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Gradient estimates for a class of elliptic equations with logarithmic terms

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Abstract

We obtain the gradient estimates of the positive solutions to a nonlinear elliptic equation on an n -dimensional complete Riemannian manifold (M, g)

$$\Delta u + au(\ln u)^p + bu \ln u = 0,$$

where $a \neq 0, b$ are two constants and $p = \frac{k_1}{2k_2+1} \geq 2$, here k_1 and k_2 are two positive integers. The gradient bound is independent of the bounds of the solution and the Laplacian of the distance function. As the applications of the estimates, we show the Harnack inequality and the upper bound of the solution.

Keywords: Complete Riemannian manifold; Nonlinear elliptic equation; Gradient estimates; Harnack inequality

1 Introduction

Let (M, g) be an n -dimensional complete Riemannian manifold. Recently, many authors studied the following elliptic differential equation

$$\Delta u + au \ln u = 0 \quad \text{on } M, \tag{1.1}$$

where a is a constant. This equation is closely related to the logarithmic Sobolev inequality [3, 6, 16]. It is also involved in the gradient Ricci solution [11, 13, 17] devoted to understanding the Ricci flow introduced by Hamilton [8].

In [1], Abolarinawa considered the following equation

$$\Delta_f u(x) + au(\ln u)^\alpha = 0 \tag{1.2}$$

on a complete smooth metric measure manifold with weight e^{-f} and Bakry-Emery Ricci tensor bounded from below, where α and α are constants. He obtained the local gradient estimates dependent on the bound of solutions. The importance of gradient estimates cannot be overemphasized in geometric analysis and mathematical physics. For instance, they can be used to find the Hölder continuity of solutions and estimate on the eigenvalues; see [4, 5, 14, 15] and references therein. In particular, Gui, Jian, and Ju [7] obtained the local

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gradient estimate and Liouville-type theorem of translating solutions to mean curvature flow.

In this paper, we study the local gradient estimate of the positive solution to the following more general nonlinear elliptic equation

$$\Delta u + au(\ln u)^p + bu \ln u = 0 \quad \text{on } M, \quad (1.3)$$

where $a \neq 0, b$ are two constants and $p = \frac{k_1}{2k_2+1} \geq 2$, here k_1 and k_2 are two positive integers.

In the case $a \equiv 0$ and $b \equiv 0$, (1.3) is the Laplace equation. The corresponding gradient estimate was established by Yau [18]. Later, Li and Yau [10] obtained the well-known Li-Yau estimate for the Schrödinger equation and derived a Harnack inequality. In the case $a \equiv 0$ and $b < 0$, Ma [11] studied the gradient estimates of the positive solutions to the above elliptic equation for $\dim(M) \geq 3$. Then, Yang [17] improved the estimate of [11] and extended it to the case $b > 0$, and M is of any dimension. Chen and Chen [2] also extended the estimate of [11] to the case $b > 0$. Later, Huang and Ma [9], Qian [13], Zhu and Li [19] also studied the gradient estimates of the positive solutions to the above elliptic or the corresponding parabolic equation in the case $a \equiv 0$ and $b \in \mathbb{R}$. Recently, Peng, Wang, and Wei [12] considered the following equation

$$\Delta u + au(\ln u)^p + bu = 0,$$

where $a, b \in \mathbb{R}$ and $p = \frac{k_1}{2k_2+1} \geq 2$, here k_1 and k_2 are positive integers. They obtained the local gradient estimates and derived a Harnack inequality.

Throughout the paper, we use the notation $\text{Ric}(g)$ to denote the Ricci curvature of (M, g) . Now we state the local gradient estimates independent of the bounds of the solution and the Laplacian of the distance function.

Theorem 1.1 (*Local gradient estimate*) *Let (M, g) be an n -dimensional complete Riemannian manifold with $\text{Ric}(g) \geq -Kg$, where the constant $K := K(2R) \geq 0$ in the geodesic ball $B_{2R}(O)$. Here, O is a point in M . Suppose $a, b \in \mathbb{R}$, $a \neq 0$, $p = \frac{k_1}{2k_2+1} \geq 2$ where k_1 and k_2 are two integers, and $1 < \lambda < 2$. Let $u(x)$ be a smooth positive solution to (1.3). Then we have, in $B_R(O)$,*

(i) *when $a > 0$,*

$$\frac{|\nabla u|^2}{u^2} + \lambda a(\ln u)^p + 2|b| \ln u \leq \max\{1, T_1\},$$

where

$$\begin{aligned} T_1 = n &\left(\frac{2\lambda A}{2-\lambda} + \max \left\{ \frac{\lambda}{\lambda-1}, \frac{4\lambda}{2-\lambda}(1+nK) \right\} K + \max \left\{ \frac{\lambda^2}{4(\lambda-1)^2} + 2, \frac{4\lambda^2}{(2-\lambda)^2} \right\} \frac{nC_1^2}{R^2} \right. \\ &\left. + S_1(n, p, a, b, \lambda, K) \right) \end{aligned}$$

with

$$\begin{aligned}
S_1 = & \left(2|b|(2p-1) + \frac{4}{n} \right) (\lambda-1)a(-Y_1)^p + H_1^p + \lambda^2 a^2 p(p-1)(Y_1^{2p-2} + Y_2^{2p-2} + H_1^{2p-2}) \\
& + \frac{12}{n} a|b|(\lambda-1)(-Y_1)^{p+1} - \frac{\lambda|b|p(p-1)}{(\lambda-1)Y_2} + \lambda a p(p-1)(-Y_2)^{p-2} + \frac{12|b|H_1}{n} \\
& + \lambda a(2K + 3|b|(p-2))(-Y_2)^p + \lambda(\lambda-1)a^2 p H_1^{2p-1} + \frac{3\lambda|b|(p-2)}{2(\lambda-1)} \\
& + \frac{2\lambda}{2-\lambda} \left(9nb^2 + \frac{p-1}{np} (12|b|) \left(\frac{ap(2-\lambda)}{2} \right)^{-\frac{1}{p-1}} + (2-\lambda)(2n(p-1))^{p-1} a \right) \\
& + ((2-\lambda) + 2\lambda|b|(p-1))ap(H_1^{p-1} + (-Y_2)^{p-1}),
\end{aligned}$$

