Boulaaras et al. *Boundary Value Problems* (2024) 2024:36
https://doi.org/10.1186/s13661-024-01843-5

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Blow-up and lifespan of solutions for elastic membrane equation with distributed delay and logarithmic nonlinearity

Salah Boulaaras[1*], Rashid Jan[2,3,4], Abdelbaki Choucha[5,6], Aderrahmane Zaraï[7] and Mourad Benzahi[7]

*Correspondence:
s.boularas@qu.edu.sa
[1]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

We examine a Kirchhoff-type equation with nonlinear viscoelastic properties, characterized by distributed delay, logarithmic nonlinearity, and Balakrishnan–Taylor damping terms (elastic membrane equation). Under appropriate hypotheses, we establish the occurrence of solution blow-up.

Mathematics Subject Classification: 35B40; 35L70; 76Exx; 93D20

Keywords: Kirchhoff equation; Partial differential equations; Blow-up; Distributed delay term; Viscoelastic term; Logarithmic nonlinearity

1 Introduction

Analyzing nonlinear mathematical problems involves a distinct set of challenges and techniques compared to linear problems [1, 2]. Nonlinear problems often manifest in various scientific and engineering domains, and their analysis is crucial for gaining insights into complex phenomena [3, 4]. The systematic examination of mathematical problems necessitates a methodical approach encompassing rigorous formulation, assessment of existence and uniqueness of solutions, linearization procedures, stability analyses, application of numerical methodologies, bifurcation investigations, phase plane analyses, sensitivity assessments, optimization strategies, and the validation and verification of outcomes [5, 6]. This multifaceted framework is imperative for elucidating the intricate dynamics inherent in various systems across diverse scientific and engineering domains [7].

The analysis of solutions to the Kirchhoff equation concerning viscoelastic materials is paramount in comprehending the mechanical characteristics of such materials. These solutions serve as a cornerstone in directing the design methodology, thereby ensuring the dependability and efficacy of materials across diverse applications. In this current study,

we examine the Kirchhoff equation provided below:

$$\begin{cases} v_{tt} - N(t)\Delta v(t) + \int_0^t f(t-\Lambda)\Delta v(\Lambda)\,d\Lambda + \mu_1 v_t + \int_{\tau_1}^{\tau_2} \mu_2(s) v_t(y,t-s)\,ds \\ \quad = v|v|^{\gamma-2}\ln|v|^l, \quad x \in \Omega, 0 < t, \\ v(y,0) = v_0(y), \qquad v_t(y,0) = v_1(y), \quad y \in \Omega, \\ v_t(y,-t) = h_0(y,t), \quad y \in \Omega, t \in (0,\tau_2), \\ v(y,t) = 0, \quad (y,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where

$$N(t) := \left(\zeta_0 + \zeta_1 \|\nabla v\|_2^2 + \sigma\left(\nabla v(t), \nabla v_t(t)\right)_{M^2(\Omega)}\right).$$

In this expression, Ω belongs to the set of bounded domains in $\mathbb{R}^N$ and possesses a boundary $\partial\Omega$ that is suitably smooth. $\gamma \geq 2$, ζ_0, ζ_1, σ, μ_1, l are positive constants. In addition to this, the time delays are indicated by τ_1, τ_2 with $0 \leq \tau_1 < \tau_2$,while μ_2 is an M^∞ function and f is a positive function.

In a physical sense, the connection between the stress and strain history in the beam is influenced by a viscoelastic damping term inspired by Boltzmann theory. The kernel of memory term in this context is represented by the function f, which is frequently discussed in the literature [8–16].

In [17], Balakrishnan and Taylor introduced a novel damping model known as Balakrishnan–Taylor damping, specifically addressing concerns related to the span problem and the plate equation. Numerous studies have explored this damping phenomenon, as documented in [11, 14, 15, 17–20, 35–37], and [21]. The occurrence of delay is a common feature in various applications and practical problems, rendering many systems worthy of investigation. Recently, several authors have directed their attention towards analyzing the asymptotic behavior and stability of evolution systems incorporating time delay, as discussed in the research [9–12, 15, 22–26], and [27].

The significance of logarithmic nonlinearity in physical system is emphasized by its involvement in a wide range of topics and theories, encompassing symmetry, cosmology, quantum mechanics, nuclear physics, and various applications including nuclear, optical, and subterranean physics. Different authors have delved into such type of problem across various domains, exploring aspects such as global solution existence, stability, blow-up, and the growth of solutions, as documented in works such as [11, 19, 28–31], and [32–34]. Taking into account various elements, damping terms including the distributed delay terms, logarithmic nonlinearity, Balakrishnan–Taylor damping, and memory term are integrated into a specific problem, along with the incorporation of $\int_{\tau 1}^{\tau 2} \mu 2(s) vt(y,t-s)\,ds$. Further investigation is required to explore such type of novel and distinctive problem. This diverges from the previously mentioned scenarios, and our objective is to shed light on this unique problem.

Our work is structured as follows: In the subsequent section, we lay out the necessary lemmas, concepts, and hypotheses. In Sect. 3, we state and prove the main blow-up solution results. We present the concluding remarks of our work in Sect. 4 of this work.

