# Infinitely many solutions for three quasilinear Laplacian systems on weighted graphs 

Yan Pang ${ }^{1}$, Junping Xie ${ }^{2 *}$ and Xingyong Zhang ${ }^{1,3}$

"Correspondence:
hnxiejunping@163.com
${ }^{2}$ Faculty of Transportation Engineering, Kunming University of Science and Technology, Kunming, Yunnan, 650500, P.R. China
Full list of author information is available at the end of the article


#### Abstract

We investigate a generalized poly-Laplacian system with a parameter on weighted finite graphs, a generalized poly-Laplacian system with a parameter and Dirichlet boundary value on weighted locally finite graphs, and a ( $p, q$ )-Laplacian system with a parameter on weighted locally finite graphs. We utilize a critical points theorem built by Bonanno and Bisci [Bonanno, Bisci, and Regan, Math. Comput. Model. 52(1-2):152-160, 2010], which is an abstract critical points theorem without compactness condition, to obtain that these systems have infinitely many nontrivial solutions with unbounded norm when the parameters locate some well-determined range.


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## 1 Introduction

Assume that $G=(V, E)$ is a graph, where $V$ is the vertex set and $E$ is the edge set. $G$ is usually known as a finite graph when $V$ and $E$ have finite elements, and $G$ is usually known as a locally finite graph when for any $x \in V$, there exist finite $y \in V$ satisfying $x y \in E$, where $x y$ represents an edge linking $x$ and $y$. The weight on any given edge $x y \in E$ is denoted by $\omega_{x y}$, which is supposed to satisfy $\omega_{x y}>0$ and $\omega_{x y}=\omega_{y x}$. Moreover, we set $\operatorname{deg}(x)=\sum_{y \sim x} \omega_{x y}$ for any fixed $x \in V$. Here, we use $y \sim x$ to represent those $y$ linked to $x . d(x, y)$ represents the distance between any two points $x, y \in V$, which is defined by the minimal number of edges linking $x$ to $y$. Suppose that $\Omega$ is a subset in $V$. If there exists a positive constant $D$ such that $d(x, y) \leq D$ for all $x, y \in \Omega$, then $\Omega$ is known as a bounded domain in $V$. Set

$$
\partial \Omega:=\{y \in V, y \notin \Omega: \exists x \in \Omega \text { satisfying } x y \in E\},
$$

which is known as the boundary of $\Omega$. The interior of $\Omega$ is represented by $\Omega^{\circ}=\Omega \backslash \partial \Omega$, which obviously satisfies $\Omega^{\circ}=\Omega$.

[^0]Thereinafter, $\mu: V \rightarrow \mathbb{R}^{+}$is supposed to be a finite measure. Set

$$
\begin{equation*}
D_{w, y} u(x):=\frac{1}{\sqrt{2}}(u(x)-u(y)) \sqrt{\frac{w_{x y}}{\mu(x)}}, \tag{1.1}
\end{equation*}
$$

which is the directional derivative of $u: V \rightarrow \mathbb{R}$, and then the gradient of $u$ is defined as

$$
\begin{equation*}
\nabla u(x):=\left(D_{w, y} u(x)\right)_{y \in V} \tag{1.2}
\end{equation*}
$$

that is a vector and is indexed by $y$. Set

$$
\begin{equation*}
\Gamma(u, v)(x)=\frac{1}{2 \mu(x)} \sum_{y \sim x} w_{x y}(u(y)-u(x))(v(y)-v(x)) . \tag{1.3}
\end{equation*}
$$

Then it is obvious that

$$
\begin{equation*}
\Gamma(u, v)=\nabla u \cdot \nabla v . \tag{1.4}
\end{equation*}
$$

Define

$$
\begin{equation*}
|\nabla u|(x)=\sqrt{\Gamma(u, u)(x)}=\left(\frac{1}{2 \mu(x)} \sum_{y \sim x} w_{x y}(u(y)-u(x))^{2}\right)^{\frac{1}{2}}, \tag{1.5}
\end{equation*}
$$

which represents the length of $\nabla u$. Furthermore, the length of $m$-order gradient of $u$ is represented by $\left|\nabla^{m} u\right|$ that is defined by

$$
\left|\nabla^{m} u\right|= \begin{cases}\left|\nabla \Delta^{\frac{m-1}{2}} u\right|, & \text { if } m \text { is an odd number }  \tag{1.6}\\ \left|\Delta^{\frac{m}{2}} u\right|, & \text { if } m \text { is an even number. }\end{cases}
$$

Here, we define $\nabla \Delta^{\frac{m-1}{2}} u$ by (1.2) with substituting $\Delta^{\frac{m-1}{2}} u$ for $u$, and $\Delta^{\frac{m}{2}} u=\Delta\left(\Delta^{\frac{m}{2}-1} u\right)$, where the Laplacian operator $\Delta$ of $u$ is defined as

$$
\begin{equation*}
\Delta u(x):=\frac{1}{\mu(x)} \sum_{y \sim x} w_{x y}(u(y)-u(x)) . \tag{1.7}
\end{equation*}
$$

For any given $l>1$, set

$$
\begin{equation*}
\Delta_{l} u(x):=\frac{1}{2 \mu(x)} \sum_{y \sim x}\left(|\nabla u|^{l-2}(y)+|\nabla u|^{l-2}(x)\right) \omega_{x y}(u(y)-u(x)), \tag{1.8}
\end{equation*}
$$

which is known as the $l$-Laplacian operator of $u$. $l$-Laplacian operator obviously becomes the Laplacian operator of $u$ as $l=1$.

For convenience, we set

$$
\begin{equation*}
\int_{V} u(x) d \mu=\sum_{x \in V} \mu(x) u(x) . \tag{1.9}
\end{equation*}
$$

For any $r \in \mathbb{R}$ with $r \geq 1$, set

$$
L^{r}(V)=\left\{u:\left.V \rightarrow \mathbb{R}\left|\int_{V}\right| u(x)\right|^{r} d \mu<\infty\right\}
$$

equipped by the norm

$$
\begin{equation*}
\|u\|_{L^{r}(V)}=\left(\int_{V}|u(x)|^{r} d \mu\right)^{\frac{1}{r}} \tag{1.10}
\end{equation*}
$$

For any $u: V \rightarrow \mathbb{R}$, according to the distributional sense, we write $\Delta_{l}$ as

$$
\begin{equation*}
\int_{V}\left(\Delta_{l} u\right) v d \mu=-\int_{V}|\nabla u|^{l-2} \Gamma(u, v) d \mu \tag{1.11}
\end{equation*}
$$

where $v \in \mathcal{C}_{c}(V)$ and $\mathcal{C}_{c}(V)$ is the set of all real functions with compact support. Furthermore, a more general operator $£_{m, l}$ could be defined as

$$
\begin{align*}
& \int_{V}\left(£_{m, l} u\right) \phi d \mu \\
& \quad= \begin{cases}\int_{V}\left|\nabla^{m} u\right|^{l-2} \Gamma\left(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \phi\right) d \mu, & \text { when } m \text { is an odd number, } \\
\int_{V}\left|\nabla^{m} u\right|^{l-2} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \phi d \mu, & \text { when } m \text { is an even number, }\end{cases} \tag{1.12}
\end{align*}
$$

for any $\phi \in \mathcal{C}_{c}(V)$, where $l \in \mathbb{R}$ with $l>1$ and $m \in \mathbb{N} . £_{m, p}$ is known as the poly-Laplacian of $u$ as $m=2$, and $£_{m, l}$ degenerates to the $l$-Laplacian operator as $m=1$. Those above concepts and more related details refer to [6] and [10].
In this paper, we focus on the existence of infinitely many solutions for the following generalized poly-Laplacian system on finite graph $G=(V, E)$ :

$$
\begin{cases}£_{m_{1}, p} u+h_{1}(x)|u|^{p-2} u=\lambda F_{u}(x, u, v), & x \in V,  \tag{1.13}\\ £_{m_{2}, q} v+h_{2}(x)|v|^{q-2} v=\lambda F_{v}(x, u, v), & x \in V\end{cases}
$$

where $m_{i} \in \mathbb{N}, h_{i}: V \rightarrow \mathbb{R}^{+}, i=1,2,1<p, q<+\infty, \lambda>0$, and $F: V \times \mathbb{R}^{2} \rightarrow \mathbb{R}$.
Moreover, if $G=(V, E)$ is a locally finite graph, we focus on the existence of infinitely many solutions for the following generalized poly-Laplacian system with Dirichlet boundary condition:

$$
\left\{\begin{array}{l}
£_{m_{1}, p} u=\lambda F_{u}(x, u, v), \quad x \in \Omega^{\circ},  \tag{1.14}\\
£_{m_{2}, q} v=\lambda F_{v}(x, u, v), \quad x \in \Omega^{\circ}, \\
\left|\nabla^{j} u\right|=0, \quad x \in \partial \Omega, 0 \leq j \leq m_{1}-1, \\
\left|\nabla^{i} v\right|=0, \quad x \in \partial \Omega, 0 \leq i \leq m_{2}-1,
\end{array}\right.
$$

where $1<p, q<+\infty, \lambda>0, m_{i} \in \mathbb{N}$ with $m_{i} \geq 1, i=1,2$, and $\Omega \subset G(V, E)$ is a bounded domain.

Finally, we are also concerned with the existence of infinitely many solutions for the following $(p, q)$-Laplacian system on locally finite $\operatorname{graph} G=(V, E)$ :

$$
\begin{cases}-\Delta_{p} u+h_{1}(x)|u|^{p-2} u=\lambda F_{u}(x, u, v), & x \in V  \tag{1.15}\\ -\Delta_{q} v+h_{2}(x)|v|^{q-2} v=\lambda F_{v}(x, u, v), & x \in V\end{cases}
$$

where $-\Delta_{p}$ and $-\Delta_{q}$ are defined by (1.8) with $l=p, q, p \geq 2$ and $q \geq 2, F: V \times \mathbb{R}^{2} \rightarrow \mathbb{R}$, $h_{i}: V \rightarrow \mathbb{R}^{+}, i=1,2$, and $\lambda>0$.

