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The Robin problems for the coupled system of reaction–diffusion equations

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Abstract

This article investigates the local well-posedness of Turing-type reaction–diffusion equations with Robin boundary conditions in the Sobolev space. Utilizing the Hadamard norm, we derive estimates for Fokas unified transform solutions for linear initial-boundary value problems subject to external forces. Subsequently, we demonstrate that the iteration map, defined by the unified transform solutions and incorporating nonlinearity instead of external forces, acts as a contraction map within an appropriate solution space. Our conclusive result is established through the application of the contraction mapping theorem.

Keywords: Coupled system of reaction; Diffusion equations; Unified transform method; The local well-posedness of the coupled system of reaction; Diffusion equations

1 Introduction and main results

1.1 Introduction

The diffusion equation is a widely used concept in contemporary science, employed to describe various phenomena in physics, chemistry, and biology. In 1952, Alan Turing used this equation to explain natural patterns in a ground-breaking way. In the realm of physics, the heat equation is a prominent example of a diffusion equation. Joseph Fourier developed it in 1822 to model the diffusion of heat within a specific area. This classical parabolic partial differential equation is a significant subject in pure mathematics and has been extensively researched. The study of the heat equation is a cornerstone of the field of partial differential equations. Additionally, considering the heat equation on Riemannian manifolds leads to many geometric applications. In the field of biology, the classical Lotka–Volterra equation system is another example of a diffusion equation system. This model provides a framework for understanding variations in predator and prey populations. In conclusion, the diffusion equation has numerous scientific applications, and its substantial contribution to the advancement of human knowledge requires further research and development.

We now present recent articles that address the existence, uniqueness, and well-posedness of solutions related to the reaction–diffusion equations. Slavík, Stehlík, and Volek [20] examine issues concerning lattice reaction–diffusion equations, utilizing maximum principles to establish results of existence, uniqueness, and continuous dependence. They establish both the local existence and global uniqueness of bounded solutions, as well as the continuous dependence of solutions on the underlying time structure and initial conditions. The weak maximum principle is applied to prove the global existence of solutions. Finally, the authors provide the strong maximum principle, revealing an intriguing dependence on the time structure. Xu, Lian, and Nin [22] study nonlinear parabolic systems with power-type source terms, dividing the study into three cases based on initial energy considerations. In the low initial energy scenario, they use the Galerkin method and the concave function method to establish the global existence and finite-time blowup of the solution. For the critical initial energy case, the global solution, the blowup solution, and the asymptotic behavior are proved by scaling the initial data. In the high initial energy case, the authors explore the potential for both the global existence and the finite-time blowup by finding the corresponding initial data with arbitrarily high initial energy and then provide proof of the global existence. Palencia and Redondo [18] investigate the existence, uniqueness, and positivity conditions for a cooperative system formulated with high-order diffusion. They demonstrate the oscillatory behavior of self-similar solutions and characterize regions of positivity for a class of high-order cooperative systems without advection. Palencia, Rahman, and Redondo [17] analyzed a Fisher-KPP nonlinear reaction equation within a framework involving higher-order diffusion and the presence of an advection term. Palencia and Rahman [16] proposed a new model to describe the behavior of flames driven by temperature and pressure variables. They used the p-Laplacian operator in flame propagation, making their model applicable to a wide range of diffusion-driven domains, and they proved the uniqueness and boundedness of the weak solution and the existence of a minimum traveling-wave speed. Palencia [15] studied a reaction–diffusion problem involving high-order operators, nonlinear advection, and Fisher-KPP reaction terms. The author introduced a novel extended operator to study the reaction within the open domain $\mathbb{R}^n$ but depart from a sequence of bounded domains. Regularity, existence, and uniqueness analyses of the solutions were performed using semigroup theory. Morgan and Tang [11] investigate the global existence of classical solutions for volume–surface reaction–diffusion systems with mass control. They introduce a novel family of L^p-energy functions and utilize a general assumption known as the intermediate sum condition to establish the global existence of classical solutions. Himonas, Mantzavinos, and Yan [8] use the unified transform method to prove the local well-posedness of the reaction–diffusion equations with the Dirichlet boundary conditions.

In contemporary scientific research, coupled systems with Robin boundary conditions are extensively applied. Well-posedness ensures the equation models' reliability and predictive accuracy in various fields, making it critical for scientific research, engineering applications, and decision making. Based on our current understanding derived from relevant studies on reaction–diffusion equations, we have looked at the local well-posedness of the coupled system of reaction–diffusion equations with Robin boundary conditions in this article.

1.2 Main results

The occurrence of patterns is ubiquitous in the natural world, appearing in diverse forms such as stripes and spots on animals, intricate branching patterns in leaves, and the remarkable structural diversity observed in both biological and nonbiological systems. The investigation and comprehension of the underlying mechanisms of pattern formation have been central subjects of scientific inquiry. In 1952, Alan Turing [21] made a substantial contribution to this field by studying Turing-type reaction–diffusion equations:

$$\begin{cases} u_t - Au_{xx} + \mathcal{F}(u,v) = 0, \\ v_t - Bv_{xx} + \mathcal{G}(u,v) = 0, \end{cases}$$

where A and B are positive constants in $\mathbb{R}$, and u and v are morphogen concentrations, $\mathcal{F}$ and $\mathcal{G}$ describe the interrelation between morgens, Au_{xx} and Bv_{xx} can move randomly with diffusivities A and B. His research suggested a possible connection between the mathematical models described by these equations and the actual processes of pattern formations. Turing's work laid the groundwork for studying the mathematical aspects of pattern emergences in various systems. Following Turing's pioneering efforts, numerous articles and studies have been devoted to advancing our understanding of Turing-type reaction–diffusion equations and their implications for pattern formations. These equations have been applied in a variety of fields, providing insights into the emergence of patterns in natural systems. References to these articles and books are available for further explorations [1, 7, 9, 10, 12–14, 19].

In this article, we establish the local well-posedness of the following Turing-type reaction–diffusion equations with the Robin boundary conditions:

$$\begin{cases} u_t - u_{xx} + u^2 + cuv = 0, & x \in (0,\infty), t \in (0,T), c \in \mathbb{R}, \\ v_t - v_{xx} + v^2 + duv = 0, & x \in (0,\infty), t \in (0,T), d \in \mathbb{R}, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in [0,\infty), \\ u_x(0,t) - \alpha u(0,t) = g_0(t), & t \in [0,T], \alpha \geq 1, \\ v_x(0,t) - \beta v(0,t) = h_0(t), & t \in [0,T], \beta \geq 1, \end{cases} \tag{1}$$

where $0 < T < 1$, and $u(x,t)$ and $v(x,t)$ are real-valued functions, and $u_0(x) \in H_x^s(0,\infty)$ and $v_0(x) \in H_x^s(0,\infty)$ are initial data, and $g_0(t) \in H_t^{(2s-1)/4}(0,T)$ and $h_0(t) \in H_t^{(2s-1)/4}(0,T)$ are boundary data, where $1/2 < s < 3/2$.

In this study, we investigate the local well-posedness of the initial boundary value problem (IBVP) given by equation (1). Our proof of the local well-posedness of (1) consists of three steps. In the first step, we replace the nonlinearities $u^2 + cuv$ and $v^2 + duv$ by the forcings and use the Unified Transform Method (UTM) to solve the corresponding linear IBVPs. (Fokas [2–5] introduced the UTM and its applications.) The second step involves deriving linear estimates using the UTM formula with data and forcing in appropriate spaces. The third step shows that the iteration map defined by the UTM formula, with the forcing replaced by the nonlinearity, is a contraction map in an appropriate solution space. Finally, the uniqueness of the solution for the IBVP (1) is established by the contraction mapping theorem. In addition, we prove the local Lipschitz continuity of the data-to-solution map, thereby confirming the local well-posedness of the IBVP (1).

Now, we provide an overview of the Sobolev space. The Sobolev spaces $H_x^s(0,\infty)$ and $H_t^{(2s-1)/4}(0,T)$ are derived as restrictions of their counterparts over the entire real line, following the general definition: For $s \in \mathbb{R}$, Sobolev spaces $H^s(\mathbb{R})$ consist of all tempered distributions F with the finite norm

$$\|F\|_{H^s(\mathbb{R})} \doteq \left(\int_{\mathbb{R}} \left(1+\xi^2\right)^s \left|\widehat{F}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}},$$

where $\widehat{F}(\xi)$ is the Fourier transform defined by

$$\widehat{F}(\xi) \doteq \int_{\mathbb{R}} e^{-ix\xi} F(x)\, dx.$$

Furthermore, for an open interval Ω in $\mathbb{R}$, the Sobolev space $H^s(\Omega)$ is defined as

$$H^s(\Omega) = \left\{ \begin{matrix} f : f = F|_{\Omega}, \text{ where } F \in H^s(\mathbb{R}) \\ \text{and } \|f\|_{H^s(\Omega)} \doteq \inf_{F\in H^s(\mathbb{R})} \|F\|_{H^s(\mathbb{R})} < \infty \end{matrix} \right\}.$$

By solving the forced linear Robin IBVP via the Fokas method, it leads us to the following Fourier transform.

Definition 1.1 (Fourier transform on the half-line) For a test function $\phi(x)$ defined on $(0,\infty)$, its half-line Fourier transform is given by the formula

$$\widehat{\phi}(k) \doteq \int_0^{\infty} e^{-ikx}\phi(x)\, dx, \tag{2}$$

where $k \in \mathbb{C}$ and $\Im(k) \le 0$. The notations $\Im(k)$ and $\Re(k)$ represent the imaginary and real parts of k.

Remark 1.2 For (2), it is obvious that if ϕ is an integrable function on $(0,\infty)$, we observe that $\widehat{\phi}(k)$ is well defined for $\Im(k) \le 0$. In fact, in the context of a more appropriate space $L^2(0,\infty)$, it is possible to define the half-line Fourier transform. Subsequently, the function ϕ in $L^2(0,\infty)$ can be extended to the entire real line by assigning $\phi(x) = 0$ for $x < 0$, yielding a function in $L^2(\mathbb{R})$. Furthermore, the half-line Fourier transform of ϕ can be defined using the same formula used for the Fourier transform of ϕ and its extension to the real line. It follows that the formula for the inverse can also be obtained, which is the inverse Fourier transform on the real line.

Let us begin by outlining the first step of our approach to solving the problem for the associated forced linear equation:

$$\begin{cases} u_t - u_{xx} = f(x,t), & x \in (0,\infty), 0 < t < T < 1, \\ u(x,0) = u_0(x) \in H_x^s(0,\infty), & x \in [0,\infty), \\ u_x(0,t) - \alpha u(0,t) = g_0(t) \in H_t^{\frac{2s-1}{4}}(0,T), & 0 \le t \le T < 1, \alpha \ge 1. \end{cases} \tag{3}$$

According to the UTM formulation, the solution to (3) is denoted by

$$\begin{aligned} u(x,t) &\doteq S[u_0,g_0;f](x,t) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-k^2t}\widehat{u_0}(k)\, dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{\alpha+ik}{\alpha-ik}\widehat{u_0}(-k))\, dk \end{aligned} \tag{4}$$

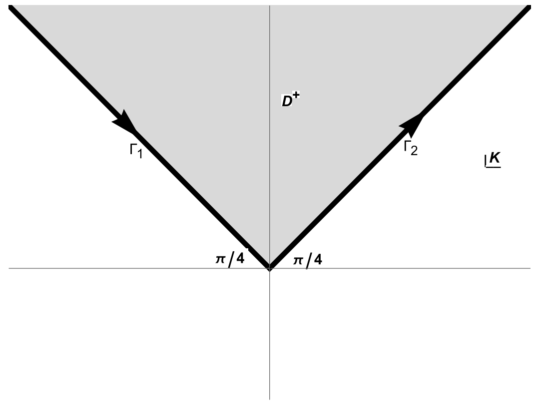

Figure 1 The region D^+ and its positively oriented boundaries ∂D

$$
\begin{aligned}
&- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} \frac{-2ik}{\alpha - ik} \left(\int_0^t e^{k^2y} g_0(y)\, dy \right) dk \\
&- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} \frac{\alpha + ik}{\alpha - ik} \left(\int_0^t e^{k^2y} \widehat{f}(-k,y)\, dy \right) dk \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t} \left(\int_0^t e^{k^2y} \widehat{f}(k,y)\, dy \right) dk,
\end{aligned}
$$

where

$$
\widehat{f}(k,y) \doteq \int_{\mathbb{R}} e^{-ikx} f(x,y)\, dx, \quad k \in \mathbb{C}
$$

is the Fourier transform of $f(x,y)$ with respect to x, and D^+ represents the domain in the complex k plane shown in Fig. 1.

For ease of calculation and presentation, we use the following notations.

Remark 1.3 For two quantities A and B depending on one or several variables, we express $A \lesssim B$ if there exists a positive constant c such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, then we denote $A \simeq B$.

Now, we delineate the second step, which involves estimating the Hadamard norm of the UTM solution formula $S[u_0, g_0; f]$ in (4) by the Sobolev norms of the data and an appropriate norm of the forcing. More precisely, we derive the following linear estimate.

Theorem 1.4 (The linear estimate for the reaction–diffusion equation) *Consider the reaction–diffusion equation* (3). *Suppose* $1/2 < s < 3/2$, $0 < T < 1$, $u_0(x) \in H_x^s(0,\infty)$, *and* $g_0(t) \in H_t^{(2s-1)/4}(0,T)$. *Then, the solution* $u = S[u_0, g_0; f]$ *of the forced linear equation IBVP* (3) *given by* (4) *satisfies the estimate*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| u(t) \right\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)} \left\| u(x) \right\|_{H_t^{\frac{2s+1}{4}}(0,T)} \\
&\quad \leq \mathfrak{C}_s \left(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \sqrt{T} \sup_{t\in[0,T]} \left\| f(t) \right\|_{H_x^s(0,\infty)} \right),
\end{aligned} \tag{5}
$$

where $\mathfrak{C}_s = \mathfrak{C}(s) > 0$ *is a constant depending on s.*

Finally, our goal is to prove the uniqueness of the solution for (1) and establish that the data-to-solution is locally Lipschitz continuous. Therefore, for $s > 1/2$ and $0 < T^* \leq T < 1$, we define two Banach spaces $\mathbb{X}$ and D as

$$\mathbb{X} = X \times X, \quad \text{where } X = C\big([0, T^*]; H_x^s(0,\infty)\big) \cap C\big([0,\infty); H_t^{\frac{2s+1}{4}}(0,T^*)\big)$$

with the norm

$$\begin{aligned}\big\|(u,v)\big\|_{\mathbb{X}} = &\sup_{t\in[0,T^*]}\big\|u(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|u(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\ &+ \sup_{t\in[0,T^*]}\big\|v(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|v(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)}.\end{aligned}$$

The data space

$$D = H_x^s(0,\infty) \times H_x^s(0,\infty) \times H_t^{\frac{2s-1}{4}}(0,T) \times H_t^{\frac{2s-1}{4}}(0,T),$$

which has the norm defined by

$$\begin{aligned}\big\|(u_0,v_0,g_0,h_0)\big\|_D = &\|u_0\|_{H_x^s(0,\infty)} + \|v_0\|_{H_x^s(0,\infty)} \\ &+ \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|h_0\|_{H_t^{\frac{2s-1}{4}}(0,T)},\end{aligned} \tag{6}$$

for $(u_0,v_0,g_0,h_0) \in D$.

Then, using the above definitions, we give the main result of this work.

Theorem 1.5 (The local well-posedness of the coupled system of reaction–diffusion equations) *Consider the coupled system of reaction–diffusion equations* (1). *Suppose* $1/2 < s < 3/2$ *and* $0 < T < 1$. *For the data* $u_0, v_0 \in H_x^s(0,\infty)$, *and* $g_0(t), h_0(t) \in H_t^{(2s-1)/4}(0,T)$.