2 Fundamental concepts

Here, to investigate our problem, we require certain materials. To begin with, we present the following assumptions regarding β_2 and f:

(H1) $f:\mathbb{R}+\to\mathbb{R}+$ represents nonincreasing B^1 functions fulfilling

$$0<f(t),\quad \zeta_0-\int_0^\infty f(\Lambda)\,d\Lambda=l>0. \tag{2.1}$$

(H2) $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is an M^∞ function in a way that

$$\left(\frac{2\delta+1}{2}\right)\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds<\mu_1,\quad \delta>\frac{1}{2}, \tag{2.2}$$

with

$$(f\circ\psi)(t):=\int_\Omega\int_0^t f(t-\Lambda)\left|\psi(t)-\psi(\Lambda)\right|^2 d\Lambda\,dy.$$

As in [27], we take the following:

$$x(y,\rho,s,t)=v_t(y,t-s\rho),\quad (y,\rho,s,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+,$$

which satisfy

$$\begin{cases} sx_t(y,\rho,s,t)+x_\rho(y,\rho,s,t)=0,\\ x(y,0,s,t)=v_t(y,t). \end{cases} \tag{2.3}$$

Then one can write (1.1) as follows:

$$\begin{cases} v_{tt}-N(t)\Delta v(t)+\int_0^t f(t-\Lambda)\Delta v(\Lambda)\,d\Lambda+\mu_1 v_t+\int_{\tau_1}^{\tau_2}\mu_2(s)x(y,1,s,t)\,ds\\ \quad =v|v|^{\gamma-2}\ln|v|^l,\\ sx_t(y,\rho,s,t)+x_\rho(y,\rho,s,t)=0,\\ v(y,0)=v_0(y),\quad v_t(y,0)=v_1(y),\quad y\in\Omega,\\ x(y,\rho,s,0)=h_0(y,s\rho),\quad \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2),\\ v(y,t)=0,\quad (x,t)\in\partial\Omega\times(0,\infty). \end{cases} \tag{2.4}$$

Here, the energy functional is introduced as follows.

Lemma 2.1 *Let Q represent the energy functional given by*

$$\begin{aligned} Q(t)={}&\frac{1}{2}\|v_t\|_2^2+\frac{1}{2}\left(\zeta_0-\int_0^t v(\Lambda)\,d\Lambda\right)\left\|\nabla v(t)\right\|_2^2+\frac{\zeta_1}{4}\left\|\nabla v(t)\right\|_2^4\\ &+\frac{1}{2}(f\circ\nabla v)(t)+\frac{l}{\gamma}\left\|v(t)\right\|_\gamma^\gamma-\frac{1}{\gamma}\int_\Omega|v|^\gamma\ln|v|^l\,dy\\ &+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2(s)\right|x^2(y,\rho,s,t)\,ds\,d\rho\,dy, \end{aligned} \tag{2.5}$$

which satisfies

$$\begin{aligned} Q'(t) &\leq -B_0\left(\|v_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} \left|\mu_2(s)\right| x^2(y,1,s,t)\,ds\,dx\right) + \frac{1}{2}\left(f' \circ \nabla v\right)(t) \\ &\quad - \frac{1}{2} f(t)\left\|\nabla v(t)\right\|_2^2 - \frac{\sigma}{4}\left(\frac{d}{dt}\left\{\left\|\nabla v(t)\right\|_2^2\right\}\right)^2 \\ &\leq 0. \end{aligned} \tag{2.6}$$

Proof By taking the inner product of $(2.4)_1$ with vt and subsequently integrating over Ω, we obtain

$$\begin{aligned} &\left(v_{tt}(t), v_t(t)\right)_{M^2(\Omega)} - \left(N(t)\Delta v(t), v_t(t)\right)_{M^2(\Omega)} \\ &\quad + \left(\int_0^t f(t-\Lambda)\Delta v(\Lambda)\,d\Lambda, v_t(t)\right)_{M^2(\Omega)} + \mu_1(v_t, v_t)_{M^2(\Omega)} \\ &\quad + \left(\int_{\tau_1}^{\tau_2} \mu_2(s) x(y,1,s,t)\,ds, v_t(t)\right)_{M^2(\Omega)} - \left(l v|v|^{\gamma-2}\ln|v|, v_t(t)\right)_{M^2(\Omega)} = 0. \end{aligned} \tag{2.7}$$

Then a straightforward calculation yields

$$\left(v_{tt}(t), v_t(t)\right)_{M^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\left(\left\|v_t(t)\right\|_2^2\right), \tag{2.8}$$

further simplification implies that

$$\begin{aligned} &-\left(N(t)\Delta v(t), v_t(t)\right)_{M^2(\Omega)} \\ &\quad = -\left(\left(\zeta_0 + \zeta_1\|\nabla v\|_2^2 + \sigma\left(\nabla v(t), \nabla v_t(t)\right)_{M^2(\Omega)}\right)\Delta v(t), v_t(t)\right)_{M^2(\Omega)} \\ &\quad = \left(\zeta_0 + \zeta_1\|\nabla v\|_2^2 + \sigma\left(\nabla v(t), \nabla v_t(t)\right)_{M^2(\Omega)}\right)\int_\Omega \nabla v(t).\nabla v_t(t)\,dy \\ &\quad = \left(\zeta_0 + \zeta_1\|\nabla v\|_2^2 + \sigma\left(\nabla v(t), \nabla v_t(t)\right)_{M^2(\Omega)}\right)\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega \left|\nabla v(t)\right|^2 dy\right\} \\ &\quad = \frac{d}{dt}\left\{\frac{1}{2}\left(\zeta_0 + \frac{\zeta_1}{2}\|\nabla v\|_2^2\right)\left\|\nabla v(t)\right\|_2^2\right\} + \frac{\sigma}{4}\left\{\frac{d}{dt}\left\|\nabla v(t)\right\|_2^2\right\}^2, \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} &\left(\int_0^t f(t-\Lambda)\Delta v(\Lambda)\,d\Lambda, v_t(t)\right)_{L^2(\Omega)} \\ &\quad = \int_0^t f(t-\Lambda)\left(\Delta v(\Lambda), v_t(t)\right)_{M^2(\Omega)} d\Lambda \\ &\quad = -\int_0^t f(t-\Lambda)\left[\int_\Omega \nabla v(y,\Lambda)\nabla v(y,t)\,dy\right] d\Lambda, \end{aligned} \tag{2.10}$$

and

$$-\nabla v(y,\Lambda).\nabla v(y,t) = \frac{1}{2}\frac{d}{dt}\left\{\left|\nabla v(y,\Lambda) - \nabla v(y,t)(t)\right|^2\right\} - \frac{1}{2}\frac{d}{dt}\left\{\left|\nabla v(y,t)\right|^2\right\}. \tag{2.11}$$