The existence and asymptotic properties of nontrivial solutions for quasilinear elliptic equations have been studied extensively on Euclidean domain (for example, [13, 14, $16,24])$. With the development of machine learning, data analysis, social network, image processing and traffic network, the analysis on graphs has attracted some attentions [1-3, 7, 20, 21]. In particular, recently, in [10] and [11], Grigor'yan, Lin, and Yang studied several nonlinear elliptic equations on graphs and first established the Sobolev spaces and the variational framework on graphs. Subsequently, there have been some works on $p$ Laplacian equations and more general poly-Laplacian equations on graphs. For example, in [15], Pinamonti and Stefani studied some semi-linear equations with the poly-Laplacian operator on locally finite graphs. They established some existence results of weak solutions via a variational method by using the continuity properties of the energy functionals. In [19], Shao studied a nonlinear $p$-Laplacian equation on a locally finite graph. Some existence results of positive solutions and positive ground state solutions are established by exploiting the mountain pass theorem and the Nehari manifold. For more related results, also refer to, for example, [8, 9, 12, 17], and [18].
In addition to the case of single equations, recently, the study of systems on graphs has also yielded some results. For example, in [25], Zhang et al. considered system (1.13) with $\lambda=1$. They supposed that $F$ takes on the super- $(p, q)$ growth and then established the existence result of a nontrivial solution by exploiting the mountain pass theorem. They also established a multiplicity result by utilizing the symmetric mountain pass theorem. In [23], Yu et al. considered (1.14) and system (1.15) with $p=q, \lambda=1$, and $F(x, u)=$ $-K(x, u)+W(x, u)$ for all $x \in V$. By utilizing the mountain pass theorem, they achieved that (1.14) has a nontrivial solution. In [22], Yang and Zhang investigated (1.15) with perturbations and two parameters $\lambda_{1}$ and $\lambda_{2}$. Under the assumptions that the nonlinearity satisfies a sub- $(p, q)$ conditions, they achieved that system has at least one nontrivial solution by Ekeland's variational principle. When the nonlinearity equipped the super- $(p, q)$ conditions, they established that system has at least one nontrivial solution with positive energy and one nontrivial solution with negative energy by exploiting mountain pass theorem and Ekeland's variational principle. In [17], when $h_{1}(x)=\lambda a+1$ and $h_{2}(x)=\lambda b+1$, Shao studied (1.15) with $p=q$. By the Nehari manifold method and some analytical techniques, under some suitable assumptions on the potentials and nonlinear terms, they proved that system possesses a ground state solution $\left(u_{\lambda}, v_{\lambda}\right)$ when the parameter $\lambda$ is large enough.

Our investigation is mainly motivated by the above mentioned works and [4, 5]. In [4], Bonanno and Bisci established the existence result of a sequence $\left\{u_{n}\right\}$ of critical points for the functional $f_{\lambda}:=\Phi-\lambda \Psi$ with $\lambda \in \mathbb{R}$, and got a well-determined interval of the parameter $\lambda$. In [5], Bonanno and Bisci obtained that a class of quasilinear elliptic system in the Euclidean framework possesses infinitely many weak solutions by the abstract theorem established in [4]. In the present paper, we also apply the critical points theorem developed
by Bonanno and Bisci [4] to system (1.13), (1.14), and (1.15), and we obtain that these systems have infinitely many nontrivial solutions with unbounded norm when the parameters $\lambda$ locate some well-determined ranges. To the best of our knowledge, there seemed to be no works to investigate the existence of infinitely many solutions for equations or systems on finite graph or locally finite graph. Our works are a preliminary attempt in this field.

## 2 Preliminaries

In this section, we recall some basic knowledge on the Sobolev space on graph. For more details, refer to [10, 22, 25]. We also recall an abstract critical point theorem built in [4], which is exploited to prove our main results.

Suppose that $G=(V, E)$ is a finite graph. For any fixed $m \in \mathbb{N}$ and any fixed $l \in \mathbb{R}$ with $l>1$, set

$$
W^{m, l}(V)=\{u: V \rightarrow \mathbb{R}\}
$$

equipped with the norm

$$
\begin{equation*}
\|u\|_{W^{m, l}(V)}=\left(\int_{V}\left(\left|\nabla^{m} u(x)\right|^{l}+h(x)|u(x)|^{l}\right) d \mu\right)^{\frac{1}{l}} \tag{2.1}
\end{equation*}
$$

where $h(x)>0$ for all $x \in V . W^{m, l}(V)$ is a Banach space with finite dimension.
Suppose that $G=(V, E)$ is a locally finite graph and $\Omega$ is a bounded domain in $V$. For any fixed $l \in \mathbb{R}$ with $l>1$ and any fixed $m \in \mathbb{N}$, set

$$
W^{m, l}(\Omega)=\{u: \Omega \rightarrow \mathbb{R}\}
$$

equipped with the norm

$$
\|u\|_{W^{m}, l(\Omega)}=\left(\sum_{k=0}^{m} \int_{\Omega \cup \partial \Omega}\left(\left|\nabla^{k} u(x)\right|^{l} d \mu\right)^{\frac{1}{l}}\right.
$$

Define

$$
C_{0}^{m}(\Omega)=\left\{u: \Omega \rightarrow \mathbb{R}\left|u=|\nabla u|=\cdots=\left|\nabla^{m-1} u\right|=0\right\} .\right.
$$

$W_{0}^{m, l}(\Omega)$ is seen as the completion of $C_{0}^{m}(\Omega)$ in $W^{m, l}(\Omega) . W_{0}^{m, l}(\Omega)$ is a finite dimensional Banach space since $\Omega$ is a finite set. On $W_{0}^{m, l}(\Omega)$, one can also equip the following norm:

$$
\|u\|_{W_{0}^{m, l}(\Omega)}=\left(\int_{\Omega \cup \partial \Omega}\left(\left|\nabla^{k} u(x)\right|^{l} d \mu\right)^{\frac{1}{l}} .\right.
$$

Then $\|u\|_{W_{0}^{m, l}(\Omega)}$ is equivalent to $\|u\|_{W^{m, l}(\Omega)}$.
Suppose that $G=(V, E)$ is a locally finite graph. $W^{1, l}(V)(l>1)$ is the completion of $\mathcal{C}_{c}(V)$ based on the norm

$$
\|u\|_{W_{h}^{1, l}(V)}=\left(\int_{V}\left(|\nabla u(x)|^{l}+h(x)|u(x)|^{l}\right) d \mu\right)^{\frac{1}{l}}
$$

where $h: V \rightarrow \mathbb{R}$ and there exists a positive constant $h_{0}$ such that $h(x) \geq h_{0}$. Set the space

$$
W_{h}^{1, l}(V)=\left\{\left.u \in W^{1, l}(V)\left|\int_{V} h(x)\right| u(x)\right|^{l} d \mu<\infty\right\}
$$

equipped with the norm $\|u\|_{W_{h}^{1, l}(V)}$.
Lemma $2.1([10,25])$ Suppose that $G=(V, E)$ is a finite graph. For any $\psi \in W^{m, l}(V)$, there exists

$$
\|\psi\|_{\infty, V} \leq K_{l}\|\psi\|_{W^{m}, l}(V)
$$

where $\|\psi\|_{\infty}=\max _{x \in V}|\psi(x)|$ and $K_{l}=\left(\frac{1}{\mu_{\min } h_{\min }}\right)^{\frac{1}{l}}$ with $\mu_{\min }=\min _{x \in V} \mu(x)$ and $h_{\min }=$ $\min _{x \in V} h(x)$.

Lemma $2.2([10,25])$ Suppose that $G=(V, E)$ is a locally finite graph and $\Omega$ is a bounded domain in $V$ satisfying $\Omega^{\circ} \neq \emptyset$. Let $m \in \mathbb{N}$ and $l>1$. Then $W_{0}^{m, l}(\Omega)$ is continuously embedded into $L^{\theta}(\Omega)$ for all $1 \leq \theta \leq+\infty$. In particular, there exists a positive constant $C(m, l, \Omega)$, which just depends on $m, l$, and $\Omega$ satisfying

$$
\begin{aligned}
& \left(\int_{\Omega}|u|^{q} d \mu\right)^{\frac{1}{q}} \leq C(m, l, \Omega)\left(\int_{\Omega \cup \partial \Omega}\left|\nabla^{m} u\right|^{p} d \mu\right) \\
& \|u\|_{\infty, \Omega} \leq \frac{C}{\mu_{\min , \Omega}^{1 / l}}\|u\|_{W_{0}^{m, l}(\Omega)}
\end{aligned}
$$

for all $1 \leq \theta \leq+\infty$ and all $u \in W_{0}^{m, l}(\Omega)$, where $\|u\|_{\Omega, \infty}=\max _{x \in \Omega}|u(x)|$ and $\mu_{\min , \Omega}=$ $\min _{x \in \Omega} \mu(x)$. Moreover, $W_{0}^{m, l}(\Omega)$ is pre-compact, that is, if $\left\{u_{n}\right\}$ is bounded in $W_{0}^{m, l}(\Omega)$, then $u p$ to a subsequence, there exists some $u \in W_{0}^{m, l}(\Omega)$ such that $u_{n} \rightarrow u$ in $W_{0}^{m, l}(\Omega)$.