(ii) when $a < 0$,

$$\frac{|\nabla u|^2}{u^2} + \lambda a(\ln u)^p + 2|b|\ln u \leq \max\{1, T_2\},$$

where

$$\begin{aligned}
T_2 = & n \left(\frac{\lambda A}{2-\lambda} + \max \left\{ \frac{\lambda}{\lambda-1}, \frac{2\lambda}{2-\lambda}(1+2nK) \right\} K \right. \\
& + \max \left\{ \frac{\lambda^2}{2(\lambda-1)^2} + 2, \frac{\lambda^2}{(2-\lambda)^2} + \frac{\lambda}{2-\lambda} \right\} \frac{nC_1^2}{R^2} \\
& \left. + S_2(n, p, a, b, \lambda) \right)
\end{aligned}$$

with

$$\begin{aligned}
S_2 = & \left(2|b|(1-2p) - \frac{4}{n} \right) (\lambda-1)a(-Y_1)^p + \lambda^2 a^2 p(p-1)(Y_1^{2p-2} + Y_2^{2p-2} + H_2^{2p-2}) \\
& + \frac{12|b|H_2}{n} \\
& - \frac{\lambda|b|p(p-1)}{(\lambda-1)Y_2} - \frac{12}{n} a|b|(\lambda-1)((-Y_1)^{p+1} + H_2^{p+1}) - \lambda a(2K + 3|b|(p-2))(-Y_2)^p \\
& - ((2-\lambda) + 2\lambda|b|(p-1))ap(-Y_2)^{p-1} - \lambda a p(p-1)(-Y_2)^{p-2} + 9nb^2 + \frac{3\lambda|b|(p-2)}{2(\lambda-1)} \\
& + \lambda(\lambda-1)a^2 p H_2^{2p-1} - (2K\lambda + 3\lambda|b|(p-2))aH_2^p - \lambda a p(p-1)H_2^{p-2}.
\end{aligned}$$

Here,

$$A = \frac{(n-1)(1+\sqrt{KR})C_1^2 + C_2 + 2C_1^2}{R^2}$$

with C_1 and C_2 are two uniform positive constants,

$$J_1 = \frac{(\lambda-2)pn - \sqrt{(2-\lambda)^2 n^2 p^2 + 8n\lambda(\lambda-1)p(p-1)}}{4(\lambda-1)},$$

$$\begin{aligned}
J_2 &= \frac{(\lambda - 2)pn + \sqrt{(2 - \lambda)^2 n^2 p^2 + 8n\lambda(\lambda - 1)p(p - 1)}}{4(\lambda - 1)}, \\
V_1 &= \frac{pn + \sqrt{n^2 p^2 + 8np(p - 1)}}{4} \frac{\lambda}{\lambda - 1}, \\
V_2 &= \frac{pn - \sqrt{n^2 p^2 + 4np(p - 1)}}{2} \frac{\lambda}{\lambda - 1}, \\
V_3 &= \frac{pn + \sqrt{n^2 p^2 + 4np(p - 1)}}{2} \frac{\lambda}{\lambda - 1}, \\
L_1 &= \frac{(\lambda - 1)(2p - 1)n + \sqrt{n^2(\lambda - 1)^2(2p - 1)^2 + 8n\lambda(\lambda - 1)p(p - 1)}}{4(\lambda - 1)}, \\
H_1 &= \max\{L_1, V_1\}, \quad Y_2 = \min\{J_1, V_2\}, \\
H_2 &= \max\left\{J_2, V_3, \sqrt[p-1]{\frac{48|b|}{-\alpha(\lambda - 1)}}, \sqrt[p]{\frac{4n(2K\lambda + 3\lambda|b|(p - 2))}{-\alpha(\lambda - 1)^2}}\right\},
\end{aligned}$$

and

$$Y_1 = \min\left\{-\sqrt[p-1]{\frac{24|b|}{|\alpha|(\lambda - 1)}}, -\sqrt[p]{\frac{4n(2p - 1)|b|}{|\alpha|(\lambda - 1)}}, V_2\right\}.$$

As a consequence of Theorem 1.1, we have the following Harnack inequality.

Corollary 1.2 (Harnack inequality) Assume that the same conditions in Theorem 1.1 hold. Then, for $a > 0$, $b = 0$, $\lambda = \frac{3}{2}$, $p = \frac{2k}{2k+1} \geq 2$ where k and k_2 are two integers, we have

$$\sup_{B_{\frac{R}{2}}(O)} u \leq e^{\max\{1, S\}R} \inf_{B_{\frac{R}{2}}(O)} u.$$

Here,

$$\begin{aligned}
S &= (n)^{\frac{1}{2}} \left[6 \left(A + 2K + 2nK^2 + \frac{a}{2} (2n(p-1))^{p-1} \right) + \frac{36nC_1^2}{R^2} + \frac{6aK(-Y_2)^p}{2} \right. \\
&\quad + \frac{9a^2p(p-1)(H_1^{2p-2} + Y_1^{2p-2} + Y_2^{2p-2})}{4} + \frac{3a^2pH_1^{2p-1}}{4} + \frac{2a}{n} (H_1^p + (-Y_1)^p) \\
&\quad \left. + \frac{ap}{2} (H_1^{p-1} + (-Y_2)^{p-1}) + \frac{3ap(p-1)(-Y_2)^{p-2}}{2} \right]^{\frac{1}{2}},
\end{aligned}$$

where H_1 , Y_1 and Y_2 are constants in Theorem 1.1 with $\lambda = \frac{3}{2}$.

As another application of Theorem 1.1, we show the upper bound of solutions, which is analogous to the result obtained by Qian [13].

Corollary 1.3 Assume that the same conditions in Theorem 1.1 hold. Then, for $a > 0$, $b \neq 0$, $\lambda = \frac{3}{2}$, $p = \frac{2k}{2k+1} \geq 2$ where k and k_2 are two integers, we see

$$u \leq e^{\frac{1}{2|b|} \max\{1, \hat{S}\}} \text{ in } B_R(O).$$

Here,

$$\begin{aligned}\hat{S} = & n \left[6 \left(A + 9nb^2 + 2K + 2nK^2 + \frac{p-1}{np} (12|b|)^{\frac{p}{p-1}} \left(\frac{ap}{4} \right)^{-\frac{1}{p-1}} + \frac{a}{2} (2n(p-1))^{p-1} \right) \right. \\ & + \frac{36nC_1^2}{R^2} \\ & + \frac{9a^2p(p-1)(H_1^{2p-2} + Y_1^{2p-2} + Y_2^{2p-2})}{4} + \frac{3a^2pH_1^{2p-1}}{4} \\ & + a(H_1^p + (-Y_1)^p) \left(\frac{2}{n} + |b|(2p-1) \right) \\ & + \left(\frac{1}{2} + 3|b|(p-1) \right) ap(H_1^{p-1} + (-Y_2)^{p-1}) + \frac{12|b|H_1}{n} \\ & + \frac{6a|b|(-Y_1)^{p+1}}{n} - \frac{3|b|p(p-1)}{Y_2} \\ & \left. + \frac{3a(2K + 3|b|(p-2))(-Y_2)^p}{2} + \frac{3ap(p-1)(-Y_2)^{p-2}}{2} + \frac{9|b|(p-2)}{2} \right],\end{aligned}$$

where H_1 , Y_1 and Y_2 are constants in Theorem 1.1 with $\lambda = \frac{3}{2}$.

