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Generalized strongly n -polynomial convex functions and related inequalities

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Abstract

This paper focuses on introducing and examining the class of generalized strongly n -polynomial convex functions. Relationships between these functions and other types of convex functions are explored. The Hermite–Hadamard inequality is established for generalized strongly n -polynomial convex functions. Additionally, new integral inequalities of Hermite–Hadamard type are derived for this class of functions using the Hölder–İşcan integral inequality. The results obtained in this paper are compared with those known in the literature, demonstrating the superiority of the new results. Finally, some applications for special means are provided.

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1 Introduction and preliminaries

The concept of convexity has played a crucial role in the advancement of the theory of inequalities.

Definition 1 Let $F \subseteq \mathbb{R}$ be an interval. Then, a function $\Lambda : F \rightarrow \mathbb{R}$ is said to be convex if

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq \mu\Lambda(\vartheta) + (1 - \mu)\Lambda(\varpi) \quad (1.1)$$

holds for all $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$.

If inequality (1.1) holds in the reverse direction, then Λ is said to be concave on interval $F \neq \emptyset$.

Numerous generally accepted results from the theory of inequalities can be obtained using the properties of functions; see [2, 8, 11, 14–17, 21, 23, 28, 33] and the references therein.

The Hermite–Hadamard (H–H) inequality is a well-explored and renowned result concerning convex functions, stating that if $\Lambda : F \rightarrow \mathbb{R}$ is a convex function in F for all

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$\vartheta, \varpi \in F$ with $\vartheta < \varpi$, then

$$\Lambda\left(\frac{\vartheta + \varpi}{2}\right) \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \leq \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2}. \quad (1.2)$$

Interested readers may refer to the monographs [1, 6, 7, 10, 12, 20, 24–27, 32, 34, 35].

Definition 2 [29] Let $F \subset \mathbb{R}$ be an interval and k be a positive number. A function $\Lambda : F \subset \mathbb{R} \rightarrow \mathbb{R}$ is called strongly convex with modulus k if

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq \mu\Lambda(\vartheta) + (1 - \mu)\Lambda(\varpi) - k\mu(1 - \mu)(\varpi - \vartheta)^2 \quad (1.3)$$

for all $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$.

It is clear that every strongly convex function is also a convex function.

Theorem 1 If a function $\Lambda : F \rightarrow \mathbb{R}$ is a strongly convex function with modulus k , then

$$\Psi\left(\frac{\vartheta + \varpi}{2}\right) + \frac{k}{12}(\varpi - \vartheta)^2 \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \leq \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{k}{6}(\varpi - \vartheta)^2 \quad (1.4)$$

for all $\vartheta, \varpi \in F$ with $\vartheta < \varpi$.

Definition 3 [31] Let $h : J \rightarrow \mathbb{R}$ be a nonnegative function, $h \neq 0$. A function $\Lambda : F \rightarrow \mathbb{R}$ is called h -convex, if Λ is nonnegative, and for all $\vartheta, \varpi \in F$, $\mu \in (0, 1)$, we have

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq h(\mu)\Lambda(\vartheta) + h(1 - \mu)\Lambda(\varpi). \quad (1.5)$$

If this inequality is reversed, then Λ is said to be h -concave. Clearly, if we substitute $h(\mu) = \mu$, then the h -convex functions reduce to the classical convex functions; see [5].

Definition 4 [3] Let $(\chi, \|\cdot\|)$ be a real normed space, and \aleph stands for a convex subset of χ , $h : (0, 1) \rightarrow (0, \infty)$ is a given function, and k is a positive constant. A function $\Lambda : \aleph \rightarrow \mathbb{R}$ is called strongly h -convex with module k if

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq h(\mu)\Lambda(\vartheta) + h(1 - \mu)\Lambda(\varpi) - k\mu(1 - \mu)\|\varpi - \vartheta\|^2 \quad (1.6)$$

for all $\vartheta, \varpi \in \aleph$ and $\mu \in (0, 1)$.

Theorem 2 [22] Let $h : (0, 1) \rightarrow (0, \infty)$ be a given function. If a function $\Lambda : F \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue integrable and strongly h -convex with module $k > 0$, then

$$\begin{aligned} & \frac{1}{2h(\frac{1}{2})} \left[\Lambda\left(\frac{\vartheta + \varpi}{2}\right) + \frac{k}{12}(\varpi - \vartheta)^2 \right] \\ & \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \leq (\Lambda(\vartheta) + \Lambda(\varpi)) \int_{\vartheta}^{\varpi} h(\mu) d\mu - \frac{k}{6}(\varpi - \vartheta)^2 \end{aligned} \quad (1.7)$$

for all $\vartheta, \varpi \in F$ with $\vartheta < \varpi$.

In [30], Tekin et al. gave the following definition and related H–H-type inequality as follows:

Definition 5 Let $n \in \mathbb{N}$. A nonnegative function $\Lambda : F \subset \mathbb{R} \rightarrow \mathbb{R}$ is called n -polynomial convex if for every $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$,

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq \frac{1}{n} \sum_{j=1}^n [1 - (1 - \mu)^j] \Lambda(\vartheta) + \frac{1}{n} \sum_{j=1}^n [1 - \mu^j] \Lambda(\varpi). \quad (1.8)$$

Theorem 3 Let $\Psi : [\vartheta, \varpi] \rightarrow \mathbb{R}$ be an n -polynomial convex function. If $a < b$ and $\Psi \in L[\vartheta, \varpi]$, then

$$\begin{aligned} & \frac{1}{2} \left(\frac{n}{n + 2^{-n} - 1} \right) \Lambda \left(\frac{\vartheta + \varpi}{2} \right) \\ & \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \leq \left(\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{n} \right) \sum_{j=1}^n \frac{j}{j+1}. \end{aligned} \quad (1.9)$$

In [4], Ataman et al. introduced the class of strongly n -polynomial convex functions and related H–H-type inequality as follows:

Definition 6 Let $n \in \mathbb{N}$, $F \subset \mathbb{R}$ be an interval and k be a positive number. A nonnegative function $\Lambda : F \subset \mathbb{R} \rightarrow \mathbb{R}$ is called strongly n -polynomial convex with modulus k if for every $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$,

$$\begin{aligned} & \Lambda(\mu\vartheta + (1 - \mu)\varpi) \\ & \leq \frac{1}{n} \sum_{j=1}^n [1 - (1 - \mu)^j] \Lambda(\vartheta) + \frac{1}{n} \sum_{j=1}^n [1 - \mu^j] \Lambda(\varpi) - k\mu(1 - \mu)(\varpi - \vartheta)^2. \end{aligned} \quad (1.10)$$

Theorem 4 Let $\Psi : [\vartheta, \varpi] \rightarrow \mathbb{R}$ be a strongly n -polynomial convex function with modulus k . If $\vartheta < \varpi$ and $\Psi \in L[\vartheta, \varpi]$, then

$$\begin{aligned} & \frac{1}{2} \left(\frac{n}{n + 2^{-n} - 1} \right) \left[\Lambda \left(\frac{\vartheta + \varpi}{2} \right) + \frac{k}{12} (\varpi - \vartheta)^2 \right] \\ & \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \leq \left(\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{n} \right) \sum_{j=1}^n \frac{j}{j+1} - \frac{k}{6} (\varpi - \vartheta)^2. \end{aligned} \quad (1.11)$$

In [18], Kadakal et al. introduced the class of generalized n -polynomial convex function and related H–H-type inequality as follows:

Definition 7 Let $n \in \mathbb{N}$ and $\alpha_j \geq 0$ ($j = \overline{1, n}$) such that $\sum_{j=1}^n \alpha_j > 0$. A nonnegative function $\Lambda : F \subset \mathbb{R} \rightarrow \mathbb{R}$ is called generalized n -polynomial convex function if for every $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$,

$$\Lambda(\mu\vartheta + (1 - \mu)\varpi) \leq \frac{\sum_{j=1}^n \alpha_j (1 - (1 - \mu)^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\vartheta) + \frac{\sum_{j=1}^n \alpha_j (1 - \mu^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\varpi). \quad (1.12)$$

Theorem 5 Let $\Lambda : [\vartheta, \varpi] \rightarrow \mathbb{R}$ be a generalized n -polynomial convex function. If $\vartheta < \varpi$ and $\Lambda \in L[\vartheta, \varpi]$, then

$$\begin{aligned} \frac{1}{2} \left(\frac{\sum_{j=1}^n \alpha_j}{\sum_{j=1}^n \alpha_j [1 - (\frac{1}{2})^j]} \right) \Lambda \left(\frac{\vartheta + \varpi}{2} \right) &\leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \\ &\leq \left[\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{\sum_{j=1}^n \alpha_j} \right] \sum_{j=1}^n \alpha_j \left(\frac{j}{j+1} \right). \end{aligned} \quad (1.13)$$

Theorem 6 (Hölder-İşcan integral inequality [13]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If Λ and Θ are real functions defined on $[\vartheta, \varpi]$ and if $|\Lambda|^p$, $|\Theta|^q$ are integrable functions on $[\vartheta, \varpi]$, then

$$\begin{aligned} &\int_{\vartheta}^{\varpi} |\Lambda(\sigma)\Theta(\sigma)| d\sigma \\ &\leq \frac{1}{\varpi - \vartheta} \left\{ \left(\int_{\vartheta}^{\varpi} (\varpi - \sigma) |\Lambda(\sigma)|^p d\sigma \right)^{\frac{1}{p}} \left(\int_{\vartheta}^{\varpi} (\varpi - \sigma) |\Theta(\sigma)|^q d\sigma \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{\vartheta}^{\varpi} (\sigma - \vartheta) |\Lambda(\sigma)|^p d\sigma \right)^{\frac{1}{p}} \left(\int_{\vartheta}^{\varpi} (\sigma - \vartheta) |\Theta(\sigma)|^q d\sigma \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (1.14)$$

Theorem 7 (Improved power-mean integral inequality [19]) Let $q \geq 1$. If Λ and Θ are real functions defined on $[\vartheta, \varpi]$ and $|\Lambda|$, $|\Lambda||\Theta|^q$ are integrable functions on $[\vartheta, \varpi]$, then

$$\begin{aligned} &\int_a^b |\Lambda(\sigma)\Theta(\sigma)| d\sigma \\ &\leq \frac{1}{\varpi - \vartheta} \left\{ \left(\int_{\vartheta}^{\varpi} (\varpi - \sigma) |\Lambda(\sigma)| d\sigma \right)^{1-\frac{1}{q}} \left(\int_{\vartheta}^{\varpi} (\varpi - \sigma) |\Lambda(\sigma)| |\Theta(\sigma)|^q d\sigma \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_{\vartheta}^{\varpi} (\sigma - \vartheta) |\Lambda(\sigma)| d\sigma \right)^{1-\frac{1}{q}} \left(\int_{\vartheta}^{\varpi} (\sigma - \vartheta) |\Lambda(\sigma)| |\Theta(\sigma)|^q d\sigma \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

2 The definition of generalized strongly n -polynomial convex functions

In this section, we will add a new definition called a generalized strongly n -polynomial convex function and its basic algebraic properties.

Definition 8 Let $n \in \mathbb{N}$, $\alpha_j \geq 0$ ($j = \overline{1, n}$) such that $\sum_{j=1}^n \alpha_j > 0$, $F \subset \mathbb{R}$ be an interval, and k be a positive number. A nonnegative function $\Lambda : F \subset \mathbb{R} \rightarrow \mathbb{R}$ is called the generalized strongly n -polynomial convex function (*GSPOLC*) if for every $\vartheta, \varpi \in F$ and $\mu \in [0, 1]$,

$$\begin{aligned} &\Lambda(\mu\vartheta + (1 - \mu)\varpi) \\ &\leq \frac{\sum_{j=1}^n \alpha_j (1 - (1 - \mu)^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\vartheta) + \frac{\sum_{j=1}^n \alpha_j (1 - \mu^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\varpi) - k\mu(1 - \mu)(\varpi - \vartheta)^2. \end{aligned} \quad (2.1)$$

We will denote by $GSPOLC(F)$ the class of all generalized strongly n -polynomial convex functions on F .