Lemma 2.3 ([22]) Suppose that $G=(V, E)$ is a locally finite graph, and $h(x)>h_{0}$ and $\mu(x)>$ $\mu_{0}$ for all $x \in V$, some $h_{0}>0$ and some $\mu_{0}>0$. If $\left(H_{1}\right)$ holds, then $W_{h}^{1, l}(V)$ is continuously embedded into $L^{r}(V)$ for all $1<l \leq r \leq \infty$, and the following inequalities hold:

$$
\|u\|_{\infty} \leq \frac{1}{h_{0}^{1 / l} \mu_{0}^{1 / l}}\|u\|_{W_{h}^{1, l}(V)}
$$

and

$$
\|u\|_{L^{r}(V)} \leq \mu_{0}^{\frac{l-r}{l r}} h_{0}^{-\frac{1}{l}}\|u\|_{W_{h}^{1, l}(V)} \quad \text { for all } l \leq r<\infty .
$$

Lemma 2.4 ([4]) Assume that $X$ is a reflexive real Banach space, $\Phi, \Psi: X \rightarrow \mathbb{R}$ are two Gâteaux differentiable functionals satisfying $\Phi$ is continuous, sequentially weakly lower semicontinuous, and coercive, and $\Psi$ is sequentially weakly upper semicontinuous. For each $r>\inf _{X} \Phi$, set

$$
\varphi(r):=\inf _{u \in \Phi^{-1}([-\infty, r])} \frac{\left(\sup _{u \in \Phi^{-1}([-\infty, r])} \Psi(u)\right)-\Psi(u)}{r-\Phi(u)}
$$

and

$$
\gamma:=\liminf _{r \rightarrow+\infty} \varphi(r) .
$$

Then,
(a) if $\gamma<+\infty$, for each $\lambda \in\left(0, \frac{1}{\gamma}\right)$, the following alternative holds: either
$\left(a_{1}\right) I_{\lambda}:=\Phi-\lambda \Psi$ admits a global minimum, or
$\left(a_{2}\right)$ there exists a sequence $\left\{u_{n}\right\}$ of critical points (local minima) of $I_{\lambda}$ satisfying $\lim _{n \rightarrow \infty} \Phi\left(u_{n}\right)=+\infty$.
(b) if $\delta<+\infty$, for each $\lambda \in\left(0, \frac{1}{\delta}\right)$, the following alternative holds: either
$\left(b_{1}\right)$ there exists a global minimum of $\Phi$ that is a local minimum of $I_{\lambda}$, or
$\left(b_{2}\right)$ there exists a sequence of pairwise distinct critical points (local minima) of $I_{\lambda}$ that weakly converges to a global minimum of $\Phi$.

## 3 Result and proofs for system (1.13)

In this section, we investigate the generalized poly-Laplacian system (1.13) and obtain the following result.
Let

$$
\begin{align*}
& \varrho_{V}=\max \left\{\frac{1}{p} \int_{V} h_{1}(x) d \mu, \frac{1}{q} \int_{V} h_{2}(x) d \mu\right\},  \tag{3.1}\\
& K_{V}=\max \left\{\frac{1}{\mu_{\min } h_{1, \min }}, \frac{1}{\mu_{\min } h_{2, \min }}\right\} .
\end{align*}
$$

Theorem 3.1 Suppose that $G=(V, E)$ is a finite graph and the following conditions hold:
(H) $h_{i}(x)>0$ for all $x \in V, i=1,2$;
$\left(F_{0}\right) F(x, s, t)$ is continuously differentiable in $(s, t) \in \mathbb{R}^{2}$ for all $x \in V$;
$\left(F_{1}\right) \int_{V} F(x, 0,0) d \mu=0$;
$\left(F_{2}\right)$

$$
0<A_{V}:=\liminf _{y \rightarrow+\infty} \frac{\int_{V} \max _{|s|+|t| \leq y} F(x, s, t) d \mu}{y^{\delta}}<\limsup _{|s|+|t| \rightarrow \infty} \frac{\int_{V} F(x, s, t) d \mu}{|s|^{p}+|t|^{q}}:=B_{V}
$$

where $\delta=\min \{p, q\}$. Then, for each $\lambda \in\left(\lambda_{1, V}, \lambda_{2, V}\right)$ with $\lambda_{1, V}=\frac{\varrho_{V}}{B_{V}}$ and $\lambda_{2, V}=\frac{1}{p 2^{p-1} K_{V} A_{V}}$, system (1.13) possesses an unbounded sequence of solutions.

To prove Theorem 3.1, we work in the space $W_{V}:=W^{m_{1}, p}(V) \times W^{m_{2}, q}(V)$ equipped with the norm $\|(u, v)\|_{V}=\|u\|_{W^{m_{1}, p}(V)}+\|v\|_{W^{m_{2}, q}(V)}$. Then $\left(W_{V},\|\cdot\|_{V}\right)$ is a finite dimensional Banach space.

Consider the functional $I_{\lambda, V}: W_{V} \rightarrow \mathbb{R}$ as

$$
\begin{aligned}
I_{\lambda, V}(u, v)= & \frac{1}{p} \int_{V}\left(\left|\nabla^{m_{1}} u\right|^{p}+h_{1}(x)|u|^{p}\right) d \mu+\frac{1}{q} \int_{V}\left(\left|\nabla^{m_{2}} v\right|^{q}+h_{2}(x)|v|^{q}\right) d \mu \\
& -\lambda \int_{V} F(x, u, v) d \mu .
\end{aligned}
$$

Then, under the assumptions of Theorem 3.1, $I_{\lambda, V} \in C^{1}\left(W_{V}, \mathbb{R}\right)$ and

$$
\begin{aligned}
\left\langle I_{\lambda, V}^{\prime}(u, v),\left(\phi_{1}, \phi_{2}\right)\right\rangle= & \int_{V}\left[\left(£_{m_{1}, p} u\right) \phi_{1}+h_{1}(x)|u|^{p-2} u \phi_{1}-\lambda F_{u}(x, u, v) \phi_{1}\right] d \mu \\
& +\int_{V}\left[\left(£_{m_{2}, q} v\right) \phi_{2}+h_{2}(x)|v|^{q-2} v \phi_{2}-\lambda F_{v}(x, u, v) \phi_{2}\right] d \mu
\end{aligned}
$$

for any $(u, v),\left(\phi_{1}, \phi_{2}\right) \in W_{V}$.
A standard argument implies that $(u, v) \in W_{V}$ is a critical point of $I_{\lambda, V}$ iff

$$
\int_{V}\left(£_{m_{1}, p} u+h_{1}(x)|u|^{p-2} u-\lambda F_{u}(x, u, v)\right) \phi_{1} d \mu=0
$$

and

$$
\int_{V}\left(£_{m_{2}, q} v+h_{2}(x)|v|^{q-2} v-\lambda F_{v}(x, u, v)\right) \phi_{2} d \mu=0
$$

for all $\left(\phi_{1}, \phi_{2}\right) \in W_{V}$. Furthermore, by the arbitrariness of $\phi_{1}$ and $\phi_{2}$, it can be achieved that

$$
\begin{aligned}
& £_{m_{1}, p} u+h_{1}(x)|u|^{p-2} u=\lambda F_{u}(x, u, v), \\
& £_{m_{2}, q} v+h_{2}(x)|v|^{q-2} v=\lambda F_{v}(x, u, v) .
\end{aligned}
$$

Therefore, seeking the solutions for system (1.13) is equivalent to seeking the critical points of $I_{\lambda, V}$ on $W_{V}$ (see [25] for example).
To apply Lemma 2.4, we shall exploit the functionals $\Phi_{V}: W_{V} \rightarrow \mathbb{R}$ and $\Psi_{V}: W_{V} \rightarrow \mathbb{R}$, which are set by

$$
\begin{aligned}
\Phi_{V}(u, v) & =\frac{1}{p} \int_{V}\left(\left|\nabla^{m_{1}} u\right|^{p}+h_{1}(x)|u|^{p}\right) d \mu+\frac{1}{q} \int_{V}\left(\left|\nabla^{m_{2}} v\right|^{q}+h_{2}(x)|v|^{q}\right) d \mu \\
& =\frac{1}{p}\|u\|_{W^{m_{1}, p}(V)}^{p}+\frac{1}{q}\|v\|_{W^{m_{2}, q}(V)}^{q}
\end{aligned}
$$

and

$$
\Psi_{V}(u, v)=\int_{V} F(x, u, v) d \mu
$$

Then $I_{\lambda, V}(u, v)=\Phi_{V}-\lambda \Psi_{V}$. For each $r>\inf _{W_{V}} \Phi_{V}$, define

$$
\varphi_{V}(r)=\inf _{(u, v) \in \Phi_{V}^{-1}([-\infty, r])} \frac{\left(\sup _{(u, v) \in \Phi_{V}^{-1}([-\infty, r])} \Psi_{V}(u, v)\right)-\Psi_{V}(u, v)}{r-\Phi_{V}(u, v)} .
$$

Lemma 3.1 Assume that $\left(F_{2}\right)$ holds. Then $\gamma_{V}:=\liminf _{r \rightarrow+\infty} \varphi_{V}(r)<+\infty$.
Proof Let $\left\{c_{n}\right\}$ be a real sequence satisfying $\lim _{n \rightarrow \infty} c_{n}=+\infty$ and

$$
\lim _{n \rightarrow \infty} \frac{\int_{V} \max _{|s|+|t| \leq c_{n}} F(x, s, t) d \mu}{c_{n}^{\delta}}=A_{V}
$$

Write

$$
r_{n}=\frac{2^{1-p} c_{n}^{\delta}}{p K_{V}} \quad \text { for every } n \in \mathbb{N}
$$

By Lemma 2.1, for all $(u, v) \in W$ with $\Phi_{V}(u, v) \leq r_{n}$, we get

$$
\begin{equation*}
\frac{\|u\|_{\infty, V}^{p}}{p}+\frac{\|v\|_{\infty, V}^{q}}{q} \leq K_{V}\left(\frac{\|u\|_{W^{m} m_{1}, p}(V)}{p}+\frac{\|v\|_{W^{m}, q}^{q}(V)}{q}\right) \leq K_{V} r_{n} . \tag{3.2}
\end{equation*}
$$

Next, we claim that there exists $n_{0} \in \mathbb{N}$ such that $|u(x)|+|v(x)| \leq c_{n}$ for all $n \geq n_{0}$, all $x \in V$, and all $(u, v) \in W_{V}$ with $\Phi_{V}(u, v) \leq r_{n}$. We prove the claim through the following three cases. Without loss of generality, we let $\delta=q$.
(1) Assume that $\|u\|_{\infty, V}<1$ and $\|v\|_{\infty, V}<1$. It is obvious that there exists $n_{1} \in \mathbb{N}$ such that $\|u\|_{\infty, V}+\|v\|_{\infty, V} \leq c_{n}$ for all $n>n_{1}$ by the fact $\lim _{n \rightarrow \infty} c_{n}=+\infty$.
(2) Assume that $\|u\|_{\infty, V} \geq 1,\|v\|_{\infty, V} \geq 1$ or $\|u\|_{\infty, V} \geq 1,\|u\|_{\infty, V}<1$. Then

$$
\begin{aligned}
\frac{\|u\|_{\infty, V}^{p}}{p}+\frac{\|v\|_{\infty, V}^{q}}{q} & \geq \frac{\|u\|_{\infty, V}^{q}+\|v\|_{\infty, V}^{q}}{p} \\
& \geq \frac{2^{1-q}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{q}}{p}
\end{aligned}
$$

which together with (3.2), implies that

$$
\begin{equation*}
c_{n}^{q} \geq 2^{p-q}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{q} \geq\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{q} \tag{3.3}
\end{equation*}
$$