The structure of this paper is as follows: In Sect. 2, we give some lemmas, which will be used in the following section. Section 3 is a proof of Theorem 1.1. The last section is devoted to the proof of Corollary 1.2 (a Harnack inequality) and Corollary 1.3.

2 Preliminaries

In this section, we first construct an auxiliary function and establish a differential inequality. Then, a lemma on cut-off functions is introduced. Suppose that an n -dimensional complete Riemannian manifold (M, g) satisfies $\text{Ric}(g) \geq -Kg$ in a geodesic ball $B_{2R}(O)$, where $K = K(2R)$ is a nonnegative constant, and O is a fixed point on M .

Lemma 2.1 *Assume that $u(x)$ is a smooth positive solution to (1.3) in a geodesic ball $B_{2R}(O)$. Setting $w = \ln u$ and*

$$G = |\nabla w|^2 + \lambda aw^p + 2|b|w,$$

where $1 < \lambda < 2$ and $b \in \mathbb{R}$, we obtain

$$\begin{aligned}\Delta G \geq & \frac{2}{n} G^2 - 2\langle \nabla w, \nabla G \rangle \\ & + \left[-\frac{4}{n}(\lambda-1)aw^p + (\lambda-2)apw^{p-1} + (2|b|-2b) + \lambda ap(p-1)w^{p-2} \right. \\ & \left. - \frac{4(2|b|-b)w}{n} - 2K \right] G \\ & + \left[\frac{2a^2(\lambda-1)^2w^{2p}}{n} - \lambda(\lambda-1)a^2pw^{2p-1} - \lambda^2a^2p(p-1)w^{2p-2} \right] \\ & + \left[\frac{4(\lambda-1)(2|b|-b)a}{n}w^{p+1} \right]\end{aligned}$$

$$\begin{aligned}
& + ((\lambda a(2|b|-b)(p-2) + 2K\lambda a + 2(\lambda-1)a|b|(1-2p))w^p \\
& - 2\lambda a|b|p(p-1)w^{p-1}) \left[+ \frac{2(2|b|-b)^2 w^2}{n} - 2|b|(2|b|-b)w + 4K|b|w. \right]
\end{aligned}$$

Proof In a normal coordinate at point O , we have

$$\Delta w = -\frac{|\nabla u|^2}{u^2} + \frac{\Delta u}{u} = -|\nabla w|^2 - aw^p - bw.$$

It follows that

$$\Delta w = -G + (\lambda - 1)aw^p + (2|b|-b)w. \quad (2.1)$$

Using the Bochner-Weitzenböck formula and $\text{Ric}(g) \geq -Kg$, we get

$$\Delta|\nabla w|^2 \geq 2|\nabla^2 w|^2 + 2\langle \nabla w, \nabla(\Delta w) \rangle - 2K|\nabla w|^2.$$

By the definition of G and the Cauchy-Schwarz inequality, we see

$$\begin{aligned}
\Delta G &= \Delta|\nabla w|^2 + \Delta(\lambda aw^p + 2|b|w) \\
&\geq \frac{2}{n}(\Delta w)^2 + 2\langle \nabla w, \nabla(\Delta w) \rangle - 2K|\nabla w|^2 + (\lambda ap(p-1)w^{p-2}|\nabla w|^2 + \lambda apw^{p-1}\Delta w \\
&\quad + 2|b|\Delta w).
\end{aligned} \quad (2.2)$$

From (2.1) and (2.2), we obtain this lemma. \square

Lemma 2.2 ([12], Lemma 2.2) *Let ϕ be a cut-off function, that is, $\phi(x)|_{B_R(O)} = 1$, $\phi(x)|_{M \setminus B_{2R}(O)} = 0$. Then, ϕ satisfies*

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{C_1^2}{R^2}, \quad (2.3)$$

and

$$\Delta \phi \geq -\frac{(n-1)(1+\sqrt{KR})C_1^2 + C_2}{R^2}, \quad (2.4)$$

where C_1 and C_2 are two positive constants independent of (M, g) .

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

Proof of Theorem 1.1 Let $x_0 \in B_{2R}(O)$ such that $\phi G(x_0) = \sup_{B_{2R}(O)}(\phi G)$. If $\phi G(x_0) < 0$, then we finish the proof. Hence, we may assume that $\phi G(x_0) > 0$. Note that $x_0 \notin \partial B_{2R}(O)$. It follows that $\nabla(\phi G)(x_0) = 0$, $\Delta(\phi G)(x_0) \leq 0$. By Lemma 2.2, we see that

$$\phi \Delta G \leq GA,$$

where $A = \frac{(n-1)(1+\sqrt{K}R)C_1^2+C_2+2C_1^2}{R^2}$, and

$$-\langle \nabla w, \nabla G \rangle \phi = G \langle \nabla w, \nabla \phi \rangle \geq -G |\nabla \phi| (G - \lambda \alpha w^p - 2|b|w)^{\frac{1}{2}}.$$

By Lemma 2.1, we have

$$\begin{aligned} AG &\geq \frac{2}{n} \phi G^2 - 2G |\nabla \phi| (G - \lambda \alpha w^p - 2|b|w)^{\frac{1}{2}} \\ &\quad + \left[-\frac{4}{n} (\lambda - 1) \alpha w^p + (\lambda - 2) \alpha p w^{p-1} + (2|b| - 2b) \right. \\ &\quad \left. + \lambda \alpha p (p-1) w^{p-2} - \frac{4(2|b|-b)w}{n} - 2K \right] \phi G \\ &\quad + \left[\frac{2\alpha^2(\lambda-1)^2 w^{2p}}{n} - \lambda(\lambda-1)\alpha^2 p w^{2p-1} \right. \\ &\quad \left. - \lambda^2 \alpha^2 p (p-1) w^{2p-2} \right] \phi \\ &\quad + \left[\frac{4(\lambda-1)(2|b|-b)\alpha}{n} w^{p+1} + (\lambda \alpha (2|b|-b)(p-2) + 2K \lambda \alpha \right. \\ &\quad \left. + 2(\lambda-1)\alpha |b|(1-2p)) w^p - 2\lambda \alpha p (p-1) |b| w^{p-1} \right] \phi + \frac{2(2|b|-b)^2 w^2 \phi}{n} \\ &\quad - 2|b|(2|b|-b) w \phi + 4K |b| w \phi. \end{aligned} \tag{3.1}$$

If $G \leq 1$, then by $0 \leq \phi \leq 1$, we have $\phi G \leq 1$. Thus, we may assume that $G \geq 1$. For the clarity, we consider the following six cases to prove Theorem 1.1.

