Yang et al. *Boundary Value Problems* (2024) 2024:51
https://doi.org/10.1186/s13661-024-01840-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Multiplicity and nonexistence of positive solutions to impulsive Sturm–Liouville boundary value problems

Xuxin Yang[1], Piao Liu[2*] and Weibing Wang[2]

*Correspondence:
2282486005@qq.com
[2]Department of Mathematics, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the existence, nonexistence, and multiplicity of positive solutions to a nonlinear impulsive Sturm–Liouville boundary value problem with a parameter. By using a variational method, we prove that the problem has at least two positive solutions for the parameter $\lambda \in (0, \Lambda)$, one positive solution for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$, where $\Lambda > 0$ is a constant.

Keywords: Impulsive differential equation; Sturm; Liouville boundary value problem; Positive solution; Critical point

1 Introduction

In this paper, we investigate the following nonlinear impulsive Sturm–Liouville boundary value problem:

$$\begin{cases} -Lu(t) = f(t, u(t)), \quad t \in J/\{t_1, t_2, \dots, t_p\}, \\ \Delta(h(t_k)u'(t_k)) = -\lambda I_k(u(t_k)), \quad k = 1, 2, \dots, p, \\ R_1(u) = R_2(u) = 0, \end{cases} \tag{1.1}$$

where $J = [0, 1]$, p is a positive integer, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $Lu = (h(t)u'(t))' - q(t)u(t)$, $R_1(u) = \alpha u'(0) - \beta u(0), R_2(u) = \gamma u'(1) + \sigma u(1)$, $\alpha, \beta, \gamma, \sigma \in \mathbb{R}$, $\Delta(h(t_k)u'(t_k)) = h(t_k^+)u'(t_k^+) - h(t_k^-)u'(t_k^-)$, $u'(t_k^+)$ and $u'(t_k^-)$ represent the right limit and the left limit of $u'(t)$ at t_k, respectively, and λ is a positive parameter.

The following conditions are assumed:

(H_1) $h \in C^1(J), q \in C(J), h > 0, q > 0$ for all $t \in J$, $\alpha, \beta, \gamma, \sigma \geq 0$, $\alpha^2 + \beta^2 > 0, \gamma^2 + \sigma^2 > 0$, and the linear equation (2.1) has only a trivial solution.

(H_2) $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, where $\mathbb{R}^+ = (0, +\infty)$. $f(t,x) = o(x)$ as $x \to 0^+$ uniformly to $t \in J$, and there exist constants $\mu > 2$ and $r > 0$ such that for $x \geq r, t \in J$,

$$\mu F(t,x) \leq xf(t,x),$$

where $F(t,x) = \int_0^x f(t,s)\,ds$.

(H_3) $I_k \in C([0,+\infty),[0,+\infty))$. For any $k \in \{1,2,\ldots,p\}$, $I_k(x) \to +\infty$ as $x \to +\infty$ or $\sup_{x>0} I_k(x) < \infty$, and there exist $1 \leq K \leq p$, $0 \leq \tau_0 < 1, \kappa > 0$ such that $I_K(x) \geq \kappa x^{\tau_0}$ for $x > 0$.

It should be noted that impulsive differential equations are important models described by phenomena with abrupt changes in their states. Such models have considerable popularity in physics, population dynamics, ecology, industrial robotics, economics, biotechnology, and so on, see [1–3].

In recent years, boundary value problems of impulsive differential equations have been researched extensively, see for example [4–18] and the references therein. In [19], Tian and Ge studied the special cases of (1.1) without parameters

$$\begin{cases} -u''(t) - \mu u(t) = g(t,u(t)), & t \in J/\{t_1,t_2,\ldots,t_p\}, \\ \Delta(h(t_k)u'(t_k)) = s_k(u(t_k)), & k = 1,2,\ldots,p, \\ R_1(u) = R_2(u) = 0, \end{cases} \tag{1.2}$$

where $\mu \in \mathbb{R}$, g and s_k are of superlinear growth or sublinear growth, the authors showed the existence of one or two positive solutions for (1.2). In [20], authors obtained the existence of a sign-changing solution and multiple solutions of

$$\begin{cases} -Lu(t) = g(t,u(t)), & t \in J/\{t_1,t_2,\ldots,t_p\}, \\ \Delta(h(t_k)u'(t_k)) = s_k(u(t_k)), & k = 1,2,\ldots,p, \\ R_1(u) = R_2(u) = 0. \end{cases} \tag{1.3}$$

The technical approach makes use of the minimax method.

In [21], by using a three critical point theorem, authors considered the following fourth order impulsive differential equations with two control parameters:

$$\begin{cases} u^{(4)}(t) + Au''(t) + Bu(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), & t \neq t_k, t \in J, \\ \Delta(u''(t_k)) = I_{1k}(u'(t_k)), & k = 1,2,\ldots,p, \\ \Delta(u'''(t_k)) = I_{2k}(u(t_k)), & k = 1,2,\ldots,p, \\ u(0) = u(T) = u''(0) = u''(T) = 0. \end{cases} \tag{1.4}$$

They established the sharp bounds of the parameters λ and μ for which problem (1.4) admits at least three solutions. For the existence and multiplicity results of solutions of impulsive boundary value problems obtained by using variational methods, we refer to [22–31].

To our knowledge, there are few studies on the connection between the number of solutions and the given parameter for impulsive differential equations. The aim of this paper is to show the multiplicity, existence, and nonexistence of positive solutions for various values of the given parameter. Our result shows that the number of positive solutions of (1.1) changes if the parameter crosses a certain threshold.

This paper is organized as follows. In Sect. 2, we recall some preliminary results. In Sect. 3, by using the mountain pass principle, we prove that (1.1) has at least two positive solutions if λ is sufficiently small, and no solution if λ is sufficiently large, see Theorem 3.1. Throughout the paper, the symbols $C_1, C_2, \ldots$ denote various positive constants whose exact values are not essential to the analysis of the problem. In addition, without loss of generality, we may assume that $f(t,x) = 0, I_k(x) = 0 (1 \leq k \leq p)$ for $x < 0$ as we only consider positive solutions.

2 Basic lemma

Let $G(t,s)$ be Green's function of the problem

$$\begin{cases} -Lu = 0, \\ R_1(u) = R_2(u) = 0. \end{cases} \tag{2.1}$$

From [32], $G(t,s)$ can be written as

$$G(t,s) = \frac{1}{\omega}\begin{cases} m(t)n(s), & 0 \leq t \leq s \leq 1, \\ m(s)n(t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

Lemma 2.1 *The function $G(t,s)$ defined by (2.2) has the following properties:*

(1) $m \in C^2(J,\mathbb{R})$ *is increasing and* $m(t) > 0, t \in (0,1]$.

(2) $n \in C^2(J,\mathbb{R})$ *is decreasing and* $n(t) > 0, t \in [0,1)$.

(3) $Lm \equiv 0, m(0) = \alpha, m'(0) = \beta$.

(4) $Ln \equiv 0, n(1) = \gamma, n'(1) = -\sigma$.