Thus $\|u\|_{\infty, V}+\|v\|_{\infty, V} \leq c_{n}$.
(3) Assume that $\|u\|_{\infty, V}<1$ and $\|v\|_{\infty, V} \geq 1$. Then

$$
\begin{align*}
\frac{\|u\|_{\infty, V}^{p}}{p}+\frac{\|v\|_{\infty, V}^{q}}{q} & \geq \min \left\{\frac{1}{p}, \frac{\|v\|_{\infty, V}^{q-p}}{q}\right\}\left(\|u\|_{\infty, V}^{p}+\|v\|_{\infty, V}^{p}\right) \\
& \geq \min \left\{\frac{1}{p}, \frac{\|v\|_{\infty, V}^{q-p}}{q}\right\} 2^{1-p}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p} \tag{3.4}
\end{align*}
$$

If $\min \left\{\frac{1}{p}, \frac{\|\nu\|_{\infty, V}^{q-p}}{q}\right\}=\frac{1}{p}$, by (3.2) and (3.4), we have

$$
\begin{aligned}
K_{V} r_{n} & =\frac{2^{1-p} c_{n}^{q}}{p} \\
& \geq \frac{2^{1-p}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p}}{p} \\
& \geq \frac{2^{1-p}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{q}}{p}
\end{aligned}
$$

Thus $\|u\|_{\infty, V}+\|v\|_{\infty, V} \leq c_{n}$.
By (3.2), we have

$$
\frac{\|v\|_{\infty, V}^{q}}{q} \leq \frac{2^{1-p} c_{n}^{q}}{p}
$$

Note that $q-p \leq 0$. Then the above inequality implies that

$$
\|v\|_{\infty, V}^{q-p} \geq\left(\frac{q}{p}\right)^{\frac{q-p}{q}} 2^{\frac{(1-p)(q-p)}{q}} c_{n}^{q-p}
$$

Thus, if $\min \left\{\frac{1}{p}, \frac{\|\nu\|_{\infty, V}^{q-p}}{q}\right\}=\frac{\|\nu\|_{\infty, V}^{q-p}}{q}$, by (3.2) and (3.4), we have

$$
\begin{aligned}
K_{V} r_{n} & =\frac{2^{1-p} c_{n}^{q}}{p} \\
& \geq \frac{\|v\|_{\infty, V}^{q-p}}{q} 2^{1-p}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p} \\
& \geq\left(\frac{q}{p}\right)^{\frac{q-p}{q}} 2^{\frac{(1-p)(q-p)}{q}} c_{n}^{q-p} \frac{2^{1-p}}{q}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p} .
\end{aligned}
$$

Hence, an easy computation implies that

$$
c_{n}^{p} \geq\left(\frac{p}{q}\right)^{\frac{p}{q}} 2^{\frac{(1-p)(q-p)}{q}}\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p} \geq\left(\|u\|_{\infty, V}+\|v\|_{\infty, V}\right)^{p} .
$$

Thus, based on the three cases, we conclude that for all $(u, v) \in W_{V}$ with $\Phi_{V}(u, v) \leq r_{n}$, we have $|u(x)|+|v(x)| \leq c_{n}$ for all $x \in V$. Therefore, it follows from $\left(F_{1}\right)$ that

$$
\begin{aligned}
& \varphi_{V}\left(r_{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{\sup _{\frac{1}{p}\|u\|_{W^{m}, p},(V)}^{p}+\frac{1}{q}\|\nu\|_{W^{m} m_{2}, q(V)}^{q} \leq r_{n}}{} \int_{V} F(x, u, v) d \mu \\
& =p K_{V} 2^{p-1} \frac{\sup _{\frac{1}{p}\|u\|_{W^{m} m_{1}, p}(V)}^{p}+\frac{1}{q}\|\nu\|_{W^{m_{2}}, q_{(V)}}^{q} \leq r_{n}}{} \int_{V} F(x, u, v) d \mu \\
& \leq p K_{V} 2^{p-1} \frac{\int_{V} \max _{|s|+|t| \leq c_{n}} F(x, s, t) d \mu}{c_{n}^{\delta}} .
\end{aligned}
$$

Hence, $\left(F_{2}\right)$ implies that

$$
\gamma_{V} \leq \liminf _{n \rightarrow \infty} \varphi_{V}\left(r_{n}\right) \leq p K_{V} 2^{p-1} A_{V}<p K_{V} 2^{p-1} B_{V} \leq+\infty
$$

This finishes the proof of the lemma.

Lemma 3.2 For any fixed $\lambda \in\left(\lambda_{1, V}, \lambda_{2, V}\right), I_{\lambda, V}(u, v)=\Phi_{V}(u, v)-\lambda \Psi_{V}(u, v)$ is unbounded from below.

Proof Assume that $\left\{\xi_{n}\right\}$ and $\left\{\eta_{n}\right\}$ are two positive real sequences satisfying $\lim _{n \rightarrow \infty}\left|\xi_{n}\right|+$ $\left|\eta_{n}\right|=+\infty$ and

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu}{\xi_{n}^{p}+\eta_{n}^{q}}=B_{V} \tag{3.5}
\end{equation*}
$$

For each $n \in \mathbb{N}$, we define

$$
u_{n}(x) \equiv \xi_{n}, \quad v_{n}(x) \equiv \eta_{n}, \quad \forall x \in V
$$

It is obvious that $\left(u_{n}, v_{n}\right) \in W_{V},\left|\nabla^{m_{i}} u_{n}\right|=0$, and $\left|\nabla^{m_{i}} v_{n}\right|=0$ for all $m_{i} \geq 1, i=1,2$. Then, for every $n \in \mathbb{N}$, we have

$$
\begin{align*}
I_{\lambda, V}\left(u_{n}, v_{n}\right) & =\Phi_{V}\left(u_{n}, v_{n}\right)-\lambda \Psi_{V}\left(u_{n}, v_{n}\right) \\
& =\frac{\xi_{n}^{p}}{p} \int_{V} h_{1}(x) d \mu+\frac{\eta_{n}^{q}}{q} \int_{V} h_{2}(x) d \mu-\lambda \int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu \\
& \leq \varrho_{V}\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\lambda \int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu, \tag{3.6}
\end{align*}
$$

where $\varrho_{V}$ is defined by (3.1).
If $B_{V}<+\infty$, choosing $\epsilon_{\lambda} \in\left(\frac{\varrho_{V}}{\lambda B}, 1\right)$, by (3.5), there exists $n_{\epsilon_{\lambda}}>0$ such that

$$
\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>\epsilon_{\lambda} B_{V}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\epsilon_{\lambda}}
$$

Then, combining with (3.6), we get

$$
\begin{aligned}
\Phi_{V}\left(u_{n}, v_{n}\right)-\lambda \Psi_{V}\left(u_{n}, v_{n}\right) & \leq \varrho_{V}\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\lambda \epsilon_{\lambda} B_{V}\left(\xi_{n}^{p}+\eta_{n}^{q}\right) \\
& =\left(\varrho_{V}-\lambda \epsilon_{\lambda} B_{V}\right)\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\epsilon_{\lambda}} .
\end{aligned}
$$

Then

$$
\lim _{n \rightarrow \infty}\left[\Phi_{V}\left(u_{n}\right)-\lambda \Psi_{V}\left(v_{n}\right)\right]=-\infty
$$

If $B_{V}=+\infty$, we consider $M_{\lambda}>\frac{\varrho_{V}}{\lambda}$. By (3.5), there exists $n_{M_{\lambda}}$ such that

$$
\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>M_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{M_{\lambda}}
$$

Then, combining with (3.6), we get

$$
\begin{aligned}
\Phi_{V}\left(u_{n}, v_{n}\right)-\lambda \Psi_{V}\left(u_{n}, v_{n}\right) & \leq \varrho_{V}\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\lambda M_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right) \\
& =\left(\varrho_{V}-\lambda M_{\lambda}\right)\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{M_{\lambda}} .
\end{aligned}
$$

Noticing the choice of $M_{\lambda}$, we also have

$$
\lim _{n \rightarrow \infty}\left[\Phi_{V}\left(u_{n}\right)-\lambda \Psi_{V}\left(v_{n}\right)\right]=-\infty
$$

Thus, we finish the proof of the lemma.

Lemma 3.3 $\Phi_{V}$ is sequentially weakly lower semi-continuous.

Proof The proof is easily finished by exploiting the weak lower semi-continuity of the norm.

Lemma 3.4 $\Psi_{V}$ is sequentially weakly upper semi-continuous.

Proof Assume that $\left(u_{n}, v_{n}\right) \rightharpoonup\left(u_{0}, v_{0}\right)$ in $W_{V}$. Note that $W_{V}$ is of finite dimension. Then $\left(u_{n}, v_{n}\right) \rightarrow\left(u_{0}, v_{0}\right)$ in $W_{V}$. By $\left(F_{0}\right)$ and the fact that $V$ is a finite set, it is easy to obtain that

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \int_{V} F\left(x, u_{k}, v_{k}\right) d \mu & =\lim _{n \rightarrow \infty} \sum_{x \in V} F\left(x, u_{k}, v_{k}\right) \mu(x) \\
& =\sum_{x \in V} F\left(x, u_{0}, v_{0}\right) \mu(x) \\
& =\int_{V} F\left(x, u_{0}, v_{0}\right) d \mu .
\end{aligned}
$$

Hence, $\Psi_{V}$ is sequentially weakly upper semi-continuous in $W_{V}$.

Proof of Theorem 3.1 It is easy to see that $\Phi_{V}: W_{V} \rightarrow \mathbb{R}$ is coercive. Lemma 3.1-Lemma 3.4 imply that all of conditions in Lemma 2.4 are satisfied. Hence, Lemma 2.4 (a) implies that for each $\left(\lambda_{1, V} \lambda_{2, V}\right)$, the functional $I_{\lambda, V}$ has a sequence $\left\{\left(u_{n}^{*}, v_{n}^{*}\right)\right\}$ of critical points that are solutions of system (1.13) such that $\lim _{n \rightarrow \infty} \Phi_{V}\left(u_{n}^{*}, v_{n}^{*}\right)=+\infty$.