Note that every *GSPOLC* is a strongly h -convex function with the function

$$h(\mu) = \frac{\sum_{j=1}^n \alpha_j (1 - (1 - \mu)^j)}{\sum_{j=1}^n \alpha_j}.$$

Therefore, if $\Lambda, \Theta \in GSPOLC(F)$, it can be easily seen that $\Lambda + \Theta \in GSPOLC(F)$ and for $b \in \mathbb{R}$ ($b \geq 0$), $b\Lambda \in GSPOLC(F)$.

Also, if $\Lambda : F \rightarrow J$ is convex and $\Theta \in GSPOLC(J)$ and nondecreasing, then $\Theta \circ \Lambda \in GSPOLC(F)$ (see [31], Theorem 15).

Remark 1 For $n = 1$ and $k = 0$, Definition 8 reduces to Definition 1.

Remark 2 For $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$, Definition 8 reduces to Definition 5.

Remark 3 For $k = 0$, Definition 8 reduces to Definition 7.

Remark 4 For $n = 1$, Definition 8 reduces to Definition 2.

Remark 5 For $\alpha_j = 1$ ($j = \overline{1, n}$), Definition 8 reduces to Definition 6.

Remark 6 Every nonnegative strongly convex function is *GSPOLC*. It is obvious from the inequalities

$$\sum_{j=1}^n \alpha_j \mu \leq \sum_{j=1}^n \alpha_j (1 - (1 - \mu)^j) \quad \text{and}$$

$$\sum_{j=1}^n \alpha_j (1 - \mu) \leq \sum_{j=1}^n \alpha_j (1 - \mu^j)$$

for all $\mu \in [0, 1]$ and $n \in \mathbb{N}$.

Example 1 Let $\Lambda(\sigma) = \sigma^2$ and $[\vartheta, \varpi] = [-1, 1]$. Then, Λ is *GSPOLC* with modulus $k = 1$.

3 H–H inequality for generalized strongly n -polynomial convex functions

This section aims to establish the H–H inequality for the new class of functions.

Theorem 8 Let $\Lambda : [\vartheta, \varpi] \rightarrow \mathbb{R}$ be *GSPOLC* with modulus k . If $\vartheta < \varpi$ and $\Lambda \in L[\vartheta, \varpi]$, then

$$\begin{aligned} & \frac{1}{2} \left(\frac{\sum_{j=1}^n \alpha_j}{\sum_{j=1}^n \alpha_j [1 - (\frac{1}{2})^j]} \right) \left[\Lambda \left(\frac{\vartheta + \varpi}{2} \right) + \frac{k}{12} (\varpi - \vartheta)^2 \right] \\ & \leq \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \\ & \leq \left[\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{\sum_{j=1}^n \alpha_j} \right] \sum_{j=1}^n \alpha_j \left(\frac{j}{j+1} \right) - \frac{k}{6} (\varpi - \vartheta)^2. \end{aligned} \tag{3.1}$$

Proof Using the generalized strongly n -polynomial convexity of Λ , one has

$$\begin{aligned}
& \Lambda\left(\frac{\vartheta + \varpi}{2}\right) \\
&= \Lambda\left(\frac{1}{2}[\mu\vartheta + (1-\mu)\varpi] + \frac{1}{2}[(1-\mu)\vartheta + \mu\varpi]\right) \\
&\leq \frac{\sum_{j=1}^n \alpha_j[1 - (1 - \frac{1}{2})^j]}{\sum_{j=1}^n \alpha_j} \Lambda(\mu\vartheta + (1-\mu)\varpi) \\
&\quad + \frac{\sum_{j=1}^n \alpha_j[1 - (\frac{1}{2})^j]}{\sum_{j=1}^n \alpha_j} \Lambda((1-\mu)\vartheta + \mu\varpi) - \frac{k}{4}[(2\mu - 1)\varpi - (1 - 2\mu)\vartheta]^2 \\
&= \frac{\sum_{j=1}^n \alpha_j[1 - (\frac{1}{2})^j]}{\sum_{j=1}^n \alpha_j} [\Lambda(\mu\vartheta + (1-\mu)\varpi) + \Lambda((1-\mu)\vartheta + \mu\varpi)] \\
&\quad - \frac{k}{4}(2\mu - 1)(\varpi - \vartheta)^2.
\end{aligned}$$

Now integrating the above inequality with respect to $\mu \in [0, 1]$, one gets

$$\begin{aligned}
& \Lambda\left(\frac{\vartheta + \varpi}{2}\right) \\
&\leq \frac{\sum_{j=1}^n \alpha_j[1 - (\frac{1}{2})^j]}{\sum_{j=1}^n \alpha_j} \left[\int_0^1 \Lambda(\mu\vartheta + (1-\mu)\varpi) d\mu + \int_0^1 \Lambda((1-\mu)\vartheta + \mu\varpi) d\mu \right] \\
&\quad - \frac{k}{4}(\varpi - \vartheta)^2 \int_0^1 (2\mu - 1)^2 d\mu \\
&\leq \frac{2}{\varpi - \vartheta} \frac{\sum_{j=1}^n \alpha_j[1 - (\frac{1}{2})^j]}{\sum_{j=1}^n \alpha_j} \int_\vartheta^\varpi \Lambda(\sigma) d\sigma - \frac{k}{12}(\varpi - \vartheta)^2,
\end{aligned}$$

which completes the left-hand side of inequality (3.1). For the right-hand side of inequality (3.1), changing the variable of integration as $\sigma = \mu\vartheta + (1 - \mu)\varpi$ and using generalized strongly n -polynomial convexity of Λ , one obtains

