problem, we obtain the following proposition.

**Proposition 17** *Given  $\kappa > 0$ , there exists  $\varepsilon_\kappa > 0$  and  $\lambda_\kappa > 0$  (depending on  $\kappa$ ) such that for any  $\varepsilon \in (0, \varepsilon_\kappa)$ ,  $\lambda \in (0, \lambda_\kappa)$ ,  $\eta_i$  and  $\tilde{x}^i$  satisfying (94) and functions  $\tilde{\varphi}_j^i$  and  $\tilde{\psi}_j^i$  satisfying (66) and (93), there exists a unique*

$(\tilde{v}_1, \tilde{v}_2)(:= (\tilde{\nu}_{1,\varepsilon,\eta_1,\eta_2,\tilde{x},\tilde{\varphi}_1^i,\tilde{\psi}_1^i}, \tilde{\nu}_{2,\varepsilon,\eta_2,\eta_3,\tilde{x},\tilde{\varphi}_2^i,\tilde{\psi}_2^i}))$  solution of (92) so that for  $v_k(k = 1, 2)$  defined by

$$\begin{aligned} v_1(x) &:= \frac{1 + \eta_1}{\gamma} G(x, \tilde{x}^1) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_1^i, \tilde{\psi}_1^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}\right) + \tilde{v}_1(x), \\ v_2(x) &:= \frac{1 + \eta_3}{\xi} G(x, \tilde{x}^3) + (1 + \eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}\left(\tilde{\varphi}_2^i, \tilde{\psi}_2^i; \frac{x - \tilde{x}^i}{r_{\varepsilon,\lambda}}\right) + \tilde{v}_2(x) \end{aligned}$$

solves (89) in  $\bar{\Omega}_{r_{\varepsilon,\lambda}}(\tilde{x})$ . In addition, we have

$$\|(\tilde{v}_1, \tilde{v}_2)\|_{C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

### 4.4 The nonlinear Cauchy-data matching

We will summarize the results of the previous sections. Using the previous notations, assume that  $\tilde{x} := (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \Omega^3$  are given close to  $x := (x^1, x^2, x^3)$ . Assume also that

$$\tau := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3,$$

are given (the values of  $\tau_l^-$  and  $\tau_l^+$ , for  $l = 1, 2, 3$  will be fixed later). First, we consider some set of boundary data  $\varphi^i := (\varphi_1^i, \varphi_2^i) \in (\mathcal{C}^{4,\alpha}(S^3))^2$  and  $\psi^i := (\psi_1^i, \psi_2^i) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ . According to the result of Proposition 14, 15 and 16 and provided  $\varepsilon \in (0, \varepsilon_\kappa)$ , we can find,  $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$  a solution of (139) in  $B_{r_{\varepsilon,\lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon,\lambda}}(\tilde{x}^3)$ , which can be decomposed as

$$u_{\text{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \tilde{x}^1) - \frac{1-\gamma}{\gamma} G(x, \tilde{x}^2) - \frac{1-\gamma}{\gamma\xi} G(x, \tilde{x}^3) - \frac{\ln \gamma}{\gamma} + H_{\varphi_1^1, \psi_1^1}^{\text{int}}\left(\frac{x - \tilde{x}^1}{r_{\varepsilon,\lambda}}\right) \\ \quad + h_1^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) + v_1^1\left(\frac{R_{\varepsilon,\lambda}^1(x - \tilde{x}^1)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \tilde{x}^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{ex}{\tau_2}, \tilde{x}^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{ex}{\tau_2}, \tilde{x}^3\right) \\ \quad + H_{\varphi_1^2, \psi_1^2}^{\text{int}}\left(\frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}}\right) + h_1^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) + v_1^2\left(\frac{R_{\varepsilon,\lambda}^2(x - \tilde{x}^2)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^2), \\ \frac{1}{\gamma} G(x, \tilde{x}^1) + G(x, \tilde{x}^2) + H_{\varphi_1^3, \psi_1^3}^{\text{int}}\left(\frac{x - \tilde{x}^3}{r_{\varepsilon,\lambda}}\right) + h_1^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) \\ \quad + v_1^3\left(\frac{R_{\varepsilon,\lambda}^3(x - \tilde{x}^3)}{r_{\varepsilon,\lambda}}\right) \\ \quad \text{in } B_{r_{\varepsilon,\lambda}}(\tilde{x}^3), \end{cases}$$