## 4 Result and proofs for system (1.14)

In this section, we investigate the generalized poly-Laplacian system (1.14) and obtain the following result.
Let

$$
K_{\Omega}=\max \left\{\frac{C^{p}\left(m_{1}, p, \Omega\right)}{\mu_{\min , \Omega}}, \frac{C^{q}\left(m_{2}, q, \Omega\right)}{\mu_{\min , \Omega}}\right\},
$$

where $C\left(m_{1}, p, \Omega\right)$ and $C\left(m_{2}, q, \Omega\right)$ are defined in Lemma 2.2.

Theorem 4.1 Assume that $G=(V, E)$ is a locally finite graph and the following conditions hold:
$(H)^{\prime} h_{i}(x)>0$ for all $x \in \Omega, i=1,2$;
$\left(F_{0}\right)^{\prime} F(x, s, t)$ is continuously differentiable in $(s, t) \in \mathbb{R}^{2}$ for all $x \in \Omega$;
$\left(F_{1}\right)^{\prime} \int_{\Omega} F(x, 0,0) d \mu=0$;
$\left(F_{2}\right)^{\prime}$

$$
0<A_{\Omega}:=\liminf _{y \rightarrow+\infty} \frac{\int_{\Omega} \max _{|s|+|t| \leq y} F(x, s, t) d \mu}{y^{\delta}}<\limsup _{|s|+|t| \rightarrow \infty} \frac{\int_{\Omega} F(x, s, t) d \mu}{|s|^{p}+|t|^{q}}:=B_{\Omega} .
$$

where $\delta=\min \{p, q\}$. Then, for each $\lambda \in\left(\lambda_{1, \Omega}, \lambda_{2, \Omega}\right)$ with $\lambda_{1, \Omega}=\frac{1}{B_{\Omega}}$ and $\lambda_{2, \Omega}=\frac{1}{p K_{\Omega} 2^{p-1} A_{\Omega}}$, system (1.14) possesses an unbounded sequence of solutions.

The proofs of Theorem 4.1 are the essentially same as Theorem 3.1 with some slight modifications. To prove Theorem 4.1, we work in the space $W_{0}:=W_{0}^{m_{1}, p}(\Omega) \times W_{0}^{m_{2}, q}(\Omega)$
equipped with the norm $\|(u, v)\|_{0}=\|u\|_{W_{0}^{m_{1}, p}(\Omega)}+\|v\|_{W_{0}^{m_{2}, q}(\Omega)}$. Then $\left(W_{0},\|\cdot\|_{0}\right)$ is a finite dimensional Banach space. Consider the functional $I_{\lambda, \Omega}: W_{0} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
I_{\lambda, \Omega}(u, v)=\frac{1}{p} \int_{\Omega \cup \partial \Omega}\left|\nabla^{m_{1}} u\right|^{p} d \mu+\frac{1}{q} \int_{\Omega \cup \partial \Omega}\left|\nabla^{m_{2}} v\right|^{q} d \mu-\lambda \int_{\Omega} F(x, u, v) d \mu . \tag{4.1}
\end{equation*}
$$

Then, under the assumptions of Theorem 4.1, $I_{\lambda, \Omega} \in C^{1}\left(W_{0}, \mathbb{R}\right)$ and

$$
\begin{align*}
\left\langle I_{\lambda, \Omega}^{\prime}(u, v),\left(\phi_{1}, \phi_{2}\right)\right\rangle= & \int_{\Omega \cup \partial \Omega}\left[\left(£_{m_{1}, p} u\right) \phi_{1}-\lambda F_{u}(x, u, v) \phi_{1}\right] d \mu \\
& +\int_{\Omega \cup \partial \Omega}\left[\left(£_{m_{2}, q} v\right) \phi_{2}-\lambda F_{v}(x, u, v) \phi_{2}\right] d \mu \tag{4.2}
\end{align*}
$$

for any $(u, v),\left(\phi_{1}, \phi_{2}\right) \in W_{0}$.
Obviously, $(u, v) \in W_{0}$ is a critical point of $I_{\lambda, \Omega}$ iff

$$
\int_{\Omega \cup \partial \Omega}\left(£_{m_{1}, p} u-\lambda F_{u}(x, u, v)\right) \phi_{1} d \mu=0
$$

and

$$
\int_{\Omega \cup \partial \Omega}\left(£_{m_{2}, q} v-\lambda F_{v}(x, u, v)\right) \phi_{2} d \mu=0
$$

for all $\left(\phi_{1}, \phi_{2}\right) \in W_{0}$. Furthermore, by the arbitrariness of $\phi_{1}$ and $\phi_{2}$, it can be achieved that system (1.14) holds. Therefore, seeking the solutions for system (1.14) is equivalent to seeking the critical points of $I_{\lambda, V}$ on $W_{0}$.

To apply Lemma 2.4, we will use the functionals $\Phi_{\Omega}: W_{0} \rightarrow \mathbb{R}$ and $\Psi_{\Omega}: W_{0} \rightarrow \mathbb{R}$ defined by

$$
\begin{aligned}
\Phi_{\Omega}(u, v) & =\frac{1}{p} \int_{\Omega \cup \partial \Omega}\left|\nabla^{m_{1}} u\right|^{p} d \mu+\frac{1}{q} \int_{\Omega \cup \partial \Omega}\left|\nabla^{m_{2}} v\right|^{q} d \mu \\
& =\frac{1}{p}\|u\|_{W_{0}^{m_{1}, p}(\Omega)}^{p}+\frac{1}{q}\|v\|_{W_{0}^{m_{2}, q}(\Omega)}^{q}
\end{aligned}
$$

and

$$
\Psi_{\Omega}(u, v)=\int_{\Omega} F(x, u, v) d u .
$$

Then $I_{\lambda, \Omega}(u, v)=\Phi_{\Omega}-\lambda \Psi_{\Omega}$ and for every $r>\inf _{W_{0}} \Phi_{\Omega}$, define

$$
\varphi_{\Omega}(r)=\inf _{(u, v) \in \Phi_{\Omega}^{-1}([-\infty, r])} \frac{\left(\sup _{(u, v) \in \Phi_{\Omega}^{-1}([-\infty, r])} \Psi_{\Omega}(u, v)\right)-\Psi_{\Omega}(u, v)}{r-\Phi_{\Omega}(u, v)} .
$$

Lemma 4.1 Assume that $\left(F_{2}\right)^{\prime}$ holds. Then $\gamma_{\Omega}:=\liminf _{r \rightarrow+\infty} \varphi_{\Omega}(r)<+\infty$.
Proof The proof is the same as that of Theorem 3.1 with substituting $\Omega, K_{\Omega}, A_{\Omega}, B_{\Omega}$, $\|u\|_{\infty, \Omega}$, and $\|v\|_{\infty, \Omega}$ for $V, K_{V}, A_{V}, B_{V},\|u\|_{\infty, V}$, and $\|v\|_{\infty, V}$, respectively. We omit the details.

Lemma 4.2 For any fixed $\lambda \in\left(\lambda_{1, \Omega}, \lambda_{2, \Omega}\right), I_{\lambda, \Omega}(u, v)=\Phi_{\Omega}(u, v)-\lambda \Psi_{\Omega}(u, v)$ is unbounded from below.

Proof Suppose that $\left\{\xi_{n}\right\}$ and $\left\{\eta_{n}\right\}$ are two positive real sequences such that $\lim _{n \rightarrow \infty}\left|\xi_{n}\right|+$ $\left|\eta_{n}\right|=+\infty$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\int_{\Omega} F\left(x, \xi_{n}, \eta_{n}\right) d \mu}{\xi_{n}^{p}+\eta_{n}^{q}}=B_{\Omega} \tag{4.3}
\end{equation*}
$$

For each $n \in \mathbb{N}$, we define

$$
u_{n}(x) \equiv \xi_{n}, \quad v_{n}(x) \equiv \eta_{n}, \quad \forall x \in \Omega
$$

It is easy to check that $\left(u_{n}, v_{n}\right) \in W_{0},\left|\nabla^{m_{i}} u_{n}\right|=0$ and $\left|\nabla^{m_{i}} v_{n}\right|=0$ for all $m_{i} \geq 1, i=1,2$. Then

$$
\begin{aligned}
I_{\lambda, \Omega}\left(u_{n}, v_{n}\right) & =\Phi_{\Omega}\left(u_{n}, v_{n}\right)-\lambda \Psi_{\Omega}\left(u_{n}, v_{n}\right) \\
& =-\int_{\Omega} F\left(x, \xi_{n}, \eta_{n}\right) d \mu .
\end{aligned}
$$

If $B_{\Omega}<+\infty$, choosing $\epsilon_{\lambda} \in\left(0, \frac{1}{\lambda B_{\Omega}}\right)$, by (4.3), there exists $n_{\epsilon_{\lambda}}$ such that

$$
\int_{\Omega} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>\epsilon_{\lambda} B_{\Omega}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\epsilon_{\lambda}}
$$

Hence

$$
\Phi_{\Omega}\left(u_{n}, v_{n}\right)-\lambda \Psi_{\Omega}\left(u_{n}, v_{n}\right) \leq-\lambda \epsilon_{\lambda} B_{\Omega}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\epsilon_{\lambda}} .
$$

Thus,

$$
\lim _{n \rightarrow \infty}\left[\Phi_{\Omega}\left(u_{n}\right)-\lambda \Psi_{\Omega}\left(v_{n}\right)\right]=-\infty
$$

If $B_{\Omega}=+\infty$, consider $M_{\lambda}>\frac{1}{\lambda}$. By (4.3), there exists $n_{M_{\lambda}}$ such that

$$
\int_{\Omega} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>M_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{M_{\lambda}}
$$

Hence

$$
\Phi_{\Omega}\left(u_{n}, v_{n}\right)-\lambda \Psi_{\Omega}\left(u_{n}, v_{n}\right) \leq-\lambda M_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{M_{\lambda}} .
$$

By the choice of $M_{\lambda}$, we also have

$$
\lim _{n \rightarrow \infty}\left[\Phi_{\Omega}\left(u_{n}\right)-\lambda \Psi_{\Omega}\left(v_{n}\right)\right]=-\infty
$$

Thus, we finish the proof of this lemma.