(5) ω *is a positive constant and* $p(t)(m'(t)n(t) - m(t)n'(t)) \equiv \omega$.

(6) $G(t,s)$ *is continuous and symmetric in* $\{(t,s) : 0 \leq t \leq s \leq 1\} \times \{(t,s) : 0 \leq s \leq t \leq 1\}$.

(7) $G(t,s)$ *has continuous partial derivatives in* $\{(t,s) : 0 \leq t \leq s \leq 1\}, \{(t,s) : 0 \leq s \leq t \leq 1\}$.

(8) *For each fixed* $s \in [0,1], G(t,s)$ *satisfies* $LG(t,s) = 0$ *for* $t \neq s, t \in J$ *and* $R_1(G) = R_2(G) = 0$.

(9) G'_t *has a discontinuous point of the first kind at* $t = s$ *and* $G'_t(s+0,s) - G'_t(s-0,s) = -\frac{1}{h(s)}, s \in (0,1)$.

Let $P = \{u \in C(J), u \geq 0\}$ and $(Tu)(t) = \int_0^1 G(t,s)u(s)\,ds$, then

$$T\big(P/\{0\}\big) \subset \operatorname{int} P.$$

By the Krein–Rutman theorem, the spectral radius $r(T) > 0$ is determined by a simple eigenvalue of T having an eigenfunction $\varphi_0(t) \in P$ with $\varphi_0 > 0, t \in (0,1)$. It is easy to check that $\lambda^* = r^{-1}(T)$ is the smallest eigenvalue of the eigenvalue problem:

$$\begin{cases} -Lu = \lambda u, \\ R_1(u) = R_2(u) = 0, \end{cases} \tag{2.3}$$

and φ_0 is eigenfunction corresponding to λ_1.

Definition 2.1 A function $u \in \Theta = \{x \in C(J) : hx' \in C(J/\{t_1,\dots,t_p\}), x'(t_k^+), x'(t_k^-)$ exists and $x'(t_k) = x'(t_k^-)\}$ is said to be a solution of (1.1) if u satisfies the equation in (1.1) for $t \in J/\{t_1,t_2,\dots,t_p\}$ and impulsive conditions, boundary conditions of (1.1). The function u is a positive solution of (1.1) if u is a solution of (1.1) and $u(t) > 0$ for $t \in (0,1)$.

Let

$$\begin{aligned}
&H^1(0,1) = \{u \in L^2(0,1) : u' \in L^2(0,1)\},\\
&H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\},\\
&\Sigma_1 = \{u \in H^1(0,1) : u(0) = 0\},\\
&\Sigma_2 = \{u \in H^1(0,1) : u(1) = 0\},\\
&A = \begin{cases} \frac{\sigma}{\gamma}, & \gamma \neq 0,\\ 0, & \gamma = 0, \end{cases} \qquad B = \begin{cases} \frac{\beta}{\alpha}, & \alpha \neq 0,\\ 0, & \alpha = 0, \end{cases}\\
&H = \begin{cases} H^1(0,1), & \alpha \neq 0, \gamma \neq 0,\\ H_0^1(0,1), & \alpha = \gamma = 0,\\ \Sigma_1, & \alpha = 0, \gamma \neq 0,\\ \Sigma_2, & \alpha \neq 0, \gamma = 0. \end{cases}
\end{aligned}$$

Define an inner product in H as follows:

$$(u,v) = \int_0^1 \bigl(h(t)u'v' + q(t)uv\bigr)\,dt + h(1)Au(1)v(1) + h(0)Bu(0)v(0).$$

This inner product induces the norm

$$\|u\| = \left(\int_0^1 \bigl(h\bigl(u'\bigr)^2 + qu^2\bigr)\,dt + h(1)Au^2(1) + h(0)Bu^2(0)\right)^{\frac{1}{2}}.$$

It is easy to check that H with the inner product $(\cdot,\cdot)$ is a Hilbert space, and $u^+ = \max\{u,0\} \in H, u^- = \max\{-u,0\} \in H$ for $u \in H$.

Define the functional Φ_λ in H by

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F\bigl(t,u(t)\bigr)\,dt - \lambda \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)\,ds.$$

Then $\Phi_\lambda \in C^1(H,R)$, and

$$\bigl\langle \Phi'_\lambda(u), v\bigr\rangle = (u,v) - \int_0^1 f\bigl(t,u(t)\bigr)v(t)\,dt - \lambda \sum_{k=1}^{p} I_k\bigl(u(t_k)\bigr)v(t_k).$$

Lemma 2.2 *If $u \in H$, then there is $C > 0$ such that*

$$|u|_0 \le C\|u\|, \quad \forall u \in H,$$

where $|u|_0 = \max_{t\in J} |u(t)|$.

Proof Let $\|\cdot\|_H$ be the usual norm of $H^1(0,1)$. From the imbedding theorem, we know that there is $C_0 > 0$ such that

$$|u|_0 \le C_0\|u\|_H.$$

Note that there exists $C > 0$ such that

$$|u|_0 \le C_0\|u\|_H \le C\|u\|.$$

The claim follows. □

Lemma 2.3 *The problem of finding a solution u of (1.1) is equivalent to that of finding a critical point of Φ_λ, that is, $\langle \Phi'_\lambda(u), v\rangle = 0$ for all $v \in H$.*

Proof Assume that $u \in \Theta$ is a solution of (1.1). It is easy to check that $u \in H$. For any $v \in H$,

$$\int_0^1 \bigl[-\bigl(hu'\bigr)' + qu\bigr]v\,dt = \int_0^1 f(t,u)v\,dt,$$

that is,

$$\begin{aligned}
&-\int_0^{t_1} v\bigl(hu'\bigr)'\,dt - \int_{t_1}^{t_2} v\bigl(hu'\bigr)'\,dt - \cdots - \int_{t_p}^{1} v\bigl(hu'\bigr)'\,dt + \int_0^1 quv\,dt = \int_0^1 f(t,u)v\,dt,\\
&\int_0^1 hu'v'\,dt + \int_0^1 huv\,dt + h(0)u'(0)v(0) - h(1)u'(1)v(1)\\
&\quad = \int_0^1 f(t,u)v\,dt + \lambda\sum_{k=1}^{p} I_k\bigl(u(t_k)\bigr)v(t_k),\\
&(u,v) = \int_0^1 f(t,u)v\,dt + \lambda\sum_{k=1}^{p} I_k\bigl(u(t_k)\bigr)v(t_k).
\end{aligned}$$

Hence, u is a critical point of Φ_λ in H.