Case 1: $\alpha > 0$, $w^{p-1} > 0$ and $w > 0$; Case 2: $\alpha > 0$, $w^{p-1} < 0$ and $w < 0$;

Case 3: $\alpha > 0$, $w^{p-1} > 0$ and $w < 0$; Case 4: $\alpha < 0$, $w^{p-1} > 0$ and $w > 0$;

Case 5: $\alpha < 0$, $w^{p-1} > 0$ and $w < 0$; Case 6: $\alpha < 0$, $w^{p-1} < 0$ and $w < 0$;

Now we discuss the above six possibilities on a case-by-case basis.

Case 1: If $\alpha > 0$, $w^{p-1} > 0$ and $w > 0$, then by the Young inequality, we have

$$-6b^2 w \phi \geq -\frac{b^2 w^2 \phi}{nG} - 9nb^2 \phi G, \tag{3.2}$$

and

$$-2G |\nabla \phi| (G - \lambda \alpha w^p - 2|b|w)^{\frac{1}{2}} \leq \frac{R |\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{1}{2}}} + \frac{C_1 G^{\frac{1}{2}} \phi^{\frac{1}{2}}}{R} (G - \lambda \alpha w^p - 2|b|w). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$2G |\nabla \phi| (G - \lambda \alpha w^p - 2|b|w)^{\frac{1}{2}} \leq \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R}. \tag{3.4}$$

From (3.1), (3.2) and (3.4), we see

$$AG \geq \frac{2}{n} G^2 \phi - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R}$$

$$\begin{aligned}
& + \left[-2K - 9nb^2 - \frac{4}{n}(\lambda - 1)aw^p + (\lambda - 2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \right. \\
& - \frac{12|b|w}{n} \Big] \phi G + \left[\frac{2(\lambda - 1)^2 w^{2p}}{n} - \lambda(\lambda - 1)pw^{2p-1} - \lambda^2 p(p-1)w^{2p-2} \right] a^2 \phi \\
& + \left[\frac{4(\lambda - 1)|b|w^{p+1}}{n} + (2K\lambda + \lambda|b|(p-2) + 2|b|(\lambda - 1)(1-2p))w^p \right. \\
& \left. - 2\lambda|b|p(p-1)w^{p-1} \right] a\phi + \frac{2b^2 w^2 \phi}{n} - \frac{b^2 w^2 \phi}{nG}. \tag{3.5}
\end{aligned}$$

Case 1.1: If $w \in [H_1, +\infty)$, where $H_1 = \max\{L_1, V_1\} > 0$ with

$$\begin{aligned}
L_1 &= \frac{(\lambda - 1)(2p - 1)n + \sqrt{n^2(\lambda - 1)^2(2p - 1)^2 + 8n\lambda(\lambda - 1)p(p - 1)}}{4(\lambda - 1)}, \\
V_1 &= \frac{pn + \sqrt{n^2p^2 + 8np(p - 1)}}{4} \frac{\lambda}{\lambda - 1},
\end{aligned}$$

then we have

$$\begin{aligned}
& - \frac{4}{n}(\lambda - 1)aw^p + (\lambda - 2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \\
& \geq -\frac{4(\lambda - 1)G}{n\lambda} + (\lambda - 2)ap \left(\frac{G}{\lambda a} \right)^{\frac{p-1}{p}}, \tag{3.6}
\end{aligned}$$

By the definition of H_1 , we see

$$\frac{2}{n}(\lambda - 1)^2 a^2 w^{2p} - \lambda(\lambda - 1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \geq 0, \tag{3.7}$$

$$\frac{4}{n}(\lambda - 1)a|b|w^{p+1} + 2a|b|(\lambda - 1)(1-2p)w^p - 2\lambda a|b|p(p-1)w^{p-1} \geq 0. \tag{3.8}$$

From (3.5), (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned}
AG &\geq \frac{2}{n}\phi G^2 - \frac{2C_1\phi^{\frac{1}{2}}G^{\frac{3}{2}}}{R} + \left[-\frac{4(\lambda - 1)G}{n\lambda} + (\lambda - 2)ap \left(\frac{G}{\lambda a} \right)^{\frac{p-1}{p}} - \frac{12|b|\left(\frac{G}{\lambda a}\right)^{\frac{1}{p}}}{n} - 2K \right. \\
&\quad \left. - 9nb^2 \right] \phi G.
\end{aligned}$$

By the Young inequality again, we have

$$\frac{2C_1\phi^{\frac{1}{2}}G^{\frac{1}{2}}}{R} \leq \frac{2n\lambda C_1^2}{(2-\lambda)R^2} + \frac{(2-\lambda)\phi G}{2n\lambda}, \tag{3.9}$$

$$\frac{(2-\lambda)p}{\lambda}(\lambda a)^{\frac{1}{p}}(\phi G)^{\frac{p-1}{p}} \leq \frac{(2-\lambda)\phi G}{2n\lambda} + (2-\lambda)(2n(p-1))^{p-1}a, \tag{3.10}$$

and

$$\frac{12|b|\left(\frac{G}{\lambda a}\right)^{\frac{1}{p}}\phi}{n} \leq \frac{(2-\lambda)\phi G}{2n\lambda} + \frac{\phi(p-1)}{np}(12|b|)^{\frac{p}{p-1}} \left(\frac{ap(2-\lambda)}{2} \right)^{-\frac{1}{p-1}}. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we get

$$\begin{aligned} \phi G &\leq \frac{2n\lambda}{2-\lambda} \left[A + 9nb^2 + 2K + \frac{2nC_1^2\lambda}{R^2(2-\lambda)} + \frac{p-1}{np} (12|b|)^{\frac{p}{p-1}} \left(\frac{ap(2-\lambda)}{2} \right)^{-\frac{1}{p-1}} \right. \\ &\quad \left. + (2-\lambda)(2n(p-1))^{p-1} a \right]. \end{aligned} \quad (3.12)$$