and

$$u_{\text{int},2}(x) := \begin{cases} \frac{1}{\xi} G(x, \tilde{x}^3) + G(x, \tilde{x}^2) + H_{\varphi_2^1, \psi_2^1}^{\text{int}}\left(\frac{x-\tilde{x}^1}{r_{\varepsilon, \lambda}}\right) + h_2^1\left(\frac{R_{\varepsilon, \lambda}^1(x-\tilde{x}^1)}{r_{\varepsilon, \lambda}}\right) \\ \quad + v_2^1\left(\frac{R_{\varepsilon, \lambda}^1(x-\tilde{x}^1)}{r_{\varepsilon, \lambda}}\right) & \text{in } B_{r_{\varepsilon, \lambda}}(\tilde{x}^1), \\ u_{\varepsilon, \tau_2}(x - \tilde{x}^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, \tilde{x}^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, \tilde{x}^3\right) \\ \quad + H_{\varphi_2^2, \psi_2^2}^{\text{int}}\left(\frac{x-\tilde{x}^2}{r_{\varepsilon, \lambda}}\right) + h_2^2\left(\frac{R_{\varepsilon, \lambda}^2(x-\tilde{x}^2)}{r_{\varepsilon, \lambda}}\right) + v_2^2\left(\frac{R_{\varepsilon, \lambda}^2(x-\tilde{x}^2)}{r_{\varepsilon, \lambda}}\right) & \text{in } B_{r_{\varepsilon, \lambda}}(\tilde{x}^2), \\ \frac{1}{\xi} u_{\varepsilon, \tau_3}(x - \tilde{x}^3) - \frac{1-\xi}{\xi} G(x, \tilde{x}^2) - \frac{1-\xi}{\gamma\xi} G(x, \tilde{x}^1) \\ \quad - \frac{\ln \xi}{\xi} + H_{\varphi_2^3, \psi_2^3}^{\text{int}}\left(\frac{x-\tilde{x}^3}{r_{\varepsilon, \lambda}}\right) \\ \quad + h_2^3\left(\frac{R_{\varepsilon, \lambda}^3(x-\tilde{x}^3)}{r_{\varepsilon, \lambda}}\right) + v_2^3\left(\frac{R_{\varepsilon, \lambda}^3(x-\tilde{x}^3)}{r_{\varepsilon, \lambda}}\right) & \text{in } B_{r_{\varepsilon, \lambda}}(\tilde{x}^3), \end{cases}$$

where for  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ ,  $R_{\varepsilon, \lambda}^i = \tau_i \frac{r_{\varepsilon, \lambda}}{\varepsilon}$  and the functions  $h_j^i$  and  $v_j^i$  satisfy

$$\begin{aligned} \| (h_1^1, h_2^1) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2, & \| (h_1^2, h_2^2) \|_{(C_\mu^{4,\alpha}(\mathbb{R}^4))^2} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2, \\ \| (h_1^3, h_2^3) \|_{(C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4))} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2 \end{aligned}$$

and

$$\begin{aligned} \| (v_1^1, v_2^1) \|_{C_\mu^{4,\alpha}(\mathbb{R}^4) \times C_\delta^{4,\alpha}(\mathbb{R}^4)} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2, & \| (v_1^2, v_2^2) \|_{(C_\mu^{4,\alpha}(\mathbb{R}^4))^2} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2, \\ \| (v_1^3, v_2^3) \|_{(C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4))} &\leq 2c_\kappa r_{\varepsilon, \lambda}^2. \end{aligned}$$