$$\begin{aligned}
& \frac{1}{\varpi - \vartheta} \int_\vartheta^\varpi \Lambda(\sigma) d\sigma \\
&= \int_0^1 \Lambda(\mu\vartheta + (1-\mu)\varpi) d\mu \\
&\leq \int_0^1 \left[\frac{\sum_{j=1}^n \alpha_j(1 - (1 - \mu)^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\vartheta) + \frac{\sum_{j=1}^n \alpha_j(1 - \mu^j)}{\sum_{j=1}^n \alpha_j} \Lambda(\varpi) - k\mu(1 - \mu)(\varpi - \vartheta)^2 \right] d\mu \\
&= \frac{\Lambda(\vartheta)}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 [1 - (1 - \mu)^j] d\mu + \frac{\Lambda(\varpi)}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 [1 - \mu^j] d\mu \\
&\quad - k(\varpi - \vartheta)^2 \int_0^1 \mu(1 - \mu) d\mu \\
&= \left[\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{\sum_{j=1}^n \alpha_j} \right] \sum_{j=1}^n \alpha_j \left(\frac{j}{j+1} \right) - \frac{k}{6}(\varpi - \vartheta)^2,
\end{aligned}$$

where

$$\begin{aligned} \int_0^1 [1 - (1 - \mu)^j] d\mu &= \int_0^1 [1 - \mu^j] d\mu = \frac{j}{j+1}, \\ \int_0^1 \mu(1 - \mu) d\mu &= \frac{1}{6}. \end{aligned}$$
□

Remark 7 For $n = 1$ and $k = 0$, inequality (3.1) reduces to inequality (1.2).

Remark 8 For $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$, inequality (3.1) reduces to inequality (1.9).

Remark 9 For $k = 0$, inequality (3.1) reduces to inequality (1.13).

Remark 10 For $n = 1$, inequality (3.1) reduces to inequality (1.4).

Remark 11 For $\alpha_j = 1$ ($j = \overline{1, n}$), inequality (3.1) reduces to inequality (1.11).

4 Refinements of H-H inequality

Let us recall the following crucial lemma that we will use in the future:

Lemma 1 Let $\Lambda : F^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$. If $\Lambda' \in L[\vartheta, \varpi]$, then

$$\frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma = \frac{\varpi - \vartheta}{2} \int_0^1 (1 - 2\mu) \Lambda'(\mu\vartheta + (1 - \mu)\varpi) d\mu.$$

Theorem 9 Let $\Lambda : F \rightarrow \mathbb{R}$ be a differentiable function on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$, and let $\Lambda' \in L[\vartheta, \varpi]$. If $|\Lambda'|$ is GSPOLC with modulus k on $[\vartheta, \varpi]$, then the following inequality holds for $\mu \in [0, 1]$:

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \left[\frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right] A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|) - \frac{k}{12} (\varpi - \vartheta)^3, \end{aligned} \tag{4.1}$$

where A is the arithmetic mean.

Proof Using Lemma 1 and the inequality

$$\begin{aligned} &|\Lambda'(\mu\vartheta + (1 - \mu)\varpi)| \\ &\leq \frac{\sum_{j=1}^n \alpha_j (1 - (1 - \mu)^j)}{\sum_{j=1}^n \alpha_j} |\Lambda'(\vartheta)| + \frac{\sum_{j=1}^n \alpha_j (1 - \mu^j)}{\sum_{j=1}^n \alpha_j} |\Lambda'(\varpi)| \\ &\quad - k\mu(1 - \mu)(\varpi - \vartheta)^2, \end{aligned}$$

one obtains

$$\begin{aligned}
& \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
& \leq \frac{\varpi - \vartheta}{2} \int_0^1 |1 - 2\mu| |\Lambda'(\mu\vartheta + (1 - \mu)\varpi)| d\mu \\
& \leq \frac{\varpi - \vartheta}{2} \int_0^1 |1 - 2\mu| \left(\frac{|\Lambda'(\vartheta)|}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1 - (1 - \mu)^j) \right. \\
& \quad \left. + \frac{|\Lambda'(\varpi)|}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1 - \mu^j) - k\mu(1 - \mu)(\varpi - \vartheta)^2 d\mu \right) \\
& = \frac{\varpi - \vartheta}{2} \left(\frac{|\Lambda'(\vartheta)|}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 |1 - 2\mu|(1 - (1 - \mu)^j) \right. \\
& \quad \left. + \frac{|\Lambda'(\varpi)|}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 |1 - 2\mu|(1 - \mu^j) - k(\varpi - \vartheta)^2 \int_0^1 \mu(1 - \mu) d\mu \right) \\
& = \frac{\varpi - \vartheta}{2} \sum_{j=1}^n \alpha_j \left[\frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right] A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|) - \frac{k}{12}(\varpi - \vartheta)^3,
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 \mu(1 - \mu) d\mu &= \frac{1}{6}, \\
\int_0^1 |1 - 2\mu|[1 - (1 - \mu)^j] d\mu &= \int_0^1 |1 - 2\mu|[1 - \mu^j] d\mu = \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}}. \quad \square
\end{aligned}$$

Corollary 1 If one takes $n = 1$ and $k = 0$ in (4.1), then

$$\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \leq \frac{\varpi - \vartheta}{4} A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|). \quad (4.2)$$

Inequality (4.2) coincides with the inequality in [9, Theorem 2.2].

Corollary 2 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$ in (4.1), then

$$\begin{aligned}
& \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \quad (4.3) \\
& \leq \frac{\varpi - \vartheta}{n} \sum_{j=1}^n \left[\frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right] A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|).
\end{aligned}$$

Inequality (4.3) coincides with the inequality in [30, Theorem 5].

Corollary 3 If one takes $k = 0$ in (4.1), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \left[\frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right] A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|). \end{aligned} \quad (4.4)$$

Inequality (4.4) coincides with the inequality in [18, Theorem 5].

