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Mixed boundary value problems involving Sturm–Liouville differential equations with possibly negative coefficients

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Abstract

This paper is devoted to the study of a mixed boundary value problem for a complete Sturm–Liouville equation, where the coefficients can also be negative. In particular, the existence of infinitely many distinct positive solutions to the given problem is obtained by using critical point theory.

Mathematics Subject Classification: 34B15

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1 Introduction

The aim of this paper is to study the multiplicity of solutions to the following mixed boundary value problem with a complete second-order Sturm–Liouville differential equation

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda f(x,u) & \text{in }]a,b[, \\ u(a) = u'(b) = 0, \end{cases} \tag{M_λ}$$

where λ is a positive real parameter, f is an L^1-Carathéodory function, and $\gamma, \delta \in L^\infty([a,b])$ are such that

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) > -\left(\frac{\pi}{2(b-a)}\right)^2. \tag{1.1}$$

Sturm–Liouville problems with mixed boundary conditions have been studied by several authors. In particular, papers [3, 16], and [2] are concerned with the existence of three solutions, [9, 11], and [10] with the existence of one or two solutions, and [7] deals with the existence of a sequence of distinct solutions. It is worth noting that, in each of the above references, the coefficients of the differential equation are nonnegative. We also observe that such problems provide a useful model for describing physical or chemical phenomena, and are used in many applied sciences, such as mechanical engineering. Examples of application of mixed boundary value problems occur in industrial processes and involve the solidification and melting of a material (see, for instance, [5] and the references therein). The main novelty of this paper is that, in contrast to the papers that are available in the literature, we assume that the coefficients γ and δ can change their sign. For example, we can consider the following differential equation, well known as Laguerre equation:

$$-u''(x) - \frac{1-x}{x}u'(x) + \frac{1}{2x}u(x) = f(x,u), \quad x \in]-2,-1[. \tag{1.2}$$

Indeed, (1.2) is a complete Sturm–Liouville differential equation with coefficients

$$\gamma(x) = -\frac{1-x}{x}, \qquad \delta(x) = \frac{1}{2x},$$

which are negative and satisfy our hypotheses. Let us recall that the Laguerre equation is one of the equations used in quantum mechanics for the hydrogen atom modeling, and its solutions are useful for describing the orbitals of this molecular chemical element (see, for example, [13]).

The main aim of this paper is to investigate the existence of infinitely many solutions to complete Sturm–Liouville equations, with possibly negative coefficients, under mixed boundary conditions, assuming a suitable oscillatory behavior of the nonlinearity. It has been proven that if the nonlinear term is nonnegative, the obtained solutions are positive. That is because of a strong maximum principle, which is emphasized in the paper (see Propositions 2.5 and 2.6). Our approach is based on variational methods. Note that the variational formulation of the corresponding problem is not natural due to the presence of the term $\gamma(x)u'$; indeed, such problems are often referred to as "nonvariational problems" since there is no simple associated minimization problem. But this paper presents a specific functional, which is different from the classical energy functional, and shows that it can be studied by variational methods (see Proposition 2.4).

The paper is organized as follows. In Sect. 2, we give some basic properties, which are our main tools, along with an infinitely many critical point theorem given in [6], which is a more precise version of the variational principle of Ricceri [15]. In Sect. 3, we present our result on the existence of a sequence of pairwise distinct generalized solutions for the problem (M_λ) and its consequence in the autonomous case, providing an explicit example.

2 Basic properties and preliminaries

In this section we introduce the functional space and we recall some preliminaries and basic properties in order to study problem (M_λ). Take the Sobolev space

$$X = \{u \in W^{1,2}([a,b]) : u(a) = 0\}$$

endowed with the following norm:

$$\|u\| = \left(\int_a^b |u(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_a^b |u'(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Moreover, for all $u \in X$, put

$$\|u\|_0 = \|u'\|_{L^2} = \left(\int_a^b |u'(x)|^2\,dx\right)^{\frac{1}{2}}.$$

The following proposition holds true.

Proposition 2.1 *The norms* $\|\cdot\|_0$ *and* $\|\cdot\|$ *are equivalent on* X.

Proof For all $u \in X$, one has

$$\|u\| = \|u\|_{L^2} + \|u'\|_{L^2} \geq \|u'\|_{L^2} = \|u\|_0.$$

Further, taking into account that u is absolutely continuous and $u' \in L^1([a,b])$, from the fundamental theorem of integral calculus, one has

$$u(x) = \int_a^x u'(t)\,dt,$$

then

$$|u(x)| \leq \int_a^x |u'(t)|\,dt \leq \int_a^b |u'(x)|\,dx.$$

Applying Hölder's inequality, we obtain

$$|u(x)| \leq (b-a)^{\frac{1}{2}} \|u'\|_{L^2}. \tag{2.1}$$

By squaring and integrating between a and b, one has

$$\|u\|_{L^2} \leq (b-a)\|u'\|_{L^2}. \tag{2.2}$$

Therefore, taking (2.2) into account yields

$$\|u\| = \|u\|_{L^2} + \|u'\|_{L^2} \leq \big(1 + (b-a)\big)\|u\|_0. \tag{2.3}$$

Thus, the proof is complete. □

Here, we point out the following result:

Proposition 2.2 (Poincaré inequalities) *For all* $u \in X$, *one has*

(*j*) $\max_{x\in[a,b]} |u(x)| \leq (b-a)^{\frac{1}{2}} \|u\|_0$,

(*jj*) $\|u\|_{L^2} \leq \frac{2(b-a)}{\pi} \|u'\|_{L^2}$.