Then we have

$$\begin{aligned}
&-\int_0^t f(t-\Lambda)\big(\nabla v(\varrho),\nabla v_t(t)\big)_{M^2(\Omega)}\,d\Lambda \\
&\quad=-\int_0^t f(t-\Lambda)\int_\Omega\left[\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla v(y,\Lambda)-\nabla v(y,t)\right|^2\right\}\right]dy\,ds. \\
&\qquad-\int_0^t f(t-\Lambda)\int_\Omega\left[\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla v(y,t)\right|^2\right\}\right]dy\,d\Lambda \\
&\quad=\frac{1}{2}\int_0^t f(t-\Lambda)\left[\frac{d}{dt}\left\{\int_\Omega\left|\nabla v(y,t)-\nabla v(y,\Lambda)\right|^2dy\right\}\right]d\Lambda \\
&\qquad-\frac{1}{2}\int_0^t f(t-\Lambda)\left[\frac{d}{dt}\left\{\left\|\nabla v(y,t)\right\|_2^2\right\}\right]dy\,d\Lambda.
\end{aligned}\tag{2.12}$$

Using (2.1), one has

$$\begin{aligned}
&\frac{1}{2}\int_0^t f(t-\Lambda)\left[\frac{d}{dt}\left\{\int_\Omega\left|\nabla v(y,t)-\nabla v(y,\Lambda)\right|^2dy\right\}\right]d\Lambda \\
&\quad=\frac{1}{2}\frac{d}{dt}\left\{\int_0^t f(t-\Lambda)\left[\int_\Omega\left|\nabla v(y,t)-\nabla v(y,\Lambda)\right|^2dy\right]\right\}d\Lambda \\
&\qquad-\frac{1}{2}\int_0^t f'(t-\Lambda)\left[\int_\Omega\left|\nabla v(y,t)-\nabla v(y,\Lambda)\right|^2dy\right]d\Lambda \\
&\quad=\frac{1}{2}\frac{d}{dt}(f\circ\nabla u)(t)-\frac{1}{2}\big(f'\circ\nabla u\big)(t)
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-\Lambda)\left[\frac{d}{dt}\left\{\left\|\nabla v(t)\right\|_2^2\right\}\right]dy\,d\Lambda \\
&\quad=-\frac{1}{2}\left(\int_0^t f(t-\Lambda)\,d\Lambda\right)\left(\frac{d}{dt}\left\{\left\|\nabla v(t)\right\|_2^2\right\}\right)dy \\
&\quad=-\frac{1}{2}\left(\int_0^t f(\Lambda)\,d\Lambda\right)\left(\frac{d}{dt}\left\{\left\|\nabla v(t)\right\|_2^2\right\}\right)dy \\
&\quad=-\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t f(\Lambda)\,d\Lambda\right)\left\|\nabla v(t)\right\|_2^2\right\}+\frac{1}{2}f(t)\left\|\nabla v(t)\right\|_2^2.
\end{aligned}\tag{2.14}$$

By substituting (2.13) and (2.14) into (2.12), we have

$$\begin{aligned}
&\left(\int_0^t f(t-\Lambda)\Delta v(\Lambda)\,d\Lambda,v_t(t)\right)_{M^2(\Omega)} \\
&\quad=\frac{d}{dt}\left\{\frac{1}{2}(f\circ\nabla v)(t)-\frac{1}{2}\left(\int_0^t f(\Lambda)\,d\Lambda\right)\left\|\nabla v(t)\right\|_2^2\right\} \\
&\qquad-\frac{1}{2}\big(f'\circ\nabla v\big)(t)+\frac{1}{2}f(t)\left\|\nabla v(t)\right\|_2^2
\end{aligned}\tag{2.15}$$

and

$$-\big(lv|v|^{\gamma-2}\ln|u|,v_t(t)\big)_{M^2(\Omega)}=\frac{d}{dt}\left\{\frac{l}{\gamma}\left\|v(t)\right\|_\gamma^\gamma-\frac{1}{\gamma}\int_\Omega|v|^\gamma\ln|v|^l\,dy\right\}.\tag{2.16}$$

Here, multiply $x|\mu_2(s)|$ with equation $(2.4)_2$ by $x|\mu_2(s)|$ and integrate over $\Omega \times (0,1) \times (\tau_1, \tau_2)$. Then applying $(2.3)_2$, the following is obtained:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\big|\mu_2(s)\big|x^2(y,\rho,s,t)\,ds\,d\rho\,dy \\
&\quad = -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2\big|\mu_2(s)\big|xx_\rho\,ds\,d\rho\,dy \\
&\quad = -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} \big|\mu_2(s)\big|\big[x^2(y,1,s,t)-x^2(y,0,s,t)\big]\,ds\,dy \\
&\quad = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)\,ds\right)\|u_t\|_2^2 - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|x^2(y,1,s,t)\,ds\,dy,
\end{aligned} \tag{2.17}$$

and through the application of Young's inequality, we obtain

$$\begin{aligned}
\left(\int_{\tau_1}^{\tau_2}\mu_2(s)x(y,1,s,t)\,ds, v_t(t)\right)_{M^2(\Omega)} &\le \delta\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)\,ds\right)\|v_t\|_2^2 \\
&\quad + \frac{1}{4\delta}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|x^2(y,1,s,t)\,ds\,dy.
\end{aligned} \tag{2.18}$$