Corollary 4 If one takes $n = 1$ in (4.1), then

$$\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \leq \frac{\varpi - \vartheta}{4} A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|) - \frac{k}{12} (\varpi - \vartheta)^3.$$

Corollary 5 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) in the inequality, then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{n} \sum_{j=1}^n \left[\frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right] A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|) - \frac{k}{12} (\varpi - \vartheta)^3. \end{aligned} \quad (4.5)$$

Inequality (4.5) coincides with the inequality in [4, Theorem 5].

Theorem 10 Let $\Lambda : F \rightarrow \mathbb{R}$ be a differentiable function on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and assume that $\Lambda' \in L[\vartheta, \varpi]$. If $|\Lambda'|^q$ is GSPOLC with modulus k on $[\vartheta, \varpi]$, then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}, \end{aligned} \quad (4.6)$$

where A is the arithmetic mean.

Proof Using Lemma 1, the Hölder integral inequality and the inequality

$$\begin{aligned} & |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q \\ & \leq \frac{\sum_{j=1}^n \alpha_j (1 - (1-\mu)^j)}{\sum_{j=1}^n \alpha_j} |\Lambda'(\vartheta)|^q + \frac{\sum_{j=1}^n \alpha_j (1 - \mu^j)}{\sum_{j=1}^n \alpha_j} |\Lambda'(\varpi)|^q \\ & \quad - k\mu(1-\mu)(\varpi - \vartheta)^2, \end{aligned}$$

one gets

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \int_0^1 |1 - 2\mu| |\Lambda'(\mu\vartheta + (1-\mu)\varpi)| d\mu \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\varpi - \vartheta}{2} \left(\int_0^1 |1 - 2\mu|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q d\mu \right)^{\frac{1}{q}} \\
&\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1 - (1-\mu)^j) d\mu \right)^{\frac{1}{q}} \\
&\quad + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1 - \mu^j) d\mu - c(\varpi - \vartheta)^2 \int_0^1 \mu(1-\mu) d\mu \Big)^{\frac{1}{q}} \\
&= \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 |1 - 2\mu|^p d\mu &= \frac{1}{p+1}, \\
\int_0^1 [1 - (1-\mu)^j] d\mu &= \int_0^1 [1 - \mu^j] d\mu = \frac{j}{j+1}, \\
\int_0^1 \mu(1-\mu) d\mu &= \frac{1}{6}.
\end{aligned}$$
□

Corollary 6 If one takes $n = 1$ and $k = 0$ in (4.6), then

$$\begin{aligned}
&\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
&\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} A^{\frac{1}{q}} (|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q).
\end{aligned} \tag{4.7}$$

Inequality (4.7) coincides with the inequality in [9, Theorem 2.3].

Corollary 7 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$ in (4.6), then

$$\begin{aligned}
&\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
&\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{j=1}^n \frac{j}{j+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q).
\end{aligned} \tag{4.8}$$

Inequality (4.8) coincides with the inequality in [30, Theorem 6].

Corollary 8 If one takes $k = 0$ in (4.6), then

$$\begin{aligned}
&\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
&\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q).
\end{aligned} \tag{4.9}$$

Inequality (4.9) coincides with the inequality in [18, Theorem 6].

Corollary 9 If one takes $n = 1$ in (4.6), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 10 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) in (4.6), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \tag{4.10} \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{j=1}^n \frac{j}{j+1} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Inequality (4.10) coincides with the inequality in [4, Theorem 5].

Theorem 11 Let $\Lambda : F \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$, $q \geq 1$, and assume that $\Lambda' \in L[\vartheta, \varpi]$. If $|\Lambda'|^q$ is GSPOLC with modulus k on $[\vartheta, \varpi]$, then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \tag{4.11} \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \times \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{16} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}, \end{aligned}$$

where A is the arithmetic mean.

Proof From Lemma 1, the Hölder integral inequality, and the property of GSPOLC of $|\Lambda'|^q$, one obtains

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\int_0^1 |1 - 2\mu| d\mu \right)^{1-\frac{1}{q}} \left(\int_0^1 |1 - 2\mu| |\Lambda'(\mu a + (1 - \mu)b)|^q d\mu \right)^{\frac{1}{q}} \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left[\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 |1 - 2\mu| [1 - (1 - \mu)^j] d\mu \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 |1 - 2\mu| [1 - \mu^j] d\mu - k(\varpi - \vartheta)^2 \int_0^1 |1 - 2\mu| \mu(1 - \mu) d\mu \right]^{\frac{1}{q}} \\ & = \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{16} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \int_0^1 |1 - 2\mu| \mu(1 - \mu) d\mu &= \frac{1}{16}, \\ \int_0^1 |1 - 2\mu| [1 - (1 - \mu)^j] d\mu &= \int_0^1 |1 - 2\mu| [1 - \mu^j] d\mu = \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}}. \end{aligned} \quad \square$$

Corollary 11 Under the assumption of Theorem 11 with $q = 1$, one gets the conclusion of Theorem 9.

Corollary 12 If one takes $n = 1, k = 0$ and $q = 1$ in (4.11), then

$$\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \leq \frac{\varpi - \vartheta}{4} A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|). \quad (4.12)$$

Inequality (4.12) coincides with the inequality in [9, Theorem 1].

Corollary 13 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$ in (4.11), then

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{2}{q}} \left(\frac{1}{n} \sum_{j=1}^n \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} \right)^{\frac{1}{q}} A^{\frac{1}{q}} (|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) \end{aligned} \quad (4.13)$$

Inequality (4.13) coincides with the inequality in [30, Theorem 1]. Also, if one takes $q = 1$ in (4.13), then one gets

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{n} \sum_{j=1}^n \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|). \end{aligned} \quad (4.14)$$

Inequality (4.14) coincides with the inequality in [30, Theorem 1].