Lemma 4.3 $\Phi_{\Omega}$ is sequentially weakly lower semi-continuous.
Proof The proof is easily completed by using the weak lower semi-continuity of the norm.

Lemma 4.4 $\Psi_{\Omega}$ is sequentially weakly upper semi-continuous.

Proof The proof is the same as that of Lemma 3.4 with replacing $W$ with $W_{0}$ and $V$ with $\Omega$.

Proof of Theorem 4.1 It is obvious that $\Phi_{\Omega}: W_{0} \rightarrow \mathbb{R}$ is coercive. Lemma 4.1-Lemma 4.4 imply that all of conditions in Lemma 2.4 are satisfied. Hence, Lemma 2.4(a) implies that for each $\left(\lambda_{1, \Omega}, \lambda_{2, \Omega}\right), I_{\lambda, \Omega}$ has a sequence $\left\{\left(u_{n}^{\star}, v_{n}^{\star}\right)\right\}$ of critical points that are solutions of system (1.14) such that $\lim _{n \rightarrow \infty} \Phi_{\Omega}\left(u_{n}^{\star}, v_{n}^{\star}\right)=+\infty$.

## 5 Result and proofs for system (1.15)

In this section, we investigate the $(p, q)$-Laplacian system (1.15). We first make the following assumptions:
$\left(M_{1}\right)$ There exists $\mu_{0}>0$ such that $\mu(x) \geq \mu_{0}$ for all $x \in V$;
$\left(M_{2}\right)$ There exists $x_{0} \in V$ such that $M_{1}\left(x_{0}\right) \leq M_{1}(x)$ and $M_{2}\left(x_{0}\right) \leq M_{2}(x)$ for all $x \in V$, where

$$
\begin{aligned}
& M_{1}(x)=\left(\frac{\operatorname{deg}(x)}{2 \mu(x)}\right)^{\frac{p}{2}} \mu(x)+h_{1}(x) \mu(x)+\sum_{y \sim x}\left(\frac{w_{x y}}{2 \mu(y)}\right)^{\frac{p}{2}} \mu(y), \quad x \in V, \\
& M_{2}(x)=\left(\frac{\operatorname{deg}(x)}{2 \mu(x)}\right)^{\frac{q}{2}} \mu(x)+h_{2}(x) \mu(x)+\sum_{y \sim x}\left(\frac{w_{x y}}{2 \mu(y)}\right)^{\frac{q}{2}} \mu(y), \quad x \in V ;
\end{aligned}
$$

$\left(H_{1}\right)$ There exists a constant $h_{0}>0$ such that $h_{i}(x) \geq h_{0}>0$ for all $x \in V, i=1,2$;
Let

$$
\begin{equation*}
\varrho=\max \left\{\frac{M_{1}\left(x_{0}\right)}{p}, \frac{M_{2}\left(x_{0}\right)}{q}\right\} \quad \text { and } \quad K=\max \left\{\frac{1}{h_{0}^{1 / p} \mu_{0}^{1 / p}}, \frac{1}{h_{0}^{1 / q} \mu_{0}^{1 / q}}\right\} . \tag{5.1}
\end{equation*}
$$

Theorem 5.1 Suppose that $G=(V, E)$ is a locally finite graph, and $\left(M_{1}\right),\left(M_{2}\right),\left(H_{1}\right)$ and the following conditions hold:
$\left(\tilde{F}_{0}\right) F(x, s, t)$ is continuously differentiable in $(s, t) \in \mathbb{R}^{2}$ for all $x \in V$, and there exist a function $a \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$and a function $b: V \rightarrow \mathbb{R}^{+}$with $b \in L^{1}(V)$ such that

$$
\begin{aligned}
& \qquad\left|F_{s}(x, s, t)\right| \leq a(|(s, t)|) b(x), \quad\left|F_{t}(x, s, t)\right| \leq a(|(s, t)|) b(x), \\
& \qquad|F(x, s, t)| \leq a(|(s, t)|) b(x) \\
& \text { for all } x \in V \text { and all }(s, t) \in \mathbb{R}^{2} ; \\
& \quad\left(\tilde{F}_{1}\right) \int_{V} F(x, 0,0) d \mu=0 ; \\
& \left(\tilde{F}_{2}\right) \\
& \qquad 0<A:=\liminf _{y \rightarrow \infty} \frac{\int_{V} \max _{|s|+|t| \leq y} F(x, s, t) d \mu}{y^{\delta}}<\limsup _{|s|+|t| \rightarrow+\infty} \frac{\int_{V} F(x, s, t) d \mu}{|s|^{p}+|t|^{q}}:=B,
\end{aligned}
$$

where $\delta=\min \{p, q\}$.

Then, for each $\lambda \in\left(\Theta_{1}, \Theta_{2}\right)$ with $\Theta_{1}=\frac{\varrho}{B}$ and $\Theta_{2}=\frac{1}{p K 2^{p-1} A}$, system (1.15) possesses an unbounded sequence of solutions.

We work in the space $W:=W_{h_{1}}^{1, p}(V) \times W_{h_{2}}^{1, q}(V)$ with the norm equipped with $\|(u, v)\|=$ $\|u\|_{W_{h_{1}}^{1, p}(V)}+\|v\|_{W_{h_{2}}^{1, q}(V)}$ and then $(W,\|\cdot\|)$ is a Banach space that is infinite dimensional.
We consider the functional $I_{\lambda}: W \rightarrow \mathbb{R}$ as

$$
\begin{align*}
I_{\lambda}(u, v)= & \frac{1}{p} \int_{V}\left(|\nabla u|^{p}+h_{1}(x)|u|^{p}\right) d \mu+\frac{1}{q} \int_{V}\left(|\nabla v|^{q}+h_{2}(x)|v|^{q}\right) d \mu \\
& -\lambda \int_{V} F(x, u, v) d \mu . \tag{5.2}
\end{align*}
$$

Then, by Appendix A. 2 in [22], under the assumptions of Theorem $5.1, I_{\lambda} \in C^{1}(W, \mathbb{R})$, and

$$
\begin{align*}
\left\langle I_{\lambda}^{\prime}(u, v),\left(\phi_{1}, \phi_{2}\right)\right\rangle= & \int_{V}\left[|\nabla u|^{p-2} \Gamma\left(u, \phi_{1}\right)+h_{1}(x)|u|^{p-2} u \phi_{1}-\lambda F_{u}(x, u, v) \phi_{1}\right] d \mu \\
& +\int_{V}\left[|\nabla v|^{q-2} \Gamma\left(v, \phi_{2}\right)+h_{2}(x)|v|^{q-2} v \phi_{2}-\lambda F_{v}(x, u, v) \phi_{2}\right] d \mu \tag{5.3}
\end{align*}
$$

for any $(u, v),\left(\phi_{1}, \phi_{2}\right) \in W$.
Obviously, $(u, v) \in W$ is a critical point of $I_{\lambda}$ iff

$$
\int_{V}\left[|\nabla u|^{p-2} \Gamma\left(u, \phi_{1}\right)+h_{1}(x)|u|^{p-2} u \phi_{1}-\lambda F_{u}(x, u, v) \phi_{1}\right] d \mu=0
$$

and

$$
\int_{V}\left[|\nabla v|^{q-2} \Gamma\left(v, \phi_{2}\right)+h_{2}(x)|v|^{q-2} v \phi_{2}-\lambda F_{v}(x, u, v) \phi_{2}\right] d \mu=0
$$

for all $\left(\phi_{1}, \phi_{2}\right) \in W$. Furthermore, by the arbitrariness of $\phi_{1}$ and $\phi_{2}$, it can be achieved that system (1.15) holds. Therefore, seeking the solutions for system (1.15) is equivalent to seeking the critical points of $I_{\lambda}$ on $W$.

Define $\Phi: W \rightarrow \mathbb{R}$ and $\Psi: W \rightarrow \mathbb{R}$ by

$$
\begin{aligned}
\Phi(u, v) & =\frac{1}{p} \int_{V}\left(|\nabla u|^{p}+h_{1}(x)|u|^{p}\right) d \mu+\frac{1}{q} \int_{V}\left(|\nabla v|^{q}+h_{2}(x)|v|^{q}\right) d \mu \\
& =\frac{1}{p}\|u\|_{W_{h}^{1, p}(V)}^{p}+\frac{1}{q}\|v\|_{W_{h}^{1, q}(V)}^{q}
\end{aligned}
$$

and

$$
\Psi(u, v)=\int_{V} F(x, u, v) d \mu
$$

Then $I_{\lambda}(u, v)=\Phi-\lambda \Psi$. For every $r>\inf \Phi$, set

$$
\varphi(r)=\inf _{(u, v) \in \Phi^{-1}([-\infty, r])} \frac{\left.\sup _{(u, v) \in \Phi^{-1}([-\infty, r])} \Psi(u, v)\right)-\Psi(u, v)}{r-\Phi(u, v)} .
$$

Lemma 5.1 Assume that $\left(\tilde{F}_{2}\right)$ holds. Then $\gamma:=\liminf _{r \rightarrow+\infty} \varphi(r)<+\infty$.

Proof The proof is essentially the same as that of Theorem 3.1 with substituting the locally finite graph $V, K, A, B,\|u\|_{\infty}$, and $\|v\|_{\infty}$ for the finite graph $V, K_{V}, A_{V}, B_{V},\|u\|_{\infty, V}$, and $\|\nu\|_{\infty, V}$, respectively. We omit the details.

Lemma 5.2 For any given $\lambda \in\left(\Theta_{1}, \Theta_{2}\right)$, the functional $I_{\lambda}(u, v)=\Phi(u, v)-\lambda \Psi(u, v)$ is unbounded from below.