Case 1.2: If $w \in (0, H_1)$, then we see

$$\begin{aligned} & -\frac{4}{n}(\lambda-1)aw^p + (\lambda-2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \\ & \geq -\frac{4}{n}(\lambda-1)aH_1^p + (\lambda-2)apH_1^{p-1}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \frac{2}{n}(\lambda-1)^2 a^2 w^{2p} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \\ & \geq -\lambda(\lambda-1)a^2 p H_1^{2p-1} - \lambda^2 a^2 p(p-1)H_1^{2p-2}, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} & \frac{4}{n}(\lambda-1)a|b|w^{p+1} + (2K\lambda a + \lambda a|b|(p-2) + 2a|b|(\lambda-1)(1-2p))w^p \\ & \quad - 2\lambda a|b|p(p-1)w^{p-1} \\ & \geq 2a|b|(\lambda-1)(1-2p)H_1^p - 2\lambda a|b|p(p-1)H_1^{p-1}. \end{aligned} \quad (3.15)$$

Employing (3.5), (3.13), (3.14), (3.15) and $G \geq 1$, we get

$$\begin{aligned} AG &\geq \frac{2}{n}G^2\phi - \frac{2C_1\phi^{\frac{1}{2}}G^{\frac{3}{2}}}{R} \\ &\quad + \left[-\frac{4}{n}(\lambda-1)aH_1^p + (\lambda-2)apH_1^{p-1} - \frac{12|b|H_1}{n} - 2K - 9nb^2 \right] \phi G \\ &\quad - [\lambda(\lambda-1)pH_1^{2p-1} + \lambda^2 p(p-1)H_1^{2p-2}]a^2\phi G + [2|b|(\lambda-1)(1-2p)H_1^p \\ &\quad - 2\lambda|b|p(p-1)H_1^{p-1}]a\phi G. \end{aligned}$$

It follows that

$$\begin{aligned} \phi G &\leq n \left[A + \frac{nC_1^2}{R^2} + \left(\frac{4}{n}(\lambda-1) + 2|b|(\lambda-1)(2p-1) \right) aH_1^p \right. \\ &\quad + ((2-\lambda) + 2\lambda|b|(p-1))apH_1^{p-1} \\ &\quad \left. + \frac{12|b|H_1}{n} + 2K + 9nb^2 + \lambda(\lambda-1)a^2 p H_1^{2p-1} + \lambda^2 a^2 p(p-1)H_1^{2p-2} \right]. \end{aligned} \quad (3.16)$$

Case 2: If $a > 0$, $w^{p-1} < 0$ and $w < 0$, then from (3.3), we have

$$2G|\nabla\phi|\left(G - \lambda aw^p - 2|b|w\right)^{\frac{1}{2}} \leq \frac{R|\nabla\phi|^2 G^{\frac{3}{2}}}{C_1\phi^{\frac{1}{2}}} + \frac{C_1 G^{\frac{1}{2}}\phi^{\frac{1}{2}}}{R} (G - \lambda aw^p - 2|b|w)$$

$$\leq \frac{2C_1 G^{\frac{3}{2}} \phi^{\frac{1}{2}}}{R} + \frac{n C_1^2 G}{R^2} + \frac{b^2 w^2 \phi}{n}. \quad (3.17)$$

By the Cauchy inequality, we see

$$4K|b|w\phi \geq -\frac{b^2 w^2 \phi}{nG} - 4nK^2 \phi G. \quad (3.18)$$

Using (3.1), (3.17), and (3.18), we find

$$\begin{aligned} \left(A + \frac{n C_1^2}{R^2} \right) G &\geq \frac{2}{n} G^2 \phi - 2 \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} \\ &+ \left[-\frac{4}{n} (\lambda - 1) aw^p + (\lambda - 2) apw^{p-1} + \lambda ap(p-1) w^{p-2} \right. \\ &- \frac{4|b|w}{n} - 2K - 4nK^2 \Big] \phi G + \left[\frac{2(\lambda - 1)^2 a^2 w^{2p}}{n} - \lambda(\lambda - 1) a^2 p w^{2p-1} \right. \\ &- \lambda^2 a^2 p(p-1) w^{2p-2} \Big] \phi + \left[\frac{12(\lambda - 1)|b|w^{p+1}}{n} + (2K\lambda + \lambda|b|(p-2) \right. \\ &+ 2|b|(\lambda - 1)(1 - 2p))w^p - 2\lambda|b|p(p-1)w^{p-1} \Big] a\phi \\ &+ \frac{b^2 w^2 \phi}{n} - \frac{b^2 w^2 \phi}{nG}. \end{aligned} \quad (3.19)$$

Case 2.1: If $w \in (-\infty, Y_1]$, where

$$Y_1 = \min \left\{ -\sqrt[p-1]{\frac{24|b|}{|\alpha|(\lambda-1)}}, -\sqrt[p]{\frac{4n(2p-1)|b|}{|\alpha|(\lambda-1)}}, V_2 \right\}$$

with

$$V_2 = \frac{pn - \sqrt{n^2 p^2 + 4np(p-1)}}{2} \frac{\lambda}{\lambda - 1},$$

then we have

$$\frac{(\lambda - 1)^2 a^2 w^{2p}}{2n} \geq \frac{-12(\lambda - 1)a|b|w^{p+1}}{n}, \quad (3.20)$$

$$\frac{(\lambda - 1)^2 a^2 w^{2p}}{2n} \geq 2(\lambda - 1)(2p - 1)a|b|w^p, \quad (3.21)$$

and

$$\begin{aligned} &-\frac{4}{n} (\lambda - 1) aw^p + (\lambda - 2) apw^{p-1} + \lambda ap(p-1) w^{p-2} \\ &\geq -\frac{4}{n} (\lambda - 1) aw^p \\ &\geq -\frac{4}{n\lambda} (\lambda - 1) G + \frac{8}{n\lambda} (\lambda - 1) |b| w, \end{aligned} \quad (3.22)$$

By the definition of Y_1 , we get

$$\frac{(\lambda-1)^2 a^2 w^{2p}}{n} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \geq 0. \quad (3.23)$$

Combining (3.19), (3.20), (3.21), (3.22), and (3.23), we have

$$\begin{aligned} \left(A + \frac{nC_1^2}{R^2} \right) G &\geq \frac{2(2-\lambda)}{n\lambda} \phi G^2 - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \left[\frac{4|b|w(\lambda-2)}{n\lambda} - 2K - 4nK^2 \right] \phi G \\ &\geq \frac{2(2-\lambda)}{n\lambda} \phi G^2 - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} - [2K + 4nK^2] \phi G. \end{aligned} \quad (3.24)$$