Proof (*j*) This Poincaré inequality is obtained as in formula (2.1). We observe that $(b-a)^{\frac{1}{2}}$ is the best constant k among those for which $\|u\|_\infty \leq k\|u\|_0$ for all $u \in X$. Indeed, arguing by a contradiction, assume that there is a positive constant $k < (b-a)^{\frac{1}{2}}$ for which $\|u\|_\infty \leq k\|u\|_0$ for all $u \in X$. In particular, choosing $\overline{u}(x) \in X$ as $\overline{u}(x) = x-a$, one has $\|\overline{u}\|_\infty = (b-a) \leq k\|\overline{u}\|_0$, that is, $(b-a) \leq k(b-a)^{\frac{1}{2}}$ and this is absurd, thus our claim is proved.

(*jj*) From (2.2), one has

$$\|u\|_{L^2} \leq (b-a)\|u'\|_{L^2}.$$

However, $k = (b-a)$ is not the best constant for Poincaré inequality in X. In order to obtain the best constant, we reason as done in [4, Lemma 8.1] for Dirichlet problems. We consider the following problem:

$$\begin{cases} -u'' = \lambda u & \text{in }]a,b[, \\ u(a) = u'(b) = 0. \end{cases} \tag{2.4}$$

Standard computations show that $\lambda_0 = \frac{\pi^2}{4(b-a)^2}$ is the first eigenvalue of the above problem. Let $A\colon X \to \mathbb{R}^+$ be defined by

$$A(u) = \left\{ \frac{\|u'\|_{L^2}^2}{\|u\|_{L^2}^2} : \|u\|_{L^2} \neq 0 \right\}.$$

We prove that A has a minimizer in X, $\inf A = \lambda_0$ and the value of the best constant in (*jj*) is $\frac{1}{\sqrt{\lambda_0}}$. First, we note that by (2.2) A is bounded from below.

Let u_n be a minimizing sequence, that is, $A(u_n) \to \inf A$, as $n \to +\infty$. Note that

$$z_n = \frac{u_n}{\|u_n'\|_{L^2}}$$

satisfies $\|z_n'\|_{L^2} = 1$. Up to a subsequence, $z_n \to z$ in $L^2([a,b])$ (the embedding being compact; see, for instance, [8, Theorem IX.16]) and weakly in X.

First, we need to prove that $\|z\|_{L^2} \neq 0$. Since A is homogeneous of degree 0, z_n is a minimizing sequence and $A(z_n)$ is a bounded sequence, i.e., there exists a positive constant C such that

$$1 = \left\|z_n'\right\|_{L^2}^2 \leq C\|z_n\|_{L^2}^2.$$

From $\|z_n\|_{L^2} \geq \sqrt{\frac{1}{C}}$, taking into account that z_n converges strongly to z in L^2 and passing to the limit as $n \to +\infty$, we get $\|z\|_{L^2} > 0$.

Moreover, we observe that

$$0 \leq \int_a^b \left(z_n'(x) - z'(x)\right)^2 dx = \int_a^b \left(z_n'(x)\right)^2 dx + \int_a^b \left(z'(x)\right)^2 dx - 2\int_a^b z_n'(x)z'(x)\, dx.$$

Therefore,

$$\int_a^b \left(z_n'(x)\right)^2 dx \geq 2\int_a^b z_n'(x)z'(x)\, dx - \int_a^b \left(z'(x)\right)^2 dx.$$

If we pass to the limit in the above inequality, we have

$$\liminf_{n\to+\infty} \int_a^b \left(z_n'(x)\right)^2 dx \geq \int_a^b \left(z'(x)\right)^2 dx.$$

Since z_n is a minimizing sequence, one has

$$\inf A = \lim_{n\to+\infty} A(z_n) = \liminf_{n\to+\infty} \frac{\int_a^b (z_n'(x))^2\, dx}{\int_a^b (z_n(x))^2\, dx} \geq \frac{\int_a^b (z'(x))^2\, dx}{\int_a^b z(x)^2\, dx} = A(z).$$

Consequently z is a minimizer for A.