Then, there exist $C_s^* = C^*(s) > 0$ *and* T^*, $0 < T^* \leq T < 1$, *with*

$$T^* = \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\} > 0$$

such that the coupled system of reaction–diffusion equations IBVP (1) *has a unique solution* $(u,v) \in \mathbb{X}$ *and the solution satisfies the size estimate*

$$\big\|(u,v)\big\|_{\mathbb{X}} \leq 2C_s^*\big\|(u_0,v_0,g_0,h_0)\big\|_D.$$

Furthermore, the data-to-solution map $(u_0,v_0,g_0,h_0) \longmapsto (u,v)$ *is locally Lipschitz continuous.*

In Sect. 2, we study a reduced pure IBVP for the linear reaction–diffusion equation to derive Theorem 2.1 and Theorem 2.4, which help to prove Theorem 1.4. In Sects. 3 and 4, we provide the proofs of Theorem 1.4 and Theorem 1.5, respectively.

2 The reduced pure IBVP for the linear reaction–diffusion equation

In this section, we analyze a basic Robin problem associated with the linear reaction–diffusion equation to establish Theorem 2.1 and Theorem 2.4. These theorems serve as crucial tools for estimating linear IBVPs (III) and (IV) in Sect. 3.

2.1 Reduced pure IBVP

We start by considering the fundamental linear reaction–diffusion equation IBVP on the half-line. This corresponds to the homogeneous IBVP with zero initial data and nonzero boundary data.

In addition, we assume that the boundary data $g \in H_t^{(2s-1)/4}(\mathbb{R})$ is a time-dependent test function with compact support in the interval $[0,2]$. This particular problem, known as the reduced pure IBVP, can be formulated as follows:

$$\begin{cases} w_t - w_{xx} = 0, & x \in (0,\infty), t \in (0,2), \\ w(x,0) = 0, & x \in [0,\infty), \\ w_x(0,t) - \alpha w(0,t) = g(t), & t \in [0,2], \alpha \geq 1. \end{cases} \tag{7}$$

Taking advantage of the compact support of g, we express its time transformation over the interval $(0,2)$ as a full Fourier transform, denoted by

$$\widetilde{g}(\zeta,2) \doteq \int_0^2 e^{\zeta t} g(t)\,dt = \int_{\mathbb{R}} e^{\zeta t} g(t)\,dt.$$

By the UTM formula, the solution of the reduced pure IBVP (7) is

$$\begin{aligned} w(x,t) = S[0,g;0](x,t) &= -\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{-2ik}{\alpha - ik}\widetilde{g}\big(k^2,t\big)\,dk \\ &= \frac{i}{\pi}\int_{\Gamma_1} e^{ikx-k^2t}\frac{k}{\alpha - ik}\widetilde{g}\big(k^2,t\big)\,dk + \frac{i}{\pi}\int_{\Gamma_2} e^{ikx-k^2t}\frac{k}{\alpha - ik}\widetilde{g}\big(k^2,t\big)\,dk, \end{aligned} \tag{8}$$

for all $x \in [0,\infty)$ and $t \in \mathbb{R}$.

Now, we compute (8). First, we calculate

$$\begin{aligned} &\frac{i}{\pi}\int_{\Gamma_1} e^{ikx-k^2t}\frac{k}{\alpha - ik}\widetilde{g}\big(k^2,t\big)\,dk \\ &\quad = -\frac{i}{\pi}\int_0^\infty e^{ia^3k'x+i(k')^2t}\frac{a^3k'}{\alpha - ia^3k'}\widetilde{g}\big(-i\big(k'\big)^2,t\big)a^3\,dk', \quad \big(\text{Let } k' = ke^{-i\frac{3}{4}\pi} \text{ and } a = e^{i\frac{\pi}{4}}.\big) \\ &\quad = -\frac{1}{\pi}\int_0^\infty e^{ir_1kx+ik^2t}\frac{k(\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}ki)}{\alpha^2 + \sqrt{2}\alpha k + k^2}\widehat{g}\big(k^2\big)\,dk, \\ &\qquad \big(\text{we know } \widetilde{g}\big(-i\big(k'\big)^2,t\big) = \widehat{g}\big(\big(k'\big)^2\big). \text{ Let } k = k' \text{ and } r_1 = a^3 = e^{i\frac{3\pi}{4}}\big) \end{aligned}$$

and on the other hand

$$\begin{aligned} &\frac{i}{\pi}\int_{\Gamma_2} e^{ikx-k^2t}\frac{k}{\alpha - ik}\widetilde{g}\big(k^2,t\big)\,dk \\ &\quad = \frac{i}{\pi}\int_0^\infty e^{i(k'e^{i\frac{\pi}{4}})x-(k'e^{i\frac{\pi}{4}})^2t}\frac{k'e^{i\frac{\pi}{4}}}{\alpha - i(k'e^{i\frac{\pi}{4}})}\widetilde{g}\big(\big(k'e^{i\frac{\pi}{4}}\big)^2,t\big)e^{i\frac{\pi}{4}}\,dk', \\ &\big(\text{Let } k' = ke^{-i\frac{\pi}{4}}, a = e^{i\frac{\pi}{4}}.\big) \\ &\quad = -\frac{1}{\pi}\int_0^\infty e^{ir_2kx-ik^2t}\frac{k(\alpha + \frac{\sqrt{2}}{2}k + \frac{\sqrt{2}}{2}ki)}{\alpha^2 + \sqrt{2}\alpha k + k^2}\widehat{g}\big(-k^2\big)\,dk \end{aligned}$$

$$\left(\text{we know } \widetilde{g}\left(-i\left(k'\right)^2, t\right) = \widehat{g}\left(\left(k'\right)^2\right). \text{ Let } k = k' \text{ and } r_2 = a = e^{i\frac{\pi}{4}}\right).$$

Therefore, we can rewrite (8) as

$$w(x,t) = S[0,g;0](x,t) = w_1(x,t) + w_2(x,t), \tag{9}$$

where

$$\begin{aligned} r_1 &= e^{i\frac{3\pi}{4}}, \qquad G_1(k,t) = -\frac{1}{\pi} e^{ik^2 t} \frac{k(\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}ki)}{\alpha^2 + \sqrt{2}\alpha k + k^2} \widehat{g}\left(k^2\right), \\ w_1(x,t) &= \int_0^\infty e^{ir_1 kx} G_1(k,t)\, dk \end{aligned} \tag{10}$$

and

$$\begin{aligned} r_2 &= e^{i\frac{\pi}{4}}, \qquad G_2(k,t) = -\frac{1}{\pi} e^{-ik^2 t} \frac{k(\alpha + \frac{\sqrt{2}}{2}k + \frac{\sqrt{2}}{2}ki)}{\alpha^2 + \sqrt{2}\alpha k + k^2} \widehat{g}\left(-k^2\right), \\ w_2(x,t) &= \int_0^\infty e^{ir_2 kx} G_2(k,t)\, dk. \end{aligned}$$

In the following result, we estimate the solution (9) in the Hadamard space.

Theorem 2.1 (*Estimates for the pure linear IBVP on the half-line*) *For* $1/2 < s < 3/2$ *and the boundary data test function* $g \in H_t^{(2s-1)/4}(\mathbb{R})$ *is compactly supported in the interval* $[0,2]$. *Then, the solution of the reduced pure IBVP* (7), *which satisfies the following Hadamard space estimates*:

$$\textit{space estimate}: \quad \sup_{t\in[0,2]} \left\| S[0,g;0](t) \right\|_{H_x^s(0,\infty)} \le C_s \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \tag{11}$$

$$\textit{time estimate}: \quad \sup_{x\in[0,\infty)} \left\| S[0,g;0](x) \right\|_{H_t^{\frac{2s+1}{4}}(0,2)} \le C_s \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \tag{12}$$

where $C_s = C(s) > 0$ *is a constant depending on* s.

Proof First, we start with the proof of the space estimate (11). We can derive the inequalities

$$\sup_{t\in[0,2]} \left\| w_1(t) \right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \tag{13}$$

and

$$\sup_{t\in[0,2]} \left\| w_2(t) \right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \tag{14}$$

but we will only present the proof for the inequality (13). Since the estimation processes of (13) and (14) are similar, we can use a process analogous to the proof of (13) to obtain (14). Therefore, by (9), (13), and (14), we establish the equation for the space estimate (11):

$$\sup_{t\in[0,2]} \left\| w(t) \right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.$$

Now, we begin the proof of the inequality (13). We use the physical space definition of the $H_x^s(0,\infty)$ norm:

$$\|w_1(t)\|_{H_x^s(0,\infty)} = \begin{cases} \sum_{j=0}^{\lfloor s\rfloor} \|\partial_x^j w_1(t)\|_{L_x^2(0,\infty)} + \|\partial_x^{\lfloor s\rfloor} w_1(t)\|_\beta & \text{for } s \in \mathbb{R}^+\backslash\mathbb{Z}^+, \\ \sum_{j=0}^{\lfloor s\rfloor} \|\partial_x^j w_1(t)\|_{L_x^2(0,\infty)} & \text{for } s \in \mathbb{Z}^+, \end{cases} \tag{15}$$

where $0 < \beta < 1$ and $\lfloor s\rfloor = s - \beta \in \mathbb{Z}^+ \cup \{0\}$. The fractional norm $\|\cdot\|_\beta$ is defined by

$$\|w_1(t)\|_\beta^2 = \int_0^\infty \int_0^\infty \frac{|w_1(x+\zeta,t) - w_1(x,t)|^2}{\zeta^{1+2\beta}}\, d\zeta\, dx, \quad \forall \beta \in (0,1).$$

We will prove that the inequality (13) holds under these three cases: Case (I): when $\beta = 0$ holds, Case (II): when $s = \beta \neq 0$ holds, and Case (III): when $\lfloor s\rfloor \neq 0$ and $\beta \neq 0$ hold.

We require the following two lemmas in [6] to assist us in proving the inequality (13) under Cases (I)–(III).

Lemma 2.2 ([6]) *If $r = r_R + ir_I$ with $r_I > 0$, then*

$$\left|e^{irkx} - e^{irk\zeta}\right| \le \sqrt{2}\left(1 + \frac{|r_R|}{r_I}\right)\left|e^{-r_I kx} - e^{-r_I k\zeta}\right|, \quad \forall k, x, \zeta \ge 0.$$

Lemma 2.3 ([6]. (L^2-boundedness of the Laplace transform)) *Suppose $\phi \in L_\tau^2(0,\infty)$. Then, the map*

$$\mathcal{L} : \phi \longmapsto \int_0^\infty e^{-\tau t}\phi(\tau)\, d\tau$$

is bounded from $L_\tau^2(0,\infty)$ into $L_t^2(0,\infty)$ with

$$\|\mathcal{L}\{\phi\}\|_{L_t^2(0,\infty)} \le \sqrt{\pi}\,\|\phi\|_{L_\tau^2(0,\infty)}.$$

Now, we begin to prove that when $\beta = 0$, then the inequality (13) holds. Case (I): Suppose $\beta = 0$. This implies that $s = \lfloor s\rfloor = 1$. Then, the definition of $\|w_1(t)\|_{H_x^s(0,\infty)}$ is

$$\|w_1(t)\|_{H_x^s(0,\infty)} = \|w_1(t)\|_{L_x^2(0,\infty)} + \|\partial_x w_1(t)\|_{L_x^2(0,\infty)}.$$

First, we calculate $\|w_1(t)\|_{L_x^2(0,\infty)}^2$:

$$\begin{aligned}
&\|w_1(t)\|_{L_x^2(0,\infty)}^2 \\
&\quad = \int_0^\infty \left|\int_0^\infty e^{ir_1 kx} G_1(k,t)\, dk\right|^2 dx \\
&\quad = \int_0^\infty \left|\int_0^1 e^{ir_1 kx} G_1(k,t)\, dk + \int_1^\infty e^{ir_1 kx} G_1(k,t)\, dk\right|^2 dx \\
&\quad \lesssim \int_0^\infty \left|\int_0^1 e^{ir_1 kx} G_1(k,t)\, dk\right|^2 dx + \int_0^\infty \left|\int_1^\infty e^{ir_1 kx} G_1(k,t)\, dk\right|^2 dx \\
&\quad \le \int_0^\infty \left(\int_0^1 e^{-\frac{\sqrt{2}}{2}kx} \frac{k}{(k^2+\alpha^2)^{\frac{1}{2}}}\left|\widehat{g}(k^2)\right| dk\right)^2 dx + \int_0^\infty \left(\int_1^\infty e^{-\frac{\sqrt{2}}{2}kx}\left|\widehat{g}(k^2)\right| dk\right)^2 dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left(\int_0^{\frac{\sqrt{2}}{2}} e^{-\xi x} \frac{\sqrt{2}\xi}{(2\xi^2+\alpha^2)^{\frac{1}{2}}} \left|\widehat{g}\left(2\xi^2\right)\right| \sqrt{2}\, d\xi \right)^2 dx \\
&\quad + \int_0^\infty \left(\int_{\frac{\sqrt{2}}{2}}^\infty e^{-\xi x} \left|\widehat{g}\left(2\xi^2\right)\right| \sqrt{2}\, d\xi \right)^2 dx \\
&\lesssim \underbrace{\int_0^{\frac{\sqrt{2}}{2}} \frac{4\xi^2}{2\xi^2+\alpha^2} \left|\widehat{g}\left(2\xi^2\right)\right|^2 d\xi}_{(A)} + \underbrace{\int_{\frac{\sqrt{2}}{2}}^\infty 2\left|\widehat{g}\left(2\xi^2\right)\right|^2 d\xi}_{(B)}, \quad \text{(by Lemma 2.3)}.
\end{aligned}
$$

To calculate (A):

$$
\begin{aligned}
(A) &= \int_0^1 \frac{2\tau}{\tau+\alpha^2} \left|\widehat{g}(\tau)\right|^2 \frac{d\tau}{4\sqrt{\frac{\tau}{2}}}, \quad \left(\text{Let } \tau = 2\xi^2\right) \\
&\simeq \int_0^1 \frac{\tau^{\frac{1}{2}}}{\tau+\alpha^2} \left|\widehat{g}(\tau)\right|^2 d\tau = \int_0^1 \frac{\tau^{\frac{1}{2}}}{\tau+\alpha^2} \left(1+\tau^2\right)^{-\frac{(2s-1)}{4}} \left(1+\tau^2\right)^{\frac{(2s-1)}{4}} \left|\widehat{g}(\tau)\right|^2 d\tau.
\end{aligned}
$$

Let $f(\tau) = (\tau^{1/2}(1+\tau^2)^{-(2s-1)/4})/(\tau+\alpha^2)$, for all $\alpha \ge 1$ and $1/2 < s < 3/2$. Since f is bounded on $[0,1]$, there exists $M > 0$ such that $f(\tau) \le M$, $\forall \tau \in [0,1]$. Therefore, we obtain the following inequality:

$$
\begin{aligned}
(A) &\simeq \int_0^1 \frac{\tau^{\frac{1}{2}}}{\tau+\alpha^2} \left(1+\tau^2\right)^{-\frac{(2s-1)}{4}} \left(1+\tau^2\right)^{\frac{(2s-1)}{4}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\le M \int_0^1 \left(1+\tau^2\right)^{\frac{(2s-1)}{4}} \left|\widehat{g}(\tau)\right|^2 d\tau \lesssim \|g\|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
\tag{16}
$$

To calculate (B):

$$
\begin{aligned}
(B) &= \int_{\frac{\sqrt{2}}{2}}^\infty 2\left|\widehat{g}\left(2\xi^2\right)\right|^2 d\xi \quad \left(\text{let } \tau = 2\xi^2.\right) \\
&= \int_1^\infty 2\left|\widehat{g}(\tau)\right|^2 \frac{d\tau}{4\sqrt{\frac{\tau}{2}}} \lesssim \int_1^\infty \left|\widehat{g}(\tau)\right|^2 d\tau \le \int_1^\infty \left(1+\tau^2\right)^{\frac{2s-1}{4}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\le \|g\|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
\tag{17}
$$

By inequalities (16) and (17), we can derive the following inequality:

$$
\left\|w_1(t)\right\|_{L_x^2(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
\tag{18}
$$

Next, we calculate $\|\partial_x w_1(t)\|^2_{L_x^2(0,\infty)}$. We differentiate the formula (10) and then have the equation

$$
\partial_x w_1(t) = \int_0^\infty (ir_1 k) e^{ir_1 kx} G_1(k,t)\, dk.
$$

Therefore, we obtain the following inequality:

$$
\left\|\partial_x w_1(t)\right\|^2_{L_x^2(0,\infty)} = \int_0^\infty \left|\partial_x w_1(x,t)\right|^2 dx = \int_0^\infty \left| \int_0^\infty (ir_1 k) e^{ir_1 kx} G_1(k,t)\, dk \right|^2 dx
$$

$$
\begin{aligned}
&\le \int_0^\infty \left(\int_0^\infty k e^{-\frac{\sqrt{2}}{2}kx} \left| \frac{k(\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}ki)}{\alpha^2 + \sqrt{2}\alpha k + k^2} \right| \left| \widehat{g}(k^2) \right| dk \right)^2 dx \\
&\le \int_0^\infty \left(\int_0^\infty e^{-sx} |\sqrt{2}s| \left| \widehat{g}(2s^2) \right| \sqrt{2}\, ds \right)^2 dx, \quad \left(\text{Let } s = \frac{k}{\sqrt{2}} \right) \\
&\lesssim \pi \int_0^\infty 4s^2 \left| \widehat{g}(2s^2) \right|^2 ds, \quad (\text{By Lemma 2.3}) \\
&\lesssim \pi \int_{\tau=0}^\infty 2\tau \left| \widehat{g}(\tau) \right|^2 \frac{d\tau}{4\sqrt{\frac{\tau}{2}}}, \quad (\text{Let } \tau = 2s^2) \\
&\lesssim \int_0^\infty \tau^{\frac{1}{2}} \left| \widehat{g}(\tau) \right|^2 d\tau \le \int_0^\infty \left(1+\tau^2\right)^{\frac{2s-1}{4}} \left| \widehat{g}(\tau) \right|^2 d\tau \le \|g\|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \\
&\quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Finally, we arrive at the following inequality:

$$
\left\| \partial_x w_1(t) \right\|_{L_x^2(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}. \tag{19}
$$

Therefore, when $s = 1$, by equations (18) and (19), we obtain the inequality (13)

$$
\sup_{t\in[0,2]} \left\| w_1(t) \right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.
$$

Hence, the inequality (13) holds under Case (I).