Let $u \in H$ be a critical point of Φ_λ, then $\langle \Phi'_\lambda(u), v\rangle = 0$ for any $v \in H$. Without generality, we have

$$\begin{aligned}
0 &= \int_{t_k}^{t_{k+1}} \bigl(hu'v' + quv - f(t,u)v\bigr)\,dt\\
&= \int_{t_k}^{t_{k+1}} hu'v'\,dt + \int_{t_k}^{t_{k+1}} v\,d\biggl(\int_t^{t_k} \bigl(q(s)u(s) - f\bigl(s,u(s)\bigr)\bigr)\,ds\biggr)\\
&= \int_{t_k}^{t_{k+1}} \biggl[h(t)u' - \int_t^{t_k} \bigl(q(s)u(s) - f\bigl(s,u(s)\bigr)\bigr)\,ds\biggr]v'(t)\,dt.
\end{aligned}$$

Hence,

$$h(t)u'(t) - \int_t^{t_k} \bigl(q(s)u(s) - f\bigl(s,u(s)\bigr)\bigr)\,ds \equiv C, \quad \text{a.e } t \in (t_k, t_{k+1}).$$

So, hu' has a weak derivative satisfying

$$\bigl(hu'\bigr)' - q(t)u(t) + f\bigl(t,u(t)\bigr) = 0, \quad \text{a.e. } t \in (t_k, t_{k+1}).$$

From the continuity of h, q, f, u, we see that $(hu')'$ exists for $t \in (t_k, t_{k+1})$. Hence u satisfies the equality in (1.1).

Noting that

$$\begin{aligned}0 &= (u,v) - \int_0^1 f\big(t,u(t)\big)v(t)\,dt - \lambda\sum_{k=1}^{p} I_k\big(u(t_k)\big)v(t_k)\\&= \sum_{k=0}^{p} h(t)u'(t)v(t)\big|_{t_k^+}^{t_{k+1}} + \int_0^1 (-Lu)v\,dt + h(1)Au(1)v(1) + h(0)Bu(0)v(0)\\&\quad - \int_0^1 f\big(t,u(t)\big)v(t)\,dt - \lambda\sum_{k=1}^{p} I_k\big(u(t_k)\big)v(t_k)\\&= \int_0^1 \big(-Lu - f(t,u)\big)v\,dt + h(1)\big[u'(1) + Au(1)\big]v(1) + h(0)\big[-u'(0) + Bu(0)\big]v(0)\\&\quad - \sum_{k=1}^{p}\big[\Delta\big(h(t_k)u'(t_k)\big) + \lambda I_k\big(u(t_k)\big)\big]v(t_k),\end{aligned}$$

we get

$$\begin{aligned}&h(1)\big[u'(1) + Au(1)\big]v(1) + h(0)\big[-u'(0) + Bu(0)\big]v(0)\\&\quad - \sum_{k=1}^{p}\big[\Delta\big(h(t_k)u'(t_k)\big) + \lambda I_k\big(u(t_k)\big)\big]v(t_k) = 0.\end{aligned} \tag{2.4}$$

Next, we show that u satisfies the impulsive conditions in (1.1). If this is not the case, without loss of generality, we may assume that there exists $i \in \{1,2,\dots,p\}$ such that

$$\Delta\big(h(t_i)u'(t_i)\big) + \lambda I_i\big(u(t_i)\big) \neq 0.$$

Let $v = \Pi_{j=0,j\neq i}^{p+1}(t - t_j)$, then by (2.4),

$$\begin{aligned}&h(1)\big[u'(1) + Au(1)\big]v(1) + h(0)\big[-u'(0) + Bu(0)\big]v(0)\\&\qquad - \sum_{k=1}^{p}\big[\Delta\big(h(t_k)u'(t_k)\big) + \lambda I_k\big(u(t_k)\big)\big]v(t)\\&\quad = -\big[\Delta\big(h(t_i)u'(t_i)\big) + \lambda I_i\big(u(t_i)\big)\big]v(t_i) \neq 0,\end{aligned}$$

which contradicts (2.4). So u satisfies the impulsive conditions in (1.1). Similarly, u satisfies the boundary conditions. □

Lemma 2.4 *The function $u \in \Theta$ is a solution of* (1.1), *then u satisfies*

$$u(t) = \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds + \lambda\sum_{k=1}^{p} G(t,t_k)I_k\big(u(t_k)\big).$$

Proof Let $g \in C(J), d_k \in \mathbb{R}$ $(1 \le k \le p)$ and consider the equation

$$\begin{cases}-Lu = g(t), & t \neq t_k,\\ \Delta(h(t_k)u'(t_k)) = -d_k, & k = 1,2,\dots,p,\\ R_1(u) = R_2(u) = 0.\end{cases} \tag{2.5}$$

We only need to show that the solution u of (2.5) satisfies

$$u(t) = \int_0^1 G(t,s)g(s)\,ds + \sum_{k=1}^{p} G(t,t_k)d_k.$$

It is easy to check that (2.5) has a unique solution. Let

$$S_1(t) = \int_0^1 G(t,s)g(s)\,ds, \qquad S_2(t) = \sum_{k=1}^{p} G(t,t_k)d_k, \qquad S(t) = S_1(t) + S_2(t).$$

It follows from Theorem 3.1.1 in [32] that $S_1 \in C^2(J)$ and

$$-LS_1 = g(t), \qquad R_1(S_1) = R_2(S_1) = 0.$$

Hence,$-\Delta(h(t_k)S_1'(t_k)) = 0$ for $1 \le k \le p$. From (8) and (9) of Lemma 2.1, one easily shows that $S_2 \in C(J)$ and

$$\begin{cases} -LS_2 = 0, \quad t \neq t_k, \\ -\Delta(h(t_k)S_2'(t_k)) = d_k, \quad k = 1,2,\dots,p, \\ R_1(u) = R_2(u) = 0, \end{cases} \tag{2.6}$$

which implies that $(hS_2)' = qS_2 \in C(J/\{t_1,\dots,t_p\})$. Hence, S is the solution of (2.5). □

Corollary 2.1 *Let $g \ge 0$ for $t \in J$ and $d_k \ge 0$ for $1 \le k \le p$, then the solution of (2.5) is positive if $g \not\equiv 0$ or $d_k \not\equiv 0$.*

3 Main result

In this section, we give our main result. Firstly, we need to prove some lemmas.

Lemma 3.1 *Let $(H_1) - (H_3)$ hold. Then the problem*

$$\begin{cases} -Lu = f(t,u(t)), \quad t \neq t_k, \\ -\Delta(h(t_k)u(t_k)) = \tau \in \mathbb{R}^+, \quad k = 1,2,\dots,p, \\ R_1(u) = R_2(u) = 0 \end{cases} \tag{3.1}$$

has a positive solution for sufficiently small $\tau > 0$.

Proof Let

$$\widetilde{C} = \inf\left\{C : \int_0^1 u^2\,dt \le C\|u\|, u \in H\right\},$$
$$\widehat{C} = \inf\left\{C : |u|_0 \le C\|u\|, \forall u \in H\right\}.$$

By the imbedding theorem, $\widetilde{C} > 0, \widehat{C} > 0$. From (H_2), there exist $a > 0, b > 0, \sigma > 0$ such that

$$F(t,\varepsilon) \le \frac{1}{4\widetilde{C}}\varepsilon^2, \quad \forall t \in J, \forall \varepsilon < \sigma, \tag{3.2}$$

$$F(t,s) \geq as^{\mu} - b, \quad \forall t \in J, \forall s \geq 0. \tag{3.3}$$

Consider the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F\bigl(t,u(t)\bigr)\,dt - \tau\sum_{k=1}^{p} u(t_k),$$

whose critical point is a solution of (3.1). If $u \neq 0$ is a solution of (3.1), from Corollary 2.1, u is a positive solution of (3.1).