By the Young inequality again, we find

$$\frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{1}{2}}}{R} \leq \frac{n\lambda C_1^2}{(2-\lambda)R^2} + \frac{(2-\lambda)\phi G}{n\lambda}. \quad (3.25)$$

It follows from (3.24) and (3.25) that

$$\phi G \leq \frac{n\lambda}{2-\lambda} \left[A + \frac{nC_1^2}{R^2} + \frac{n\lambda C_1^2}{(2-\lambda)R^2} + 2K + 4nK^2 \right]. \quad (3.26)$$

Case 2.2: If $w \in (Y_1, 0)$, then we get

$$-\frac{4}{n}(\lambda-1)aw^p + (\lambda-2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \geq -\frac{4}{n}(\lambda-1)aY_1^p, \quad (3.27)$$

$$\begin{aligned} \frac{2(\lambda-1)^2 a^2 w^{2p}}{n} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \\ \geq -\lambda^2 a^2 p(p-1)Y_1^{2p-2}, \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} &\frac{12(\lambda-1)a|b|w^{p+1}}{n} + (2K\lambda + \lambda|b|(p-2) + 2|b|(\lambda-1)(1-2p))aw^p \\ &- 2\lambda a|b|p(p-1)w^{p-1} \\ &\geq \frac{12(\lambda-1)a|b|Y_1^{p+1}}{n} + 2a|b|(\lambda-1)(1-2p)Y_1^p. \end{aligned} \quad (3.29)$$

Employing (3.19), (3.27), (3.28), (3.29), and $G \geq 1$, we obtain

$$\begin{aligned} \phi G &\leq n \left[A + \frac{2nC_1^2}{R^2} + \left(2|b|(2p-1) + \frac{4}{n} \right) (\lambda-1)aY_1^p + 2K + 4K^2n + \lambda^2 a^2 p(p-1)Y_1^{2p-2} \right. \\ &\quad \left. - \frac{12}{n} a|b|(\lambda-1)Y_1^{p+1} \right]. \end{aligned} \quad (3.30)$$

Case 3: If $a > 0$, $w^{p-1} > 0$ and $w < 0$, then by the Young inequality, we see

$$\frac{\lambda aw^p C_1 \phi^{\frac{1}{2}} G^{\frac{1}{2}}}{R} \geq -\frac{nC_1^2 \lambda^2 G}{4R^2(\lambda-1)^2} - \frac{a^2 w^{2p} (\lambda-1)^2 \phi}{n},$$

Combining (3.1), (3.17), (3.18), and the above inequality, we get

$$\begin{aligned}
& \left(A + \frac{nC_1^2\lambda^2}{4R^2(\lambda-1)^2} + \frac{nC_1^2}{R^2} \right) G \\
& \geq \frac{2}{n} G^2 \phi - 2 \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} \\
& \quad + \left[-\frac{4}{n} (\lambda-1)aw^p + (\lambda-2)apw^{p-1} + \lambda ap(p-1)w^{p-2} - \frac{4|b|w}{n} - 2K \right. \\
& \quad \left. - 4nK^2 \right] \phi G + \left[\frac{(\lambda-1)^2 a^2 w^{2p}}{n} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \right] \phi \\
& \quad + \left[\frac{4(\lambda-1)|b|w^{p+1}}{n} + (2K\lambda + 3\lambda|b|(p-2) + 2|b|(\lambda-1)(1-2p))w^p \right. \\
& \quad \left. - 2\lambda|b|p(p-1)w^{p-1} \right] aw \\
& \quad + \frac{b^2 w^2 \phi}{n} - \frac{b^2 w^2 \phi}{nG}. \tag{3.31}
\end{aligned}$$

Case 3.1: If $w \in (-\infty, Y_2]$, where $Y_2 = \min\{J_1, V_2\}$ with

$$\begin{aligned}
J_1 &= \frac{(\lambda-2)pn - \sqrt{(2-\lambda)^2 n^2 p^2 + 8n\lambda(\lambda-1)p(p-1)}}{4(\lambda-1)}, \\
V_2 &= \frac{pn - \sqrt{n^2 p^2 + 4np(p-1)}}{2} \frac{\lambda}{\lambda-1},
\end{aligned}$$

then we have

$$-\frac{4(\lambda-1)aw^p}{n} + (\lambda-2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \geq -\frac{2(\lambda-1)aw^p}{n}, \tag{3.32}$$

$$\frac{(\lambda-1)^2 a^2 w^{2p}}{n} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \geq 0, \tag{3.33}$$

and

$$\begin{aligned}
& \left[\frac{4(\lambda-1)|b|w^{p+1}}{n} + (2K\lambda + 3\lambda|b|(p-2) + 2|b|(\lambda-1)(1-2p))w^p \right. \\
& \quad \left. - 2\lambda|b|p(p-1)w^{p-1} \right] aw \\
& \geq (2K\lambda w + 3\lambda|b|(p-2)w - 2\lambda|b|p(p-1))aw^{p-1}. \tag{3.34}
\end{aligned}$$

From (3.31), (3.32), (3.33), and (3.34), we find

$$\begin{aligned}
& \left(A + \frac{n\lambda^2 C_1^2}{4R^2(\lambda-1)^2} + \frac{nC_1^2}{R^2} \right) G \\
& \geq \frac{2}{n} \phi G^2 - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} - \left(\frac{2(\lambda-1)aw^p}{n} + 2K + 4nK^2 \right) \phi G \\
& \quad + \left(2K\lambda + 3\lambda|b|(p-2) - \frac{2\lambda|b|p(p-1)}{w} \right) aw^p \phi.
\end{aligned}$$

By the Young inequality again, we see

$$\frac{2C_1\phi^{\frac{1}{2}}G^{\frac{1}{2}}}{R} \leq \frac{\phi G}{n} + \frac{nC_1^2}{R^2}. \quad (3.35)$$

If $-\frac{2(\lambda-1)G}{n} + 2K\lambda + 3\lambda|b|(p-2) - \frac{2\lambda|b|p(p-1)}{w} < 0$, then from (3.35), we have

$$\phi G \leq n \left[A + \frac{2nC_1^2}{R^2} + \frac{n\lambda^2 C_1^2}{4R^2(\lambda-1)^2} + 2K + 4nK^2 \right]. \quad (3.36)$$