By replacement (2.8)–(2.9) and (2.15)–(2.18) into (2.7), we find (2.5) and

$$\begin{aligned}
Q'(t) &\le -\left(\mu_1 - \left(\delta+\frac{1}{2}\right)\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|\,ds\right)\|v_t\|_2^2 \qquad (2.19)\\
&\quad - \left(\frac{2\delta-1}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|x^2(y,1,s,t)\,ds\,dy + \frac{1}{2}\big(f'\circ\nabla v\big)(t) \\
&\quad - \frac{1}{2}f(t)\big\|\nabla v(t)\big\|_2^2 - \frac{\sigma}{4}\left(\frac{d}{dt}\big\{\big\|\nabla v(t)\big\|_2^2\big\}\right)^2 \\
&\le 0. \qquad (2.20)
\end{aligned}$$

Thus, according to (2.2), we get(2.6), where $B_0 > 0$. Hence, the proof is completed. □

Theorem 2.2 *Let us consider that* (2.1)–(2.2) *hold true. For any* $v_0, u_1 \in F^1 0(\Omega) \cap M^2(\Omega)$ *and* $f0 \in M^2(\Omega,(0,1))$, *one can find a weak solution* v *to* (2.4) *such that*

$$\begin{aligned}
&v \in B\big(]0,P[,F_0^1(\Omega)\big) \cap Q^1\big(]0,P[,M^2(\Omega)\big), \\
&v_t \in B\big(]0,T[,F_0^1(\Omega)\big) \cap M^2\big(]0,P[,M^2\big(\Omega,(0,1)\big)\big).
\end{aligned}$$

Lemma 2.3 [34] *One can find a constant* $b(\Omega) > 0$ *in a manner that*

$$\left(\int_\Omega |v|^\gamma \ln|v|^l\,dy\right)^{\frac{s}{\gamma}} \le b\left(\int_\Omega |v|^\gamma \ln|v|^l\,dy + \|\nabla v\|_2^2\right)$$

for any $2 \le s \le \gamma$, *provided that* $0 \le \int_\Omega |v|^\gamma \ln|v|^l\,dy$.

Corollary 2.4 [34] *One can find a constant $b(\Omega) > 0$ in a way that*

$$\|v\|_2^2 \le c\left[\left(\int_\Omega |v|^\gamma \ln |v|^l \, dy\right)^{\frac{2}{\gamma}} + \|\nabla v\|_2^{\frac{4}{\gamma}}\right],$$

provided that $0 \le \int_\Omega |v|^\gamma \ln |v|^l \, dy$.

Lemma 2.5 [34] *Let us take a constant $b(\Omega) > 0$ in a way that*

$$\|v\|_\gamma^s \le b\big(\|v\|_\gamma^\gamma + \|\nabla v\|_2^2\big)$$

for any $v \in M^\gamma(\Omega)$ and $2 \le s \le \gamma$.

3 Blow-up result

Here, we establish the blow-up results for the solution of (2.4). First of all, the functional is introduced as

$$\begin{aligned}\mathbb{F}(t) = -Q(t) = {} & -\frac{1}{2}\|v_t\|_2^2 - \frac{1}{2}\left(\zeta_0 - \int_0^t f(\Lambda)\, d\Lambda\right)\|\nabla v(t)\|_2^2 - \frac{\zeta_1}{4}\|\nabla v(t)\|_2^4 \\ & - \frac{1}{2}(f \circ \nabla v)(t) - \frac{l}{\gamma}\|v(t)\|_\gamma^\gamma + \frac{1}{\gamma}\int_\Omega |v|^\gamma \ln |v|^l \, dy \\ & - \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\big|\mu_2(s)\big|x^2(y,\rho,s,t)\, ds\, d\rho\, dy. \end{aligned} \tag{3.1}$$

Theorem 3.1 *Assuming that* (2.1)–(2.2) *are satisfied, and given that $Q(0) < 0$, the solution to problem* (2.4) *experiences a finite time blow-up.*

Proof For the required proof, the following is obtained from (2.6):

$$Q(t) \le Q(0) \le 0; \tag{3.2}$$

thus, we have

$$\mathbb{F}'(t) = -Q'(t) \ge B_0\left(\|v_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} \big|\mu_2(s)\big|x^2(y,1,s,t)\, ds\, dy\right), \tag{3.3}$$

which implies that

$$\begin{aligned} &\mathbb{F}'(t) \ge B_0\|v_t(t)\|_2^2 \ge 0 \\ &\mathbb{F}'(t) \ge B_0 \int_\Omega \int_{\tau_1}^{\tau_2} \big|\mu_2(s)\big|x^2(y,1,s,t)\, ds\, dy \ge 0. \end{aligned} \tag{3.4}$$

By (3.1), we have

$$0 \le \mathbb{F}(0) \le \mathbb{F}(t) \le \frac{1}{\gamma}\int_\Omega |v|^\gamma \ln |v|^l \, dy. \tag{3.5}$$

We set

$$\mathcal{L}(t) = \mathbb{F}^{1-\alpha}(t) + \varepsilon \int_\Omega v v_t \, dy + \frac{\varepsilon \mu_1}{2}\int_\Omega v^2 \, dy + \frac{\sigma}{4}\|\nabla v\|_2^4, \tag{3.6}$$

where $\varepsilon > 0$ will be assigned a specific value later, and

$$\frac{2(\gamma-1)}{\gamma^2} < \alpha < \frac{\gamma-2}{2\gamma} < 1. \tag{3.7}$$