Now, we begin to prove that when $s = \beta \neq 0$, then the inequality (13) holds. Case (II): Suppose $s = \beta \neq 0$. This implies $s = \beta \in (1/2, 1)$. Then, the definition of $\|w_1(t)\|_{H_x^s(0,\infty)}$ is

$$
\left\| w_1(t) \right\|_{H_x^s(0,\ell)} = \left\| w_1(t) \right\|_{L_x^2(0,\infty)} + \left\| w_1(t) \right\|_{\beta}.
$$

We need to estimate $\|w_1(t)\|_{L_x^2(0,\infty)}$ and $\|w_1(t)\|_\beta$. By (18), we have the following inequality of the first norm:

$$
\left\| w_1(t) \right\|_{L_x^2(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
$$

Now, we calculate $\|w_1(t)\|_\beta^2$.

$$
\begin{aligned}
\left\| w_1(t) \right\|_\beta^2 &= \int_0^\infty \int_0^\infty \frac{|w_1(x+\zeta,t) - w_1(x,t)|^2}{\zeta^{1+2\beta}}\, d\zeta\, dx \\
&= \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left| \int_0^\infty e^{ir_1 k(x+\zeta)} G_1(k,t)\, dk - \int_0^\infty e^{ir_1 kx} G_1(k,t)\, dk \right|^2 dx\, d\zeta \\
&\le \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_0^\infty \left| e^{ir_1 k(x+\zeta)} - e^{ir_1 kx} \right| \left| G_1(k,t) \right| dk \right)^2 dx\, d\zeta.
\end{aligned}
$$

For the estimation of $\|w_1(x,t)\|_\beta$, we use Lemma 2.2. This gives us the following inequality:

$$
\left\| w_1(t) \right\|_\beta^2
$$

$$
\begin{aligned}
&\le \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_0^\infty \left| e^{ir_1 k(x+\zeta)} - e^{ir_1 kx} \right| \left| G_1(k,t) \right| dk \right)^2 dx\, d\zeta \\
&\le \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_0^\infty 2\sqrt{2} \left| e^{-\frac{\sqrt{2}}{2}k(x+\zeta)} - e^{-\frac{\sqrt{2}}{2}kx} \right| \left| G_1(k,t) \right| dk \right)^2 dx\, d\zeta \\
&\le \int_0^\infty \int_0^\infty \frac{1}{(\sqrt{2}\mathcal{Z})^{1+2\beta}} \left(\int_0^\infty 2\sqrt{2} \left| e^{-\frac{\sqrt{2}}{2}k(\sqrt{2}\mathcal{X}+\sqrt{2}\mathcal{Z})} \right.\right. \\
&\quad \left.\left. - e^{-\frac{\sqrt{2}}{2}k(\sqrt{2}\mathcal{X})} \right| \left| G_1(k,t) \right| dk \right)^2 2\, d\mathcal{X}\, d\mathcal{Z} \\
&\left(\text{Let } \mathcal{X} = \frac{\sqrt{2}}{2}x,\ \mathcal{Z} = \frac{\sqrt{2}}{2}\zeta. \right) \\
&\simeq \int_0^\infty \int_0^\infty \frac{1}{\mathcal{Z}^{1+2\beta}} \left(\int_0^\infty \left| e^{-k(\mathcal{X}+\mathcal{Z})} - e^{-k\mathcal{X}} \right| \left| G_1(k,t) \right| dk \right)^2 d\mathcal{X}\, d\mathcal{Z}.
\end{aligned}
$$

We use Lemma 2.3 to obtain the following inequality:

$$
\begin{aligned}
&\left\| w_1(t) \right\|_\beta^2 \\
&\quad \lesssim \int_0^\infty \int_0^\infty \frac{1}{\mathcal{Z}^{1+2\beta}} \left(\int_0^\infty \left| e^{-k(\mathcal{X}+\mathcal{Z})} - e^{-k\mathcal{X}} \right| \left| G_1(k,t) \right| dk \right)^2 d\mathcal{X}\, d\mathcal{Z} \\
&\quad = \int_0^\infty \frac{1}{\mathcal{Z}^{1+2\beta}} \int_0^\infty \left(\int_0^\infty e^{-k\mathcal{X}} \left(1 - e^{-k\mathcal{Z}}\right) \left| G_1(k,t) \right| dk \right)^2 d\mathcal{X}\, d\mathcal{Z} \\
&\quad \lesssim \int_0^\infty \frac{1}{\mathcal{Z}^{1+2\beta}} \left(\pi \int_{k=0}^\infty \left(1 - e^{-k\mathcal{Z}}\right)^2 \left| G_1(k,t) \right|^2 dk \right) d\mathcal{Z} \\
&\quad \simeq \int_0^\infty \left| G_1(k,t) \right|^2 \left(\int_0^\infty \frac{(1 - e^{-k\mathcal{Z}})^2}{\mathcal{Z}^{1+2\beta}}\, d\mathcal{Z} \right) dk \\
&\quad = \int_0^\infty \left| G_1(k,t) \right|^2 \left(k^{2\beta} \int_0^\infty \frac{(1 - e^{-\xi})^2}{\xi^{1+2\beta}}\, d\xi \right) dk, \quad (\text{Let } \xi = k\mathcal{Z}) \\
&\quad \simeq \int_0^\infty \left| G_1(k,t) \right|^2 k^{2\beta}\, dk \lesssim \int_0^\infty \left| \frac{\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}ki}{\alpha^2 + \sqrt{2}\alpha k + k^2} \right|^2 k^{2+2\beta} \left| \widehat{g}\left(k^2\right) \right|^2 dk \\
&\quad \lesssim \int_0^\infty k^{2\beta} \left| \widehat{g}\left(k^2\right) \right|^2 dk \\
&\quad = \int_0^\infty \tau^\beta \left| \widehat{g}(\tau) \right|^2 \frac{d\tau}{2\sqrt{\tau}}, \quad \left(\text{Let } \tau = k^2\right) \\
&\quad = \frac{1}{2} \int_0^\infty \tau^{\beta - \frac{1}{2}} \left| \widehat{g}(\tau) \right|^2 d\tau \le \int_0^\infty \left(\tau^2\right)^{\frac{\beta - 1/2}{2}} \left| \widehat{g}(\tau) \right|^2 d\tau \\
&\quad \le \int_0^\infty \left(1 + \tau^2\right)^{\frac{2\beta - 1}{4}} \left| \widehat{g}(\tau) \right|^2 d\tau \le \|g\|^2_{H_t^{\frac{2\beta-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s = \beta < 1.
\end{aligned}
$$

Hence, we obtain the following inequality:

$$
\left\| w_1(x,t) \right\|_\beta \lesssim \|g\|_{H_t^{\frac{2\beta-1}{4}}(\mathbb{R})} = \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s = \beta < 1. \tag{20}
$$

Therefore, when $s = \beta \in (1/2, 1)$, by equations (18) and (20), we obtain the inequality (13):

$$\sup_{t\in[0,2]} \left\| w_1(t) \right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.$$

Hence, the inequality (13) holds under Case (II).

Finally, we begin to prove that when $\lfloor s \rfloor \neq 0$ and $\beta \neq 0$, then the inequality (13) holds. Case (III): Suppose $\lfloor s \rfloor \neq 0$ and $\beta \neq 0$. This implies that $s \in (1, 3/2)$ and $\beta \in (0, 1/2)$. Then, the definition of $\|w_1(t)\|_{H_x^s(0,\infty)}$ is

$$\left\| w_1(t) \right\|_{H_x^s(0,\infty)} = \left\| w_1(t) \right\|_{L_x^2(0,\infty)} + \left\| \partial_x w_1(t) \right\|_{L_x^2(0,\infty)} + \left\| \partial_x w_1(t) \right\|_{\beta}.$$

We need to estimate $\|w_1(t)\|_{L_x^2(0,\infty)}$, $\|\partial_x w_1(t)\|_{L_x^2(0,\infty)}$, and $\|\partial_x w_1(t)\|_\beta$. By (18) and (19), we have the following inequalities of the first norm and the second norm:

$$\begin{aligned}
&\left\| w_1(t) \right\|_{L_x^2(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}, \\
&\left\| \partial_x w_1(t) \right\|_{L_x^2(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}$$

Now, we calculate $\|\partial_x^{\lfloor s \rfloor} w_1(t)\|_\beta^2$, for $s \in (1, 3/2)$ and $\beta \in (0, 1/2)$:

$$\begin{aligned}
&\left\| \partial_x^{\lfloor s \rfloor} w_1(t) \right\|_\beta^2 \\
&\quad = \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left| \int_{k=0}^\infty (ir_1 k)^{\lfloor s \rfloor} e^{ir_1 k(x+\zeta)} G_1(k,t)\, dk \right. \\
&\qquad \left. - \int_{k=0}^\infty (ir_1 k)^{\lfloor s \rfloor} e^{ir_1 kx} G_1(k,t)\, dk \right|^2 d\zeta\, dx \\
&\quad = \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left| \int_0^\infty (ir_1 k)^{\lfloor s \rfloor} \left(e^{ir_1 k(x+\zeta)} - e^{ir_1 kx} \right) G_1(k,t)\, dk \right|^2 d\zeta\, dx \\
&\quad \le \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_0^\infty k^{\lfloor s \rfloor} \left| e^{ir_1 k(x+\zeta)} - e^{ir_1 kx} \right| \left| \frac{k(\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}i)}{\alpha^2 + \sqrt{2}\alpha k + k^2} \right| \left| \widehat{g}(k^2) \right| dk \right)^2 d\zeta\, dx \\
&\quad \lesssim \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_{k=0}^\infty k^{\lfloor s \rfloor} \left| e^{-\frac{\sqrt{2}}{2}k(x+\zeta)} \right. \right. \\
&\qquad \left. \left. - e^{-\frac{\sqrt{2}}{2}kx} \right| \left| \frac{k(\alpha + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}i)}{\alpha^2 + \sqrt{2}\alpha k + k^2} \right| \left| \widehat{g}(k^2) \right| dk \right)^2 d\zeta\, dx \quad \text{(by Lemma 2.2)} \\
&\quad \le \int_0^\infty \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\int_0^\infty k^{\lfloor s \rfloor} e^{-\frac{\sqrt{2}}{2}kx} \left(1 - e^{-\frac{\sqrt{2}}{2}k\zeta} \right) \left| \widehat{g}(k^2) \right| dk \right)^2 d\zeta\, dx \\
&\quad = \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \int_0^\infty \left(\int_0^\infty e^{-\ell x} (\sqrt{2}\ell)^{\lfloor s \rfloor} \left(1 - e^{-\ell\zeta} \right) \left| \widehat{g}(2\ell^2) \right| \sqrt{2}\, d\ell \right)^2 dx\, d\zeta, \\
&\left(\text{Let } \ell = \frac{\sqrt{2}}{2}k. \right) \\
&\quad \lesssim \int_0^\infty \frac{1}{\zeta^{1+2\beta}} \left(\pi \int_0^\infty (\sqrt{2}\ell)^{2\lfloor s \rfloor} \left(1 - e^{-\ell\zeta} \right)^2 \cdot 2 \left| \widehat{g}(2\ell^2) \right|^2 d\ell \right) d\zeta, \quad \text{(by Lemma 2.3)}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^\infty \ell^{2\lfloor s\rfloor} \left|\widehat{g}(2\ell^2)\right|^2 \left(\int_0^\infty \frac{(1-e^{-\ell\zeta})^2}{\zeta^{1+2\beta}}\, d\zeta\right) d\ell \\
&= \int_0^\infty \ell^{2\lfloor s\rfloor} \left|\widehat{g}(2\ell^2)\right|^2 \left(\ell^{2\beta}\int_0^\infty \frac{(1-e^{-\xi})^2}{\xi^{1+2\beta}}\, d\xi\right) d\ell,
\end{aligned}
$$

(Let $\xi = \ell\zeta$.)

$$
\begin{aligned}
&\simeq \int_0^\infty \ell^{2\lfloor s\rfloor} \left|\widehat{g}(2\ell^2)\right|^2 \ell^{2\beta}\, d\ell = \int_0^\infty \ell^{2\lfloor s\rfloor+2\beta} \left|\widehat{g}(2\ell^2)\right|^2 d\ell \\
&= \int_0^\infty \left(\frac{\tau}{2}\right)^{\lfloor s\rfloor+\beta} \left|\widehat{g}(\tau)\right|^2 \frac{d\tau}{4\sqrt{\frac{\tau}{2}}}, \quad (\text{Let } \tau = 2\ell^2.) \\
&\lesssim \int_0^\infty \tau^{\lfloor s\rfloor+\beta-\frac{1}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau = \int_0^\infty \left(\tau^2\right)^{\frac{\lfloor s\rfloor+\beta-\frac{1}{2}}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\le \int_{-\infty}^\infty \left(1+\tau^2\right)^{\frac{\lfloor s\rfloor+\beta-\frac{1}{2}}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau = \int_{-\infty}^\infty \left(1+\tau^2\right)^{\frac{2s-1}{4}} \left|\widehat{g}(\tau)\right|^2 d\tau = \|g\|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.
\end{aligned}
$$

Hence, we obtain the following inequality:

$$
\left\|\partial_x^{\lfloor s\rfloor} w_1(t)\right\|_\beta \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } s \in \left(1, \frac{3}{2}\right) \text{ and } \beta \in \left(0, \frac{1}{2}\right). \tag{21}
$$

Therefore, when $s \in (1, 3/2)$ and $\beta \in (0, 1/2)$, by equations (18), (19), and (21), we obtain the inequality (13):

$$
\sup_{t\in[0,2]} \left\|w_1(t)\right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.
$$

Hence, the inequality (13) holds under Case (III).

We establish the validity of the inequality (13) under these three cases. Thus, we derive the space estimate for w_1 (13):

$$
\sup_{t\in[0,2]} \left\|w_1(t)\right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
$$

Additionally, we can use a process similar to the proof of equation (13) to obtain the space estimate for w_2 (14):

$$
\sup_{t\in[0,2]} \left\|w_2(t)\right\|_{H_x^s(0,\infty)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}.
$$

Then, we can obtain the space estimation (11).