If $-\frac{2(\lambda-1)G}{n} + 2K\lambda + 3\lambda|b|(p-2) - \frac{2\lambda|b|p(p-1)}{w} \geq 0$, then we get

$$\phi G \leq \frac{n}{2(\lambda-1)} \left(2K\lambda + 3\lambda|b|(p-2) - \frac{2\lambda|b|p(p-1)}{Y_2} \right). \quad (3.37)$$

It follows from (3.36) and (3.37) that

$$\begin{aligned} \phi G \leq n & \left[A + \frac{2nC_1^2}{R^2} + \frac{n\lambda^2 C_1^2}{4R^2(\lambda-1)^2} + \frac{3\lambda|b|(p-2)}{2(\lambda-1)} - \frac{\lambda|b|p(p-1)}{(\lambda-1)Y_2} \right. \\ & \left. + K \cdot \max \left\{ \frac{\lambda}{\lambda-1}, 2 + 4nK \right\} \right]. \end{aligned} \quad (3.38)$$

Case 3.2: If $w \in (Y_2, 0)$, then we find

$$\begin{aligned} & -\frac{4(\lambda-1)aw^p}{n} + (\lambda-2)apw^{p-1} + \lambda ap(p-1)w^{p-2} \\ & \geq (\lambda-2)apY_2^{p-1} + \lambda ap(p-1)Y_2^{p-2}, \end{aligned} \quad (3.39)$$

$$\frac{(\lambda-1)^2 a^2 w^{2p}}{n} - \lambda(\lambda-1)a^2 p w^{2p-1} - \lambda^2 a^2 p(p-1)w^{2p-2} \geq -\lambda^2 a^2 p(p-1)Y_2^{2p-2}, \quad (3.40)$$

and

$$\begin{aligned} & \left[\frac{4(\lambda-1)|b|w^{p+1}}{n} + (2K\lambda + 3\lambda|b|(p-2) + 2|b|(\lambda-1)(1-2p))w^p \right. \\ & \left. - 2\lambda|b|p(p-1)w^{p-1} \right] a \\ & \geq [2K\lambda Y_2 + 3\lambda|b|(p-2)Y_2 - 2\lambda|b|p(p-1)]aY_2^{p-1}. \end{aligned} \quad (3.41)$$

Employing (3.31), (3.39), (3.40), (3.41), and $G \geq 1$, we have

$$\begin{aligned} \phi G \leq n & \left[A + \frac{nC_1^2\lambda^2}{4R^2(\lambda-1)^2} + \frac{2nC_1^2}{R^2} + \lambda^2 a^2 p(p-1)Y_2^{2p-2} - \lambda a(2K + 3|b|(p-2))Y_2^p \right. \\ & \left. + (2 - \lambda + 2|b|\lambda(p-1))apY_2^{p-1} - \lambda ap(p-1)Y_2^{p-2} + 2K + 4nK^2 \right]. \end{aligned} \quad (3.42)$$

From (3.12), (3.16), (3.26), (3.30), (3.38), and (3.42), we obtain the upper bound of ϕG in the case $a > 0$.

We are able to get the estimates for Cases 4–6 along a similar line to Cases 1–3. Thus, we only state the results of Cases 4–6.

Case 4: If $a < 0$, $w^{p-1} > 0$ and $w > 0$, then we have

$$\begin{aligned}\phi G \leq n & \left[A + \frac{nC_1^2\lambda^2}{2R^2(\lambda-1)^2} + \frac{nC_1^2}{R^2} - \lambda ap(p-1)H_2^{p-2} + \frac{12|b|H_2}{n} + \lambda(\lambda-1)a^2pH_2^{2p-1} \right. \\ & + \lambda^2a^2p(p-1)H_2^{2p-2} - \frac{12(\lambda-1)a|b|H_2^{p+1}}{n} \\ & \left. - (2K\lambda + 3\lambda|b|(p-2))aH_2^p + 2K + 9nb^2 \right].\end{aligned}\quad (3.43)$$

Case 5: If $a < 0$, $w^{p-1} > 0$ and $w < 0$, then we see

$$\begin{aligned}\phi G \leq n & \left[\frac{\lambda}{2-\lambda} \left(A + \frac{nC_1^2}{R^2} + \frac{n\lambda C_1^2}{(2-\lambda)R^2} + 2K + 4nK^2 \right) + \left(2|b|(2p-1) + \frac{4}{n} \right)(\lambda-1)aY_1^p \right. \\ & \left. + \lambda^2a^2p(p-1)Y_1^{2p-2} - \frac{12}{n}a|b|(\lambda-1)Y_1^{p+1} \right].\end{aligned}\quad (3.44)$$

Case 6: If $a < 0$, $w^{p-1} < 0$ and $w < 0$, we get

$$\begin{aligned}\phi G \leq n & \left[A + \frac{nC_1^2\lambda^2}{4R^2(\lambda-1)^2} + \frac{2nC_1^2}{R^2} + \lambda^2a^2p(p-1)Y_2^{2p-2} - \lambda a(2K + 3|b|(p-2))Y_2^p \right. \\ & + (2-\lambda + 2|b|\lambda(p-1))apY_2^{p-1} - \lambda ap(p-1)Y_2^{p-2} + \frac{3\lambda|b|(p-2)}{2(\lambda-1)} - \frac{\lambda|b|p(p-1)}{(\lambda-1)Y_2} \\ & \left. + K \cdot \max \left\{ \frac{\lambda}{\lambda-1}, 2 + 4nK \right\} \right].\end{aligned}\quad (3.45)$$

From (3.43), (3.44), and (3.45), we see that the upper bound of ϕG in the case $a < 0$. \square

4 Proof of Corollaries

In this section, we will prove Corollary 1.2 (the Harnack inequality) and Corollary 1.3.