Taking $\tau < \frac{\sigma}{8\widehat{C}p}$, for $\|u\| = 8p\tau$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 F(t,u)\,dt - \tau\sum_{k=1}^{p} u(t_k) \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4\widetilde{C}}\int_0^1 u^2\,dt - \tau p|u|_0 \\ &\geq \|u\|\left(\frac{1}{4}\|u\| - \tau p\widehat{C}\right) = 8(p\widehat{C}\tau)^2 > 0. \end{aligned}$$

For any $u^+ \neq 0$,

$$\begin{aligned} J\bigl(tu^+\bigr) &= \frac{t^2}{2}\bigl\|u^+\bigr\|^2 - \int_0^1 F\bigl(s,tu^+\bigr)\,ds - \tau t\sum_{k=1}^{p} u^+(t_k) \\ &= \frac{t^2}{2}\bigl\|u^+\bigr\|^2 - at^{\mu}\int_0^1 \bigl(u^+\bigr)^{\mu}\,ds + b - \tau t\sum_{k=1}^{p} u^+(t_k) \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$ since $\mu > 2$. Hence, there exists $e \in H$ with $\|e\| > 8p\tau$ such that $J(e) < 0$.

It remains to check that J satisfies the PS condition. Let $\{u_n\} \subset H$ such that $|J(u_n)| \leq M$ and $|J'(u_n)| \to$ as $n \to \infty$. Then

$$\begin{aligned} &M + \frac{1}{\mu}\|u_n\| \\ &\quad\geq J(u_n) - \frac{1}{\mu}J'(u_n)u_n \\ &\quad= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \int_0^1 \left[F(t,u_n) - \frac{1}{\mu}f(t,u_n)u_n\right]dt - \tau\left(1 - \frac{1}{\mu}\right)\sum_{k=1}^{p} u_n(t_k) \\ &\quad\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - C_2\|u_n\| - C_1 \end{aligned}$$

for some constants $C_1 > 0$ and $C_2 > 0$. So, $\{u_n\}$ is bounded in H. Considering a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in H. Thus,

$$\begin{aligned} 0 &\leftarrow (J'(u_n) - J'(u), u_n - u) \\ &= \|u_n - u\|^2 - \int_0^1 [f(t,u_n) - f(t,u)](u_n - u)\,dt - \tau\sum_{k=1}^{p}(u_n(t_k) - u(t_k)). \end{aligned}$$

Noting that $u_n \rightharpoonup u$ in H implies $u_n \to u$ in $C(J)$, we have

$$\int_0^1 \big[f(t,u_n)-f(t,u)\big](u_n-u)\,dt \to 0, \sum_{k=1}^{p}\big(u_n(t_k)-u(t_k)\big) \to 0$$

as $n \to \infty$. Hence, $\|u_n - u\| \to 0$, that is, $u_n \to u$ in H. Then the mountain pass theorem implies that J has the critical point u with $J(u) > 0$. Clearly, $u \not\equiv 0$. □

Let $\Lambda_1 = \{\lambda > 0, (1.1)$ has positive solution$\}$ and $\Lambda = \sup \Lambda_1$.

Lemma 3.2 *There exists $\bar{\lambda} > 0$ such that (1.1) has at least a positive solution u_λ for $\lambda \in (0,\bar{\lambda})$ and u_λ is the local minimum of Φ_λ with $\Phi_\lambda(u_\lambda) < 0$. Moreover, $u_{\lambda_1} \le u_{\lambda_2}$ for $0 < \lambda_1 \le \lambda_2 < \bar{\lambda}$.*

Proof Let $\tau > 0$ be sufficiently small and x_τ be the positive solution of (3.1) in Lemma 3.1. Consider the equation

$$\begin{cases} -Lu = f(t, T_{x_\tau}(u)), \quad t \neq t_k, \\ -\Delta h(t_k)u(t_k) = \lambda_0 I_k(T_{x_\tau(t_k)}(u(t_k))), \quad k = 1,2,\dots,p, \\ R_1(u) = R_2(u) = 0, \end{cases} \tag{3.4}$$

where

$$\lambda_0 = \frac{\tau}{\max_{1\le k\le p}(I_k(x_\tau(t_k)))},$$

$$T_\alpha(u) = \begin{cases} \alpha, & u > \alpha, \\ u, & 0 \le u \le \alpha, \quad \text{if } \alpha > 0 \\ 0, & u < 0. \end{cases}$$

Obviously, the solution of (3.4) is equivalent to the critical point of $\phi_{\lambda_0,T_{x_\tau}}$, where

$$\phi_{\lambda,T_{x_\tau}}(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F_{T_{x_\tau}}\big(t,u(t)\big)\,dt - \lambda\sum_{k=1}^{p} I_k^{T_{x_\tau(t_k)}}\big(u(t_k)\big),$$

$$F_{T_{x_\tau}}(t,u) = \int_0^u f\big(t, T_{x_\tau}(s)\big)\,ds, \qquad I_k^{T_{x_\tau(t_k)}}(u) = \int_0^u I_k\big(T_{x_\tau(t_k)}(s)\big)\,ds.$$

Since $T_{x_\tau}(s), T_{x_\tau(t_k)}(s)$ are bounded, we obtain that for any $u \in H$,

$$\big|F_{T_{x_\tau}}(t,u)\big| \le C_3|u|_0 \le C_4\|u\|,$$

$$\big|I_K^{T_{x_\tau(t_k)}}\big(u(t_k)\big)\big| \le C_5|u|_0 \le C_6\|u\|,$$

which imply that $\phi_{\lambda_0,T_{x_\tau}(u)} \to +\infty$ as $\|u\| \to +\infty$. In addition, $\phi_{\lambda_0,T_{x_\tau}(u)}$ is sequentially weakly lower semicontinuous. It follows that there exists $u_{\lambda_0} \in H$ such that

$$\phi_{\lambda_0,T_{x_\tau}}(u_{\lambda_0}) = \inf\big\{\phi_{\lambda_0,T_{x_\tau}}(u) : u \in H\big\} = m_{\lambda_0}.$$

Let $\xi > 0$ be sufficiently small such that $\xi\varphi_0(t_K) \le x_\tau(t_K)$. We obtain

$$\begin{aligned}\phi_{\lambda_0,T_{x_\tau}}(\xi\varphi_0) &= \frac{\xi^2}{2}\|u\|^2 - \int_0^1 F_{T_{x_\tau}}(t,\xi\varphi_0)\,dt - \lambda_0\sum_{k=0}^{p} I_k^{T_{x_\tau}}\big(\xi\varphi_0(t_k)\big),\\ &\le \frac{\xi^2}{2}\|u\|^2 - \lambda_0\int_0^{\xi\varphi_0(t_K)} I_K\big(T_{x_\tau(t_K)}(s)\big)\,ds,\\ &\le \frac{\xi^2}{2}\|u\|^2 - \frac{\kappa\lambda_0}{1+\tau_0}\big(\xi\varphi_0(t_K)\big)^{\tau_0+1} := g(\xi),\end{aligned}$$

where we use the fact that

$$F_{T_{x_\tau}}(t,\xi\varphi_0) \ge 0, \qquad I_k^{T_{x_\tau(t_k)}}\big(\xi\varphi_0(t_k)\big) \ge 0 \quad (k \ne K).$$