Finally, we start with the proof of the space estimate (12). Now, we calculate (9):

$$
\begin{aligned}
w(x,t) &= \int_0^\infty e^{ir_1kx} G_1(k,t)\, dk + \int_0^\infty e^{ir_2kx} G_2(k,t)\, dk \qquad (22) \\
&\simeq \int_0^\infty e^{ia^3\sqrt{\tau}x+i\tau t} \left(\frac{\alpha + \frac{\sqrt{2}}{2}\sqrt{\tau} - \frac{\sqrt{2}}{2}\sqrt{\tau}i}{\alpha^2 + \sqrt{2}\alpha\sqrt{\tau} + \tau}\right) \widehat{g}(\tau)\, d\tau \\
&\quad + \int_0^\infty e^{ia\sqrt{\tau}x-i\tau t} \left(\frac{\alpha + \frac{\sqrt{2}}{2}\sqrt{\tau} + \frac{\sqrt{2}}{2}\sqrt{\tau}i}{\alpha^2 + \sqrt{2}\alpha\sqrt{\tau} + \tau}\right) \widehat{g}(-\tau)\, d\tau, \quad (\text{Let } k = \sqrt{\tau}.)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\tau=0}^{\infty} e^{ia^3\sqrt{\tau}x+i\tau t}\left(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{\tau}-\frac{\sqrt{2}}{2}\sqrt{\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{\tau}+\tau}\right)\widehat{g}(\tau)\,d\tau \\
&\quad + \int_{-\infty}^{0} e^{ia\sqrt{-\tau}x+i\tau t}\left(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{-\tau}+\frac{\sqrt{2}}{2}\sqrt{-\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{-\tau}-\tau}\right)\widehat{g}(\tau)\,d\tau,
\end{aligned}
$$

where $a = e^{i\pi/4}$. Hence, by equation (22), we infer that the temporal Fourier transform of w is given by

$$
\widehat{w}(x,t) \simeq \begin{cases} e^{ia^3\sqrt{\tau}x}(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{\tau}-\frac{\sqrt{2}}{2}\sqrt{\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{\tau}+\tau})\widehat{g}(\tau) & \tau \geq 0, \\ e^{ia\sqrt{-\tau}x}(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{-\tau}+\frac{\sqrt{2}}{2}\sqrt{-\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{-\tau}-\tau})\widehat{g}(\tau) & \tau \leq 0. \end{cases}
$$

Therefore, we obtain the following inequality:

$$
\begin{aligned}
&\left\|w(x)\right\|^2_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left|\widehat{w}(x,\tau)\right|^2 d\tau \\
&\simeq \int_{-\infty}^{0} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left| e^{ia\sqrt{-\tau}x}\left(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{-\tau}+\frac{\sqrt{2}}{2}\sqrt{-\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{-\tau}-\tau}\right)\widehat{g}(\tau)\right|^2 d\tau \\
&\quad + \int_{0}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left| e^{ia^3\sqrt{\tau}x}\left(\frac{\alpha+\frac{\sqrt{2}}{2}\sqrt{\tau}-\frac{\sqrt{2}}{2}\sqrt{\tau}i}{\alpha^2+\sqrt{2}\alpha\sqrt{\tau}+\tau}\right)\widehat{g}(\tau)\right|^2 d\tau \\
&\leq \int_{-\infty}^{0} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{\alpha^2+\sqrt{2}\alpha\sqrt{-\tau}-\tau}{(\alpha^2+\sqrt{2}\alpha\sqrt{-\tau}-\tau)^2} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\quad + \int_{0}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{\alpha^2+\sqrt{2}\alpha\sqrt{\tau}+\tau}{(\alpha^2+\sqrt{2}\alpha\sqrt{\tau}+\tau)^2} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\leq \int_{-\infty}^{0} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{1}{((1+|\tau|)^2)^{\frac{1}{2}}} \left|\widehat{g}(\tau)\right|^2 d\tau + \int_{0}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{1}{((1+\tau)^2)^{\frac{1}{2}}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&= \int_{-\infty}^{0} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{1}{(1+2|\tau|+|\tau|^2)^{\frac{1}{2}}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\quad + \int_{0}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \frac{1}{(1+2\tau+\tau^2)^{\frac{1}{2}}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&\leq \int_{-\infty}^{0} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left(1+\tau^2\right)^{-\frac{1}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau + \int_{0}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left(1+\tau^2\right)^{-\frac{1}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau \\
&= \int_{-\infty}^{\infty} \left(1+\tau^2\right)^{\frac{2s+1}{4}} \left(1+\tau^2\right)^{-\frac{1}{2}} \left|\widehat{g}(\tau)\right|^2 d\tau = \|g\|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.
\end{aligned}
$$

Hence, we obtain the time estimate (12):

$$
\sup_{x\in[0,\infty)} \left\|w(x)\right\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \quad \Rightarrow \quad \sup_{x\in[0,\infty)} \left\|w(x)\right\|_{H_t^{\frac{2s+1}{4}}(0,2)} \lesssim \|g\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.
$$

Thus, we conclude the demonstration of Theorem 2.1. □

2.2 Homogeneous IBVP with zero initial data

In this subsection, we consider the pure IBVP:

$$\begin{cases} z_t - z_{xx} = 0, & x \in (0,\infty), t \in (0,T), \\ z(x,0) = 0, & x \in [0,\infty), \\ z_x(0,t) - \alpha z(0,t) = \varphi(t), & t \in [0,T], \alpha \geq 1, \end{cases} \tag{23}$$

where $0 < T < 1$. We shall extend the boundary data $\varphi(t)$ from the interval $[0,T]$ to the entire real line $\mathbb{R}$. We aim to define a function $\varphi^*(t) \in H_t^{(2s-1)/4}(\mathbb{R})$ as an extension of $\varphi(t) \in H_t^{(2s-1)/4}(0,T)$ with $\mathrm{supp}(\varphi^*) \subset (0,2)$. For $1/2 < s < 3/2$, the definition of φ^* is given by

$$\varphi^*(t) = \begin{cases} E_\theta(t), & t \in (0,2), \\ 0, & t \in (0,2)^c, \end{cases}$$

where $E_\theta = \theta(t)E(t)$, where $\theta \in C_0^\infty(\mathbb{R})$ is a smooth cutoff function satisfying $|\theta(t)| \leq 1$ for all $t \in \mathbb{R}$, $\theta(t) = 1$ for all $|t| \leq 1$, and $\theta(t) = 0$ for all $|t| \geq 2$. Here, $E \in H_t^{(2s-1)/4}(\mathbb{R})$ is an extension of $\varphi \in H_t^{(2s-1)/4}(0,T)$ such that

$$\|E\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \leq D_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}.$$

Consequently, we have $\mathrm{supp}(\varphi^*) \subset (0,2)$ and

$$\|\varphi^*\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \leq D_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}, \tag{24}$$

where $D_s = D(s) > 0$ is a constant depending on s.

Hence, for IBVP (23), we can obtain the following two inequalities (space estimate and time estimate):

$$\begin{aligned} \sup_{t\in[0,T]} \|z(t)\|_{H_x^s(0,\infty)} &\leq C_s \|\varphi^*\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \big(\text{by Theorem 2.1}\big) \\ &\leq \widetilde{C}_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}, \ \big(\text{by (24)}\big) \end{aligned}$$

and

$$\begin{aligned} \sup_{x\in[0,\infty)} \|z(x)\|_{H_t^{\frac{2s+1}{4}}(0,2)} &\leq C_s \|\varphi^*\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}, \quad \big(\text{by Theorem 2.1}\big) \\ &\leq \widetilde{C}_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{for } \frac{1}{2} < s < \frac{3}{2}, \ \big(\text{by (24)}\big), \end{aligned}$$

where $\widetilde{C}_s = C_s D_s$.

Therefore, we have the following result.

Theorem 2.4 *For $1/2 < s < 3/2$ and the boundary data test function $\varphi \in H_t^{(2s-1)/4}(\mathbb{R})$. The solution for the IBVP (23) that satisfies the following Hadamard space:*

$$\textit{space estimate}: \quad \sup_{t\in[0,T]} \|S[0,\varphi;0](t)\|_{H_x^s(0,\infty)} \leq \widetilde{C}_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \tag{25}$$

$$\textit{time estimate:}\quad \sup_{x\in[0,\infty)} \left\| S[0,\varphi;0](x) \right\|_{H_t^{\frac{2s+1}{4}}(0,2)} \le \widetilde{C}_s \|\varphi\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \tag{26}$$

where $\widetilde{C}_s = \widetilde{C}(s) > 0$ *is a constant depending on* s.

3 Proof of forced linear IBVP estimates (Theorem 1.4)

In this section, we aim to prove Theorem 1.4 by decomposing the Robin problem for the forced linear reaction–diffusion equation into four simpler problems. In particular, two of these problems are linear initial value problems (IVPs), and they are estimated directly; their proofs can be found in [8]. The remaining two problems are IBVPs, and their estimates are obtained using the theorems presented in Sect. 2.

3.1 Decomposition into simple problems

To establish the proof of Theorem 1.4, we begin the process by decomposing the forced linear IBVP (3) into the superposition of the following problems.

(I) The homogeneous linear initial value problem:

$$\begin{cases} U_t - U_{xx} = 0, & x \in \mathbb{R}, t \in (0,T), \\ U(x,0) = U_0(x) \in H_x^s(\mathbb{R}), & x \in \mathbb{R}, \end{cases} \tag{27}$$

where $U_0 \in H_x^s(\mathbb{R})$ is an extension of the initial data $u_0 \in H_x^s(0,\infty)$ such that

$$\|U_0\|_{H_x^s(\mathbb{R})} \le 2\|u_0\|_{H_x^s(0,\infty)}, \tag{28}$$

with the solution to IVP (27) given by the Duhamel formula

$$U(x,t) = S[U_0;0](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x-\xi^2 t}\widehat{U_0}(\xi)\,d\xi, \tag{29}$$

where $\widehat{U_0}(\zeta)$ is the Fourier transform with respect to the spatial variable, i.e.,

$$\widehat{U_0}(\xi,t) = \int_{\mathbb{R}} e^{-i\xi x} U_0(x,t)\,dx, \quad \xi \in \mathbb{R}.$$

(II) The forced linear IVP with zero initial condition:

$$\begin{cases} W_t - W_{xx} = F(x,t), & x \in \mathbb{R}, t \in (0,T), \\ W(x,0) = 0, & x \in \mathbb{R}, \end{cases} \tag{30}$$

where $F(x,t) \in C([0,T];H_x^s(\mathbb{R}))$ is the extension of $f(x,t) \in C([0,T];H_x^s(0,\infty))$ and satisfies

$$\sup_{t\in[0,T]} \|F\|_{H_x^s(\mathbb{R})} \le 2 \sup_{t\in[0,T]} \|f\|_{H_x^s(0,\infty)}. \tag{31}$$

The solution to IVP (30) is given by

$$W(x,t) = S[0;F](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_0^t e^{i\xi x-\xi^2(t-t')}\widehat{F}(\xi,t')\,dt' \right) d\xi \tag{32}$$

$$= \int_0^t S\big[F(\cdot,t');0\big](x,t-t')\,dt',$$

which is found by the whole-line Fourier transform

$$\widehat{W}(\xi,t) = \int_{\mathbb{R}} e^{-i\xi x} W(x,t)\,dx, \quad \xi \in \mathbb{R},$$

where $\widehat{F}$ is the Fourier transform of F with respect to x.

(III) The linear IBVP on the half-line:

$$\begin{cases} u_t^{\#} - u_{xx}^{\#} = 0 & x \in (0,\infty), t \in (0,T), \\ u^{\#}(x,0) = 0, & x \in [0,\infty), \\ u_x^{\#}(0,t) - \alpha u^{\#}(0,t) = G_0(t), & t \in [0,T], \alpha \geq 1, \end{cases} \tag{33}$$

where $G_0(t) \doteq g_0(t) - U_x(0,t) - W_x(0,t)$ and $u^{\#} = S[0,G_0;0]$ is the solution (33).

(IV) The homogeneous linear IBVP with zero initial condition:

$$\begin{cases} u_t^{*} - u_{xx}^{*} = 0 & x \in (0,\infty), t \in (0,T), \\ u^{*}(x,0) = 0, & x \in [0,\infty), \\ u_x^{*}(0,t) - \alpha u^{*}(0,t) = H_0(t), & t \in [0,T], \alpha \geq 1, \end{cases} \tag{34}$$

where $H_0(t) \doteq \alpha(U(0,t) + W(0,t))$ and $u^* = S[0,H_0;0]$ is the solution (34).

In summary, the UTM solution (4) of the linear IBVP (3) has been expressed as

$$S[u_0,g_0;f] = S[U_0;0]|_{x>0} + S[0;F]|_{x>0} + S[0,G_0;0] + S[0,H_0;0], \tag{35}$$

where the four quantities on the right-hand side correspond to the solutions of problems (27), (30), (33), and (34), respectively.

3.2 The estimates for the linear IVPs

In this subsection, we prove Theorem 3.3 and apply Theorem 2.1, Theorem 2.4, Theorem 3.1, Theorem 3.2, and Theorem 3.3 to establish Theorem 1.4. Now, applying Theorem 3.1 and Theorem 3.2, we obtain the space and time estimates of $S[U_0;0]|x>0$ and $S[0;F]|x>0$ that are components of (35).

Theorem 3.1 ([8]. Estimates for the homogeneous linear IVP (I)) *The solution* $U = S[U_0;0]$ *of the linear IVP* (27) *given by the formula* (29) *admits the following space and time estimates*:

$$\sup_{t\in[0,T]} \big\| U(t) \big\|_{H_x^s(\mathbb{R})} \leq \|U_0\|_{H_x^s(\mathbb{R})}, \quad \textit{for } s \in \mathbb{R}, \tag{36}$$

$$\sup_{x\in\mathbb{R}} \big\| U(x) \big\|_{H_t^{\frac{2s+1}{4}}(0,T)} \leq C_s \|U_0\|_{H_x^s(\mathbb{R})}, \quad \textit{for } -\frac{1}{2} \leq s < \frac{3}{2}, \tag{37}$$

where $C_s = C(s) > 0$ *is a constant depending on* s.

Theorem 3.2 ([8]. Sobolev-type estimates for the homogeneous linear IVP (II)) *The solution $W = S[0;F]$ of the forced linear IVP (30) given by the formula (32) admits the following space and time estimates*:

$$\sup_{t\in[0,T]} \left\| W(t) \right\|_{H_x^s(\mathbb{R})} \leq T \sup_{t\in[0,T]} \left\| F(t) \right\|_{H_x^s(\mathbb{R})}, \quad \textit{for } s \in \mathbb{R}, \tag{38}$$

$$\sup_{x\in\mathbb{R}} \left\| W(x) \right\|_{H_t^{\frac{2s+1}{4}}(0,T)} \leq C_s\sqrt{T} \sup_{t\in[0,T]} \left\| F(t) \right\|_{H_x^s(\mathbb{R})}, \quad \textit{for } \frac{1}{2} < s < \frac{3}{2}, \tag{39}$$

where $C_s = C(s) > 0$ is a constant depending on s.

The proofs of the above two Theorems are provided in [8].

Thanks to the superposition principle, Theorem 2.1, Theorem 3.1, and Theorem 3.2 can be combined to derive Theorem 1.4 for the forced linear IBVP (3). Furthermore, we consider the time estimates with G_0 and H_0 are in (33) and (34), respectively. We obtain the following two inequalities:

$$\begin{aligned} \|G_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} &= \left\| g_0(t) - U_x(0,t) - W_x(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \\ &\leq \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \left\| U_x(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \left\| W_x(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \end{aligned} \tag{40}$$

and

$$\begin{aligned} \|H_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} &= \left\| \alpha U(0,t) + \alpha W(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \\ &\leq \alpha \left\| U(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \alpha \left\| W(0,t) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)}. \end{aligned} \tag{41}$$

Therefore, we must estimate the $\|U_x(0,t)\|_{H_t^{(2s-1)/4}(0,T)}$ and $\|W_x(0,t)\|_{H_t^{(2s-1)/4}(0,T)}$.

Theorem 3.3 (Sobolev-type estimates) *For $1/2 < s < 3/2$, the solution $U = S[U_0;0]$ of the linear IVP (27) given by the formula (29) and the solution $W = S[0;F]$ of the forced linear IVP (30) given by the formula (32) admit the following estimates*:

$$\sup_{x\in[0,\infty)} \left\| U_x(x) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \leq C_s \|U_0\|_{H_x^s(\mathbb{R})}, \tag{42}$$

$$\sup_{x\in[0,\infty)} \left\| W_x(x) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \leq C_s\sqrt{T} \sup_{t\in[0,T]} \left\| F(t) \right\|_{H_x^s(\mathbb{R})}, \tag{43}$$

where $T \in (0,1)$, and $C_s = C(s) > 0$ is a constant depending on s.