Proof By $\left(\tilde{F}_{2}\right)$, we can assume that $\left\{\xi_{n}\right\}$ and $\left\{\eta_{n}\right\}$ are two positive real sequences satisfying $\lim _{n \rightarrow \infty}\left|\xi_{n}\right|+\left|\eta_{n}\right|=+\infty$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu}{\xi_{n}^{p}+\eta_{n}^{q}}=B \tag{5.4}
\end{equation*}
$$

For each $n \in \mathbb{N}$, define

$$
u_{n}(x)=\left\{\begin{array}{ll}
\xi_{n}, & x=x_{0}, \\
0, & x \neq x_{0},
\end{array} \quad v_{n}(x)= \begin{cases}\eta_{n}, & x=x_{0} \\
0, & x \neq x_{0}\end{cases}\right.
$$

where $x_{0}$ is given in assumption $\left(M_{2}\right)$. Then a simple calculation implies that

$$
\left|\nabla u_{n}\right|(x)= \begin{cases}\sqrt{\frac{\operatorname{deg}\left(x_{0}\right)}{2\left(x_{0}\right)}} \xi_{n}, & x=x_{0} \\ \sqrt{\frac{w_{x_{0} y}}{2 \mu(y)}} \xi_{n}, & x=y \text { with } y \sim x_{0} \\ 0, & \text { otherwise }\end{cases}
$$

and

$$
\left|\nabla v_{n}\right|(x)= \begin{cases}\sqrt{\frac{\operatorname{deg}\left(x_{0}\right)}{2 \mu\left(x_{0}\right)}} \eta_{n}, & x=x_{0} \\ \sqrt{\frac{w_{x_{0} y} y}{2 \mu(y)}} \eta_{n}, & x=y \text { with } y \sim x_{0} \\ 0, & \text { otherwise }\end{cases}
$$

Then

$$
\begin{aligned}
\int_{V} & \left(\left|\nabla u_{n}\right|^{p}+h_{1}(x)\left|u_{n}\right|^{p}\right) d \mu \\
& =\sum_{x \in V}\left(\left|\nabla u_{n}(x)\right|^{p}+h_{1}(x)\left|u_{n}(x)\right|^{p}\right) \mu(x) \\
& =\left(\left.\left|\nabla u_{n}\left(\left.x_{0}\right|^{p}\right)+h_{1}\left(x_{0}\right)\right| u_{n}\left(x_{0}\right)\right|^{p}\right) \mu\left(x_{0}\right)+\sum_{y \sim x_{0}}\left(\left|\nabla u_{n}(y)\right|^{p}+h_{1}(y)\left|u_{n}(y)\right|^{p}\right) \mu(y) \\
& =\left(\frac{\operatorname{deg}\left(x_{0}\right)}{2 \mu\left(x_{0}\right)}\right)^{\frac{p}{2}} \xi_{n}^{p} \mu\left(x_{0}\right)+h_{1}\left(x_{0}\right) \xi_{n}^{p} \mu\left(x_{0}\right)+\xi_{n}^{p} \sum_{y \sim x_{0}}\left(\frac{w_{x_{0} y}}{2 \mu(y)}\right)^{\frac{p}{2}} \mu(y) \\
& =\xi_{n}^{p} M_{1}\left(x_{0}\right),
\end{aligned}
$$

and similarly,

$$
\int_{V}\left(\left|\nabla v_{n}\right|^{q}+h_{2}(x)\left|v_{n}\right|^{q}\right) d \mu
$$

$$
\begin{aligned}
& =\left(\frac{\operatorname{deg}\left(x_{0}\right)}{2 \mu\left(x_{0}\right)}\right)^{\frac{q}{2}} \eta_{n}^{q} \mu\left(x_{0}\right)+h_{2}\left(x_{0}\right) \eta_{n}^{q} \mu\left(x_{0}\right)+\eta_{n}^{q} \sum_{y \sim x_{0}}\left(\frac{w_{x_{0} y}}{2 \mu(y)}\right)^{\frac{q}{2}} \mu(y) \\
& =\eta_{n}^{q} M_{2}\left(x_{0}\right)
\end{aligned}
$$

where $M_{1}\left(x_{0}\right)$ and $M_{2}\left(x_{0}\right)$ are given in assumption $\left(M_{2}\right)$. Then $\left\{\left(u_{n}, v_{n}\right)\right\} \subset W$ and for every $n \in \mathbb{N}$, we have

$$
\begin{align*}
I_{\lambda}\left(u_{n}, v_{n}\right) & =\Phi\left(u_{n}, v_{n}\right)-\lambda \Psi\left(u_{n}, v_{n}\right) \\
& =\frac{\xi_{n}^{p} M_{1}\left(x_{0}\right)}{p}+\frac{\eta_{n}^{q} M_{2}\left(x_{0}\right)}{q}-\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu \\
& \leq \varrho\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu, \tag{5.5}
\end{align*}
$$

where $\varrho$ is given in (5.1).
If $B<+\infty$, choosing $\tilde{\epsilon}_{\lambda} \in\left(\frac{Q}{\lambda B}, 1\right)$, by (5.4), there exists $n_{\tilde{\epsilon}_{\lambda}}$ such that

$$
\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>\tilde{\epsilon}_{\lambda} B\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\tilde{\epsilon}_{\lambda}} .
$$

Thus, combining with (5.5), we have

$$
\begin{aligned}
\Phi\left(u_{n}, v_{n}\right)-\lambda \Psi\left(u_{n}, v_{n}\right) & \leq \varrho\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\lambda \tilde{\epsilon}_{\lambda} B\left(\xi_{n}^{p}+\eta_{n}^{q}\right) \\
& =\left(\varrho-\lambda \tilde{\epsilon}_{\lambda} B\right)\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\tilde{\epsilon}_{\lambda}} .
\end{aligned}
$$

Hence

$$
\lim _{n \rightarrow \infty}\left[\Phi\left(u_{n}\right)-\lambda \Psi\left(v_{n}\right)\right]=-\infty .
$$

If $B=+\infty$, let us consider $\tilde{M}_{\lambda}>\frac{\varrho}{\lambda}$. $\operatorname{By}$ (5.4), there exists $n_{\tilde{M}_{\lambda}}$ such that

$$
\int_{V} F\left(x, \xi_{n}, \eta_{n}\right) d \mu>\tilde{M}_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\tilde{M}_{\lambda}} .
$$

Thus

$$
\begin{aligned}
\Phi\left(u_{n}, v_{n}\right)-\lambda \Psi\left(u_{n}, v_{n}\right) & \leq \varrho\left(\xi_{n}^{p}+\eta_{n}^{q}\right)-\lambda \tilde{M}_{\lambda}\left(\xi_{n}^{p}+\eta_{n}^{q}\right) \\
& =\left(\varrho-\lambda \tilde{M}_{\lambda}\right)\left(\xi_{n}^{p}+\eta_{n}^{q}\right), \quad \forall n>n_{\tilde{M}_{\lambda}}
\end{aligned}
$$

Combining the choice of $\tilde{M}_{\lambda}$, in this case, we also have

$$
\lim _{n \rightarrow \infty}\left[\Phi\left(u_{n}\right)-\lambda \Psi\left(v_{n}\right)\right]=-\infty
$$

Thus we complete the proof of this lemma.

Lemma 5.3 $\Phi$ is sequentially weakly lower semi-continuous.

Proof The proof is easily completed by using the weak lower semi-continuity of the norm.

Lemma 5.4 $\Psi$ is sequentially weakly upper semi-continuous.

Proof Assume that $\left(u_{k}, v_{k}\right) \rightharpoonup\left(u_{0}, v_{0}\right)$ for some $\left(u_{0}, v_{0}\right) \in W$. Then

$$
\lim _{k \rightarrow \infty} \int_{V} u_{k} \varphi d \mu=\int_{V} u_{0} \varphi d \mu, \quad \forall \varphi \in C_{c}(V)
$$

which implies that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} u_{k}(x)=u_{0}(x) \quad \text { for any fixed } x \in V \tag{5.6}
\end{equation*}
$$

by choosing

$$
\varphi(y)= \begin{cases}1, & y=x \\ 0, & y \neq x\end{cases}
$$

Similarly, we have

$$
\lim _{k \rightarrow \infty} v_{k}(x)=v_{0}(x) \quad \text { for any fixed } x \in V .
$$

By $\left(\tilde{F}_{0}\right)$ and Lebesgue dominated convergence theorem, it is easy to obtain that

$$
\lim _{n \rightarrow \infty} \int_{V} F\left(x, u_{k}, v_{k}\right) d \mu=\int_{V} F\left(x, u_{0}, v_{0}\right) d \mu .
$$

Hence, $\Psi$ is sequentially weakly upper semi-continuous in $W$.

Proof of Theorem 5.1 Obviously, $\Phi: W \rightarrow \mathbb{R}$ is coercive. Lemma 5.1-Lemma 5.4 imply that all of conditions in Lemma 2.4(a) hold for $I_{\lambda}$. Hence, Lemma 2.4(b) implies that for each $\lambda \in\left(\Theta_{1}, \Theta_{2}\right), I_{\lambda}$ has a sequence $\left\{\left(u_{n}, v_{n}\right)\right\}$ of critical points that are solutions of system (1.15) such that $\lim _{n \rightarrow \infty} \Phi\left(u_{n}, v_{n}\right)=+\infty$.

## 6 The results for the scalar equations

By using similar arguments as those of Theorem 3.1, we can also obtain similar results for the following scalar equation on finite graph $G=(V, E)$ :

$$
\begin{equation*}
£_{m, p} u+h(x)|u|^{p-2} u=\lambda f(x, u), \quad x \in V, \tag{6.1}
\end{equation*}
$$

where $m \geq 1$ is an integer, $h: V \rightarrow \mathbb{R}, p>1, \lambda>0$, and $f: V \times \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 6.1 Let $G=(V, E)$ be a finite graph and $F(x, s)=\int_{0}^{s} f(x, \tau) d \tau$ for all $x \in V$. Assume that the following conditions hold:
(h) $h(x)>0$ for all $x \in V$;
$\left(f_{0}\right) F(x, s)$ is continuously differentiable in $s \in \mathbb{R}$ for all $x \in V$;
$\left(f_{1}\right) \int_{V} F(x, 0) d \mu=0 ;$
$\left(f_{2}\right)$

$$
0<\tilde{A}_{V}:=\liminf _{y \rightarrow \infty} \frac{\int_{V} \max _{|s| \leq y} F(x, s, t) d \mu}{y^{p}}<\limsup _{|s| \rightarrow \infty} \frac{\int_{V} F(x, s) d \mu}{|s|^{p}}:=\tilde{B}_{V} .
$$

Then, for each $\lambda \in\left(\tilde{\lambda}_{1, V}, \tilde{\lambda}_{2, V}\right)$ with $\tilde{\lambda}_{1, V}=\frac{\tilde{e} V}{\tilde{B}_{V}}$ and $\tilde{\lambda}_{2, V}=\frac{1}{p \tilde{K}_{V} \tilde{A}_{V}}$, where $\tilde{K}_{V}=\frac{1}{h_{\min } \mu_{\min }}$ and $\tilde{\varrho}_{V}=\frac{1}{p} \int_{V} h(x) d \mu$, equation (6.1) possesses an unbounded sequence of solutions.