Clearly,

$$m_{\lambda_0} \le \min_{\xi>0} g(\xi) = \frac{\tau_0 - 1}{2}\big(\kappa\lambda_0\varphi_0^{1+\tau_0}(t_K)\big)^{\frac{1}{1-\tau_0}} < 0 = \phi_{\lambda_0,T_{x_\tau}}(0),$$

which implies that $u_{\lambda_0} \not\equiv 0$. It is easy to show that $u_{\lambda_0} \ge 0$ and

$$(u_{\lambda_0}, v) = \int_0^1 f\big(t, T_{x_\tau}\big(u_{\lambda_0}(t)\big)\big)v(t) + \lambda_0\sum_{k=1}^{p} I_k\big(T_{x_\tau(t_k)}\big(u_{\lambda_0}(t_k)\big)\big)v(t_k). \tag{3.5}$$

Choosing $v = (u_{\lambda_0} - x_\tau)^+ \in H$ as a test function, we have

$$\begin{aligned}&\big(u_{\lambda_0}, (u_{\lambda_0} - x_\tau)^+\big)\\ &\quad= \int_0^1 f\big(t, T_{x_\tau}(u_{\lambda_0})\big)(u_{\lambda_0} - x_\tau)^+\,dt + \lambda_0\sum_{k=1}^{p} I_k\big(T_{x_\tau(t_k)}\big(u_{\lambda_0}(t_k)\big)\big)(u_{\lambda_0} - x_\tau)^+(t_k)\\ &\quad\le \int_0^1 f(t,x_\tau)(u_{\lambda_0} - x_\tau)^+\,dt + \lambda_0\sum_{k=1}^{p} I_k\big(x_\tau(t_k)\big)(u_{\lambda_0} - x_\tau)^+(t_k)\\ &\quad\le \int_0^1 f(t,x_\tau)(u_{\lambda_0} - x_\tau)^+\,dt + \tau\sum_{k=1}^{p}(u_{\lambda_0} - x_\tau)^+(t_k) = \big(x_\tau, (u_{\lambda_0} - x_\tau)^+\big)\\ &\quad\Rightarrow\quad \big(u_{\lambda_0} - x_\tau, (u_{\lambda_0} - x_\tau)^+\big) \le 0,\\ &\quad\Rightarrow\quad \big\|(u_{\lambda_0} - x_\tau)^+\big\| \le 0,\\ &\quad\Rightarrow\quad u_{\lambda_0} \le x_\tau,\\ &\quad\Rightarrow\quad \begin{cases} -Lu_{\lambda_0} = f(t, u_{\lambda_0}),\\ -\Delta(h(t_k)u'_{\lambda_0}(t_k)) = \lambda_0 I_k(u_{\lambda_0}(t_k)),\\ R_1(u_{\lambda_0}) = R_2(u_{\lambda_0}) = 0.\end{cases}\end{aligned}$$

Next we show that u_{λ_0} is a positive solution of (1.1) with $\lambda = \lambda_0$. Since $u_{\lambda_0} \not\equiv 0$ and $u_{\lambda_0} \ge 0$, we may assume that there exists $t^* \in (0,1)$ such that $u_{\lambda_0}(t^*) > 0$. Because u_{λ_0} is continuous, there exists an open interval $D \subset J$ with $t^* \in D$ such that $u_{\lambda_0}(t) > 0$ for all $t \in D$. Hence

$f(t,u_{\lambda_0}(t)) > 0$ for $t \in D$. From Lemma 2.3, we obtain

$$\begin{aligned} u_{\lambda_0}(t) &= \int_0^1 G(t,s)f\big(s,u_{\lambda_0}(s)\big)\,ds + \lambda_0 \sum_{k=1}^{p} I_k\big(u_\lambda(t_k)\big) \\ &\geq \int_D G(t,s)f\big(s,u_\lambda(s)\big)\,ds > 0 \end{aligned}$$

for $t \in (0,1)$.

Assume that $\mu \in (0,\lambda)$ and u_λ is a positive solution of (1.1) with the parameter λ. We consider the functional $\phi_{\mu,T_{u_\lambda}}$. By using a similar reasoning as above, one may obtain that $\phi_{\mu,T_{u_\lambda}}$ has the critical point $u_\mu \leq u_\lambda$, which is a positive solution of (1.1) with μ and the local minimum of Φ_μ with $\Phi_\mu(u_\mu) < 0$. □

Lemma 3.3 $0 < \Lambda < +\infty$.

Proof Clearly, $\Lambda_1 \neq \emptyset$. Let u_λ be a positive solution of (1.1), then

$$(u_\lambda, v) = \int_0^1 f(t,u_\lambda)v\,dt + \lambda \sum_{k=1}^{p} I_k\big(u_\lambda(t_k)\big)v(t_k), \quad \forall v \in H. \tag{3.6}$$

Note that φ_0 is the solution of (2.3) with $\lambda = \lambda^*$, which satisfies

$$(\varphi_0, v) = \lambda^* \int_0^1 \varphi_0(t)v(t)\,dt, \quad \forall v \in H. \tag{3.7}$$

From (3.6) and (3.7), we have

$$\lambda^* \int_0^1 \varphi_0 u_\lambda\,dt = \int_0^1 f(t,u_\lambda)\varphi_0\,dt + \lambda \sum_{k=1}^{p} I_k\big(u_\lambda(t_k)\big)\varphi_0(t_k). \tag{3.8}$$

By (H_2) and (3.3), there exists $C_7 > 0$ such that

$$f(t,s) \geq \big(1 + \lambda^*\big)s - C_7, \quad \forall t \in J, \forall s \geq 0.$$

Hence,

$$\int_0^1 \varphi_0 u_\lambda\,dt \leq C_8, \tag{3.9}$$

$$\lambda\kappa u_\lambda^{\tau_0}(t_K)\varphi_0(t_K) \leq \lambda^* \int_0^1 \varphi_0 u_\lambda\,dt \leq \lambda^* C_8. \tag{3.10}$$

In addition, by Lemma 2.3 and (H_3), we have

$$u_\lambda(t_K) \geq \lambda \sum_{k=1}^{p} G(t_k,t_k)I_k\big(u_\lambda(t_k)\big) \geq \lambda\kappa G(t_K,t_K)u_\lambda^{\tau_0}(t_K),$$

$$u_\lambda(t_K) \geq \big(\lambda\kappa G(t_K,t_K)\big)^{\frac{1}{1-\tau_0}},$$

which implies that

$$\lambda \leq \kappa^{-1}\left(\frac{\lambda^* C_8}{\varphi_0(t_K)}\right)^{1-\tau_0} G^{-1}(t_K,t_K) < +\infty.$$

Hence, $\Lambda < +\infty$. □

Remark 3.1 Since $\lambda^* \int_0^1 \varphi_0 u_\lambda \, dt \geq \lambda \sum_{k=1}^p I_k(u_\lambda(t_k))\varphi_0(t_k)$, there exists $M > 0$ independent of λ such that $I_k(u_\lambda(t_k)) < M$ for $\forall 1 \leq k \leq p$. By (H_3), there exists $\overline{C} > 0$ independent of λ such that for any $1 \leq k \leq p$,

$$\int_0^{u_\lambda(t_k)} I_k(t)\, dt \leq \overline{C}\|u_\lambda\|.$$

Lemma 3.4 $\Lambda \in \Lambda_1$.