Similarly, given some boundary data  $\tilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$ ,  $\tilde{\psi}_j^i \in C^{2,\alpha}(S^3)$  satisfying (66),  $(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$

satisfying (94), provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ , by proposition 17, we find a solution  $u_{\text{ext}} := (u_{\text{ext},1}, u_{\text{ext},2})$  of (139) in  $\bar{\Omega} \setminus (B_{r_{\varepsilon, \lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon, \lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon, \lambda}}(\tilde{x}^3))$  which can be decomposed as

$$\begin{cases} u_{\text{ext},1}(x) := \frac{1+\eta_1}{\gamma} G(x, \tilde{x}^1) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}(\tilde{\varphi}_1^i, \tilde{\psi}_1^i; \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}}) + \tilde{v}_1(x), \\ u_{\text{ext},2}(x) := \frac{1+\eta_3}{\xi} G(x, \tilde{x}^3) + (1+\eta_2) G(x, \tilde{x}^2) + \sum_{i=1}^3 \chi_{r_0}(x - \tilde{x}^i) H^{\text{ext}}(\tilde{\varphi}_2^i, \tilde{\psi}_2^i; \frac{x-\tilde{x}^i}{r_{\varepsilon, \lambda}}) + \tilde{v}_2(x) \end{cases}$$

with  $\tilde{v}_1, \tilde{v}_2 \in C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x}))$  satisfying

$$\| (\tilde{v}_1, \tilde{v}_2) \|_{(C_v^{4,\alpha}(\bar{\Omega}^*(\tilde{x})))^2} \leq 2c_\kappa r_{\varepsilon, \lambda}^2.$$

It remains to determine the parameters and the boundary data in such a way that the function equal to  $u_{\text{int}}$  in  $B_{r_{\varepsilon, \lambda}}(\tilde{x}^1) \cup B_{r_{\varepsilon, \lambda}}(\tilde{x}^2) \cup B_{r_{\varepsilon, \lambda}}(\tilde{x}^3)$  and equal to  $u_{\text{ext}}$  in  $\bar{\Omega}_{r_{\varepsilon, \lambda}}(\tilde{x})$  is a smooth function. This amounts to find the boundary data and the parameters so that, for each  $j = 1, 2$

$$\begin{aligned} u_{\text{int},j} &= u_{\text{ext},j}, & \partial_r u_{\text{int},j} &= \partial_r u_{\text{ext},j}, \\ \Delta u_{\text{int},j} &= \Delta u_{\text{ext},j} & \text{and} & \partial_r \Delta u_{\text{int},j} = \partial_r \Delta u_{\text{ext},j} \end{aligned} \tag{163}$$

on  $\partial B_{r_{\varepsilon, \lambda}}(\tilde{x}^1)$ ,  $\partial B_{r_{\varepsilon, \lambda}}(\tilde{x}^2)$  and  $\partial B_{r_{\varepsilon, \lambda}}(\tilde{x}^3)$ .

Suppose that (163) is verified, this provides that for each  $\varepsilon$  and  $\lambda$  small enough  $u_{\varepsilon,\lambda} \in \mathcal{C}^{4,\alpha}$  (which is obtained by patching together the functions  $u_{\text{int}}$  and the function  $u_{\text{ext}}$ ), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof, since  $\varepsilon$  and  $\lambda$  tend to 0, the sequence of solutions we have obtained satisfies the required singular limit behavior.

Before we proceed, the following remarks are due. First it is convenient to observe that the function  $u_{\varepsilon,\tau_i}$  can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -4 \ln \tau_i - 8 \ln |x| + \mathcal{O}\left(\frac{\varepsilon^2 \tau_i^{-2}}{|x|^2}\right) \quad \text{on } \partial B_{r_{\varepsilon,\lambda}}(0). \quad (164)$$

- On  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^1)$ , according to the proof of Theorem 5 and since when  $\varepsilon$  and  $\lambda$  tend to 0, its enough to choose  $\tau_1^-$  and  $\tau_1^+$  is such a way that

$$4 \ln(\tau_1^-) < -\ln \gamma - \mathcal{E}_1(x^1, \mathbf{x}) < 4 \ln(\tau_1^+).$$