Now, we prove that $\lambda_0 = \min_{u\in X, u\neq 0} A(u)$. Let $u \in X$ be a minimizer for A. It follows that $A'(u) = 0$, that is,

$$\frac{2\int_a^b u'(x)v'(x)\,dx \cdot \int_a^b u(x)^2\,dx - 2\int_a^b (u'(x))^2\,dx \cdot \int_a^b u(x)v(x)\,dx}{(\int_a^b u(x)^2\,dx)^2} = 0.$$

Hence,

$$\int_a^b u'(x)v'(x)\,dx = \frac{\int_a^b (u'(x))^2\,dx}{\int_a^b u(x)^2\,dx} \cdot \int_a^b u(x)v(x)\,dx \quad \forall v \in X.$$

Therefore, if u minimizes A, u is an eigenfunction of (2.4) and $\frac{\int_a^b u'(x)^2\,dx}{\int_a^b u(x)^2\,dx} = A(u) = \inf(A)$ is the corresponding eigenvalue.

Let us prove that $\inf A = \lambda_0$. Since λ_0 is the smallest eigenvalue of (2.4), one has

$$\lambda_0 \le \inf A.$$

We prove the opposite inequality. Let w be the eigenfunction corresponding to λ_0.

One has

$$\int_a^b w'(x)u'(x)\,dx = \lambda_0 \int_a^b w(x)u(x)\,dx \quad \forall u \in X.$$

Consequently, choosing $u = w$, we obtain

$$\int_a^b w'(x)^2\,dx = \lambda_0 \int_a^b w(x)^2\,dx \quad \forall u \in X.$$

Therefore, one has

$$\inf A = \inf \frac{\int_a^b u'(x)^2\,dx}{\int_a^b u(x)^2\,dx} \le \frac{\int_a^b w'(x)^2\,dx}{\int_a^b w(x)^2\,dx} = \lambda_0 \frac{\int_a^b w(x)^2\,dx}{\int_a^b w(x)^2\,dx} = \lambda_0.$$ □

Remark 2.1 We observe that the Poincarè inequalities hold true in the Sobolev space $W_0^{1,2}([a,b])$, as given in [1], with different constants.

Now, let us introduce another norm in the space X, given by

$$\|u\|_X = \left(\int_a^b e^{-\Gamma(x)}\left|u'(x)\right|^2 dx + \int_a^b e^{-\Gamma(x)}\delta(x)\left|u(x)\right|^2 dx\right)^{\frac{1}{2}},$$

where

$$\Gamma(x) = \int_a^x \gamma(\xi)\,d\xi \quad \forall x \in [a,b]. \tag{2.5}$$

Proposition 2.3 *Assume* (1.1). *Then* $\|\cdot\|_X$ *is a norm on the space* X *and it is equivalent to* $\|\cdot\|_0$. *In particular, one has*

$$m\|u\|_0 \le \|u\|_X \le M\|u\|_0, \tag{2.6}$$

for all $u \in X$, where m, M, with $M \geq m > 0$, are given by

$$
\begin{aligned}
&m = \begin{cases} (\min_{x\in[a,b]} e^{-\Gamma(x)})^{\frac{1}{2}}, \\ \quad \textit{if } \operatorname{ess\,inf}_{x\in[a,b]} \delta(x) \geq 0, \\ [\min_{x\in[a,b]} e^{-\Gamma(x)}(1 + \operatorname{ess\,inf}_{x\in[a,b]} \delta(x)(\frac{2(b-a)}{\pi})^2)]^{\frac{1}{2}}, \\ \quad \textit{if } \operatorname{ess\,inf}_{x\in[a,b]} \delta(x) < 0, \end{cases} \\
&M = \begin{cases} [\max_{x\in[a,b]} e^{-\Gamma(x)}(1 + \operatorname{ess\,sup}_{x\in[a,b]} \delta(x)(\frac{2(b-a)}{\pi})^2)]^{\frac{1}{2}}, \\ \quad \textit{if } \operatorname{ess\,sup}_{x\in[a,b]} \delta(x) \geq 0, \\ (\max_{x\in[a,b]} e^{-\Gamma(x)})^{\frac{1}{2}}, \\ \quad \textit{if } \operatorname{ess\,sup}_{x\in[a,b]} \delta(x) < 0. \end{cases}
\end{aligned}
\tag{2.7}
$$

Proof First, note that condition (1.1) ensures $\|u\|_X \geq 0$ and, by standard computations, can be easily proved that $\|u\|_X$ is a norm on X.