Proof Let $\{\lambda_n\} \in \Lambda_1$ be an increasing sequence such that $\lambda_n \to \Lambda$ as $n \to \infty$. For every $n \geq 1$, one can find $u_n \in H$ such that u_n is a positive solution of (1.1) with $\lambda = \lambda_n$. Since $f \geq 0, I_k \geq 0$ and λ is increasing, if $m > n$,

$$\begin{cases} -Lu_m = f(t,u_m), & t \neq t_k, \\ -\Delta(h(t_k)u_m'(t_k)) = \lambda_m I_k(u_m(t_k)) \geq \lambda_n I_k(u_m(t_k)), & k = 1,2,\ldots,p. \end{cases}$$

Consider the functional $\phi_{\lambda_n,T_{u_m}}$. Similar to Lemma 3.2, we obtain that $\phi_{\lambda_n,T_{u_m}}$ has a critical point $u_{\lambda_n} \leq u_m$, which is a local minimum of ϕ_{λ_n} with $\Phi_{\lambda_n}(u_{\lambda_n}) < 0$. Hence, without loss of generality, we may assume that for all $n \geq 1$,

$$\Phi_{\lambda_n}(u_n) \leq \frac{\tau_0 - 1}{2}\left(\kappa\lambda_n\varphi_0^{1+\tau_0}(t_K)\right)^{\frac{1}{1-\tau_0}} < 0.$$

Hence,

$$(u_n,u_n) = \int_0^1 f(t,u_n)u_n\, dt + \lambda_n \sum_{k=1}^p I_k\big(u_n(t_k)\big)u_n(t_k),$$

$$(u_n,u_n) \leq 2\int_0^1 F(t,u_n)\, dt + 2\lambda_n \sum_{k=1}^p \int_0^{u_n(t_k)} I_k(s)\, ds,$$

$$\int_0^1 \big[f(t,u_n)u_n - 2F(t,u_n)\big]\, dt \leq \lambda_n\left[\sum_{k=1}^p 2\int_0^{u_n(t_k)} I_k(s)\, ds - I_k\big(u_n(t_k)\big)u_n(t_k)\right].$$

From Remark 3.1 and (H_2), we have

$$\begin{aligned} &\left(1 - \frac{2}{\mu}\right)\int_0^1 f(t,u_n)u_n\, dt + 2\int_{u_n \geq r}\left[\frac{f(t,u_n)u_n}{\mu} - F(t,u_n)\right] dt \\ &\quad + 2\int_{u_n \leq r}\left[\frac{f(t,u_n)u_n}{\mu} - F(t,u_n)\right] dt \leq C_9\|u_n\|, \\ &\int_0^1 f(t,u_n)u_n \leq C_{10}\|u_n\| + C_{11}. \end{aligned}$$

Hence

$$\|u\|^2 = (u_n, u_n) \le C_{12}\|u_n\| + C_{13},$$

which implies that $\{u_n\}$ is bounded in H. Up to a subsequence, we have

$$u_n \rightharpoonup \hat{u} \in H \quad \text{in } H, \qquad u_n \to \hat{u} \in H \quad \text{in } C(J).$$

It follows that for any $v \in H$,

$$\begin{aligned}
&(u_n, v) \to (\hat{u}, v), \\
&\int_0^1 f(t,u_n)v\,dt \to \int_0^1 f(t,\hat{u})v\,dt, \qquad I_k\big(u_n(t_k)\big) \to I_k\big(\hat{u}(t_k)\big).
\end{aligned}$$

Combining with $\langle \Phi'_{\lambda_n}(u_n), v\rangle = 0$ and $\lambda_n \to \Lambda$, we have

$$\big\langle \phi'_\lambda(\hat{u}), v\big\rangle = (\hat{u}, v) - \int_0^1 f(t,\hat{u})v\,dt - \Lambda \sum_{k=1}^p I_k\big(\hat{u}(t_k)\big)v(t_k) = 0.$$

Hence, $\hat{u}$ is a solution of (1.1) with $\lambda = \Lambda$. Finally, we show that $\hat{u} > 0$ for $t \in (0,1)$. Clearly, $\hat{u} \ge 0$ since $u_n \ge 0$. In addition,

$$\begin{aligned}
(u_n, u_n - \hat{u}) &= \int_0^1 f(t,u_n)(u_n - \hat{u})\,dt + \lambda_n \sum_{k=1}^p I_k\big(u_n(t_k)\big)\big(u_n(t_k) - \hat{u}(t_k)\big) \\
&\to 0, \\
(u_n, \hat{u}) &\to (\hat{u}, \hat{u}),
\end{aligned}$$

and therefore, $\|u_n\| \to \|\hat{u}\|$. Hence,

$$\Phi_\Lambda(\hat{u}) \leftarrow \Phi_{\lambda_n}(u_n) \le \big(\kappa\lambda_n\varphi_0^{1+\tau_0}(t_k)\big)^{\frac{1}{1-\tau_0}}\frac{\tau_0 - 1}{2} < 0,$$

which implies that $\hat{u} \not\equiv 0$, which is the positive solution of (1.1) with $\lambda = \Lambda$. □

Define

$$\begin{aligned}
&f_0(t,x) = \begin{cases} f(t,u_\lambda), & x < u_\lambda, \\ f(t,x), & x \ge u_\lambda, \end{cases} \qquad i_k(x) = \begin{cases} I_k(u_\lambda), & x < u_\lambda, \\ I_k(x), & x \ge u_\lambda. \end{cases} \\
&\Phi_0(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F_0(t,u)\,dt - \lambda \sum_{k=1}^p \widetilde{I}_k\big(u(t_k)\big), \\
&F_0(t,x) = \int_0^x f_0(t,s)\,ds, \qquad \widetilde{I}_k(x) = \int_0^x i_k(s)\,ds,
\end{aligned}$$

where u_λ is the local minimum of Φ_λ with $\Phi_\lambda(u_\lambda) < 0$ obtained in Lemma 3.2.