Corollary 14 If one takes $n = 1$ in (4.11), then

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{4} \left(A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{8} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned} \quad (4.15)$$

Also, if one takes $q = 1$ in (4.15), then

$$\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \leq \frac{\varpi - \vartheta}{4} \left(A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|) - \frac{k}{8} (\varpi - \vartheta)^2 \right).$$

Theorem 12 Let $\Lambda : F \rightarrow \mathbb{R}$ be a differentiable function on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and assume that $\Lambda' \in L[\vartheta, \varpi]$. If $|\Lambda'|^q$ is GSPOLC with modulus k on $[\vartheta, \varpi]$, then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned} \tag{4.16}$$

Proof Using Lemma 1, the Hölder-İşcan integral inequality, and the property of GSPOLC of $|\Lambda'|^q$, one has

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\int_0^1 (1-\mu) |1-2\mu|^p d\mu \right)^{\frac{1}{p}} \left(\int_0^1 (1-\mu) |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q d\mu \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\int_0^1 \mu |1-2\mu|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \mu |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q d\mu \right)^{\frac{1}{q}} \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1-\mu) [1 - (1-\mu)^j] d\mu \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1-\mu) [1 - \mu^j] d\mu - k(\varpi - \vartheta)^2 \int_0^1 \mu (1-\mu)^2 d\mu \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 \mu [1 - (1-\mu)^j] d\mu \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 \mu [1 - \mu^j] d\mu - k(\varpi - \vartheta)^2 \int_0^1 \mu^2 (1-\mu) d\mu \right)^{\frac{1}{q}} \\ & = \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} \right. \\ & \quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} \right. \\
& \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-\mu) |1-2\mu|^p d\mu &= \int_0^1 \mu |1-2\mu|^p d\mu = \frac{1}{2(p+1)}, \\
\int_0^1 \mu (1-\mu)^2 d\mu &= \int_0^1 \mu^2 (1-\mu) d\mu = \frac{1}{12}, \\
\int_0^1 (1-\mu) [1-(1-\mu)^j] d\mu &= \int_0^1 \mu [1-\mu^j] d\mu = \frac{j}{2(j+2)}, \\
\int_0^1 (1-\mu) [1-\mu^j] d\mu &= \int_0^1 \mu [1-(1-\mu)^j] d\mu = \frac{j(j+3)}{2(j+1)(j+2)}. \quad \square
\end{aligned}$$

Corollary 15 If one takes $n = 1$ and $k = 0$ in (4.16), then

$$\begin{aligned}
& \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
& \leq \frac{\varpi - \vartheta}{2(p+1)^{\frac{1}{p}}} \left[\left(\frac{|\Lambda'(\vartheta)|^q + 2|\Lambda'(\varpi)|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{2|\Lambda'(\vartheta)|^q + |\Lambda'(\varpi)|^q}{3} \right)^{\frac{1}{q}} \right]. \tag{4.17}
\end{aligned}$$

Inequality (4.17) coincides with the inequality in [13, Theorem 3.2].

Corollary 16 If one takes $\alpha_j = 1$ ($j = \overline{1, n}$) and $k = 0$ in (4.16), then

$$\begin{aligned}
& \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\
& \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n \frac{j}{2(j+2)} + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{j=1}^n \frac{j(j+3)}{2(j+1)(j+2)} \right)^{\frac{1}{q}} \\
& + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n \frac{j(j+3)}{2(j+1)(j+2)} + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{j=1}^n \frac{j}{2(j+2)} \right)^{\frac{1}{q}}. \tag{4.18}
\end{aligned}$$

Inequality (4.18) coincides with the inequality in [30, Theorem 3.2].

Corollary 17 If one takes $k = 0$ in (12), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 18 If one takes $n = 1$ in (4.16), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q + 2|\Lambda'(\varpi)|^q}{6} - \frac{k}{12}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{|\Lambda'(\vartheta)|^q + 2|\Lambda'(\varpi)|^q}{6} - \frac{k}{12}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 19 If one take $\alpha_j = 1$ ($j = \overline{1, n}$) in (4.16), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n \frac{j}{2(j+2)} + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{i=1}^n \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n \frac{j(j+3)}{2(j+1)(j+2)} + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{j=1}^n \frac{j}{2(j+2)} - \frac{k}{12}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Remark 12 Inequality (4.16) gives better results than inequality (4.6). Let us show that

$$\begin{aligned} & \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\
& \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}.
\end{aligned}$$

From the concavity of the function $h : [0, \infty) \rightarrow \mathbb{R}$, $h(\sigma) = \sigma^r$, $0 < r \leq 1$, one gets

$$\begin{aligned}
& \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\
& + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\
& \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\
& \times 2 \left[\frac{1}{2} \frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} + \frac{1}{2} \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} - \frac{1}{2} \frac{k}{6} (\varpi - \vartheta)^2 \right]^{\frac{1}{q}} \\
& = \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}.
\end{aligned}$$

This is the desired result.

Theorem 13 Let $\Lambda : F \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on F° , $\vartheta, \varpi \in F^\circ$ with $\vartheta < \varpi$, $q \geq 1$, and assume that $\Lambda' \in L[\vartheta, \varpi]$. If $|\Lambda'|^q$ is GSPOLC with modulus k on the $[\vartheta, \varpi]$, then

$$\begin{aligned}
& \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \tag{4.19} \\
& \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\
& + \frac{\varpi - \vartheta}{2} \\
& \times \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned} M_1(j) &= \frac{(j^2 + j + 2)2^j - 2}{2^{j+2}(j+2)(j+3)}, \\ M_2(j) &= \frac{(j+5)[(j^2 + j + 2)2^j - 2]}{2^{j+2}(j+1)(j+2)(j+3)}. \end{aligned}$$

Proof From Lemma 1, the improved power-mean integral inequality, and the property of GSPOLC of $|\Lambda'|^q$, one obtains