Proof We will use Lemmas 3.4–3.7 to prove Theorem 3.3. In the first step, we will use Lemma 3.4 and Lemma 3.5 to prove (42) of Theorem 3.3. In the second step, we will use Lemma 3.6 and Lemma 3.7 to prove (43) of Theorem 3.3.

For $1/2 < s < 3/2$, we assume $m = (2s-1)/4$, then $0 < m < \frac{1}{2}$. Based on equations (15) and (29), we obtain the following two equations:

$$\left\| U_x(x) \right\|_{H_t^m(0,T)} = \left\| U_x(x) \right\|_{L_t^2(0,T)} + \left\| U_x(x) \right\|_m,$$

$$U_x(x,t) = \frac{1}{2\pi}\int_{\xi\in\mathbb{R}} (i\xi)e^{i\xi x-\xi^2 t}\widehat{U}_0(\xi)\,d\xi,$$

where the fraction norm $\|\cdot\|_m$ is defined by

$$\begin{aligned}\|U_x(x)\|_m^2 &= \int_0^T\int_0^T \frac{|U_x(x,t)-U_x(x,t')|^2}{|t-t'|^{1+2m}}\,dt'\,dt \\ &\simeq \int_0^T\int_0^{T-t}\frac{|U_x(x,t+\zeta)-U_x(x,t)|^2}{\zeta^{1+2m}}\,d\zeta\,dt.\end{aligned}$$

Therefore, we must estimate $\|U_x(x)\|_{L_t^2(0,T)}$ and $\|U_x(x)\|_m$ to obtain (42). In the following lemma, we provide the estimate for $\|U_x(x)\|_{L_t^2(0,T)}$.

Lemma 3.4 *For* $1/2 < s < 3/2$. *The solution* $U = S[U_0;0]$ *of the linear IVP* (27) *given by the formula* (29) *admits the following estimate*:

$$\|U_x(x)\|_{L_t^2(0,T)} \lesssim \|U_0\|_{H_x^s(\mathbb{R})}, \quad \forall x\in[0,\infty). \tag{44}$$

Proof To estimate $\|U_x(x)\|_{L_t^2(0,T)}$, we obtain the following inequality:

$$\begin{aligned}&\|U_x(x)\|_{L_t^2(0,T)}^2 \\ &= \int_0^T\left|\frac{1}{2\pi}\int_{\xi\in\mathbb{R}}(i\xi)e^{i\xi x-\xi^2 t}\widehat{U}_0(\xi)\,d\xi\right|^2 dt \\ &\le \int_0^T\left(\int_{\xi\in\mathbb{R}} e^{-\xi^2 t}|\xi|\left|\widehat{U}_0(\xi)\right|d\xi\right)^2 dt \\ &\le 2\int_0^T\left(\int_{-\infty}^0 e^{-\xi^2 t}|\xi|\left|\widehat{U}_0(\xi)\right|d\xi\right)^2 dt + 2\int_0^T\left(\int_0^\infty e^{-\xi^2 t}|\xi|\left|\widehat{U}_0(\xi)\right|d\xi\right)^2 dt \\ &= 2\int_0^T\left(\int_\infty^0 e^{-\tau t}|\sqrt{\tau}|\left|\widehat{U}_0(-\sqrt{\tau})\right|\frac{d\tau}{-2\sqrt{\tau}}\right)^2 dt \\ &\quad + 2\int_0^T\left(\int_0^\infty e^{-\tau t}\sqrt{\tau}\left|\widehat{U}_0(\sqrt{\tau})\right|\frac{d\tau}{2\sqrt{\tau}}\right)^2 dt \\ &\left(\text{Let } \tau=\xi^2\right) \\ &= \frac{1}{2}\int_0^T\left(\int_0^\infty e^{-\tau t}\left|\widehat{U}_0(-\sqrt{\tau})\right|d\tau\right)^2 dt + \frac{1}{2}\int_0^T\left(\int_0^\infty e^{-\tau t}\left|\widehat{U}_0(\sqrt{\tau})\right|d\tau\right)^2 dt \\ &\lesssim \pi\underbrace{\int_0^\infty\left|\widehat{U}_0(-\sqrt{\tau})\right|^2 d\tau}_{(A)} + \pi\underbrace{\int_0^\infty\left|\widehat{U}_0(\sqrt{\tau})\right|^2 d\tau}_{(B)}, \quad \text{(by Lemma 2.3)}.\end{aligned}$$

Now, we calculate equations (A) and (B) to obtain the following two inequalities:

$$\begin{aligned}(A) &= \int_0^{-\infty}\left|\widehat{U}_0(r)\right|^2 2r\,dr, \quad \left(\text{Let } r=-\sqrt{\tau}\right) \\ &= \int_{-\infty}^0 2|r|\left|\widehat{U}_0(r)\right|^2 dr \le 2\int_{-\infty}^0\left(1+r^2\right)^{\frac{1}{2}}\left|\widehat{U}_0(r)\right|^2 dr \\ &\lesssim \|U_0\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})} \le \|U_0\|^2_{H_x^s(\mathbb{R})}, \quad \forall x\in[0,\infty) \text{ and for } \frac{1}{2}<s<\frac{3}{2}\end{aligned}$$

and

$$
\begin{aligned}
(B) &= \int_0^\infty \left|\widehat{U}_0(r)\right|^2 2r\,dr, \quad (\text{Let } r=\sqrt{\tau}) \\
&\leq 2\int_0^\infty \left(1+r^2\right)^{\frac{1}{2}} \left|\widehat{U}_0(r)\right|^2 dr \lesssim \|U_0\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})} \leq \|U_0\|^2_{H_x^s(\mathbb{R})}, \\
&\qquad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Hence, we obtain the inequality (44):

$$
\left\|U_x(x)\right\|_{L_t^2(0,T)} \lesssim \|U_0\|_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}. \qquad \square
$$

Next, we provide the estimate for $\|U_x(x)\|_m$ in the following lemma.

Lemma 3.5 *For* $1/2 < s < 3/2$ *and* $m = (2s-1)/4$. *The solution* $U = S[U_0; 0]$ *of the linear IVP* (27) *given by the formula* (29) *admits the following estimate,*

$$
\left\|U_x(x)\right\|_m \lesssim \|U_0\|_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty). \tag{45}
$$

Proof To estimate $\|U_x(x,t)\|_m$, we obtain the following inequality:

$$
\begin{aligned}
&\left\|U_x(x,t)\right\|_m^2 \\
&\quad \simeq \int_0^T \int_0^{T-t} \frac{|U_x(x,t+\zeta) - U_x(x,t)|^2}{\zeta^{1+2m}}\,d\zeta\,dt \\
&\quad = \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}} \left|\frac{1}{2\pi}\int_{\xi\in\mathbb{R}} (i\xi)e^{i\xi x}\left(e^{-\xi^2(t+\zeta)} - e^{-\xi^2 t}\right)\widehat{U}_0(\xi)\,d\xi\right|^2 d\zeta\,dt \\
&\quad \leq \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}} \left(\int_{\xi\in\mathbb{R}} |\xi| e^{-\xi^2 t}\left(1 - e^{-\xi^2\zeta}\right)\left|\widehat{U}_0(\xi)\right| d\xi\right)^2 d\zeta\,dt \\
&\quad \leq \int_0^T \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_{-\infty}^0 |\xi| e^{-\xi^2 t}\left(1 - e^{-\xi^2\zeta}\right)\left|\widehat{U}_0(\xi)\right| d\xi \right.\\
&\qquad \left. + \int_0^\infty |\xi| e^{-\xi^2 t}\left(1 - e^{-\xi^2\zeta}\right)\left|\widehat{U}_0(\xi)\right| d\xi\right)^2 dt\,d\zeta \\
&\quad \lesssim 2\int_0^T \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_{-\infty}^0 |\xi| e^{-\xi^2 t}\left(1 - e^{-\xi^2\zeta}\right)\left|\widehat{U}_0(\xi)\right| d\xi\right)^2 dt\,d\zeta \\
&\qquad + 2\int_0^T \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_0^\infty |\xi| e^{-\xi^2 t}\left(1 - e^{-\xi^2\zeta}\right)\left|\widehat{U}_0(\xi)\right| d\xi\right)^2 dt\,d\zeta \\
&\quad = 2\int_0^T \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_\infty^0 e^{-\tau t}|\sqrt{\tau}|\left(1 - e^{-\tau\zeta}\right)\left|\widehat{U}_0(-\sqrt{\tau})\right| \frac{d\tau}{-2\sqrt{\tau}}\right)^2 dt\,d\zeta \\
&\qquad + 2\int_0^T \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_0^\infty e^{-\tau t}|\sqrt{\tau}|\left(1 - e^{-\tau\zeta}\right)\left|\widehat{U}_0(\sqrt{\tau})\right| \frac{d\tau}{2\sqrt{\tau}}\right)^2 dt\,d\zeta, \\
&\qquad \left(\text{let } \tau = \xi^2.\right) \\
&\quad \lesssim \int_0^T \frac{1}{\zeta^{1+2m}} \left(\int_0^\infty \left(1 - e^{-\tau\zeta}\right)^2 \left|\widehat{U}_0(-\sqrt{\tau})\right|^2 d\tau\right) d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T \frac{1}{\zeta^{1+2m}}\left(\int_0^\infty \left(1-e^{-\tau\zeta}\right)^2 \left|\widehat{U}_0(\sqrt{\tau})\right|^2 d\tau\right) d\zeta, \quad \text{(by Lemma 2.3)}\\
&= \int_0^\infty \left|\widehat{U}_0(-\sqrt{\tau})\right|^2 \left(\int_0^T \frac{(1-e^{-\tau\zeta})^2}{\zeta^{1+2m}}\, d\zeta\right) d\tau\\
&\quad + \int_0^\infty \left|\widehat{U}_0(\sqrt{\tau})\right|^2 \left(\int_0^T \frac{(1-e^{-\tau\zeta})^2}{\zeta^{1+2m}}\, d\zeta\right) d\tau\\
&\le \int_0^\infty \left|\widehat{U}_0(-\sqrt{\tau})\right|^2 \left(\tau^{2m}\int_0^\infty \frac{(1-e^{-\eta})^2}{\eta^{1+2m}}\, d\eta\right) d\tau\\
&\quad + \int_0^\infty \left|\widehat{U}_0(\sqrt{\tau})\right|^2 \left(\tau^{2m}\int_0^\infty \frac{(1-e^{-\eta})^2}{\eta^{1+2m}}\, d\eta\right) d\tau\\
&\quad \text{(let } \eta = \tau\zeta.)\\
&\simeq \underbrace{\int_0^\infty \left|\widehat{U}_0(-\sqrt{\tau})\right|^2 \tau^{2m}\, d\tau}_{(C)} + \underbrace{\int_0^\infty \tau^{2m}\left|\widehat{U}_0(\sqrt{\tau})\right|^2 d\tau}_{(D)}.
\end{aligned}
$$

Now, we calculate equations (C) and (D) to obtain the following two inequalities:

$$
\begin{aligned}
(C) &= \int_0^{-\infty} r^{4m}\left|\widehat{U}_0(r)\right|^2 2r\, dr, \quad \text{(let } r = -\sqrt{\tau}.)\\
&= \int_{-\infty}^0 r^{4m}\left|\widehat{U}_0(r)\right|^2 \cdot 2|r|\, dr = 2\int_{-\infty}^0 \left(|r|^2\right)^{2m+\frac{1}{2}} \left|\widehat{U}_0(r)\right|^2 dr\\
&\lesssim \int_{-\infty}^\infty \left(1+r^2\right)^{2m+\frac{1}{2}} \left|\widehat{U}_0(r)\right|^2 dr = \int_{-\infty}^\infty \left(1+r^2\right)^s \left|\widehat{U}_0(r)\right|^2 dr\\
&= \|U_0\|^2_{H^s_x(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
(D) &= 2\int_0^\infty \left(r^2\right)^{2m+\frac{1}{2}} \left|\widehat{U}_0(r)\right|^2 dr, \quad \text{(let } r = \sqrt{\tau}.)\\
&\lesssim \int_{-\infty}^\infty \left(1+r^2\right)^{2m+\frac{1}{2}} \left|\widehat{U}_0(r)\right|^2 dr = \int_{-\infty}^\infty \left(1+r^2\right)^s \left|\widehat{U}_0(r)\right|^2 dr\\
&= \|U_0\|^2_{H^s_x(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Therefore, we obtain the inequality (45):

$$
\begin{aligned}
\left\|U_x(x)\right\|_m^2 &\lesssim \|U_0\|^2_{H^s_x(\mathbb{R})} + \|U_0\|^2_{H^s_x(\mathbb{R})}\\
&= 2\|U_0\|^2_{H^s_x(\mathbb{R})} \lesssim \|U_0\|^2_{H^s_x(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}. \qquad \square
\end{aligned}
$$

Now, we can prove (42) of Theorem 3.3. By Lemma 3.4 and Lemma 3.5, we obtain the following inequality:

$$
\begin{aligned}
\left\|U_x(x)\right\|_{H^m_t(0,T)} &= \left\|U_x(x)\right\|_{L^2_t(0,T)} + \left\|U_x(x)\right\|_m\\
&\lesssim \|U_0\|_{H^s_x(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Hence, we obtain the inequality (42):

$$\sup_{x\in[0,\infty)} \left\| U_x(x) \right\|_{H_t^{\frac{2s-1}{4}}(0,T)} \lesssim \| U_0 \|_{H_x^s(\mathbb{R})}.$$

Next, we will use Lemma 3.6 and Lemma 3.7 to prove (43) of Theorem 3.3.

For $1/2 < s < 3/2$, we assume $m = (2s-1)/4$, then $0 < m < \frac{1}{2}$. Based on equations (15) and (32), we obtain that

$$\begin{aligned}
&\left\| W_x(x) \right\|_{H_t^m(0,T)} = \left\| W_x(x) \right\|_{L_t^2(0,T)} + \left\| W_x(x) \right\|_m, \\
&W_x(x,t) = \frac{1}{2\pi} \int_{\xi\in\mathbb{R}} \int_0^t (i\xi) e^{i\xi x - \xi^2(t-t')} \widehat{F}(\xi,t')\, dt'\, d\xi,
\end{aligned}$$

where the fraction norm $\|\cdot\|_m$ is defined by

$$\begin{aligned}
\left\| W_x(x) \right\|_m^2 &= \int_0^T \int_0^T \frac{|W_x(x,t) - W_x(x,t')|^2}{|t-t'|^{1+2m}}\, dt'\, dt \\
&\simeq \int_0^T \int_0^{T-t} \frac{|W_x(x,t+\zeta) - W_x(x,t)|^2}{\zeta^{1+2m}}\, d\zeta\, dt.
\end{aligned}$$

Therefore, we must estimate $\| W_x(x) \|_{L_t^2(0,T)}$ and $\| W_x(x) \|_m$ to obtain (43). In the following lemma, we provide the estimate for $\| W_x(x) \|_{L_t^2(0,T)}$.