Definition 3.1 Let $\Xi \subseteq H$ be a closed set and $\varphi \in C^1(H,\mathbb{R})$. We say that a sequence $\{v_n\} \subset H$ is a $(PS)_{\Xi,c}$ sequence of φ if

$$\operatorname{dist}(v_n, \Xi) \to 0, \qquad \varphi(v_n) \to c, \qquad \|\varphi'(v_n)\| \to 0$$

as $n \to \infty$. φ satisfies the $(PS)_{\Xi,c}$ condition if every $(PS)_{\Xi,c}$ sequence of φ has a convergent subsequence.

Lemma 3.5 [33] *Let $\varphi \in C^1(H,\mathbb{R})$. Consider the number*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi\bigl(\gamma(t)\bigr),$$

where Γ is the set of all continuous paths joining two points u and v in H. Suppose that Ξ is a closed subset of H such that

$$\Xi \cap \bigl\{w \in H : \varphi(w) \geq c\bigr\}$$

separates u and v. If φ satisfies the $(PS)_{\Xi,c}$ condition, then φ has a critical point of level c on Ξ.

Lemma 3.6 *Suppose that Ξ is a close subset of H, then Φ_0 satisfies the $(PS)_{\Xi,c}$ condition for any $c \in \mathbb{R}$.*

Proof Clearly, $\Phi_0 \in C^1(H,\mathbb{R})$, and there exist $C_{14} > 0, C_{15} > 0$ such that

$$xf_0(t,x)x - \mu F_0(t,x) > -C_{14}, \quad \forall t \in J, \forall x \in \mathbb{R}, \tag{3.11}$$

$$f_0(t,x) \geq \bigl(1 + \lambda^*\bigr)x - C_{14}, \quad \forall t \in J, \forall x \in \mathbb{R}, \tag{3.12}$$

$$F_0(t,x) \geq C_{15}x^{\mu} - C_{14}, \quad \forall t \in J, \forall x > 0, \tag{3.13}$$

$$f_0(t,x) \geq 0, \qquad i_k(x) \geq 0 \quad (1 \leq i \leq p), \forall t \in J, x \in \mathbb{R}, \tag{3.14}$$

$$i_k(x) \to +\infty \quad \text{as } x \to +\infty \quad \text{or} \quad \sup_{x>0} i_k(x) < +\infty, \quad \forall 1 \leq k \leq p. \tag{3.15}$$

Assume that $\{u_n\} \subset H$ is a $(PS)_{\Xi,c}$ sequence of Φ_0, we have

$$(u_n, v) = \int_0^1 f_0(t,u_n)v\,dt + \lambda \sum_{k=1}^{p} i_k\bigl(u_n(t_k)\bigr)v(t_k) + o_n(1), \quad \forall v \in H. \tag{3.16}$$

Similar to (3.8), using (3.7) and (3.16), we have

$$\lambda^* \int_0^1 u_n\varphi_0\,dt = \int_0^1 f_0(t,u_n)\varphi_0\,dt + \lambda \sum_{k=1}^{p} i_k\bigl(u_n(t_k)\bigr)\varphi_0(t_k) + o_n(1). \tag{3.17}$$

Let $\Omega_n^1 = \{t \in J : u_n(t) \geq 0\}, \Omega_n^2 = \{t \in J : u_n(t) < 0\}$, then if n is sufficiently large,

$$1 + \lambda^* \int_{\Omega_n^1} u_n\varphi_0\,dt \geq \int_{\Omega_n^1} f_0(t,u_n)\varphi_0\,dt - \lambda^* \int_{\Omega_n^2} u_n\varphi_0\,dt \geq \int_{\Omega_n^1} f_0(t,u_n)\varphi_0\,dt. \tag{3.18}$$

From (3.12), there exists $C_{16} > 0$ such that

$$0 \le \int_{\Omega_n^1} u_n \varphi_0 \, dt \le C_{16} \quad \text{if } n \text{ is sufficiently large.} \tag{3.19}$$

It follows that there exist $C_{17} > 0, C_{18} > 0$ such that if n is sufficiently large,

$$0 \le i_k\big(u_n(t_k)\big) \le C_{17}, \quad 0 \le \widetilde{I}_n\big(u_n(t_k)\big) \le \int_0^{u_n(t_k)} i_k(s)\, ds \le C_{18}\|u_n\|.$$

Hence, if n is sufficiently large, we have

$$\begin{aligned} 1 + c + \frac{1}{\mu}\|u_n\| \ge{}& \Phi_0(u_n) - \frac{1}{\mu}\langle \Phi_0(u_n), u_n\rangle \\ ={}& \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\| - \int_0^1 \left(F_0(t,u_n) - \frac{1}{\mu} f_0(t,u_n)u_n\right) dt \\ &- \lambda \sum_{k=1}^{p} \left[\widetilde{I}_k\big(u_n(t_k)\big) - \frac{1}{\mu} i_k\big(u_n(t_k)\big)u_n(t_k))\right] \\ \ge{}& \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - C_{19} - C_{20}\|u_n\|. \end{aligned}$$

This implies that $\{u_n\} \subset H$ is bounded. By a standard argument, one can show that $\{u_n\}$ has a convergent subsequence. □

Remark 3.2 For Φ_0,

$$\langle \Phi_0'(u_\lambda), v\rangle = \langle \Phi_\lambda'(u_\lambda), v\rangle = 0, \quad \forall v \in H, \tag{3.20}$$

$$\Phi_0\big(u_\lambda + v^+\big) = \Phi_\lambda\big(u_\lambda + v^+\big) \ge \Phi_\lambda(u_\lambda) = \Phi_0(u_\lambda), \quad \forall v \in H. \tag{3.21}$$

Theorem 3.1 *There exists* $0 < \Lambda < +\infty$ *such that* (1.1) *has at least two positive solutions for all* $\lambda \in (0,\Lambda)$, *one positive solution for* $\lambda = \Lambda$, *and no positive solutions for* $\lambda > \Lambda$.

Proof From Lemma 3.2 and Lemma 3.3, (1.1) has no solution for $\lambda > \Lambda$, at least one positive solution for $\lambda = \Lambda$ and a positive solution u_λ with $\Phi_\lambda(u_\lambda) < 0$ for $0 < \lambda < \Lambda$.

It is easy to show that $\Phi_0(u_\lambda + s\varphi_0) \to -\infty$ as $s \to +\infty$. Noting that

$$(u_\lambda, \varphi_0) = \int_0^1 f(t,u_\lambda)\varphi_0\, dt + \lambda \sum_{k=1}^{p} I_k\big(u_\lambda(t_k)\big)\varphi_0(t_k) > 0,$$

$$\|u_\lambda + s\varphi_0\|^2 = (u_\lambda + s\varphi_0, u_\lambda + s\varphi_0) = \|u_\lambda\|^2 + s(u_\lambda,\varphi_0) + s^2\|\varphi_0\|^2 \ge \|u_\lambda\|^2 + s^2\|\varphi_0\|^2,$$

we fix $s_0 > 0$ such that $R_2 =: \|u_\lambda + s_0\varphi_0\| > R_1 =: \|u_\lambda\|$, and

$$\Phi_0(u_\lambda + s_0\varphi_0) < \Phi_\lambda(u_\lambda) - 1.$$

Let $\Gamma = \{\xi \in C([0,1],H) | \xi(0) = u_\lambda, \xi(1) = u_\lambda + s_0\varphi_0\}$, and