Multiplying v with $(2.4)_1$ and taking the derivative of (3.6), the following is obtained:

$$\begin{aligned}\mathcal{L}'(t) &= (1-\alpha)\mathbb{F}^{-\alpha}\mathbb{F}'(t) + \varepsilon\|v_t\|_2^2 + \varepsilon\int_\Omega |v|^\gamma \ln|v|^l\,dy \\ &\quad - \varepsilon\zeta_0\|\nabla v\|_2^2 - \varepsilon\zeta_1\|\nabla v\|_2^4 + \underbrace{\varepsilon\int_\Omega \nabla v\int_0^t f(t-\Lambda)\nabla v(\Lambda)\,d\varrho\,dy}_{J_1} \\ &\quad \underbrace{-\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)ux(y,1,s,t)\,ds\,dy}_{J_2}.\end{aligned} \tag{3.8}$$

Next, we have

$$\begin{aligned}J_1 &= \varepsilon\int_0^t h(t-\Lambda)\,d\Lambda\int_\Omega \nabla v.\big(\nabla v(\Lambda)-\nabla v(t)\big)\,dy\,d\Lambda + \varepsilon\int_0^t f(\Lambda)\,d\Lambda\|\nabla v\|_2^2 \\ &\geq \frac{\varepsilon}{2}\left(\int_0^t f(\Lambda)\,d\Lambda\right)\|\nabla v\|_2^2 - \frac{\varepsilon}{2}(f\circ\nabla v)\end{aligned} \tag{3.9}$$

and, for $\delta_1 > 0$,

$$J_2 \geq -\varepsilon\mu_1\delta_1\|u\|_2^2 - \frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|x^2(y,1,s,t)\,ds\,dy. \tag{3.10}$$

From (3.8), we find

$$\begin{aligned}\mathcal{L}'(t) &\geq (1-\alpha)\mathbb{F}^{-\alpha}\mathbb{F}'(t) + \varepsilon\|v_t\|_2^2 + \varepsilon\int_\Omega |v|^\gamma \ln|v|^l\,dy \\ &\quad - \varepsilon\zeta_1\|\nabla v\|_2^4 - \varepsilon[\left(\zeta_0 - \frac{1}{2}\int_0^t f(\Lambda)\,d\Lambda\right)\|\nabla v\|_2^2 - \frac{\varepsilon}{2}(f\circ\nabla v) \\ &\quad - \varepsilon\mu_1\delta_1\|v\|_2^2 - \frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}\big|\mu_2(s)\big|x^2(y,1,s,t)\,ds\,dy.\end{aligned} \tag{3.11}$$

At this point, by setting δ_1 so that, for large κ to be specified later

$$\frac{1}{4\delta_1 B_0} = \kappa\mathbb{F}^{-\alpha}(t),$$

by (3.4) and putting in (3.11), we get

$$\begin{aligned}\mathcal{L}'(t) &\geq \big[(1-\alpha)-\varepsilon\kappa\big]\mathbb{F}^{-\alpha}\mathbb{F}'(t) + \varepsilon\|v_t\|_2^2 \\ &\quad - \frac{\varepsilon}{2}(f\circ\nabla v) - \varepsilon\zeta_1\|\nabla v\|_2^4 - \varepsilon\left(\zeta_0 - \frac{1}{2}\int_0^t f(\Lambda)\,d\Lambda\right)\|\nabla v\|_2^2 \\ &\quad - \varepsilon\left(\frac{\mu_1\mathbb{F}^\alpha(t)}{4B_0\kappa}\right)\|v\|_2^2 + \varepsilon\int_\Omega |v|^\gamma \ln|v|^l\,dy.\end{aligned} \tag{3.12}$$

Now, for $0 < a < 1$ and from (3.1), we have

$$\begin{aligned}\varepsilon\int_\Omega |v|^\gamma \ln|v|^l\,dy &= \varepsilon a\int_\Omega |v|^\gamma \ln|v|^l\,dy + \frac{\varepsilon\gamma(1-a)}{2}\|v_t\|_2^2 + \varepsilon\gamma(1-a)\mathbb{F}(t)\\ &\quad + \varepsilon\frac{\gamma(1-a)}{2}\left(\zeta_0 - \int_0^t f(\Lambda)\,d\Lambda\right)\|\nabla v\|_2^2 + \varepsilon l(1-a)\|v\|_\gamma^\gamma\\ &\quad + \varepsilon\frac{\zeta_1\gamma(1-a)}{2}\|\nabla v\|_2^4 - \varepsilon\frac{\gamma(1-a)}{2}((f\circ\nabla v)\\ &\quad + \frac{\varepsilon\gamma(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\big|\mu_2(s)\big|x^2(y,\rho,s,t)\,ds\,d\rho\,dy. \end{aligned}\tag{3.13}$$

Putting in (3.12), one has

$$\begin{aligned}\mathcal{L}'(t) &\geq \big\{(1-\alpha)-\varepsilon\kappa\big\}\mathbb{F}^{-\alpha}\mathbb{F}'(t) + \varepsilon a\int_\Omega |v|^\gamma \ln|u|^l\,dy\\ &\quad + \varepsilon\left\{\frac{\gamma(1-a)}{2}+1\right\}\|v_t\|_2^2 - \varepsilon\left(\frac{\mu_1\mathbb{F}^\alpha(t)}{4B_0\kappa}\right)\|v\|_2^2\\ &\quad + \varepsilon\left\{\frac{\gamma(1-a)}{2}\left(\zeta_0 - \int_0^t f(\Lambda)\,d\Lambda\right) - \left(\zeta_0 - \frac{1}{2}\int_0^t f(\Lambda)\,d\Lambda\right)\right\}\|\nabla v\|_2^2\\ &\quad + \varepsilon\zeta_1\left\{\frac{\gamma(1-a)}{2}-1\right\}\|\nabla v\|_2^4 + \varepsilon\left\{\frac{\gamma(1-a)}{2}-\frac{1}{2}\right\}(f\circ\nabla v)\\ &\quad + \varepsilon l(1-a)\|v\|_\gamma^\gamma + \varepsilon\gamma(1-a)\mathbb{F}(t)\\ &\quad + \frac{\varepsilon\gamma(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\big|\mu_2(s)\big|x^2(y,\rho,s,t)\,ds\,d\rho\,dy. \end{aligned}\tag{3.14}$$