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{2} \left(\int_0^1 (1-\mu) |1-2\mu| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 (1-\mu) |1-2\mu| |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \frac{\varpi - \vartheta}{2} \left(\int_0^1 \mu |1-2\mu| d\mu \right)^{1-\frac{1}{q}} \left(\int_0^1 \mu |1-2\mu| |\Lambda'(\mu\vartheta + (1-\mu)\varpi)|^q d\mu \right)^{\frac{1}{q}} \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1-\mu) |1-2\mu| [1-(1-\mu)^j] d\mu \right. \\ &\quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 (1-\mu) |1-2\mu| [1-\mu^j] d\mu - k(\varpi - \vartheta)^2 \right. \\ &\quad \left. \times \int_0^1 \mu (1-\mu)^2 |1-2\mu| d\mu \right)^{\frac{1}{q}} \\ &\quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 \mu |1-2\mu| [1-(1-\mu)^j] d\mu \right. \\ &\quad \left. + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \int_0^1 \mu |1-2\mu| [1-\mu^j] d\mu - k(\varpi - \vartheta)^2 \right. \\ &\quad \left. \times \int_0^1 \mu^2 (1-\mu) |1-2\mu| d\mu \right)^{\frac{1}{q}} \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ &\quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ &\quad \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \int_0^1 \mu(1-\mu)^2|1-2\mu|d\mu &= \int_0^1 \mu^2(1-\mu)|1-2\mu|d\mu = \frac{1}{32}, \\ M_1(j) &= \int_0^1 (1-\mu)|1-2\mu|\left[1-(1-\mu)^j\right]d\mu = \int_0^1 \mu|1-2\mu|\left[1-\mu^j\right]d\mu \\ &= \frac{(j^2+j+2)2^j-2}{2^{j+2}(j+2)(j+3)}, \\ M_2(j) &= \int_0^1 \mu|1-2\mu|\left[1-(1-\mu)^j\right]d\mu = \int_0^1 (1-\mu)|1-2\mu|\left[1-\mu^j\right]d\mu \\ &= \frac{(j+5)[(j^2+j+2)2^j-2]}{2^{j+2}(j+1)(j+2)(j+3)}. \end{aligned} \quad \square$$

Corollary 20 If one takes $n = 1$ in (4.19), then

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma)d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q + 3|\Lambda'(\varpi)|^q}{16} - \frac{k}{32}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ &\quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{3|\Lambda'(\vartheta)|^q + |\Lambda'(\varpi)|^q}{16} - \frac{k}{32}(\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 21 If one takes $n = 1$ and $k = 0$ in (4.19), then

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma)d\sigma \right| \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left[\left(\frac{|\Lambda'(\vartheta)|^q + 3|\Lambda'(\varpi)|^q}{16} \right)^{\frac{1}{q}} + \left(\frac{3|\Lambda'(\vartheta)|^q + |\Lambda'(\varpi)|^q}{16} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Also, if one takes $q = 1$ in (4.19), then

$$\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma)d\sigma \right| \leq \frac{\varpi - \vartheta}{4} A(|\Lambda'(\vartheta)|, |\Lambda'(\varpi)|).$$

Corollary 22 If one takes $a_j = 1$ ($j = \overline{1, n}$) and $k = 0$ in (4.19), then

$$\begin{aligned} &\left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma)d\sigma \right| \tag{4.20} \\ &\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n M_1(j) + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{i=1}^n M_2(i) \right)^{\frac{1}{q}} \\ &\quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n M_2(j) + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{i=1}^n M_1(i) \right)^{\frac{1}{q}}. \end{aligned}$$

Inequality (4.20) coincides with the inequality in [30, Theorem 9].

Corollary 23 If one takes $a_j = 1$ ($j = \overline{1, n}$) in (4.19), then

$$\begin{aligned} & \left| \frac{\Lambda(\vartheta) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \vartheta} \int_{\vartheta}^{\varpi} \Lambda(\sigma) d\sigma \right| \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n M_1(j) + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{i=1}^n M_2(i) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \left(\frac{|\Lambda'(\vartheta)|^q}{n} \sum_{j=1}^n M_2(j) + \frac{|\Lambda'(\varpi)|^q}{n} \sum_{i=1}^n M_1(i) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned} \quad (4.21)$$

Inequality (4.21) coincides with the inequality in [4, Theorem 9].

Remark 13 Inequality (4.19) gives better result than the inequality (4.11).

$$\begin{aligned} & \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \times \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{16} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Using concavity of the function $h : [0, \infty) \rightarrow \mathbb{R}$, $h(\sigma) = \sigma^\lambda$, $0 < \lambda \leq 1$, one gets

$$\begin{aligned} & \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \times \left(\frac{|\Lambda'(\vartheta)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) + \frac{|\Lambda'(\varpi)|^q}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{2}{q}} \\ \times \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(|\Lambda'(\vartheta)|^q, |\Lambda'(\varpi)|^q) - \frac{k}{16} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}},$$

where

$$M_1(j) + M_2(j) = \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}}.$$

This completes the proof.

5 Applications for special means

Throughout this section, the following notations will be used for special means of two nonnegative numbers ϑ, ϖ with $\varpi > \vartheta$:

1. The arithmetic mean

$$A(\vartheta, \varpi) = \frac{\vartheta + \varpi}{2}, \quad \vartheta, \varpi \geq 0.$$

2. The geometric mean

$$G(\vartheta, \varpi) = \sqrt{\vartheta \varpi}, \quad \vartheta, \varpi \geq 0.$$

3. The harmonic mean

$$H(\vartheta, \varpi) = \frac{2\vartheta \varpi}{\vartheta + \varpi}, \quad \vartheta, \varpi > 0.$$

4. The logarithmic mean

$$L(\vartheta, \varpi) = \begin{cases} \frac{\varpi - \vartheta}{\ln \varpi - \ln \vartheta}, & \vartheta \neq \varpi, \\ \vartheta, & \vartheta = \varpi; \end{cases} \quad \vartheta, \varpi > 0.$$

5. The p -logarithmic mean

$$L_p(\vartheta, \varpi) = \begin{cases} \left(\frac{\varpi^{p+1} - \vartheta^{p+1}}{(p+1)(\varpi - \vartheta)} \right)^{\frac{1}{p}}, & \vartheta \neq \varpi, p \in \mathbb{R} \setminus \{-1, 0\}, \\ \vartheta, & \vartheta = \varpi. \end{cases} \quad \vartheta, \varpi > 0.$$