Lemma 3.6 *For $1/2 < s < 3/2$. The solution $W = S[0;F]$ of the forced linear IVP (30) given by the formula (32) admits the following estimate*:

$$\left\| W_x(x,t) \right\|_{L_t^2(0,T)} \lesssim 2T \Big(\sup_{t\in[0,T]} \left\| F(t) \right\|_{H_x^s(\mathbb{R})} \Big), \quad \forall x \in [0,\infty). \tag{46}$$

Proof To estimate $\| W_x(x) \|_{L_t^2(0,T)}$, we obtain the following inequality:

$$\begin{aligned}
&\left\| W_x(x) \right\|_{L_t^2(0,T)}^2 \\
&\quad = \int_0^T \left| \frac{1}{2\pi} \int_{\xi\in\mathbb{R}} \int_0^t (i\xi) e^{i\xi x - \xi^2(t-t')} \widehat{F}(\xi,t')\, dt'\, d\xi \right|^2 dt \\
&\quad \lesssim \int_0^T \left(\int_{\xi\in\mathbb{R}} \int_0^t |\xi| e^{-\xi^2(t-t')} \left| \widehat{F}(\xi,t') \right| dt'\, d\xi \right)^2 dt \\
&\quad = \int_0^T \left(\int_0^t \int_{\xi\in\mathbb{R}} |\xi| e^{-\xi^2(t-t')} \left| \widehat{F}(\xi,t') \right| d\xi\, dt' \right)^2 dt \\
&\quad \leq \int_0^T \left(\int_0^t \left[\int_{\xi\in\mathbb{R}} \left(1+|\xi|^2\right)^{\frac{1}{4}} \left| \widehat{F}(\xi,t') \right| |\xi|^{\frac{1}{2}} e^{-\xi^2(t-t')}\, d\xi \right] dt' \right)^2 dt \\
&\quad \leq \int_0^T \left(\int_0^t \left[\int_{\xi\in\mathbb{R}} \left(1+|\xi|^2\right)^{\frac{1}{2}} \left| \widehat{F}(\xi,t') \right|^2 d\xi \right]^{\frac{1}{2}} \left[\int_{\xi\in\mathbb{R}} |\xi| e^{-2\xi^2(t-t')}\, d\xi \right]^{\frac{1}{2}} dt' \right)^2 dt \\
&\quad = \int_0^T \left(\int_0^t \left\| F(t') \right\|_{H_x^{\frac{1}{2}}(\mathbb{R})} \left[\int_{\xi\in\mathbb{R}} |\xi| e^{-2\xi^2(t-t')}\, d\xi \right]^{\frac{1}{2}} dt' \right)^2 dt
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^T \Big(\sup_{t'\in[0,T]} \|F(t')\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\Big)\bigg(\int_0^t \bigg[\int_{\xi\in\mathbb{R}} |\xi| e^{-2\xi^2(t-t')}\,d\xi\bigg]^{\frac{1}{2}} dt'\bigg)^2 dt \\
&= \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t \bigg[\int_{\xi\in\mathbb{R}} |\xi| e^{-2\xi^2(t-t')}\,d\xi\bigg]^{\frac{1}{2}} dt'\bigg)^2 dt\bigg) \\
&= 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t \bigg[\int_0^\infty \xi e^{-2\xi^2(t-t')}\,d\xi\bigg]^{\frac{1}{2}} dt'\bigg)^2 dt\bigg) \\
&= 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t \bigg[\int_0^\infty e^{-2r(t-t')}\frac{1}{2}\,dr\bigg]^{\frac{1}{2}} dt'\bigg)^2 dt\bigg), \big(\text{Let } r = \xi^2\big) \\
&= 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t \bigg[\frac{1}{2}\frac{1}{-2(t-t')}e^{-2r(t-t')}\big|_0^\infty\bigg]^{\frac{1}{2}} dt'\bigg)^2 dt\bigg) \\
&= 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t \bigg(\frac{1}{4(t-t')}\bigg)^{\frac{1}{2}} dt'\bigg)^2 dt\bigg) \\
&\simeq 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T \bigg(\int_0^t (t-t')^{-\frac{1}{2}}\,dt'\bigg)^2 dt\bigg) \\
&= 2\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\bigg(\int_0^T 4t\,dt\bigg) = 4T^2\Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})}\Big) \\
&\le 4T^2\Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^{s}(\mathbb{R})}\Big), \quad \forall x\in[0,\infty) \text{ and for } \frac{1}{2}<s<\frac{3}{2}.
\end{aligned}$$

Hence, we obtain the inequality (46):

$$\|W_x(x,t)\|_{L_t^2(0,T)} \lesssim 2T\Big(\sup_{t\in[0,T]} \|F(t)\|_{H_x^s(\mathbb{R})}\Big), \quad \forall x\in[0,\infty) \text{ and for } \frac{1}{2}<s<\frac{3}{2}. \qquad \square$$

Next, we provide the estimate for $\|W_x(x)\|_m$ in the following lemma.

Lemma 3.7 *For* $1/2 < s < 3/2$ *and* $m = (2s-1)/4$. *The solution* $W = S[0;F]$ *of the forced linear IVP* (30) *given by the formula* (32) *admits the following estimate*:

$$\|W_x(x)\|_m \lesssim \sqrt{T}\sup_{t\in[0,T]} \|F(t)\|_{H_x^s(\mathbb{R})}, \quad \forall x\in[0,\infty). \tag{47}$$

Proof To estimate $\|W_x(x,t)\|_m$, we obtain the following inequality:

$$\begin{aligned}
&\|W_x(x,t)\|_m^2 \\
&\simeq \int_0^T \int_0^{T-t} \frac{|W_x(x,t+\zeta)-W_x(x,t)|^2}{\zeta^{1+2m}}\,d\zeta\,dt \\
&= \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}}\bigg|\frac{1}{2\pi}\int_{\xi\in\mathbb{R}}\int_0^{t+\xi} (i\xi)e^{i\xi x-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\,d\xi \\
&\quad - \frac{1}{2\pi}\int_{\xi\in\mathbb{R}}\int_0^t (i\xi)e^{i\xi x-\xi^2(t-t')}\widehat{F}(\xi,t')\,dt'\,d\xi\bigg|^2 d\zeta\,dt \\
&= \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}}\bigg|\frac{1}{2\pi}\int_{\xi\in\mathbb{R}}\int_0^t (i\xi)e^{i\xi x}\big(e^{-\xi^2(t+\zeta-t')}-e^{-\xi^2(t-t')}\big)\widehat{F}(\xi,t')\,dt'\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\int_{\xi\in\mathbb{R}}\int_{t'=t}^{t+\zeta}(i\xi)e^{i\xi x-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\,d\xi\bigg|^2 d\zeta\,dt\\
&\lesssim \underbrace{\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg|\int_0^t\int_{\xi\in\mathbb{R}}(i\xi)e^{i\xi x}\cdot e^{-\xi^2(t-t')}\big(e^{-\xi^2\zeta}-1\big)\widehat{F}(\xi,t')\,d\xi\,dt'\bigg|^2 d\zeta\,dt}_{(E)}\\
&\quad+\underbrace{\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg|\int_t^{t+\zeta}\int_{\xi\in\mathbb{R}}(i\xi)e^{i\xi x-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,d\xi\,dt'\bigg|^2 d\zeta\,dt}_{(F)}.
\end{aligned}
$$

Now, we estimate (F) to obtain the following inequality:

$$
\begin{aligned}
(F)&=\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg|\int_t^{t+\zeta}\int_{\xi\in\mathbb{R}}(i\xi)e^{i\xi x-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,d\xi\,dt'\bigg|^2 d\zeta\,dt \qquad (48)\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg|\int_{\xi\in\mathbb{R}}e^{i\xi x}\bigg(\int_t^{t+\zeta}(i\xi)e^{-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\bigg)d\xi\bigg|^2 d\zeta\,dt\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg\|\int_{\xi\in\mathbb{R}}e^{i\xi x}\bigg(\int_t^{t+\zeta}(i\xi)e^{-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\bigg)d\xi\bigg\|^2_{L_x^\infty(\mathbb{R})} d\zeta\,dt\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg\|\int_{\xi\in\mathbb{R}}e^{i\xi x}\bigg(\int_t^{t+\zeta}(i\xi)e^{-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\bigg)d\xi\bigg\|^2_{H_x^s(\mathbb{R})} d\zeta\,dt\\
&\quad\bigg(\text{since } s>\frac{1}{2}, \text{ by the Sobolev Embedding Theorem}\bigg)\\
&\lesssim\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg(\int_{\xi\in\mathbb{R}}\big(1+\xi^2\big)^s\bigg|\int_t^{t+\zeta}(i\xi)e^{-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,dt'\bigg|^2 d\xi\bigg)d\zeta\,dt\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg(\int_t^{t+\zeta}\bigg[\int_{\xi\in\mathbb{R}}\big(1+\xi^2\big)^s\xi^2e^{-2\xi^2(t+\zeta-t')}\big|\widehat{F}(\xi,t')\big|^2 d\xi\bigg]^{\frac{1}{2}}dt'\bigg)^2 d\zeta\,dt\\
&\quad\big(\text{by Minkowski's Integral Inequality}\big).
\end{aligned}
$$

To estimate the above inequality, we consider the function $p(\xi)=\xi^2e^{-2\xi^2(t+\zeta-t')}$ for $\zeta\ge 0$, $t'\ge 0$, and $t\ge t'$. We know that $p(\xi)$ has a maximum as $\xi=(\frac{1}{2(t+\zeta-t')})^{\frac{1}{2}}$. Hence, the function $p(\xi)$ satisfies the following inequality:

$$
p(\xi)\le\frac{1}{2(t+\zeta-t')}\cdot\frac{1}{e},\quad \forall\xi\in\mathbb{R}. \qquad (49)
$$

We substitute inequality (49) into inequality (48). We obtain the following inequality:

$$
\begin{aligned}
(F)&=\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg|\int_t^{t+\zeta}\int_{\xi\in\mathbb{R}}(i\xi)e^{i\xi x-\xi^2(t+\zeta-t')}\widehat{F}(\xi,t')\,d\xi\,dt'\bigg|^2 d\zeta\,dt\\
&\lesssim\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg(\int_t^{t+\zeta}\bigg[\int_{\xi\in\mathbb{R}}\big(1+\xi^2\big)^s p(\xi)\big|\widehat{F}(\xi,t')\big|^2 d\xi\bigg]^{\frac{1}{2}}dt'\bigg)^2 d\zeta\,dt\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg(\int_t^{t+\zeta}\bigg[\int_{\xi\in\mathbb{R}}\big(1+\xi^2\big)^s\frac{1}{e}\frac{1}{2(t+\zeta-t')}\big|\widehat{F}(\xi,t')\big|^2 d\xi\bigg]^{\frac{1}{2}}dt'\bigg)^2 d\zeta\,dt\\
&\le\int_0^T\int_0^{T-t}\frac{1}{\zeta^{1+2m}}\bigg(\int_{t'=t}^{t+\zeta}\big(t+\zeta-t'\big)^{-\frac{1}{2}}\big\|F(t')\big\|_{H_x^s(\mathbb{R})}dt'\bigg)^2 d\zeta\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}} \Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) \Big(\int_t^{t+\zeta} (t+\zeta-t')^{-\frac{1}{2}}\,dt'\Big)^2 d\zeta\,dt \\
&= \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}} \Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) 4\zeta\,d\zeta\,dt \\
&= \Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) \int_0^T \int_0^{T-t} 4\zeta^{-2m}\,d\zeta\,dt \\
&= 4\Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) \Big(\int_0^T \Big(\frac{\zeta^{1-2m}}{1-2m}\Big|_0^{T-t}\Big) dt\Big) \\
&= 4\Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) \Big(\int_0^T \frac{(T-t)^{1-2m}}{1-2m}\,dt\Big) \\
&= 4\Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big) \frac{T^{2-2m}}{(1-2m)(2-2m)}.
\end{aligned}
$$

Therefore, we obtain the following inequality:

$$
(F) \lesssim \frac{4T^{2-2m}}{(1-2m)(2-2m)} \Big(\sup_{t\in[0,T]} \|F(t)\|^2_{H^s_x(\mathbb{R})}\Big), \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}. \tag{50}
$$

Next, we estimate (E) to obtain the following inequality:

$$
\begin{aligned}
(E) &= \int_0^T \int_0^{T-t} \frac{1}{\zeta^{1+2m}} \Big| \int_0^t \int_{\mathbb{R}} (i\xi) e^{i\xi x} \cdot e^{-\xi^2(t-t')} \big(e^{-\xi^2\zeta} - 1\big) \widehat{F}(\xi,t')\,d\xi\,dt' \Big|^2 d\zeta\,dt \\
&\le \int_0^T \Big(\int_0^t \int_{\mathbb{R}} \Big[\int_0^{T-t} \frac{1}{\zeta^{1+2m}} \big|(i\xi) e^{i\xi x} \cdot e^{-\xi^2(t-t')} \big(e^{-\xi^2\zeta}-1\big)\widehat{F}(\xi,t')\big|^2 d\zeta\Big]^{\frac{1}{2}} d\xi\,dt'\Big)^2 dt \\
&\quad \big(\text{by Minkowski's integral inequality}\big) \\
&= \int_0^T \Big(\int_0^t \int_{\mathbb{R}} \Big[\int_0^{T-t} \frac{1}{\zeta^{1+2m}} \xi^2 e^{-2\xi^2(t-t')} \big(1-e^{-\xi^2\zeta}\big)^2 \big|\widehat{F}(\xi,t')\big|^2 d\zeta\Big]^{\frac{1}{2}} d\xi\,dt'\Big)^2 dt \\
&\le \int_0^T \Big(\int_0^t \int_{\mathbb{R}} |\xi| e^{-\xi^2(t-t')} \big|\widehat{F}(\xi,t')\big| \Big(\int_0^{T-t} \frac{(1-e^{-\xi^2\zeta})^2}{\zeta^{1+2m}} d\zeta\Big)^{\frac{1}{2}} d\xi\,dt'\Big)^2 dt \\
&\le \int_0^T \Big(\int_0^t \int_{\mathbb{R}} |\xi| e^{-\xi^2(t-t')} \big|\widehat{F}(\xi,t')\big| \Big(\xi^{4m} \int_{\eta=0}^{\infty} \frac{(1-e^{-\eta})^2}{\eta^{1+2m}} d\eta\Big)^{\frac{1}{2}} d\xi\,dt'\Big)^2 dt \\
&\simeq \int_0^T \Big(\int_0^t \int_{\mathbb{R}} |\xi| e^{-\xi^2(t-t')} \big|\widehat{F}(\xi,t')\big| \xi^{2m}\,d\xi\,dt'\Big)^2 dt \\
&= \int_0^T \Big(\int_0^t \int_{\mathbb{R}} e^{-\xi^2(t-t')} |\xi|^{\frac{2}{3}} \cdot |\xi|^{2m+1-\frac{2}{3}} \big|\widehat{F}(\xi,t')\big|\,d\xi\,dt'\Big)^2 dt \\
&\le \int_0^T \Big(\int_0^t \Big[\int_{\mathbb{R}} e^{-2\xi^2(t-t')} |\xi|^{\frac{4}{3}} d\xi\Big]^{\frac{1}{2}} \Big[\int_{\mathbb{R}} |\xi|^{4m+\frac{2}{3}} \big|\widehat{F}(\xi,t')\big|^2 d\xi\Big]^{\frac{1}{2}} dt'\Big)^2 dt \\
&= \int_0^T \Big(\int_0^t \sqrt{2}\Big[\int_0^{\infty} e^{-2\xi^2(t-t')} \xi^{\frac{4}{3}} d\xi\Big]^{\frac{1}{2}} \Big[\int_{\mathbb{R}} |\xi|^{4m+\frac{2}{3}} \big|\widehat{F}(\xi,t')\big|^2 d\xi\Big]^{\frac{1}{2}} dt'\Big)^2 dt \\
&= \int_0^T \Big(\int_0^t \sqrt{2}\Big[\int_0^{\infty} e^{-2r^2} r^{\frac{4}{3}} (t-t')^{-\frac{2}{3}} (t-t')^{-\frac{1}{2}}\,dr\Big]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\int_{\mathbb{R}} (\xi^2)^{2m+\frac{1}{3}} \left|\widehat{F}(\xi,t')\right|^2 d\xi\right]^{\frac{1}{2}} dt'\Bigg)^2 dt \\
&\left(\text{let } r = \sqrt{t-t'}\xi.\right) \\
\leq& \int_0^T \Bigg(\int_0^t \sqrt{2}\left[\int_0^\infty r^{\frac{4}{3}} (t-t')^{-\frac{7}{6}} e^{-2r^2} dr\right]^{\frac{1}{2}} \\
&\times \left[\int_{-\infty}^{\infty} (1+\xi^2)^{2m+\frac{1}{3}} \left|\widehat{F}(\xi,t')\right|^2 d\xi\right]^{\frac{1}{2}} dt'\Bigg)^2 dt \\
\leq& \int_0^T \left(\int_0^t \sqrt{2}\left[\int_0^\infty r^{\frac{4}{3}} (t-t')^{-\frac{7}{6}} e^{-2r^2} dr\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} (1+\xi^2)^s \left|\widehat{F}(\xi,t')\right|^2 d\xi\right]^{\frac{1}{2}} dt'\right)^2 dt \\
=& \int_0^T \left(\int_0^t \sqrt{2}(t-t')^{-\frac{7}{12}} \left[\int_0^\infty r^{\frac{4}{3}} e^{-2r^2} dr\right]^{\frac{1}{2}} \|F(t')\|_{H_x^s(\mathbb{R})} dt'\right)^2 dt \\
=& \int_0^T \left(\int_0^t \sqrt{2}(t-t')^{-\frac{7}{12}} \left(\Gamma\left(\frac{7}{6}\right)\right)^{\frac{1}{2}} \|F(t')\|_{H_x^s(\mathbb{R})} dt'\right)^2 dt,
\end{aligned}
$$

we continue the estimation of the above inequality, and we obtain the following inequality:

$$
\begin{aligned}
(E) &\lesssim \int_0^T \left(\int_0^t \sqrt{2}(t-t')^{-\frac{7}{12}} \left(\Gamma\left(\frac{7}{6}\right)\right)^{\frac{1}{2}} \|F(t')\|_{H_x^s(\mathbb{R})} dt'\right)^2 dt \\
&\lesssim \int_0^T \left(\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})}\right) \left(\int_0^t (t-t')^{-\frac{7}{12}} dt'\right)^2 dt \\
&= \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})} \int_0^T \left(\int_0^t (t-t')^{-\frac{7}{12}} dt'\right)^2 dt \\
&= \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})} \left(\int_0^T \left(\frac{12}{5} t^{\frac{5}{12}}\right)^2 dt\right) = \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})} \left(\int_0^T \frac{144}{25} t^{\frac{5}{6}} dt\right) \\
&= \frac{864}{275} T^{\frac{11}{6}} \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Therefore, we obtain the following inequality:

$$
(E) \lesssim \frac{864}{275} T^{\frac{11}{6}} \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}. \tag{51}
$$