The proofs of Theorem 6.1 are almost the same as those of Theorem 3.1 and even simpler because there is no couple term. Here, we just present the proof that $\tilde{\gamma}_{V}:=$ $\liminf _{r \rightarrow+\infty} \tilde{\varphi}_{V}(r)<+\infty$, which is related to the range of the parameter of $\lambda$ and also show that the proof for single equation is indeed simpler, where

$$
\begin{aligned}
& \tilde{\varphi}_{V}(r)=\inf _{u \in \tilde{\Phi}_{V}^{-1}([-\infty, r])} \frac{\left(\sup _{(u, v) \in \tilde{\Phi}_{V}^{-1}([-\infty, r])} \tilde{\Psi}_{V}(u)\right)-\tilde{\Psi}_{V}(u)}{r-\tilde{\Phi}_{V}(u)} \\
& \tilde{\Phi}_{V}(u)=\frac{1}{p} \int_{V}\left(\left|\nabla^{m} u\right|^{p}+h(x)|u|^{p}\right) d \mu=\frac{1}{p}\|u\|_{W^{m, p}(V)}^{p} \\
& \tilde{\Psi}_{V}(u)=\int_{V} F(x, u) d \mu
\end{aligned}
$$

and $\tilde{I}_{\lambda, V}=\tilde{\Phi}_{V}-\lambda \tilde{\Psi}_{V}$ is the corresponding variational functional of (6.1).
In fact, let $\left\{c_{n}\right\}$ be a real sequence satisfying $\lim _{n \rightarrow \infty} c_{n}=+\infty$ and

$$
\lim _{n \rightarrow \infty} \frac{\int_{V} \max _{|s| \leq c_{n}} F(x, s) d \mu}{c_{n}^{\delta}}=\tilde{A}_{V}
$$

Write

$$
r_{n}=\frac{c_{n}^{p}}{p \tilde{K}_{V}} \quad \text { for every } n \in \mathbb{N}
$$

By Lemma 2.1, for all $(u, v) \in W$ with $\tilde{\Phi}_{V}(u) \leq r_{n}$, we get

$$
\frac{\|u\|_{\infty, V}^{p}}{p} \leq \tilde{K}_{V} \frac{\|u\|_{W^{m, p}(V)}^{p}}{p} \leq \tilde{K}_{V} r_{n}
$$

Hence, $|u(x)| \leq c_{n}$ for all $x \in V$. Therefore, it follows from $\left(f_{1}\right)$ that

$$
\begin{aligned}
& \tilde{\varphi}_{V}\left(r_{n}\right) \\
& =\inf _{\frac{1}{p}\|u\|_{W^{m, p}(V)}^{p} \leq r_{n}}^{p} \frac{\sup _{\frac{1}{p}\|u\|_{W^{m_{1}, p}(V)}^{p} \leq r_{n}}^{p} \int_{V} F(x, u) d \mu-\int_{V} F(x, u) d \mu}{r_{n}-\frac{1}{p}\|u\|_{W^{m_{1}, p}(V)}^{p}} \\
& \leq \frac{\sup _{\frac{1}{p}\|u\|_{W^{m_{1}, p}(V)}^{p} \leq r_{n}} \int_{V} F(x, u) d \mu}{r_{n}} \\
& =p \tilde{K}_{V} \frac{\sup _{\frac{1}{p}\|u\|_{W^{m_{1}, p}(V)}^{p} \leq r_{n}}^{p} \int_{V} F(x, u) d \mu}{c_{n}^{\delta}}
\end{aligned}
$$

$$
\leq p \tilde{K}_{V} \frac{\int_{V} \max _{|s| \leq c_{n}} F(x, s) d \mu}{c_{n}^{\delta}}
$$

Hence, $\left(f_{2}\right)$ implies that

$$
\tilde{\gamma}_{V} \leq \liminf _{n \rightarrow \infty} \varphi_{V}\left(r_{n}\right) \leq p \tilde{K}_{V} \tilde{A}_{V}<p \tilde{K}_{V} \tilde{B}_{V} \leq+\infty
$$

Thus we finish the proof.
By using similar arguments as those of Theorem 4.1, we can also obtain similar results for the following scalar equation with Dirichlet boundary value on a locally finite graph $G=(V, E)$ :

$$
\begin{cases}£_{m, p} u=\lambda f(x, u), & x \in \Omega^{\circ}  \tag{6.2}\\ \left|\nabla^{j} u\right|=0, & x \in \partial \Omega, 0 \leq j \leq m-1\end{cases}
$$

where $p>1, m \in \mathbb{N}, \lambda>0$, and $f: V \times \mathbb{R} \rightarrow \mathbb{R}$ and $\Omega \subset G(V, E)$ is a bounded domain.

Theorem 6.2 Suppose that $G=(V, E)$ is a locally finite graph, $F(x, s)=\int_{0}^{s} f(x, \tau) d \tau$ for all $x \in \Omega, \Omega^{\circ} \neq \emptyset$, and the following conditions hold:
(h)' $h(x)>0$ for all $x \in \Omega$;
$\left(f_{0}\right)^{\prime} F(x, s)$ is continuously differentiable in $s \in \mathbb{R}$ for all $x \in \Omega$;
$\left(f_{1}\right)^{\prime} \int_{\Omega} F(x, 0) d \mu=0$;
$\left(f_{2}\right)^{\prime}$

$$
0<\tilde{A}_{\Omega}:=\liminf _{y \rightarrow \infty} \frac{\int_{\Omega} \max _{|s| \leq y} F(x, s, t) d \mu}{y^{p}}<\limsup _{|s| \rightarrow \infty} \frac{\int_{\Omega} F(x, s) d \mu}{|s|^{p}}:=\tilde{B}_{\Omega} .
$$

Then, for each $\lambda \in\left(\tilde{\lambda}_{1, \Omega}, \tilde{\lambda}_{2, \Omega}\right)$ with $\tilde{\lambda}_{1, \Omega}=\frac{1}{\tilde{B}_{\Omega}}$ and $\tilde{\lambda}_{2, \Omega}=\frac{1}{p \tilde{K}_{\Omega} \tilde{A}_{\Omega}}$, where $\tilde{K}_{\Omega}=\frac{C^{p}(m, p, \Omega)}{\mu_{\min , \Omega}}$, equation (6.2) possesses an unbounded sequence of solutions.

By using similar arguments as those of Theorem 5.1, we can also obtain similar results for the following scalar equation on locally finite graph $G=(V, E)$ :

$$
\begin{equation*}
-\Delta_{p} u+h(x)|u|^{p-2} u=\lambda f(x, u), \quad x \in V, \tag{6.3}
\end{equation*}
$$

where $h: V \rightarrow \mathbb{R}, p \geq 2, \lambda>0$, and $f: V \times \mathbb{R} \rightarrow \mathbb{R}$. We make the following assumptions:
(h) There exists a constant $h_{0}>0$ such that $h(x) \geq h_{0}>0$ for all $x \in V$;
$(M)$ There exists $x_{0} \in V$ such that $M\left(x_{0}\right) \leq M(x)$ for all $x \in V$, where

$$
M(x)=\left(\frac{\operatorname{deg}(x)}{2 \mu(x)}\right)^{\frac{p}{2}} \mu(x)+h(x) \mu(x)+\sum_{y \sim x_{0}}\left(\frac{w_{x y}}{2 \mu(y)}\right)^{\frac{p}{2}} \mu(y), \quad x \in V .
$$

Let

$$
\tilde{\varrho}=\frac{M\left(x_{0}\right)}{p} \quad \text { and } \quad \tilde{K}=\frac{1}{h_{0}^{1 / p} \mu_{0}^{1 / p}} .
$$

Theorem 6.3 Let $G=(V, E)$ be a locally finite graph and $F(x, s)=\int_{0}^{s} f(x, \tau) d \tau$ for all $x \in V$. Assume that ( $h$ ), (M) and the following conditions hold:
$\left(\tilde{f}_{0}\right) F(x, s)$ is continuously differentiable in $s \in \mathbb{R}$ for all $x \in V$, and there exist a function $a \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$and a function $b: V \rightarrow \mathbb{R}^{+}$with $b \in L^{1}(V)$ such that

$$
\left|F_{s}(x, s)\right| \leq a(|s|) b(x), \quad|F(x, s)| \leq a(|s|) b(x)
$$

for all $x \in V$ and all $s \in \mathbb{R}$;
$\left(\tilde{f}_{1}\right) \int_{V} F(x, 0) d \mu=0 ;$
$\left(\tilde{f}_{2}\right)$

$$
0<\tilde{A}:=\liminf _{y \rightarrow \infty} \frac{\int_{V} \max _{|s| \leq y} F(x, s) d \mu}{y^{p}}<\limsup _{|s| \rightarrow \infty} \frac{\int_{V} F(x, s) d \mu}{|s|^{p}}:=\tilde{B} .
$$

Then, for each $\lambda \in\left(\tilde{\Theta}_{1}, \tilde{\Theta}_{2}\right)$ with $\tilde{\Theta}_{1}=\frac{\tilde{\varrho}}{\tilde{B}}$ and $\tilde{\Theta}_{2}=\frac{1}{p \tilde{K} \tilde{A}}$, equation (6.3) possesses an unbounded sequence of solutions.

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## Data Availability

No datasets were generated or analysed during the current study.

## Declarations

## Ethics approval and consent to participate

Not applicable

## Competing interests

The authors declare no competing interests.

## Author contributions

The authors contribute the manuscript equally.

## Author details

${ }^{1}$ Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan, 650500, P.R. China. ${ }^{2}$ Faculty of Transportation Engineering, Kunming University of Science and Technology, Kunming, Yunnan, 650500, P.R. China.
${ }^{3}$ Research Center for Mathematics and Interdisciplinary Sciences, Kunming University of Science and Technology, Kunming, Yunnan, 650500, P.R. China.

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