$$\rho = \inf_{\xi \in \Gamma} \max_{t \in [0,1]} \Phi_0\big(\xi(t)\big).$$

It follows that $\rho \geq \Phi_0(u_\lambda) = \Phi_\lambda(u_\lambda)$. If $\rho = \Phi_\lambda(u_\lambda)$, from (3.21), there exists $0 < \delta < R_2 - R_1$ such that $\inf\{\Phi_0(u) | \|u\| = R\} = \rho$ for all $R \in (R_1, R_1 + \delta)$. Let $\Xi = H$ if $\rho > \Phi_\lambda(u_\lambda)$ and $\Xi = \{u : \|u\| = R_1 + \delta/2\}$ if $\rho = \Phi_\lambda(u_\lambda)$. Clearly,

$$\Xi \cap \left\{w \in H, \Phi_0(w) \geq \rho\right\}$$

separates u_λ and $u_\lambda + s_0\varphi_0$. Hence, Φ_0 has a critical point v_λ such that $\Phi_0(v_\lambda) = \rho$ and $v_\lambda \in \Xi$. If $\rho = \Phi_\lambda(u_\lambda)$, $\|v_\lambda\| = R_1 + \delta/2 > \|u_\lambda\|$, if $\rho > \phi_\lambda(u_\lambda)$, $\Phi_0(v_\lambda) = \rho > \Phi_\lambda(u_\lambda) = \Phi_0(u_\lambda)$. Hence, $v_\lambda \not\equiv u_\lambda$ and

$$(v_\lambda, w) = \int_0^1 f_0(t, v_\lambda) w \, dt + \lambda \sum_{k=1}^{p} i_k\bigl(v_\lambda(t_k)\bigr) w(t_k), \quad \forall w \in H,$$
$$(u_\lambda, w) = \int_0^1 f(t, u_\lambda) w \, dt + \lambda \sum_{k=1}^{p} I_k\bigl(u_\lambda(t_k)\bigr) w(t_k), \quad \forall w \in H.$$

Choosing $w = (u_\lambda - v_\lambda)^+$, we have

$$\begin{aligned}\bigl(u_\lambda - v_\lambda, (u_\lambda - v_\lambda)^+\bigr) = {}& \int_0^1 \bigl[f(t, u_\lambda) - f_0(t, v_\lambda)\bigr](u_\lambda - v_\lambda)^+ \, dt \\ &+ \lambda \sum_{k=1}^{p} \Bigl(I_k\bigl(u_\lambda(t_k)\bigr) - i_k\bigl(v_\lambda(t_k)\bigr)\Bigr)(u_\lambda - v_\lambda)^+(t_k) = 0,\end{aligned}$$

which implies that $\|(u_\lambda - v_\lambda)^+\| = 0$ and $u_\lambda \leq v_\lambda$. Hence,

$$\begin{aligned}&f_0(t, v_\lambda) = f(t, v_\lambda), \qquad i_k\bigl(v_\lambda(t_k)\bigr) = I_k\bigl(v(t_k)\bigr), \\ &\Phi_0(v_\lambda) = \Phi_\lambda(v_\lambda), \qquad \bigl\langle \Phi_0'(v_\lambda), w\bigr\rangle = \bigl\langle \Phi_\lambda(v_\lambda), w\bigr\rangle = 0, \quad \forall w \in H,\end{aligned}$$

and v_λ is the second positive solution of (1.1). □

Remark 3.3 In fact, the function f satisfying (H_2) is of superlinear growth, and the impulsive function affecting the number of positive solutions is of sublinear growth.

Example 3.1 Consider the differential equation

$$\begin{cases} -x''(t) + x(t) = x^2(t), & t \neq 0.5, \\ x'(0.5^+) = x'(0.5) - \lambda, \\ x'(0) = x'(1) = 0. \end{cases} \tag{3.22}$$

Clearly, the nonimpulsive differential equation corresponding (3.22) has a positive solution $x \equiv 1$. The results in [19] cannot be applied to (3.22) since the nonlinear function and impulsive functions in [19] are of superlinear growth or of sublinear growth.

Assume that x is a positive solution of (3.22), then

$$-\int_0^{0.5} x''(t) \, dt - \int_{0.5}^1 x''(t) \, dt + \int_0^1 x(t) \, dt = \int_0^1 x^2(t) \, dt$$

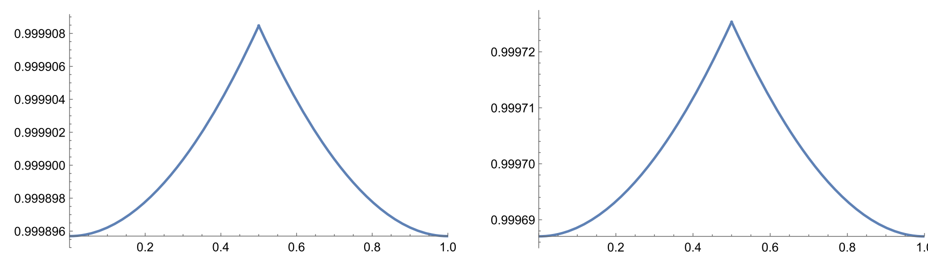

Figure 1 Positive solutions to the equation of Example 3.22

and

$$\int_0^1 x(t)\,dt = \int_0^1 \big(x^2(t) + \lambda\big)\,dt.$$

Clearly, $x \not\equiv 1$. If $\lambda \geq 1/4$, then we have

$$\int_0^1 x(t)\,dt > 2\sqrt{\lambda}\int_0^1 x(t)\,dt \geq \int_0^1 x(t)\,dt.$$

Hence, (3.22) has no positive solution if $\lambda \geq 1/4$. From Theorem 3.1, (3.22) has two positive solutions for sufficiently small $\lambda > 0$. When $\lambda = 0.0001$, two positive solutions of (3.22) can be found in Fig. 1.

4 Conclusion

In this paper, we discussed the existence, nonexistence, and multiplicity of positive solutions for a class of impulsive Sturm–Liouville boundary value problems with a parameter. Using the mountain pass principle, we show that the number of positive solutions depends on the change of parameters, in which sublinear impulsive perturbation plays an important role. In fact, one can prove that the nonimpulsive case of (1.1), that is, $I_k \equiv 0 (1 \leq k \leq p)$, has at least a positive solution. How does the combination of impulsive perturbation and parameter affect the behavior of the equation? We will discuss the issue in follow-up research.

Funding
The third author is supported by Hunan Provincial Natural Science Foundation of China (No 2022JJ30236).

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed to the study conception and technical content. The first draft of the manuscript was written by Piao Liu and Weibing Wang, and all authors commented on further versions of the manuscript. All authors have read and approved the final manuscript.

Author details
[1]Department of Mathematics and Statistics, Hunan First Normal University, Changsha, Hunan 410205, P.R. China.
[2]Department of Mathematics, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China.

Received: 14 June 2023 Accepted: 15 February 2024 Published online: 08 April 2024

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