Where

$$\mathcal{E}_1(\cdot, \tilde{\mathbf{x}}) := H(\cdot, \tilde{x}^1) + G(\cdot, \tilde{x}^2) + \frac{1-\gamma}{\xi} G(\cdot, \tilde{x}^3).$$

Also using the fact that

$$\begin{aligned} \varphi_1^1 &= \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, & \psi_1^1 &= 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp}, \\ \tilde{\varphi}_1^1 &= \tilde{\varphi}_{1,0}^1 + \tilde{\varphi}_{1,1}^1 + \tilde{\varphi}_1^{1,\perp} & \text{and} & \tilde{\psi}_1^1 = \tilde{\psi}_{1,1}^1 + \tilde{\psi}_1^{1,\perp}. \end{aligned}$$

Where  $\varphi_{1,0}^1, \tilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_{1,1}^1, \tilde{\varphi}_{1,1}^1, \tilde{\psi}_{1,1}^1$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi_1^{1,\perp}, \tilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \tilde{\psi}_1^{1,\perp}$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

We can prove that

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, & \partial_r(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, \\ \Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0 \quad \text{and} \quad \partial_r \Delta(u_{\text{int},1} - u_{\text{ext},1})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, \end{aligned} \quad (165)$$

on  $S^3$  yield to

$$\begin{aligned} T_{1,\varepsilon}^1 &= (t_1, \eta_1, \varphi_{1,0}^1, \tilde{\varphi}_{1,0}^1, \varphi_{1,1}^1, \tilde{\varphi}_{1,1}^1, \tilde{\psi}_{1,1}^1, \bar{\nabla} \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}}), \varphi_1^{1,\perp}, \tilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \tilde{\psi}_1^{1,\perp}) \\ &= \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (166)$$

Where

$$t_1 = \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_1 + \ln \gamma + \mathcal{E}_1(\tilde{x}^1, \tilde{\mathbf{x}})].$$

Finally, using the fact that

$$\begin{aligned} \varphi_2^1 &= \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, & \psi_2^1 &= 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp}, \\ \tilde{\varphi}_2^1 &= \tilde{\varphi}_{2,0}^1 + \tilde{\varphi}_{2,1}^1 + \tilde{\varphi}_2^{1,\perp} & \text{and} & \tilde{\psi}_2^1 = \tilde{\psi}_{2,1}^1 + \tilde{\psi}_2^{1,\perp}, \end{aligned}$$

with  $\varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1 \in \mathbb{E}_0$ ,  $\varphi_{2,1}^1, \tilde{\varphi}_{2,1}^1, \tilde{\psi}_{2,1}^1 \in \mathbb{E}_1$  and  $\varphi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}$  belong to  $(L^2(S^3))^{\perp}$ .

We can prove that

$$\begin{aligned} (u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, & \partial_r(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0, \\ \Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) &= 0 \quad \text{and} \quad \partial_r \Delta(u_{\text{int},2} - u_{\text{ext},2})(\tilde{x}^1 + r_{\varepsilon,\lambda} \cdot) = 0, \end{aligned} \quad (167)$$

on  $S^3$  yield to

$$T_{2,\varepsilon}^1 = (\varphi_{2,0}^1, \tilde{\varphi}_{2,0}^1, \varphi_{2,1}^1, \tilde{\psi}_{2,1}^1, \psi_2^{1,\perp}, \tilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \tilde{\psi}_2^{1,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad (168)$$

where as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

- On  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^2)$ , we have

$$\begin{aligned} (u_{\text{int},1} - u_{\text{ext},1})(x) &= -4 \ln \tau_2 + 8\eta_2 \ln |x - \tilde{x}^2| + h_1^2 \left( R_{\varepsilon,\lambda}^2 \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) \\ &\quad + H^{\text{int}} \left( \varphi_1^2, \psi_1^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) - H^{\text{ext}} \left( \tilde{\varphi}_1^2, \tilde{\psi}_1^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) \\ &\quad - \left[ (2 - \gamma - \xi)H(x, \tilde{x}^2) + \frac{1 - \xi}{\gamma} G(x, \tilde{x}^1) + \frac{1 - \gamma}{\xi} G(x, \tilde{x}^3) \right] \\ &\quad + \mathcal{O} \left( \frac{\varepsilon^2 \tau_2^{-2}}{|x - \tilde{x}^2|^2} \right) + \mathcal{O}(r_{\varepsilon,\lambda}^2) \end{aligned} \quad (169)$$