According to (3.5), Corollary 2.4, and Young's inequality, we get

$$\begin{aligned}\mathbb{F}^\alpha(t)\|v\|_2^2 &\leq \left(\int_\Omega |v|^\gamma \ln|v|^l\,dy\right)^\alpha \|v\|_2^2\\ &\leq c\left[\left(\int_\Omega |v|^\gamma \ln|v|^l\,dx\right)^{\alpha+\frac{2}{\gamma}} + \left(\int_\Omega |v|^\gamma \ln|v|^l\,dy\right)^\alpha \|\nabla v\|_2^{\frac{4}{\gamma}}\right]\\ &\leq b\left[\left(\int_\Omega |v|^\gamma \ln|v|^l\,dy\right)^{\frac{(\alpha\gamma+2)}{\gamma}} + \left(\int_\Omega |v|^\gamma \ln|v|^l\,dy\right)^{\frac{\alpha\gamma}{(\gamma-2)}} + \|\nabla v\|_2^2\right]. \end{aligned}$$

(3.7) yields

$$2 < \alpha\gamma + 2 \leq \gamma \quad \text{and} \quad 2 < \frac{\alpha\gamma^2}{\gamma-2} \leq \gamma.$$

Hence, Lemma 2.3 gives

$$\mathbb{F}^\alpha(t)\|v\|_2^2 \leq c\left(\int_\Omega |v|^\gamma \ln|v|^l\,dy + \|\nabla v\|_2^2\right). \tag{3.15}$$

From (3.14) and (3.15), we have

$$\mathcal{L}'(t) \geq \big\{(1-\alpha)-\varepsilon\kappa\big\}\mathbb{F}^{-\alpha}\mathbb{F}'(t) + \varepsilon\left(a - \frac{b\mu_1}{4B_0\kappa}\right)\int_\Omega |u|^\gamma \ln|v|^l\,dy$$

$$
\begin{aligned}
&+\varepsilon\left\{\frac{\gamma(1-a)}{2}+1\right\}\|v_t\|_2^2+\varepsilon l(1-a)\|v\|_\gamma^\gamma+\varepsilon\gamma(1-a)\mathbb{F}(t)\\
&+\varepsilon\left\{\frac{\gamma(1-a)}{2}\left(\zeta_0-\int_0^t f(\Lambda)\,d\Lambda\right)-\left(\zeta_0-\frac{1}{2}\int_0^t f(\Lambda)\,d\varrho\right)-\frac{c\mu_1}{2B_0\kappa}\right\}\|\nabla v\|_2^2\\
&+\varepsilon\zeta_1\left\{\frac{\gamma(1-a)}{2}-1\right\}\|\nabla v\|_2^4+\varepsilon\left\{\frac{\gamma(1-a)}{2}-\frac{1}{2}\right\}(f\circ\nabla v)\\
&+\frac{\varepsilon\gamma(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|x^2(y,\rho,s,t)\,ds\,d\rho\,dy.
\end{aligned}
\tag{3.16}
$$

At this point, we take $a>0$ small enough so that

$$
\varrho_1=\frac{\gamma(1-a)}{2}-1>0,
$$

and we assume that

$$
\int_0^\infty f(\Lambda)\,d\Lambda<\frac{\frac{\gamma(1-a)}{2}-1}{\frac{\gamma(1-a)}{2}-\frac{1}{2}}=\frac{2\lambda_1}{2\lambda_1+1}
\tag{3.17}
$$

gives

$$
\varrho_2=\left\{\left(\frac{\gamma(1-a)}{2}-1\right)-\left(\int_0^t f(\Lambda)\,d\Lambda\right)\left(\frac{\gamma(1-a)}{2}-\frac{1}{2}\right)\right\}>0,
$$

then we select κ in a way that

$$
\begin{aligned}
\varrho_3&=a-\frac{c\mu_1}{4C_0\kappa}>0,\\
\varrho_4&=\varrho_2-\frac{c\mu_1}{4B_0\kappa}>0.
\end{aligned}
$$

Finally, we set a and κ as fixed values and select ε to be sufficiently small fulfilling

$$
\varrho_5=(1-\alpha)-\varepsilon\kappa>0
$$

and

$$
\mathcal{L}(0)>0.
$$

This implies that for some $\eta>0$ estimate (3.14) becomes

$$
\begin{aligned}
\mathcal{L}'(t)\geq\eta\Bigg\{&\mathbb{H}(t)+\|v_t\|_2^2+\|\nabla v\|_2^2+(f\circ\nabla v)+\|v\|_\gamma^\gamma+\int_\Omega|v|^\gamma\ln|v|^l\,dy\\
&+\|\nabla v\|_2^4+\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|\left\|(x^2(y,\rho,s,t)\right\|_2^2\,ds\,d\rho\Bigg\}.
\end{aligned}
\tag{3.18}
$$