Let us prove (2.6). We recall that $\|u\|_0 = \|u'\|_{L^2}$, therefore in the following we use the notation $\|u'\|_{L^2}$. One has

$$
\begin{aligned}
\|u\|_X^2 &= \int_a^b e^{-\Gamma(x)} \left|u'(x)\right|^2 dx + \int_a^b e^{-\Gamma(x)} \delta(x) \left|u(x)\right|^2 dx \\
&\geq \min_{x\in[a,b]} e^{-\Gamma(x)} \left(\int_a^b \left|u'(x)\right|^2 dx + \operatorname*{ess\,inf}_{x\in[a,b]} \delta \int_a^b \left|u(x)\right|^2 dx \right).
\end{aligned}
$$

We have two possibilities: if $\operatorname{ess\,inf}_{x\in[a,b]} \delta(x) \geq 0$, one has

$$
\|u\|_X^2 \geq \min_{x\in[a,b]} e^{-\Gamma(x)} \left(\int_a^b \left|u'(x)\right|^2 dx \right) = \min_{x\in[a,b]} e^{-\Gamma(x)} \left\|u'\right\|_{L^2}^2,
$$

while if $\operatorname{ess\,inf}_{x\in[a,b]} \delta(x) < 0$, taking into account (1.1) and *(jj)* in Proposition 2.2, one has

$$
\begin{aligned}
\|u\|_X^2 &\geq \min_{x\in[a,b]} e^{-\Gamma(x)} \left(\int_a^b \left|u'(x)\right|^2 dx + \operatorname*{ess\,inf}_{x\in[a,b]} \delta \int_a^b \left|u(x)\right|^2 dx \right) \\
&\geq \min_{x\in[a,b]} e^{-\Gamma(x)} \left(\left\|u'\right\|_{L^2}^2 + \operatorname*{ess\,inf}_{x\in[a,b]} \delta \left(\frac{2(b-a)}{\pi} \right)^2 \left\|u'\right\|_{L^2}^2 \right) \\
&= \min_{x\in[a,b]} e^{-\Gamma(x)} \left(1 + \operatorname*{ess\,inf}_{x\in[a,b]} \delta \left(\frac{2(b-a)}{\pi} \right)^2 \right) \left\|u'\right\|_{L^2}^2,
\end{aligned}
$$

with

$$
\min_{x\in[a,b]} e^{-\Gamma(x)} \left(1 + \operatorname*{ess\,inf}_{x\in[a,b]} \delta \left(\frac{2(b-a)}{\pi} \right)^2 \right) > 0.
$$

To prove the other inequality, we observe that

$$
\|u\|_X^2 \leq \max_{x\in[a,b]} e^{-\Gamma(x)} \left(\int_a^b \left|u'(x)\right|^2 dx + \operatorname*{ess\,sup}_{x\in[a,b]} \delta \int_a^b \left|u(x)\right|^2 dx \right).
$$

So, if $\operatorname{ess\,sup}_{x\in[a,b]} \delta(x) \geq 0$, taking again (*jj*) in Proposition 2.2 into account, one has

$$\begin{aligned}\|u\|_X^2 &\leq \max_{x\in[a,b]} e^{-\Gamma(x)}\left(\|u'\|_{L^2}^2 + \operatorname*{ess\,sup}_{x\in[a,b]} \delta\left(\frac{2(b-a)}{\pi}\right)^2 \|u'\|_{L^2}^2\right) \\ &= \max_{x\in[a,b]} e^{-\Gamma(x)}\left(1 + \operatorname*{ess\,sup}_{x\in[a,b]} \delta\left(\frac{2(b-a)}{\pi}\right)^2\right)\|u'\|_{L^2}^2,\end{aligned}$$

while if $\operatorname{ess\,sup}_{x\in[a,b]} \delta(x) < 0$, one has

$$\|u\|_X^2 \leq \max_{x\in[a,b]} e^{-\Gamma(x)}\|u'\|_{L^2}^2.$$

Hence, (2.6) is proved. □

Remark 2.2 We observe that, since $\|u\|_0$ is equivalent to $\|u\|$, as proved in Proposition 2.1, thanks to the transitivity property, we obtain the equivalence between $\|u\|_X$ and $\|u\|$.

Remark 2.3 The space X is a Hilbert space with the dot product

$$\langle u, v\rangle = \int_a^b e^{-\Gamma(x)} u'(x)v'(x)\,dx + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,dx,$$

that clearly induces the norm $\|\cdot\|_X$.

Remark 2.4 Taking into account (2.1) and (2.6), the following inequality holds:

$$\max_{x\in[a,b]}|u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{m}\|u\|_X \quad \forall u \in X.$$

Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, that is,

(i) $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$;

(ii) $t \to f(x,t)$ is continuous for almost every $x \in [a,b]$;

(iii) for all $\rho > 0$, the function $\sup_{|t|\leq\rho} |f(\cdot,t)|$ belongs to $L^1([a,b])$.

Now we recall the definition of classical and generalized solution for problem (M_λ):

- We say that $u : [a,b] \to \mathbb{R}$ is a classical solution if $u \in C^2([a,b])$, $u(a) = u'(b) = 0$, $-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x,u(x))$ for all $x \in [a,b]$.
- We say that $u : [a,b] \to \mathbb{R}$ is a generalized solution if $u \in C^1([a,b]), u' \in AC([a,b])$, $u(a) = u'(b) = 0$, $-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x,u(x))$ for almost every $x \in [a,b]$.