Proof of Corollary 1.2 When $a > 0$, $b = 0$ and $p = \frac{2k}{2k_2+1} \geq 2$, where k and k_2 are two integers, setting $\lambda = \frac{3}{2}$, then

$$\frac{3}{2}a(\ln u)^p \geq 0.$$

Choose y, z in $B_{\frac{R}{2}}(O)$ such that $u(y) = \sup_{B_{\frac{R}{2}}(O)} u(x)$, $u(z) = \inf_{B_{\frac{R}{2}}(O)} u(x)$. Let $\gamma(t) \in [0, l]$ be a shortest curve connecting y and z with $\gamma(0) = y$, $\gamma(l) = z$. By the triangle inequality, we have $\gamma \in B_R(O)$ and $l \leq R$. It follows from Theorem 1.1 that

$$\ln u(y) - \ln u(z) \leq \int_{\gamma} \frac{|\nabla u|}{u} \leq \max\{1, S\}R,$$

where

$$S = (n)^{\frac{1}{2}} \left[6 \left(A + 2K + 2nK^2 + \frac{a}{2} (2n(p-1))^{p-1} \right) + \frac{36nC_1^2}{R^2} + \frac{6aK(-Y_2)^p}{2} \right]$$

$$\begin{aligned}
& + \frac{9a^2p(p-1)(H_1^{2p-2} + Y_1^{2p-2} + Y_2^{2p-2})}{4} + \frac{3a^2pH_1^{2p-1}}{4} + \frac{2a}{n}(H_1^p + (-Y_1)^p) \\
& + \frac{ap}{2}(H_1^{p-1} + (-Y_2)^{p-1}) + \frac{3ap(p-1)(-Y_2)^{p-2}}{2} \Big]^\frac{1}{2}
\end{aligned}$$

Therefore, we obtain

$$\sup_{B_{\frac{R}{2}}(O)} u \leq e^{\max\{1, S\}R} \inf_{B_{\frac{R}{2}}(O)} u.$$

□

Proof of Corollary 1.3 When $a > 0, b \neq 0$ and $p = \frac{2k}{2k+1} \geq 2$, where k and k_2 are two integers, setting $\lambda = \frac{3}{2}$, then

$$\frac{3}{2}a(\ln u)^p \geq 0,$$

and

$$\frac{|\nabla u|^2}{u^2} > 0.$$

It follows from Theorem 1.1 that

$$2|b| \ln u \leq \max\{1, \hat{S}\},$$

that is,

$$u \leq e^{\frac{1}{2|b|} \max\{1, \hat{S}\}}.$$

Here, \hat{S} is the constant T_1 in Theorem 1.1 with $\lambda = \frac{3}{2}$, i.e.,

$$\begin{aligned}
& n \left[6 \left(A + 9nb^2 + 2K + 2nK^2 + \frac{p-1}{np} (12|b|)^{\frac{p}{p-1}} \left(\frac{ap}{4} \right)^{-\frac{1}{p-1}} + \frac{a}{2} (2n(p-1))^{p-1} \right) \right. \\
& + \frac{36nC_1^2}{R^2} \\
& + \frac{9a^2p(p-1)(H_1^{2p-2} + Y_1^{2p-2} + Y_2^{2p-2})}{4} + \frac{3a^2pH_1^{2p-1}}{4} \\
& + a(H_1^p + (-Y_1)^p) \left(\frac{2}{n} + |b|(2p-1) \right) \\
& + \left(\frac{1}{2} + 3|b|(p-1) \right) ap(H_1^{p-1} + (-Y_2)^{p-1}) + \frac{12|b|H_1}{n} \\
& + \frac{6a|b|(-Y_1)^{p+1}}{n} - \frac{3|b|p(p-1)}{Y_2} \\
& \left. + \frac{3a(2K + 3|b|(p-2))(-Y_2)^p}{2} + \frac{3ap(p-1)(-Y_2)^{p-2}}{2} + \frac{9|b|(p-2)}{2} \right].
\end{aligned}$$

□

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No datasets were generated or analysed during the current study.

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References

1. Abolarinawa, A.: Gradient estimates for a weighted nonlinear elliptic equation and Liouville type theorems. *J. Geom. Phys.* **155**, 103737 (2020)
2. Chen, L., Chen, W.Y.: Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds. *Ann. Glob. Anal. Geom.* **35**(4), 397–404 (2009)
3. Chung, F., Yau, S.T.: Logarithmic Harnack inequalities. *Math. Res. Lett.* **3**, 793–812 (1996)
4. Du, F., Hou, L., Mao, J., Wu, C.: Eigenvalue inequalities for the buckling problem of the drifting Laplacian of arbitrary order. *Adv. Nonlinear Anal.* **12**(1), 20220278 (2023)
5. Du, F., Mao, J., Wang, Q., Xia, C., Zhao, Y.: Estimates for eigenvalues of the Neumann and Steklov problems. *Adv. Nonlinear Anal.* **12**(1), 20220321 (2023)
6. Gross, L.: Logarithmic Sobolev inequalities. *Am. J. Math.* **97**(4), 1061–1083 (1976)
7. Gui, C.F., Jian, H.Y., Ju, H.J.: Properties of translating solutions to mean curvature flow. *Discrete Contin. Dyn. Syst.* **28**(2), 441–453 (2010)
8. Hamilton, R.S.: The information of singularities in the Ricci flow. In: Survey in Differential Geometry, vol. 2, pp. 7–136. International Press, Boston (1995)
9. Huang, G.Y., Ma, B.Q.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. *Arch. Math. (Basel)* **94**(3), 265–275 (2010)
10. Li, P., Yau, S.T.: On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156**(3), 153–201 (1986)
11. Ma, L.: Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds. *J. Funct. Anal.* **241**(1), 374–382 (2006)
12. Peng, B., Wang, Y.D., Wei, G.D.: Yau type gradient estimates for $\Delta u + au(\log u)^p + bu = 0$ on Riemannian manifolds. *J. Math. Anal. Appl.* **498**(1), 124963 (2021)
13. Qian, B.: A uniform bound for the solutions to a simple nonlinear equation on Riemannian manifolds. *Nonlinear Anal.* **73**(6), 1538–1542 (2010)
14. Sanhaji, A., Dakkak, A., Moussaoui, M.: The first eigencurve for a Neumann boundary problem involving p -Laplacian with essentially bounded weights. *Opusc. Math.* **43**(4), 559–574 (2023)
15. Taheri, A., Vahidifar, V.: Gradient estimates for nonlinear elliptic equations involving the Witten Laplacian on smooth metric measure spaces and implications. *Adv. Nonlinear Anal.* **12**(1), 20220288 (2023)
16. Wang, F.Y.: Harnack inequalities for log-Sobolev functions and estimates of log-Sobolev constants. *Ann. Probab.* **27**(2), 653–663 (1999)
17. Yang, Y.Y.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. *Proc. Am. Math. Soc.* **136**(11), 4095–4102 (2008)
18. Yau, S.T.: Harmonic functions on complete Riemannian manifolds. *Commun. Pure Appl. Math.* **28**(2), 201–228 (1975)
19. Zhu, X.R., Li, Y.: Li-Yau estimates for a nonlinear parabolic equation on manifolds. *Math. Phys. Anal. Geom.* **17**, 273–288 (2014)

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