and

$$\begin{aligned} (u_{\text{int},2} - u_{\text{ext},2})(x) &= -4 \ln \tau_2 + 8\eta_2 \ln |x - \tilde{x}^2| + h_2^2 \left( R_{\varepsilon,\lambda}^2 \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) \\ &\quad + H^{\text{int}} \left( \varphi_2^2, \psi_2^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) - H^{\text{ext}} \left( \tilde{\varphi}_2^2, \tilde{\psi}_2^2; \frac{x - \tilde{x}^2}{r_{\varepsilon,\lambda}} \right) \\ &\quad - \left[ (2 - \gamma - \xi)H(x, \tilde{x}^2) + \frac{1 - \xi}{\gamma} G(x, \tilde{x}^1) + \frac{1 - \gamma}{\xi} G(x, \tilde{x}^3) \right] \\ &\quad + \mathcal{O} \left( \frac{\varepsilon^2 \tau_2^{-2}}{|x - \tilde{x}^2|^2} \right) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (170)$$

Next, even though all functions are defined on  $\partial B_{r_{\varepsilon,\lambda}}(\tilde{x}^2)$  in (163), it is more convenient to solve on  $S^3$ , for  $i = 1, 2$ , the following set of equations

$$\begin{aligned} (u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^2 + r_{\varepsilon,\lambda} \cdot) &= 0, & \partial_r(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^2 + r_{\varepsilon,\lambda} \cdot) &= 0, \\ \Delta(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^2 + r_{\varepsilon,\lambda} \cdot) &= 0 \quad \text{and} \quad \partial_r \Delta(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^2 + r_{\varepsilon,\lambda} \cdot) = 0. \end{aligned} \quad (171)$$

Since the boundary data are chosen to satisfy (65) or (66). We decompose

$$\varphi_i^2 = \varphi_{i,0}^2 + \varphi_{i,1}^2 + \varphi_i^{2,\perp}, \quad \psi_i^2 = 8\varphi_{i,0}^2 + 12\varphi_{i,1}^2 + \psi_i^{2,\perp},$$

$$\tilde{\varphi}_i^2 = \tilde{\varphi}_{i,0}^2 + \tilde{\varphi}_{i,1}^2 + \tilde{\varphi}_i^{2,\perp} \quad \text{and} \quad \tilde{\psi}_i^2 = \tilde{\psi}_{i,1}^2 + \tilde{\psi}_i^{2,\perp},$$

where  $\varphi_{i,0}^1, \tilde{\varphi}_{i,0}^1 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_{i,1}^1, \tilde{\varphi}_{i,1}^1, \tilde{\psi}_{i,1}^1$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi_i^{1,\perp}, \tilde{\varphi}_i^{1,\perp}, \psi_i^{1,\perp}, \tilde{\psi}_i^{1,\perp}$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

We insist that, for  $x \in S^3$ , both equations (169) and (170) involve the same relation of the parameter  $\tau_2$  and the appropriate energy  $\mathcal{E}_2$ . Then we have

$$\begin{aligned} & (u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^2 + r_{\varepsilon,\lambda}x) \\ &= -4 \ln \tau_2 + 8\eta_2 \ln r_{\varepsilon,\lambda} |x| + H^{\text{int}}(\varphi_i^2, \psi_i^2, x) - H^{\text{ext}}(\tilde{\varphi}_i^2, \tilde{\psi}_i^2, x) \\ &\quad - \left[ (2 - \gamma - \xi)H(\tilde{x}^2, \tilde{x}^2) + \frac{1-\xi}{\gamma}G(\tilde{x}^2, \tilde{x}^1) + \frac{1-\gamma}{\xi}G(\tilde{x}^2, \tilde{x}^3) \right] + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (172)$$