Classical and generalized solutions coincide when f, γ, δ are continuous functions.

Now, put

$$F(x,t) = \int_0^t f(x,\xi)\,d\xi \quad \forall (x,t) \in [a,b] \times \mathbb{R} \tag{2.8}$$

and introduce two functions $\Psi, \Phi : X \to \mathbb{R}$ defined by

$$\Psi(u) = \int_a^b e^{-\Gamma(x)} F\bigl(x, u(x)\bigr)\,dx,$$

$$\Phi(u) = \frac{1}{2}\|u\|_X^2,$$

for all $u \in X$. It is well known (see, for instance, [1]) that Ψ is well defined, Ψ and Φ are Gâteaux differentiable, and one has

$$\Psi'(u)(v) = \int_a^b e^{-\Gamma(x)} f\bigl(x, u(x)\bigr) v(x)\,dx,$$
$$\Phi'(u)(v) = \int_a^b e^{-\Gamma(x)} u'(x)v'(x)\,dx + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,dx,$$

for all $u, v \in X$. Moreover, Φ, Ψ are C^1-functions.

Finally, put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \quad \forall u \in X.$$

Now we point out the following result.

Proposition 2.4 *Function u is a generalized solution of (M_λ) $\Longleftrightarrow$ u is a critical point of I_λ.*

Proof Assume that u is a generalized solution of (M_λ). In particular, one has $-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x, u(x))$ for a.a. $x \in [a, b]$. Fixing $v \in X$ and multiplying by $e^{-\Gamma}v$, which again belongs to X, integrating and then integrating by parts the first term, it follows

$$\begin{aligned}
&\int_a^b -u''(x)e^{-\Gamma(x)}v(x)\,dx + \int_a^b \gamma(x)u'(x)e^{-\Gamma(x)}v(x)\,dx + \int_a^b \delta(x)u(x)e^{-\Gamma(x)}v(x)\,dx \\
&\quad = \lambda\int_a^b f\bigl(x, u(x)\bigr)e^{-\Gamma(x)}v(x)\,dx; \\
&\bigl[u'(b)e^{-\Gamma(b)}v(b) - u'(a)e^{-\Gamma(a)}v(a)\bigr] - \int_a^b \gamma(x)u'(x)e^{-\Gamma(x)}v(x)\,dx \\
&\qquad + \int_a^b \gamma(x)u'(x)e^{-\Gamma(x)}v(x)\,dx + \int_a^b u'(x)e^{-\Gamma(x)}v'(x)\,dx + \int_a^b \delta(x)u(x)e^{-\Gamma(x)}v(x)\,dx \\
&\quad = \lambda\int_a^b f\bigl(x, u(x)\bigr)e^{-\Gamma(x)}v(x)\,dx.
\end{aligned}$$

Taking into account that $u'(b) = 0$, since u is a solution of (M_λ), and $v(a) = 0$, since $v \in X$, one has

$$\begin{aligned}
&\int_a^b u'(x)e^{-\Gamma(x)}v'(x)\,dx + \int_a^b \delta(x)u(x)e^{-\Gamma(x)}v(x)\,dx \\
&\quad = \lambda\int_a^b f\bigl(x, u(x)\bigr)e^{-\Gamma(x)}v(x)\,dx
\end{aligned} \tag{2.9}$$

that is, u is a critical point of I_λ.

Now assume that u is critical point of I_λ. Fix $w \in X$. Clearly, $v = e^{\Gamma}w \in X$. So, $I'_\lambda(u)(v) = 0$, that is,

$$\int_a^b e^{-\Gamma(x)}u'(x)v'(x)\,dx + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,dx = \lambda\int_a^b e^{-\Gamma(x)} f\bigl(x, u(x)\bigr)v(x)\,dx.$$

Therefore, taking into account that $e^{-\Gamma(x)}v'(x) = (e^{-\Gamma(x)}v(x))' + \gamma(x)e^{-\Gamma(x)}v(x)$, one has

$$\int_a^b u'(x)\big(e^{-\Gamma(x)}v(x)\big)'\,dx + \int_a^b u'(x)\gamma(x)e^{-\Gamma(x)}v(x)\,dx + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,dx$$
$$= \lambda \int_a^b e^{-\Gamma(x)} f\big(x, u(x)\big)v(x)\,dx,$$

that is,

$$\int_a^b u'(x)w'(x)\,dx + \int_a^b \gamma(x)u'(x)w(x)\,dx + \int_a^b \delta(x)u(x)w(x)\,dx$$
$$= \lambda \int_a^b f\big(x, u(x)\big)w(x)\,dx.$$

Hence, u' admits a weak derivative, namely $\gamma u' + \delta u - \lambda f(\cdot, u(\cdot))$, which is an L^1-function, and so, by standard arguments, the conclusion is achieved. □

From the 1-dimensional strong maximum principle (see [14, Theorem 3]), we obtain here the following result for mixed boundary problems, which guarantees that if the operator is nonnegative then the solution is either zero or strictly positive.