6. The identric mean

$$I := I(\vartheta, \varpi) = \frac{1}{e} \left(\frac{\varpi^\varpi}{\vartheta^\vartheta} \right)^{\frac{1}{\varpi - \vartheta}}, \quad \vartheta, \varpi > 0.$$

The following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

Proposition 1 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$. Then,

$$\begin{aligned} & \frac{1}{2} \left(\frac{\sum_{j=1}^n \alpha_j}{\sum_{j=1}^n \alpha_j [1 - (\frac{1}{2})^j]} \right) \left[A^2(\vartheta, \varpi) + \frac{k}{12} (\varpi - \vartheta)^2 \right] \\ & \leq L_2^2(\vartheta, \varpi) \\ & \leq A(\vartheta^2, \varpi^2) \frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \left(\frac{j}{j+1} \right) - \frac{k}{6} (\varpi - \vartheta)^2. \end{aligned}$$

Proof The assertion follows from (3.1) for the function

$$\Lambda(\sigma) = \sigma^2, \quad \sigma \in [-1, 1].$$

□

Proposition 2 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$. Then,

$$\begin{aligned} & \frac{2}{3} [A(\vartheta^{\frac{3}{2}}, \varpi^{\frac{3}{2}}) - L_{\frac{3}{2}}^{\frac{3}{2}}(\vartheta, \varpi)] \\ & \leq \frac{\varpi - \vartheta}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(\vartheta^2, \varpi^2) - \frac{k}{12} (\varpi - \vartheta)^3. \end{aligned}$$

Proof The proof is easily obtained from (3.1) for the function $\Lambda(\sigma) = \frac{2}{3}\sigma^{\frac{3}{2}}$ because the function

$$\Lambda'(\sigma) = \sigma^2, \quad \sigma \in [-1, 1]$$

is GSPOLC.

□

Proposition 3 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$ and $q > 1$. Then,

$$\begin{aligned} & \frac{q}{2+q} [A(\vartheta^{\frac{2+q}{q}}, \varpi^{\frac{2+q}{q}}) - L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(\vartheta, \varpi)] \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{j+1} A(\vartheta^2, \varpi^2) - \frac{k}{6} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Proof The proof is easily seen from (3.1) for the function $\Lambda(\sigma) = \frac{q}{2+q}\sigma^{\frac{2+q}{q}}$ because the function

$$|\Lambda'(\sigma)|^q = \sigma^2 \in [-1, 1]$$

is GSPOLC.

□

Proposition 4 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$ and $q \geq 1$. Then,

$$\begin{aligned} & \frac{q}{2+q} [A(\vartheta^{\frac{2+q}{q}}, \varpi^{\frac{2+q}{q}}) - L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(\vartheta, \varpi)] \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{(j^2 + j + 2)2^j - 2}{(j+1)(j+2)2^{j+1}} A(\vartheta^2, \varpi^2) - \frac{k}{16} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Proof The proof is obvious from (3.1) for the function $\Lambda(\sigma) = \frac{q}{2+q}\sigma^{\frac{2+q}{q}}$ because the function

$$|\Lambda'(\sigma)|^q = \sigma^2 \in [-1, 1]$$

is GSPOLC. \square

Proposition 5 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$ and $q \geq 1$. Then,

$$\begin{aligned} & \frac{q}{2+q} [A(\vartheta^{\frac{2+q}{q}}, \varpi^{\frac{2+q}{q}}) - L_{\frac{2+q}{q}}(\vartheta, \varpi)] \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{\vartheta^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} + \frac{\varpi^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{\vartheta^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j(j+3)}{2(j+1)(j+2)} + \frac{\varpi^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j \frac{j}{2(j+2)} - \frac{k}{12} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Proof The proof is easily seen from (3.1) for the function $\Lambda(\sigma) = \frac{q}{2+q}\sigma^{\frac{2+q}{q}}$ because the function

$$|\Lambda'(\sigma)|^q = \sigma^2 \in [-1, 1]$$

is GSPOLC. \square

Proposition 6 Let $\vartheta, \varpi \in [-1, 1]$ with $\vartheta < \varpi$ and $q \geq 1$. Then,

$$\begin{aligned} & \frac{q}{2+q} [A(\vartheta^{\frac{2+q}{q}}, \varpi^{\frac{2+q}{q}}) - L_{\frac{2+q}{q}}(\vartheta, \varpi)] \\ & \leq \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \quad \times \left(\frac{\vartheta^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) + \frac{\varpi^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}} \\ & \quad + \frac{\varpi - \vartheta}{2} \left(\frac{1}{2} \right)^{2-\frac{2}{q}} \\ & \quad \times \left(\frac{\vartheta^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_2(j) + \frac{\varpi^2}{\sum_{j=1}^n \alpha_j} \sum_{j=1}^n \alpha_j M_1(j) - \frac{k}{32} (\varpi - \vartheta)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

Proof The proof is easily obtained from (3.1) for the function $\Lambda(\sigma) = \frac{q}{2+q}\sigma^{\frac{2+q}{q}}$ because the function

$$|\Lambda'(\sigma)|^q = \sigma^2 \in [-1, 1]$$

is *GSPOLC*. \square

6 Conclusion

This paper introduces a novel class of functions called generalized strongly n -polynomial convex functions. The relationships between these functions and other types of convex functions are investigated. The Hermite–Hadamard inequality is established for generalized strongly n -polynomial convex functions. Additionally, new integral inequalities of the Hermite–Hadamard type are derived for this specific function class using the Hölder–İşcan integral inequality. The results obtained in this paper are compared with existing findings in the literature, highlighting the superiority of the newly acquired results. Furthermore, some applications of these results to special means are explored.

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Author contributions

SÖ and MK gave the idea and initiated the writing of this paper. ii and HK followed up this with some complementary ideas. All authors read and approved the final manuscript.

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