By (50) and (51), we derive the following inequality:

$$
\begin{aligned}
\|W_x(x)\|_m^2 &\lesssim \frac{864}{275} T^{\frac{11}{6}} \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})} + \frac{4T^{2-2m}}{(1-2m)(2-2m)} \left(\sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})}\right) \\
&\lesssim T \sup_{t\in[0,T]} \|F(t)\|^2_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty) \text{ and for } \frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

Hence, we obtain the inequality (47):

$$
\|W_x(x)\|_m \lesssim \sqrt{T} \sup_{t\in[0,T]} \|F(t)\|_{H_x^s(\mathbb{R})}, \quad \forall x \in [0,\infty).
$$

□

Now, we can prove (43) of Theorem 3.3. By Lemma 3.6 and Lemma 3.7, we obtain the following inequality:

$$\begin{aligned}\|W_x(x)\|_{H_t^m(0,T)} &= \|W_x(x)\|_{L_t^2(0,T)} + \|W_x(x)\|_m \\ &\lesssim 2T\Big(\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})}\Big) + \sqrt{T}\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})} \\ &\lesssim \sqrt{T}\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})}, \quad \forall x\in[0,\infty).\end{aligned}$$

Therefore, we obtain the inequality (43):

$$\sup_{x\in[0,\infty)}\|W_x(x)\|_{H_t^m(0,T)} \lesssim \sqrt{T}\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})}, \quad \forall x\in[0,\infty).$$

Hence, we finish the proof of Theorem 3.3. □

Now, we begin the proof of Theorem 1.4. For $1/2 < s < 3/2$, to prove Theorem 1.4, we need to estimate the equation (35):

$$S[u_0,g_0;f] = S[U_0;0]|_{x>0} + S[0;F]|_{x>0} + S[0,G_0;0] + S[0,H_0;0],$$

to obtain its space and time estimates.

First, we commence the derivation of space estimates for (35). For $\frac{1}{2} < s < \frac{3}{2}$ and $T \in (0,1)$, we obtain the following two space estimates:

$$\begin{aligned}&\sup_{t\in[0,T]}\|S[0,G_0;0](t)\|_{H_x^s(0,\infty)} \\ &\quad\le \widetilde{C}_s\|G_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{(by Theorem 2.4 and (25))} \\ &\quad\le \widetilde{C}_s\big(\|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|U_x(0,t)\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|W_x(0,t)\|_{H_t^{\frac{2s-1}{4}}(0,T)}\big), \quad \text{(by (40))} \\ &\quad\lesssim \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|U_0\|_{H_x^s(\mathbb{R})} + \sqrt{T}\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})}, \quad \text{(by (42) and (43))} \\ &\quad\lesssim \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|u_0\|_{H_x^s(0,\infty)} + \sqrt{T}\sup_{t\in[0,T]}\|f(t)\|_{H_x^s(0,\infty)}, \quad \text{(by (28) and (31))}\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in[0,T]}\|S[0,H_0;0](t)\|_{H_x^s(0,\infty)} \\ &\quad\le \widetilde{C}_s\|H_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{(by Theorem 2.4 and (25))} \\ &\quad\lesssim \|U(0,t)\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \|W(0,t)\|_{H_t^{\frac{2s-1}{4}}(0,T)}, \quad \text{(by (41))} \\ &\quad\le \|U(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)} + \|W(0,t)\|_{H_t^{\frac{2s+1}{4}}(0,T)} \\ &\quad\lesssim \|U_0\|_{H_x^s(\mathbb{R})} + \sqrt{T}\sup_{t\in[0,T]}\|F(t)\|_{H_x^s(\mathbb{R})}, \quad \text{(by (37) and (39))} \\ &\quad\lesssim \|u_0\|_{H_x^s(0,\infty)} + \sqrt{T}\sup_{t\in[0,T]}\|f(t)\|_{H_x^s(0,\infty)}, \quad \text{(by (28) and (31))}.\end{aligned}$$

Hence, we obtain the following two space estimates:

$$\begin{aligned}&\sup_{t\in[0,T]}\big\|S[0,G_0;0](t)\big\|_{H_x^s(0,\infty)}\\&\quad\lesssim\|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\|u_0\|_{H_x^s(0,\infty)}+\sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}\end{aligned}\tag{52}$$

and

$$\sup_{t\in[0,T]}\big\|S[0,H_0;0](t)\big\|_{H_x^s(0,\infty)}\lesssim\|u_0\|_{H_x^s(0,\infty)}+\sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}.\tag{53}$$

Now, we estimate $\|S[u_0,g_0;f](t)\|_{H_x^s(0,\infty)}$ to obtain the following inequality:

$$\begin{aligned}&\big\|S[u_0,g_0;f](t)\big\|_{H_x^s(0,\infty)}\\&\quad=\big\|S[U_0;0]|_{x>0}+S[0;F]|_{x>0}+S[0,G_0;0]+S[0,H_0;0]\big\|_{H_x^s(0,\infty)}\\&\quad\le\big\|S[U_0;0]|_{x>0}\big\|_{H_x^s(0,\infty)}+\big\|S[0;F]|_{x>0}\big\|_{H_x^s(0,\infty)}\\&\qquad+\big\|S[0,G_0;0]\big\|_{H_x^s(0,\infty)}+\big\|S[0,H_0;0]\big\|_{H_x^s(0,\infty)}\\&\quad\lesssim\|U_0\|_{H_x^s(\mathbb{R})}+T\sup_{t\in[0,T]}\big\|F(t)\big\|_{H_x^s(\mathbb{R})}+\|G_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\|H_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}\\&\big(\text{by (25), (36), and (38)}\big)\\&\quad\lesssim\|u_0\|_{H_x^s(0,\infty)}+\|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}\\&\big(\text{by (28), (31), (52), and (53)}\big).\end{aligned}$$

Hence, we obtain the following space estimates of (35):

$$\big\|S[u_0,g_0;f](t)\big\|_{H_x^s(0,\infty)}\lesssim\|u_0\|_{H_x^s(0,\infty)}+\|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}.\tag{54}$$

Finally, we commence the derivation of time estimates for (35). For $1/2<s<3/2$ and $T\in(0,1)$, we estimate $\|S[u_0,g_0;f](t)\|_{H_x^s(0,\infty)}$ to obtain the following inequality:

$$\begin{aligned}&\big\|S[u_0,g_0;f](t)\big\|_{H_t^{\frac{2s+1}{4}}(0,T)}\\&\quad=\big\|S[U_0;0]|_{x>0}+S[0;F]|_{x>0}+S[0,G_0;0]+S[0,H_0;0]\big\|_{H_t^{\frac{2s-1}{4}}(0,T)}\\&\quad\le\big\|S[U_0;0]|_{x>0}\big\|_{H_t^{\frac{2s+1}{4}}(0,T)}+\big\|S[0;F]|_{x>0}\big\|_{H_t^{\frac{2s+1}{4}}(0,T)}\\&\qquad+\big\|S[0,G_0;0]\big\|_{H_t^{\frac{2s+1}{4}}(0,T)}+\big\|S[0,H_0;0]\big\|_{H_t^{\frac{2s+1}{4}}(0,T)}\\&\quad\lesssim\|U_0\|_{H_x^s(\mathbb{R})}+\sqrt{T}\sup_{t\in[0,T]}\big\|F(t)\big\|_{H_x^s(\mathbb{R})}+\|G_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\|H_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}\\&\big(\text{by (26), (37), and (39)}\big)\\&\quad\lesssim\|u_0\|_{H_x^s(0,\infty)}+\|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)}+\sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}\end{aligned}$$

$\big($by (28), (31), (52), and (53)$\big)$.

Hence, we obtain the following time estimates of (35):

$$\big\|S[u_0,g_0;f](t)\big\|_{H_t^{\frac{2s+1}{4}}(0,T)} \lesssim \|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}. \tag{55}$$

Accoring to (54) and (55), we obtain the inequality (5):

$$\begin{aligned}
&\sup_{t\in[0,T]}\big\|u(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|u(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T)} \\
&\quad \le \mathfrak{C}_s\Big(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \sqrt{T}\sup_{t\in[0,T]}\big\|f(t)\big\|_{H_x^s(0,\infty)}\Big),
\end{aligned}$$

where $\mathfrak{C}_s = \mathfrak{C}(s) \ge \max\{C_s, \widetilde{C}_s\}$ is a constant depending on s. Hence, we have concluded the proof of Theorem 1.4.

To discuss the local well-posedness of (1), it is essential to consider the following IBVPs: For $\frac{1}{2} < s < \frac{3}{2}$,

$$\begin{cases}
v_t - v_{xx} = p(x,t), & x\in(0,\infty), 0<t<T<1,\\
v(x,0) = v_0(x) \in H_x^s(0,\infty), & x\in[0,\infty),\\
v_x(0,t) - \beta v(0,t) = h_0(t) \in H_t^{\frac{2s-1}{4}}(0,T), & 0\le t\le T<1, \beta\ge 1.
\end{cases} \tag{56}$$

Since the estimation processes of (3) and (56) are similar, we can also obtain the following result:

$$\begin{aligned}
&\sup_{t\in[0,T]}\big\|v(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|v(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T)} \\
&\quad \le \widetilde{d}_s\Big(\|v_0\|_{H_x^s(0,\infty)} + \|h_0\|_{H_t^{\frac{2s-1}{4}}(0,T)} + \sqrt{T}\sup_{t\in[0,T]}\big\|p(t)\big\|_{H_x^s(0,\infty)}\Big),
\end{aligned} \tag{57}$$

where $\widetilde{d}_s > 0$ is a constant depending on s.

In the upcoming section, we will establish the proof for Theorem 1.5. Inequalities (5) and (57) play a crucial role in the proof of Theorem 1.5.

4 The local well-posedness of the coupled system of the reaction–diffusion equations (the proof of Theorem 1.5)

In this section, we define the iteration map. Subsequently, in Lemma 4.1 and Lemma 4.2, we prove that the iteration map is a contraction and onto a closed ball. We then utilize the contraction mapping theorem to establish the uniqueness of the solution. In the following Lemma 4.3, we prove that the data-to-solution is locally Lipschitz continuous. Therefore, we can use Lemma 4.1, Lemma 4.2, and Lemma 4.3 to complete the proof of Theorem 1.5.

Now, we define the iteration map:

$$(u,v) \longmapsto \Phi_{T^*}\times\Psi_{T^*}(u,v) \doteq \big(\Phi_{T^*}(u,v), \Psi_{T^*}(u,v)\big),$$

which is obtained from the UTM solution formulas:

$$u(x,t) \doteq S[u_0,g_0;f](x,t)$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-k^2t}\widehat{u_0}(k)\,dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{\alpha+ik}{\alpha-ik}\widehat{u_0}(-k))\,dk \\
&\quad - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{-2ik}{\alpha-ik}\left(\int_0^t e^{k^2y}g_0(y)\,dy\right)dk \\
&\quad - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{\alpha+ik}{\alpha-ik}\left(\int_0^t e^{k^2y}\widehat{f}(-k,y)\,dy\right)dk \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left(\int_0^t e^{k^2y}\widehat{f}(k,y)\,dy\right)dk
\end{aligned}
$$

and

$$
\begin{aligned}
v(x,t) &\doteq S[v_0,h_0;p](x,t) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx-k^2t}\widehat{v_0}(k)\,dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{\beta+ik}{\beta-ik}\widehat{v_0}(-k))\,dk \\
&\quad - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{-2ik}{\beta-ik}\left(\int_0^t e^{k^2y}h_0(y)\,dy\right)dk \\
&\quad - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\frac{\beta+ik}{\beta-ik}\left(\int_0^t e^{k^2y}\widehat{p}(-k,y)\,dy\right)dk \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left(\int_0^t e^{k^2y}\widehat{p}(k,y)\,dy\right)dk.
\end{aligned}
$$

More precisely, we have

$$
\begin{aligned}
&\Phi_{T^*}(u,v) \doteq S\big[u_0,g_0;-u^2-cuv\big], \\
&\Psi_{T^*}(u,v) \doteq S\big[v_0,h_0;-v^2-duv\big],
\end{aligned}
$$

i.e., the iteration map

$$
E(u,v) = \Phi_{T^*}\times\Psi_{T^*}(u,v) \doteq \big(S\big[u_0,g_0;-u^2-cuv\big],S\big[v_0,h_0;-v^2-duv\big]\big). \tag{58}
$$

We will demonstrate that our iteration map (58) is a contraction in the complete metric space

$$
\mathbb{X} = X\times X, \quad \text{where } X = C\big(\big[0,T^*\big];H_x^s(0,\infty)\big)\cap C\big([0,\infty);H_t^{\frac{2s+1}{4}}\big(0,T^*\big)\big), \tag{59}
$$

with the norm

$$
\begin{aligned}
\big\|(u,v)\big\|_{\mathbb{X}} &= \sup_{t\in[0,T^*]}\big\|u(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|u(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\
&\quad + \sup_{t\in[0,T^*]}\big\|v(t)\big\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\big\|v(x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)}.
\end{aligned}
$$

Next, we consider a closed ball $B(0,r) = \{(u,v)\in\mathbb{X} : \|(u,v)\|_{\mathbb{X}} \le r\}$, where $C_s^* = \max\{\mathfrak{C}_s,\widetilde{d}_s\}$ and $r = 2C_s^*\|(u_0,v_0,g_0,h_0)\|_D$. In the following lemma, we determine the constraint on T^* such that E is onto $B(0,r)$.