Proposition 2.5 *Fix $u \in W^{1,2}(\Omega)$ such that $u(x) \geq 0$ for all $x \in [a,b]$ and*

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u \geq 0 & \text{in }]a,b[, \\ u(a) = u'(b) = 0. \end{cases}$$

Then, one has either $u(x) \equiv 0$ or $u(x) > 0$ for all $x \in]a,b]$.

Proof First, assume $\delta(x) \geq 0$ for a.a. $x \in [a,b]$. Assume also that $u \not\equiv 0$ and fix $\bar{x} \in [a,b]$ such that $u(\bar{x}) = \min_{[a,b]} u$. We claim that $\bar{x} \notin]a,b]$. Indeed, if $\bar{x} = b$, one has $u'(b) < 0$ (see [14, Theorem 4]) and this is absurd. If $\bar{x} \in]a,b[$, one has $u \equiv \text{const}$ (see Theorem 3 of [14]) and, so, taking into account that $u(a) = 0$, one has $u \equiv 0$ and this is absurd. Hence, our claim is proved, and one has $u(x) > u(a) = 0$ for all $x \in]a,b]$.

Now assume $\delta < 0$ in a nonzero measure subset of $[a,b]$ and put $\tilde{\delta}(x) = \delta(x)$ if $\delta(x) \geq 0$ and $\tilde{\delta}(x) = 0$ if $\delta(x) < 0$. Taking into account that $u(x) \geq 0$ for all $x \in [a,b]$, one has

$$-u'' + \gamma(x)u' + \tilde{\delta}(x)u \geq -u'' + \gamma(x)u' + \delta(x)u \geq 0$$

and, based on the above, the conclusion follows. □

Now, put

$$f^+(x,t) = \begin{cases} f(x,0) & \text{if } t \leq 0, \\ f(x,t) & \text{if } t > 0, \end{cases}$$

and consider the following problem:

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda f^+(x,u) & \text{in }]a,b[, \\ u(a) = u'(b) = 0. \end{cases} \tag{M_λ^+}$$

We have the following result which, owing to the previous formulation of the 1-dimensional strong maximum principle, guarantees the positivity of the solutions when the nonlinear term is nonnegative.

Proposition 2.6 *Assume* (1.1) *and*

$$f(x,0) \geq 0 \quad a.e. \text{ in } [a,b].$$

Then, any generalized solution of problem (M_λ^+) *is nonnegative and it is also a generalized solution of* (M_λ).

Further, if in addition we assume

$$f(x,t) \geq 0 \quad a.e. \text{ in } [a,b], \forall t \geq 0,$$

then, any generalized solution of problem (M_λ^+) *is positive in* $]a,b]$ *and it is also a generalized solution of* (M_λ).

Proof Let u be a generalized solution of (M_λ^+). So, from (2.9), one has

$$\int_a^b e^{-\Gamma(x)}\big[u'(x)v'(x) + \delta(x)u(x)v(x)\big]\,dx = \lambda \int_a^b e^{-\Gamma(x)} f^+\big(x,u(x)\big)v(x)\,dx$$

for all $v \in X$. Now, put $u^- = \min\{u,0\}$ and $A = \{x \in [a,b] : u(x) < 0\}$. Clearly, $u^- \in X$ (see, for instance, [12, Lemma 7.6]) and, by choosing $v = u^-$ in the previous equality, one has

$$\begin{aligned} 0 \leq \|u^-\|_X^2 &= \int_a^b e^{-\Gamma(x)}\big[\big(u^-\big)'(x)\big(u^-\big)'(x) + \delta(x)u^-(x)u^-(x)\big]\,dx \\ &= \int_A e^{-\Gamma(x)}\big[\big(u^-\big)'(x)\big(u^-\big)'(x) + \delta(x)u^-(x)u^-(x)\big]\,dx \\ &= \int_A e^{-\Gamma(x)}\big[(u)'(x)\big(u^-\big)'(x) + \delta(x)u(x)u^-(x)\big]\,dx \\ &= \int_a^b e^{-\Gamma(x)}\big[(u)'(x)\big(u^-\big)'(x) + \delta(x)u(x)u^-(x)\big]\,dx \\ &= \lambda \int_a^b e^{-\Gamma(x)} f^+\big(x,u(x)\big)u^-(x)\,dx \\ &= \lambda \int_A e^{-\Gamma(x)} f^+\big(x,u(x)\big)u^-(x)\,dx = \lambda \int_A e^{-\Gamma(x)} f(x,0)u^-(x)\,dx \leq 0. \end{aligned}$$

Hence, one has $\|u^-\|_X = 0$, that is, $u^-(x) = 0$ for all $x \in [a,b]$, for which $u(x) \geq 0$ for all $x \in [a,b]$. At this point, it is easy to verify that it is also a solution of (M_λ) since $f^+(x,u(x)) = f(x,u(x))$, and so the first conclusion is achieved.