Projecting the set of equations (171) over  $\mathbb{E}_0$ , we get

$$\begin{cases} -4 \ln \tau_2 + 8\eta_2 \ln r_{\varepsilon,\lambda} + \varphi_{i,0}^2 - \tilde{\varphi}_{i,0}^2 - \mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 8\eta_2 + 2\varphi_{i,0}^2 + 2\tilde{\varphi}_{i,0}^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 16\eta_2 + 8\varphi_{i,0}^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ -32\eta_2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \end{cases} \quad (173)$$

where

$$\mathcal{E}_2(\cdot, \tilde{\mathbf{x}}) := (2 - \gamma - \xi)H(\cdot, \tilde{x}^2) + \frac{1-\xi}{\gamma}G(\cdot, \tilde{x}^1) + \frac{1-\gamma}{\xi}G(\cdot, \tilde{x}^3).$$

The system (173) can be simply written as

$$\begin{aligned} \eta_2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \varphi_{i,0}^2 = \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\varphi}_{i,0}^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_2 + \mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}})] = \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned}$$

We are now in a position to define  $\tau_2^-$  and  $\tau_2^+$ . In fact, according to the above analysis, as  $\varepsilon$  and  $\lambda$  tend to 0, we expect  $\tilde{x}^i$  to converge to  $x^i$  for  $i \in \{1, 2, 3\}$  and  $\tau_2$  to converge to  $\tau_2^*$ , satisfying

$$4 \ln \tau_2^* = -\mathcal{E}_2(x^2, \mathbf{x}).$$

Hence it is enough to choose  $\tau_2^-$  and  $\tau_2^+$  in such a way that

$$4 \ln(\tau_2^-) < -\mathcal{E}_2(x^2, \mathbf{x}) < 4 \ln(\tau_2^+).$$

Consider now the projection of (171) over  $\mathbb{E}_1$ . Given a smooth function  $f$  defined in  $\Omega$ , we identify its gradient  $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$  with the element of  $\mathbb{E}_1$

$$\bar{\nabla} f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

With these notations in mind, we obtain the system of equations

$$\begin{cases} \varphi_{i,1}^2 - \tilde{\varphi}_{i,1}^2 - \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 3\varphi_{i,1}^2 + 3\tilde{\varphi}_{i,1}^2 + \frac{1}{2}\tilde{\psi}_{i,1}^2 - \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{i,1}^2 - 3\tilde{\varphi}_{i,1}^2 - \tilde{\psi}_{i,1}^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 15\varphi_{i,1}^2 + 15\tilde{\varphi}_{i,1}^2 + \frac{18}{4}\tilde{\psi}_{i,1}^2 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (174)$$

Which can be simplified as follows

$$\begin{aligned} \varphi_{i,1}^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), & \tilde{\varphi}_{i,1}^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \tilde{\psi}_{i,1}^2 &= \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}) &= \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \quad (175)$$

Finally, we consider the projection onto  $L^2(S^3)^\perp$ . This yields the system

$$\begin{cases} \varphi_i^{2,\perp} - \tilde{\varphi}_i^{2,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r(H_{\varphi_i^{2,\perp}, \psi_i^{2,\perp}}^{\text{int}} - H_{\tilde{\varphi}_i^{2,\perp}, \tilde{\psi}_i^{2,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \psi_i^{2,\perp} - \tilde{\psi}_i^{2,\perp} + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ \partial_r\Delta(H_{\varphi_i^{2,\perp}, \psi_i^{2,\perp}}^{\text{int}} - H_{\tilde{\varphi}_i^{2,\perp}, \tilde{\psi}_i^{2,\perp}}^{\text{ext}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{cases} \quad (176)$$

Thanks to the result of Lemma 4, this last system can be rewritten as

$$\varphi_i^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \tilde{\varphi}_i^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2), \quad \psi_i^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \tilde{\psi}_i^{2,\perp} = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