Subsequently, employing the inequalities of Holder and Young, we obtain

$$
\left|\int_\Omega vv_t\,dy\right|^{\frac{1}{1-\alpha}}\leq c\left[\|v\|_\gamma^{\frac{\theta}{1-\alpha}}+\|v_t\|_2^{\frac{\mu}{1-\alpha}}\right],
\tag{3.19}
$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\mu=2(1-\alpha)$ to get

$$\frac{\theta}{1-\alpha}=\frac{2}{2(1-\alpha)-1}\leq\gamma.$$

Further, for $s=\frac{2}{2(1-\alpha)-1}$, estimate (3.19) gives

$$\left|\int_\Omega vv_t\,dx\right|^{\frac{1}{1-\alpha}}\leq b\big[\|v\|_\gamma^s+\|v_t\|_2^2\big].$$

Then, Lemma 2.5 yields

$$\left|\int_\Omega vv_t\,dx\right|^{\frac{1}{1-\alpha}}\leq c\big[\|v\|_\gamma^\gamma+\|v_t\|_2^2+\|\nabla v\|_2^2\big]. \tag{3.20}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{\frac{1}{1-\alpha}}(t)&=\left(\mathbb{F}^{1-\alpha}+\varepsilon\int_\Omega vv_t\,dy+\frac{\varepsilon\mu_1}{2}\|v\|_2^2+\varepsilon\frac{\sigma}{4}\|\nabla v\|_2^4\right)^{\frac{1}{1-\alpha}}\\
&\leq c\left(\mathbb{F}(t)+\left|\int_\Omega vv_t\,dy\right|^{\frac{1}{1-\alpha}}+\|v\|_2^{\frac{2}{1-\alpha}}+\|\nabla v\|_2^{\frac{4}{1-\alpha}}\right)\\
&\leq c\big(\mathbb{F}(t)+\|v\|_\gamma^\gamma+\|v_t\|_{p+2}^{p+2}+\|\nabla v\|_2^2+\|\nabla v\|_2^4+\|\nabla v_t\|_2^2\big)\\
&\leq c\left(\mathbb{F}(t)+\|v\|_\gamma^\gamma+\|v_t\|_2^2+\|\nabla v\|_2^2+\|\nabla v\|_2^4+(f\circ\nabla v)\right.\\
&\quad\left.+\int_0^1\int_{\tau_1}^{\tau_2}s\big|\mu_2(s)\big|\big\|x^2(y,\rho,s,t)\big\|_2^2\right)ds\,d\rho+\int_\Omega|v|^\gamma\ln|v|^l\,dy.
\end{aligned}\tag{3.21}$$

Next, (3.18) and (3.21) implies

$$\mathcal{L}'(t)\geq\Gamma\mathcal{L}^{\frac{1}{1-\alpha}}(t) \tag{3.22}$$

with $\Gamma>0$, this relies on η and b only.

By integration of (3.22), we have

$$\mathcal{L}^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{\mathcal{L}^{\frac{-\alpha}{1-\alpha}}(0)-\Gamma\frac{\alpha}{(1-\alpha)}t}.$$

Hence, $\mathcal{L}(t)$ blows up in time

$$P\leq P^*=\frac{1-\alpha}{\Gamma\alpha\mathcal{L}^{\alpha/(1-\alpha)}(0)}.$$

The proof is completed. □

4 Conclusion

The examination of solutions to the Kirchhoff equation in the context of viscoelastic materials holds paramount significance. In our investigation, a Kirchhoff-type equation featuring nonlinear viscoelastic properties, distinguished by distributed delay, logarithmic

nonlinearity, and Balakrishnan–Taylor damping terms, was examined. Subsequent to verifying pertinent hypotheses, the manifestation of solution blow-up was conclusively established.

Funding
There is no applicable fund.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
SB: writing original draft, Methodology, RJ: Resources, Methodology, AC: formal analysis, Conceptualization; AZ, MB conceptualized, investigated, analyzed and validated the research while; SB: formulated, investigated, reviewed and, Corresponding author, Supervision.

Author details
[1]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia. [2]Institute of Energy Infrastructure (IEI), Department of Civil Engineering, College of Engineering, Universiti Tenaga Nasional (UNITEN), Putrajaya Campus, Jalan IKRAM-UNITEN, 43000 Kajang, Selangor, Malaysia. [3]Mathematics Research Center, Near East University North Cyprus, Mersin 10, 99138, Turkey. [4]Mathematics in Applied Sciences and Engineering Research Group, Scientific Research Center, Al-Ayen University, Nasiriyah, 64001, Iraq. [5]Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Laghouat, Algeria. [6]Department of Mathematics, Faculty of Exact Sciences, University of El Oued, El Oued, Algeria. [7]Laboratory of Mathematics, Informatics and Systems (LAMIS), Department of Mathematics and Computer Science, Larbi Tebessi University, 12002 Tebessa, Algeria.

Received: 21 November 2023 Accepted: 29 February 2024 Published online: 08 March 2024