Now we prove the second claim. Since $u(x) \geq 0$, from our additional assumption, one has $f(x, u(x)) \geq 0$ for which $-u'' + \gamma(x)u' + \delta(x)u \geq 0$ in $[a, b]$. Therefore, Proposition 2.5 ensures that u is positive in $]a, b]$ and the proof is complete. □

In order to be able to prove the existence and multiplicity of solutions to problem (M_λ), we use a critical point theorem, that we recall in the following.

Let X be a nonempty set and let $\Phi, \Psi : X \to \mathbb{R}$ be two functionals. For all $r > \inf_X \Phi$, we put

$$\varphi(r) = \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{(\sup_{v \in (\Phi^{-1}(-\infty, r))} \Psi(v)) - \Psi(u)}{r - \Phi(u)}, \tag{2.10}$$

and

$$\alpha := \liminf_{r \to +\infty} \varphi(r), \qquad \beta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r). \tag{2.11}$$

Theorem 2.5 (See [6]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable functional, and $\Psi : X \to \mathbb{R}$ a sequentially weakly upper semicontinuous and Gâteaux differentiable functional. Then one has*

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.*

(b) *If $\alpha < +\infty$ then, for each $\lambda \in]0, \frac{1}{\alpha}[$, the following alternative holds: either*
  - (b_1) *$\Phi - \lambda\Psi$ possesses a global minimum, or*
  - (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(c) *If $\beta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\beta}[$, the following alternative holds: either*
  - (c_1) *there is a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or*
  - (c_2) *there is a sequence of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3 Existence of infinitely many solutions

In this section we study the existence of infinitely many solutions for problem (M_λ). In particular, by requiring an appropriate oscillation of the primitive of the nonlinear term, we obtain a sequence of pairwise distinct solutions.

Put

$$K = \frac{1}{2} \frac{m^2}{M^2} \frac{\min_{x\in[a,b]} e^{-\Gamma(x)}}{\max_{x\in[a,b]} e^{-\Gamma(x)}},$$

where Γ is defined in (2.5) and m, M are given in (2.7).

Furthermore, taking into account the definition of F given in (2.8), put

$$A = \liminf_{\xi\to+\infty} \frac{\int_a^b \max_{|t|\leq\xi} F(x,t)\,dx}{\xi^2}, \qquad B = \limsup_{\xi\to+\infty} \frac{\int_{\frac{a+b}{2}}^b F(x,\xi)\,dx}{\xi^2},$$

and, if $0 < A, B < +\infty$,

$$\lambda_1 = \frac{M^2}{B(b-a)\min_{x\in[a,b]} e^{-\Gamma(x)}}, \qquad \lambda_2 = \frac{m^2}{2A(b-a)\max_{x\in[a,b]} e^{-\Gamma(x)}}. \tag{3.1}$$

Note that $A = +\infty$ is not possible (see the next hypothesis (j_2)). Moreover, if $A = 0$ and $B = +\infty$, we read $\lambda_1 = 0$ and $\lambda_2 = +\infty$.

Now we can give our main result.

Theorem 3.1 *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that*

(j_1) $F(x,t) \geq 0$ *for a.a.* $x \in [a, \frac{a+b}{2}]$ *and for all* $t \in [0,+\infty[$;

(j_2) $\liminf_{\xi\to+\infty} \frac{\int_a^b \max_{|t|\leq\xi} F(x,t)\,dx}{\xi^2} < K \limsup_{\xi\to+\infty} \frac{\int_{\frac{a+b}{2}}^b F(x,\xi)\,dx}{\xi^2}$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, problem ($M_\lambda$) admits a sequence of pairwise distinct generalized solutions.

Proof Our goal is to apply Theorem 2.5. Take $(X, \|\cdot\|_X)$, and $\Phi, \Psi : X \to \mathbb{R}$ as defined in Sect. 2. Thanks to this choice, Φ and Ψ satisfy the hypotheses requested in Theorem 2.5. Moreover, as we have seen in Sect. 2, the critical points in X of the functional $I_\lambda = \Phi - \lambda\Psi$ are exactly the generalized solutions of the considered problem (M_λ).

We provide the proof in two steps.

Claim 1 $\alpha < +\infty$, *where α is given in* (2.11).