If we define the parameter  $t_2 \in \mathbb{R}$  by

$$t_2 = \frac{1}{\ln r_{\varepsilon,\lambda}} [4 \ln \tau_2 + \mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}})],$$

then the systems found by projecting (171) gather in this equality

$$\begin{aligned} T_{c,\varepsilon}^2 &= (t_2, \eta_2, \varphi_{i,0}^2, \tilde{\varphi}_{i,0}^2, \varphi_{i,1}^2, \tilde{\varphi}_{i,1}^2, \tilde{\psi}_{i,1}^2, \bar{\nabla}\mathcal{E}_2(\tilde{x}^2, \tilde{\mathbf{x}}), \varphi_i^{2,\perp}, \tilde{\varphi}_i^{2,\perp}, \psi_i^{2,\perp}, \tilde{\psi}_i^{2,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \\ \text{for } i &= 1, 2. \end{aligned} \quad (177)$$

where, as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but are bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

- On  $\partial B_{r_\varepsilon}(\tilde{x}^3)$ , according to the proof of Theorem 5, using the fact that

$$\begin{aligned} \varphi_1^3 &= \varphi_{1,0}^3 + \varphi_{1,1}^3 + \varphi_1^{3,\perp}, & \psi_1^3 &= 8\varphi_{1,0}^3 + 12\varphi_{1,1}^3 + \psi_1^{3,\perp}, \\ \tilde{\varphi}_1^3 &= \tilde{\varphi}_{1,0}^3 + \tilde{\varphi}_{1,1}^3 + \tilde{\varphi}_1^{3,\perp} \quad \text{and} \quad \tilde{\psi}_1^3 = \tilde{\psi}_{1,1}^3 + \tilde{\psi}_1^{3,\perp}, \end{aligned}$$

with  $\varphi_{1,0}^3, \tilde{\varphi}_{1,0}^3 \in \mathbb{E}_0$ ,  $\varphi_{1,1}^3, \tilde{\varphi}_{1,1}^3, \tilde{\psi}_{1,1}^3 \in \mathbb{E}_1$  and  $\varphi_1^{3,\perp}, \tilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}$  belong to  $(L^2(S^3))^\perp$ .

We can prove that

$$T_{1,\varepsilon}^3 = (\varphi_{1,0}^3, \tilde{\varphi}_{1,0}^3, \varphi_{1,1}^3, \tilde{\varphi}_{1,1}^3, \tilde{\psi}_{1,1}^3, \varphi_1^{3,\perp}, \tilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \tilde{\psi}_1^{3,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2). \quad (178)$$

On other hand, according to the proof of Theorem 4 and since when  $\varepsilon$  and  $\lambda$  tend to 0, its enough to choose  $\tau_1^-$  and  $\tau_1^+$  is such a way that

$$4\ln(\tau_3^-) < -\ln\xi - \mathcal{E}_3(x^3, \mathbf{x}) < 4\ln(\tau_3^+).$$

Where

$$\mathcal{E}_3(\cdot, \tilde{\mathbf{x}}) := H(\cdot, \tilde{x}^3) + G(\cdot, \tilde{x}^2) + \frac{1-\xi}{\gamma}G(\cdot, \tilde{x}^1).$$

Also using the fact that

$$\begin{aligned}\varphi_2^3 &= \varphi_{2,0}^3 + \varphi_{2,1}^3 + \varphi_2^{3,\perp}, & \psi_2^3 &= 8\varphi_{2,0}^3 + 12\varphi_{2,1}^3 + \psi_2^{3,\perp}, \\ \tilde{\varphi}_2^3 &= \tilde{\varphi}_{2,0}^3 + \tilde{\varphi}_{2,1}^3 + \tilde{\varphi}_2^{3,\perp} & \text{and} & \tilde{\psi}_2^3 = \tilde{\psi}_{2,0}^3 + \tilde{\psi}_{2,1}^3 + \tilde{\psi}_2^{3,\perp},\end{aligned}$$

where  $\varphi_{2,0}^3, \tilde{\varphi}_{2,0}^3 \in \mathbb{E}_0 = \mathbb{R}$  are constant on  $S^3$ ,  $\varphi_{2,1}^3, \tilde{\varphi}_{2,1}^3, \tilde{\psi}_{2,1}^3$  belong to  $\mathbb{E}_1 = \text{Span}\{e_1, e_2, e_3, e_4\}$  and  $\varphi_2^{3,\perp}, \tilde{\varphi}_2^{3,\perp}, \psi_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}$  are  $L^2(S^3)$  orthogonal to  $\mathbb{E}_0$  and  $\mathbb{E}_1$ .