References

1. Ahmad, I., Ali, I., Jan, R., Idris, S.A., Mousa, M.: Solutions of a three-dimensional multi-term fractional anomalous solute transport model for contamination in groundwater. PLoS ONE **18**, e0294348 (2023)
2. Ragusa, M.A., Tachikawa, A.: On continuity of minimizers for certain quadratic growth functionals. J. Math. Soc. Jpn. **57**, 691–700 (2005)
3. Arnous, A.H., Hashemi, M.S., Nisar, K.S., Shakeel, M., Ahmad, J., Ahmad, I., Jan, R., Ali, A., Kapoor, M., Shah, N.A.: Investigating solitary wave solutions with enhanced algebraic method for new extended Sakovich equations in fluid dynamics. Results Phys. **57**, 107369 (2024)
4. Jan, R., Razak, N.N.A., Boulaaras, S., Rajagopal, K., Khan, Z., Almalki, Y.: Fractional perspective evaluation of Chikungunya infection with saturated incidence functions. Alex. Eng. J. **83**, 35–42 (2023)
5. Guariglia, E., Guido, R.C.: Chebyshev wavelet analysis. J. Funct. Spaces **2022**, 1–17 (2022)
6. Guariglia, E., Silvestrov, S.: Fractional-wavelet analysis of positive definite distributions and wavelets on $\mathcal{D}'(\mathbb{C})$. In: Engineering Mathematics II: Algebraic, Stochastic and Analysis Structures for Networks, Data Classification and Optimization, pp. 337–353. Springer, Berlin (2019)
7. Ragusa, M.A., Tachikawa, A.: Regularity for minimizers for functionals of double phase with variable exponents. Adv. Nonlinear Anal. **9**, 710–728 (2019)
8. Bland, D.R.: The Theory of Linear Viscoelasticity. Dover, Mineola (2016)
9. Choucha, A., Boulaaras, S.: Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan–Taylor damping. Bound. Value Probl. **2021**, 77 (2021)
10. Choucha, A., Ouchenane, D., Zennir, K., Feng, B.: Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term. Math. Methods Appl. Sci. **20**, 1–26 (2020)
11. Choucha, A., Boulaaras, S., Ouchenane, D., Beloul, S.: General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms. Math. Methods Appl. Sci. **2020**, 1–22 (2021)
12. Choucha, A., Boulaaras, S.M., Ouchenane, D., Cherif, B.B., Abdalla, M.: Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term. J. Funct. Spaces **2021**, 1–8 (2021)
13. Coleman, B.D., Noll, W.: Foundations of linear viscoelasticity. Rev. Mod. Phys. **33**, 239 (1961)
14. Feng, B., Soufyane, A.: Existence and decay rates for a coupled Balakrishnan-Taylor viscoelastic system with dynamic boundary conditions. Math. Methods Appl. Sci. **43**, 3375–3391 (2020)
15. Gheraibia, B., Boumaza, N.: General decay result of solutions for viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term. Z. Angew. Math. Phys. **71**, 198 (2020)
16. Mesloub, F., Boulaaras, S.: General decay for a viscoelastic problem with not necessarily decreasing kernel. J. Appl. Math. Comput. **58**, 647–665 (2018)

17. Balakrishnan, A.V., Taylor, L.W.: Distributed parameter nonlinear damping models for flight structures. In: Proceedings Damping, vol. 89 (1989)
18. Bass, R.W., Zes, D.: Spillover, nonlinearity, and flexible structures. In: NASA. Langley Research Center, Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems, Part 1, pp. 1–14 (1991, March)
19. Boulaaras, S., Draifia, A., Zennir, K.: General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity. Math. Methods Appl. Sci. **42**, 4795–4814 (2019)
20. Liu, W., Zhu, B., Li, G., Wang, D.: General decay for a viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term. Evol. Equ. Control Theory **6**, 239–260 (2017)
21. Mu, C., Ma, J.: On a system of nonlinear wave equations with Balakrishnan–Taylor damping. Z. Angew. Math. Phys. **65**, 91–113 (2014)
22. Boulaaras, S., Choucha, A., Ouchenane, D., Cherif, B.: Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms. Adv. Differ. Equ. **2020**, 310 (2020)
23. Choucha, A., Ouchenane, D., Boulaaras, S.: Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term. Math. Methods Appl. Sci. **43**, 9983–10004 (2020)
24. Choucha, A.B., Ouchenane, D.J., Boulaaras, S.A.: Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms. J. Nonlinear Funct. Anal. **2020**, 1–10 (2020)
25. Choucha, A., Ouchenane, D., Zennir, K.: Exponential growth of solution with L_p-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data. Open J. Math. Anal. **4**, 76–83 (2020)
26. Djebabla, A., Choucha, A., Ouchenane, D., Zennir, K.: Explicit stability for a porous thermoelastic system with second sound and distributed delay term. Int. J. Appl. Comput. Math. **7**, 1–16 (2021)
27. Nicaise, S., Pignotti, C.: Stabilization of the wave equation with boundary or internal distributed delay. Differ. Integral Equ. **21**, 935–958 (2008)
28. Barrow, J.D., Parsons, P.: Inflationary models with logarithmic potentials. Phys. Rev. D **52**, 5576–5587 (1995)
29. Bartkowski, K., Gorka, P.: One-dimensional Klein-Gordon equation with logarithmic nonlinearities. J. Phys. A **41**, 355201 (2008)
30. Bialynicki-Birula, I., Mycielski, J.: Wave equations with logarithmic nonlinearities. Bull. Acad. Pol. Sci., Sér. Sci. Phys. Astron. **23**, 461–466 (1975)
31. Chen, H., Luo, P., Liu, G.: Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. J. Math. Anal. Appl. **422**, 84–98 (2015)
32. Gorka, P.: Logarithmic Klein-Gordon equation. Acta Phys. Pol. B **40**, 59–66 (2009)
33. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)
34. Kafini, M., Messaoudi, S.: Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay. Appl. Anal. **99**, 530–547 (2020)
35. Cavalcanti, M.M., Cavalcanti, V.D., Ferreira, J.: Existence and uniform decay for a non-linear viscoelastic equation with strong damping. Math. Methods Appl. Sci. **24**, 1043–1053 (2001)
36. Guo, L., Yuan, Z., Lin, G.: Blow up and global existence for a nonlinear viscoelastic wave equation with strong damping and nonlinear damping and source terms. Appl. Math. **6**, 806 (2015)
37. Agre, K., Rammaha, M.A.: Systems of nonlinear wave equations with damping and source terms. Differ. Integral Equ. **19**, 1235–1270 (2007)

Publisher's Note