Lemma 4.1 *For $C_s^* = \max\{\mathfrak{C}_s, \widetilde{d}_s\}$ and $r = 2C_s^*\|(u_0,v_0,g_0,h_0)\|_D$. The iteration map $E(u,v)$ is onto $B(0,r)$, when the following condition on T^* holds*:

$$0 < T^* \le \min\left\{T, \frac{1}{4^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\}. \tag{60}$$

Proof For $(u,v) \in B(0,r)$, we obtain the following inequality,

$$\begin{aligned}
&\|E(u,v)\|_{\mathbb{X}} \\
&\quad= \left\|\left(S\left[u_0,g_0;-u^2-cuv\right], S\left[v_0,h_0;-v^2-duv\right]\right)\right\|_{\mathbb{X}} \\
&\quad= \sup_{t\in[0,T^*]}\left\|S\left[u_0,g_0;-u^2-cuv\right](t)\right\|_{H_x^s(0,\infty)} + \sup_{x\in[0,\infty)}\left\|S\left[u_0,g_0;-u^2-cuv\right](x)\right\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\
&\qquad+ \sup_{t\in[0,T^*]}\left\|S\left[v_0,h_0;-v^2-duv\right](t)\right\|_{H_x^s(0,\infty)} \\
&\qquad+ \sup_{x\in[0,\infty)}\left\|S\left[v_0,h_0;-v^2-duv\right](x)\right\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\
&\quad\le \mathfrak{C}_s\left(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T^*)} + \sqrt{T^*}\sup_{t\in[0,T^*]}\left\|u^2+cuv\right\|_{H_x^s(0,\infty)}\right) \\
&\qquad+ \widetilde{d}_s\left(\|v_0\|_{H_x^s(0,\infty)} + \|h_0\|_{H_t^{\frac{2s-1}{4}}(0,T^*)} + \sqrt{T^*}\sup_{t\in[0,T^*]}\left\|v^2+duv\right\|_{H_x^s(0,\infty)}\right), \\
&\qquad\big(\text{by (5) and (57)}\big) \\
&\quad\le C_s^*\left(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s-1}{4}}(0,T^*)} + \sqrt{T^*}\sup_{t\in[0,T^*]}\left\|u^2+cuv\right\|_{H_x^s(0,\infty)}\right) \\
&\qquad+ C_s^*\left(\|v_0\|_{H_x^s(0,\infty)} + \|h_0\|_{H_t^{\frac{2s-1}{4}}(0,T^*)} + \sqrt{T^*}\sup_{t\in[0,T^*]}\left\|v^2+duv\right\|_{H_x^s(0,\infty)}\right) \\
&\quad\le \frac{r}{2} + C_s^*\sqrt{T^*}\left(\sup_{t\in[0,T^*]}\|u\|_{H_x^s(0,\infty)}^2 + |c|\sup_{t\in[0,T^*]}\|u\|_{H_x^s(0,\infty)}\|v\|_{H_x^s(0,\infty)}\right) \\
&\qquad+ C_s^*\sqrt{T^*}\left(\sup_{t\in[0,T^*]}\|v\|_{H_x^s(0,\infty)}^2 + |d|\sup_{t\in[0,T^*]}\|u\|_{H_x^s(0,\infty)}\|v\|_{H_x^s(0,\infty)}\right) \\
&\qquad\big(\text{by the algebra property for } H^s(\mathbb{R}) \text{ with } s > 1/2\big) \\
&\quad\le \frac{r}{2} + C_s^*\sqrt{T^*}\big(2+|c|+|d|\big)r^2.
\end{aligned}$$

Hence, we obtain the inequality

$$\|E(u,v)\|_{\mathbb{X}} \le \frac{r}{2} + C_s^*\sqrt{T^*}\big(2+|c|+|d|\big)r^2.$$

Now, we want to choose T^* such that

$$\frac{r}{2} + C_s^*\sqrt{T^*}\big(2+|c|+|d|\big)r^2 \le r$$

holds, i.e., the condition

$$0 < T^* \le \min\left\{T, \frac{1}{4^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\}$$

is satisfied. Hence, we obtain that E is onto $B(0,r)$ when T^* satisfies (60). □

Next, we determine the constraint on T^* such that E is a contraction on $B(0,r)$, for $C_s^* = \max\{\mathfrak{C}_s, \widetilde{d}_s\}$ and $r = 2C_s^*\|(u_0,v_0,g_0,h_0)\|_D$, and it is stated in the following lemma.

Lemma 4.2 *For $C_s^* = \max\{\mathfrak{C}_s, \widetilde{d}_s\}$ and $r = 2C_s^*\|(u_0,v_0,g_0,h_0)\|_D$. The iteration map $E(u,v)$ is a contraction on $B(0,r)$, when the following condition on T^* holds*:

$$0 < T^* \le \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\}. \tag{61}$$

Proof For $(u_1,v_1),(u_2,v_2) \in B(0,r)$, we obtain the following inequality:

$$\begin{aligned}
&\big\|E(u_1,v_1)-E(u_2,v_2)\big\|_{\mathbb{X}} \\
&\quad= \big\|\big(S\big[0,0;-u_1^2+u_2^2-cu_1v_1+cu_2v_2\big], S\big[0,0;-v_1^2+v_2^2-du_1v_1+du_2v_2\big]\big)\big\|_{\mathbb{X}} \\
&\quad= \sup_{t\in[0,T^*]}\big\|S\big[0,0;-u_1^2+u_2^2-cu_1v_1+cu_2v_2\big](t)\big\|_{H_x^s(0,\infty)} \\
&\qquad+ \sup_{x\in[0,\infty)}\big\|S\big[0,0;-u_1^2+u_2^2-cu_1v_1+cu_2v_2\big](x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\
&\qquad+ \sup_{t\in[0,T^*]}\big\|S\big[0,0;-v_1^2+v_2^2-du_1v_1+du_2v_2\big](t)\big\|_{H_x^s(0,\infty)} \\
&\qquad+ \sup_{x\in[0,\infty)}\big\|S\big[0,0;-v_1^2+v_2^2-du_1v_1+du_2v_2\big](x)\big\|_{H_t^{\frac{2s+1}{4}}(0,T^*)} \\
&\quad\le \mathfrak{C}_s\sqrt{T^*}\sup_{t\in[0,T^*]}\big\|-u_1^2+u_2^2-cu_1v_1+cu_2v_2\big\|_{H_x^s(0,\infty)} \\
&\qquad+ \widetilde{d}_s\sqrt{T^*}\sup_{t\in[0,T^*]}\big\|-v_1^2+v_2^2-du_1v_1+du_2v_2\big\|_{H_x^s(0,\infty)}, \quad \big(\text{by (5) and (57)}\big) \\
&\quad\le C_s^*\sqrt{T^*}\Big(\sup_{t\in[0,T^*]}\big\|(u_1-u_2)(u_1+u_2)\big\|_{H_x^s(0,\infty)} + |c|\sup_{t\in[0,T^*]}\big\|(u_1-u_2)v_1\big\|_{H_x^s(0,\infty)} \\
&\qquad+ |c|\sup_{t\in[0,T^*]}\big\|u_2(v_1-v_2)\big\|_{H_x^s(0,\infty)} + \sup_{t\in[0,T^*]}\big\|(v_1-v_2)(v_1+v_2)\big\|_{H_x^s(0,\infty)} \\
&\qquad+ |d|\sup_{t\in[0,T^*]}\big\|(u_1-u_2)v_1\big\|_{H_x^s(0,\infty)} + |d|\sup_{t\in[0,T^*]}\big\|u_2(v_1-v_2)\big\|_{H_x^s(0,\infty)}\Big) \\
&\quad\le 2C_s^*\sqrt{T^*}\big(2+|c|+|d|\big)r\big\|(u_1-u_2,v_1-v_2)\big\|_{\mathbb{X}}.
\end{aligned}$$

Therefore, we aim for E to be a contraction, which requires the fulfillment of the condition

$$2C_s^*\sqrt{T^*}\big(2+|c|+|d|\big)r \le \frac{1}{2},$$

i.e., the condition

$$0 < T^* \le \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\}$$

is satisfied. Hence, we obtain E is a contraction on $B(0,r)$ when T^* satisfies (61). □

Now, if we choose the lifespan

$$T^* = \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\}, \tag{62}$$

then T^* satisfies (60) and (61). Thus, the iteration map is a contraction and onto $B(0,r)$. Hence, $(u,v)=E(u,v)$ has a unique solution $(u,v)\in B(0,r)\subset\mathbb{X}$.

Next, we will show the data-to-solution map $(u_0,v_0,g_0,h_0)\longmapsto(u,v)$ is locally Lipschitz continuous.

Let (u_0,v_0,g_0,h_0) and $(\mathcal{U}_0,\mathcal{V}_0,\mathcal{G}_0,\mathcal{H}_0)$ be two different data lying inside a ball $B_\rho\subset D$ of radius $\rho>0$ centered at a distance $\mathcal{R}$ from the origin, where

$$D = H_x^s(0,\infty)\times H_x^s(0,\infty)\times H_t^{\frac{2s-1}{4}}(0,T)\times H_t^{\frac{2s-1}{4}}(0,T),$$

with the norm defined by (6).

Furthermore, we denote the corresponding solutions to the reaction–diffusion system IBVP (1) by

$$(u,v) = \big(S\big[u_0,g_0;-u^2-cuv\big], S\big[v_0,h_0;-v^2-duv\big]\big).$$

Then, by the contraction condition (62), we observe that the lifespan

$$T_\eta = \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2(\mathcal{R}+\rho)^2}\right\}$$

is common, and we have that both solutions (u,v) and $(\mathcal{U},\mathcal{V})$ exist for any $0<t\leq T_\eta$. Additionally, we denote $\mathbb{X}_\eta$ as the solution space $\mathbb{X}$ (59) with $T^*=T_\eta$.

In the following lemma, we prove the data-to-solution map $(u_0,v_0,g_0,h_0)\longmapsto(u,v)$ is locally Lipschitz continuous.

Lemma 4.3 *Given* $C_s^*=\max\{\mathfrak{C}_s,\widetilde{d}_s\}$ *and* $r_\eta = 1/(4C_s^*(2+|c|+|d|)T_\eta^{1/2})$. *For any* (u,v), $(\mathcal{U},\mathcal{V})\in B(0,r_\eta)\subset\mathbb{X}_\eta$ *with data in the ball* B_ρ, *then we obtain the following inequality*:

$$\big\|(u,v)-(\mathcal{U},\mathcal{V})\big\|_{\mathbb{X}_\eta}\leq 2C_s^*\big\|(u_0,v_0,g_0,h_0)-(\mathcal{U}_0,\mathcal{V}_0,\mathcal{G}_0,\mathcal{H}_0)\big\|_D. \tag{63}$$

Hence, the data-to-solution map $(u_0,v_0,g_0,h_0)\longmapsto(u,v)$ *is locally Lipschitz continuous.*

Proof For any $(u,v),(\mathcal{U},\mathcal{V})\in B(0,r_\eta)\subset\mathbb{X}_\eta$ with data in the ball B_ρ, then we obtain the following inequality:

$$\begin{aligned}
&\big\|(u,v)-(\mathcal{U},\mathcal{V})\big\|_{\mathbb{X}_\eta}\\
&\quad=\big\|E(u,v)-E(\mathcal{U},\mathcal{V})\big\|_{\mathbb{X}_\eta}\\
&\quad=\big\|\big(S\big[u_0-\mathcal{U}_0,g_0-\mathcal{G}_0;-u^2+\mathcal{U}^2-cuv+c\mathcal{U}\mathcal{V}\big],\\
&\qquad S\big[v_0-\mathcal{V}_0,h_0-\mathcal{H}_0;-v^2+\mathcal{V}^2-duv+d\mathcal{U}\mathcal{V}\big]\big)\big\|_{\mathbb{X}_\eta}\\
&\quad=\sup_{t\in[0,T_\eta]}\big\|S\big[u_0-\mathcal{U}_0,g_0-\mathcal{G}_0;-u^2+\mathcal{U}^2-cuv+c\mathcal{U}\mathcal{V}\big](t)\big\|_{H_x^s(0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{x\in[0,\infty)}\left\|S\left[u_0-\mathcal{U}_0,g_0-\mathcal{G}_0;-u^2+\mathcal{U}^2-cuv+c\mathcal{U}\mathcal{V}\right](x)\right\|_{H_t^{\frac{2s+1}{4}}(0,T_\eta)}\\
&+\sup_{t\in[0,T_\eta]}\left\|S\left[v_0-\mathcal{V}_0,h_0-\mathcal{H}_0;-v^2+\mathcal{V}^2-duv+d\mathcal{U}\mathcal{V}\right](t)\right\|_{H_x^s(0,\infty)}\\
&+\sup_{x\in[0,\infty)}\left\|S\left[v_0-\mathcal{V}_0,h_0-\mathcal{H}_0;-v^2+\mathcal{V}^2-duv+d\mathcal{U}\mathcal{V}\right](x)\right\|_{H_t^{\frac{2s+1}{4}}(0,T_\eta)}\\
&\le \mathfrak{C}_s\Big(\|u_0-\mathcal{U}_0\|_{H_x^s(0,\infty)}+\|g_0-\mathcal{G}_0\|_{H_t^{\frac{2s-1}{4}}(0,T_\eta)}\\
&+\sqrt{T_\eta}\sup_{t\in[0,T_\eta]}\left\|-u^2+\mathcal{U}^2-cuv+c\mathcal{U}\mathcal{V}\right\|_{H_x^s(0,\infty)}\Big)\\
&+\widetilde{d}_s\Big(\|v_0-\mathcal{V}_0\|_{H_x^s(0,\infty)}+\|h_0-\mathcal{H}_0\|_{H_t^{\frac{2s-1}{4}}(0,T_\eta)}\\
&+\sqrt{T_\eta}\sup_{t\in[0,T_\eta]}\left\|-v^2+\mathcal{V}^2-duv+d\mathcal{U}\mathcal{V}\right\|_{H_x^s(0,\infty)}\Big)
\end{aligned}
$$

(by (5) and (57))

$$
\begin{aligned}
&\le C_s^*\left\|(u_0,v_0,g_0,h_0)-(\mathcal{U}_0,\mathcal{V}_0,\mathcal{G}_0,\mathcal{H}_0)\right\|_D\\
&+C_s^*\sqrt{T_\eta}\Big(\sup_{t\in[0,T_\eta]}\left\|(u-\mathcal{U})(u+\mathcal{U})\right\|_{H_x^s(0,\infty)}+|c|\sup_{t\in[0,T_\eta]}\left\|(u-\mathcal{U})v\right\|_{H_x^s(0,\infty)}\\
&+|c|\sup_{t\in[0,T_\eta]}\left\|\mathcal{U}(v-\mathcal{V})\right\|_{H_x^s(0,\infty)}\Big)\\
&+C_s^*\sqrt{T_\eta}\Big(\sup_{t\in[0,T_\eta]}\left\|(v-\mathcal{V})(v+\mathcal{V})\right\|_{H_x^s(0,\infty)}+|d|\sup_{t\in[0,T_\eta]}\left\|(u-\mathcal{U})v\right\|_{H_x^s(0,\infty)}\\
&+|d|\sup_{t\in[0,T_\eta]}\left\|\mathcal{U}(v-\mathcal{V})\right\|_{H_x^s(0,\infty)}\Big)\\
&\le C_s^*\left\|(u_0,v_0,g_0,h_0)-(\mathcal{U}_0,\mathcal{V}_0,\mathcal{G}_0,\mathcal{H}_0)\right\|_D\\
&+C_s^*\sqrt{T_\eta}\big(4+2|c|+2|d|\big)r_\eta\left\|(u,v)-(\mathcal{U},\mathcal{V})\right\|_{\mathbb{X}_\eta}.
\end{aligned}
$$

Therefore, we obtain the following inequality:

$$
\left\|(u,v)-(\mathcal{U},\mathcal{V})\right\|_{\mathbb{X}_\eta}\le\frac{C_s^*}{1-2C_s^*(2+|c|+|d|)\sqrt{T_\eta}r_\eta}\left\|(u_0,v_0,g_0,h_0)-(\mathcal{U}_0,\mathcal{V}_0,\mathcal{G}_0,\mathcal{H}_0)\right\|_D.
$$

Hence, when we set

$$
r_\eta=\frac{1}{4C_s^*(2+|c|+|d|)\sqrt{T_\eta}},
$$

then the following two inequalities

$$
\frac{1}{1-2C_s^*(2+|c|+|d|)\sqrt{T_\eta}r_\eta}\le 2\quad\text{and}\quad 1-2C_s^*\big(2+|c|+|d|\big)\sqrt{T_\eta}r_\eta>0
$$

hold. Therefore, we establish (63), which implies that the data-to-solution map is locally Lipschitz continuous. This completes the proof of Lemma 4.3. □

Now, we can prove Theorem 1.5. We set the lifespan

$$T^* = \min\left\{T, \frac{1}{8^2(C_s^*)^4(2+|c|+|d|)^2\|(u_0,v_0,g_0,h_0)\|_D^2}\right\},$$

and then, by employing Lemma 4.1 to Lemma 4.3, we can complete the proof of Theorem 1.5.

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Author contributions
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