We can prove that

$$\begin{aligned}(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^3 + r_{\varepsilon,\lambda}\cdot) &= 0, & \partial_r(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^3 + r_{\varepsilon,\lambda}\cdot) &= 0, \\ \Delta(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^3 + r_{\varepsilon,\lambda}\cdot) &= 0 & \text{and} & \partial_r\Delta(u_{\text{int},i} - u_{\text{ext},i})(\tilde{x}^3 + r_{\varepsilon,\lambda}\cdot) = 0,\end{aligned}\tag{179}$$

on  $S^3$  yield to

$$T_{2,\varepsilon}^3 = (t_3, \eta_3, \varphi_{2,0}^3, \tilde{\varphi}_{2,0}^3, \varphi_{2,1}^3, \tilde{\varphi}_{2,1}^3, \tilde{\psi}_{2,1}^3, \bar{\nabla}\mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}}), \varphi_2^{3,\perp}, \tilde{\varphi}_2^{3,\perp}, \psi_2^{3,\perp}, \tilde{\psi}_2^{3,\perp}) = \mathcal{O}(r_{\varepsilon,\lambda}^2), \tag{180}$$

where

$$t_3 = \frac{1}{\ln r_{\varepsilon,\lambda}} [4\ln\tau_3 + \ln\xi + \mathcal{E}_3(\tilde{x}^3, \tilde{\mathbf{x}})],$$

where, as usual, the terms  $\mathcal{O}(r_{\varepsilon,\lambda}^2)$  depend nonlinearly on all the variables on the left side, but are bounded (in the appropriate norm) by a constant (independent of  $\varepsilon$  and  $\kappa$ ) times  $r_{\varepsilon,\lambda}^2$ , provided  $\varepsilon \in (0, \varepsilon_\kappa)$  and  $\lambda \in (0, \lambda_\kappa)$ .

We recall that  $\mathbf{d} = r_{\varepsilon,\lambda}(\tilde{\mathbf{x}} - \mathbf{x})$ , in addition the previous systems can be written as for  $i = 1, 2, 3$ :

$$(\mathbf{d}, t_i, \eta_i, \varphi^i, \tilde{\varphi}^i, \psi^i, \tilde{\psi}^i, \bar{\nabla}\mathcal{E}_i) = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

Combining (166), (168), (177), (178), and (180), we have

$$T_{i,\varepsilon} = (T_{i,\varepsilon}^1, T_{i,\varepsilon}^2, T_{i,\varepsilon}^3) = (\mathcal{O}(r_{\varepsilon,\lambda}^2), \mathcal{O}(r_{\varepsilon,\lambda}^2), \mathcal{O}(r_{\varepsilon,\lambda}^2)), \quad \text{for } i = 1, 2. \tag{181}$$

Then the nonlinear mapping that appears on the right-hand side of (181) is continuous and compact. In addition, reducing  $\varepsilon_\kappa$  and  $\lambda_\kappa$  if necessary, this nonlinear mapping sends the ball of radius  $\kappa r_{\varepsilon,\lambda}^2$  (for the natural product norm) into itself, provided  $\kappa$  is fixed large enough. Applying Schauder's fixed point Theorem in the ball of radius  $\kappa r_{\varepsilon,\lambda}^2$  in the product space where the entries live, we obtain the existence of a solution of equation (181).

This completes the proof of Theorem 6.  $\square$

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**Data Availability**

No datasets were generated or analysed during the current study.

**Declarations****Competing interests**

The authors declare no competing interests.

**Author contributions**

All authors wrote the main manuscript text and reviewed the manuscript.

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