First, we observe that hypothesis (j_2) ensures that $\lambda_1 < \lambda_2$. Fix $\lambda \in]\lambda_1, \lambda_2[$. Let $\{c_n\}$ be a real sequence such that, due to $\lim_{n\to+\infty} c_n = +\infty$,

$$A = \lim_{n\to+\infty} \frac{\int_a^b \max_{|t|\leq c_n} F(x,t)\,dx}{c_n^2}.$$

Put $r_n = \frac{m^2}{2(b-a)} c_n^2$ for all $n \in \mathbb{N}$. Taking Remark 2.4 into account, for each $u \in X$ such that $\Phi(u) = \frac{1}{2}\|u\|_X^2 < r_n$, one has

$$|u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{m}\|u\|_X \leq \frac{(b-a)^{\frac{1}{2}}}{m}(2r_n)^{\frac{1}{2}} = \left(\frac{2(b-a)}{m^2} r_n\right)^{\frac{1}{2}} = c_n,$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$. Therefore, from (2.10) and since $\Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_n) &= \inf_{\Phi(u)<r_n} \frac{(\sup_{\Phi(v)<r_n} \Psi(v)) - \Psi(u)}{r_n - \frac{\|u\|_X^2}{2}} \leq \frac{\sup_{\|v\|_X^2<2r_n} \Psi(v)}{r_n} \\ &= \frac{\sup_{\|v\|_X^2<2r_n} \int_a^b e^{-\Gamma(x)} F(x,v(x))\,dx}{\frac{m^2}{2(b-a)} c_n^2}.\end{aligned}$$

Taking into account that $\max_{x\in[a,b]} |v(x)| \leq c_n$ for all $v \in X$ such that $\|v\|_X^2 < 2r_n$, hence, for all $n \in \mathbb{N}$,

$$\varphi(r_n) \leq \frac{2(b-a)}{m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} \frac{\int_a^b \max_{|t|\leq c_n} F(x,t)\,dx}{c_n^2}.$$

Then, from (2.11) one has

$$\alpha = \liminf_{n\to+\infty} \varphi(r_n) \le \frac{2(b-a)}{m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} A < +\infty. \tag{3.2}$$

Claim 2 *The functional* $I_\lambda = \Phi - \lambda\Psi$ *is unbounded from below.*

Let $\{d_n\}$ be a real sequence such that, due to $\lim_{n\to+\infty} d_n = +\infty$,

$$B = \lim_{n\to+\infty} \frac{\int_{\frac{a+b}{2}}^{b} F(x,d_n)\,dx}{d_n^2}. \tag{3.3}$$

Now, for each $n \in \mathbb{N}$, consider the function

$$w_n(x) = \begin{cases} \frac{2d_n}{b-a}(x-a) & \text{if } x \in [a, \frac{a+b}{2}[, \\ d_n & \text{if } x \in [\frac{a+b}{2}, b]. \end{cases}$$

Clearly, $w_n \in X$ and one has

$$\|w_n\|_0^2 = \int_a^b \left|w_n'(x)\right|^2 dx = \int_a^{\frac{a+b}{2}} \left(\frac{2d_n}{b-a}\right)^2 dx = \frac{2d_n^2}{b-a}.$$

Hence, taking Proposition 2.3 into account, one has

$$\Phi(w_n) = \frac{1}{2}\|w_n\|_X^2 \le \frac{1}{2}M^2\|w_n\|_0^2 = \frac{M^2 d_n^2}{b-a}.$$

Moreover, from (j_1), one has

$$\Psi(w_n) = \int_a^b e^{-\Gamma(x)} F\bigl(x, w_n(x)\bigr)\,dx \ge \min_{x\in[a,b]} e^{-\Gamma(x)} \int_{\frac{a+b}{2}}^{b} F(x,d_n)\,dx.$$

Therefore,

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) \le \frac{M^2 d_n^2}{b-a} - \lambda \min_{x\in[a,b]} e^{-\Gamma(x)} \int_{\frac{a+b}{2}}^{b} F(x,d_n)\,dx.$$

Now, two possibilities arise:

1. $B < +\infty$,
2. $B = +\infty$.

If $B < +\infty$, let $\varepsilon \in]0, B - \frac{M^2}{\lambda(b-a)\min_{x\in[a,b]} e^{-\Gamma(x)}}[$.

We observe that, from hypothesis (j_2), this interval is not empty.

From (3.3), there exists ν_ε such that

$$\int_{\frac{a+b}{2}}^{b} F(x,d_n)\,dx > (B-\varepsilon)d_n^2 \quad \forall n > \nu_\varepsilon.$$

Hence, one has

$$I_\lambda(w_n) \le \frac{M^2 d_n^2}{b-a} - \lambda \min_{x\in[a,b]} e^{-\Gamma(x)}(B-\varepsilon)d_n^2 = d_n^2\left(\frac{M^2}{b-a} - \lambda \min_{x\in[a,b]} e^{-\Gamma(x)}(B-\varepsilon)\right).$$

From the choice of ε, one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Besides, if $B = +\infty$, we fix $N > \frac{M^2}{\lambda(b-a)\min_{x\in[a,b]} e^{-\Gamma(x)}}$; from (3.3) there exists ν_N such that

$$\int_{\frac{a+b}{2}}^{b} F(x, d_n)\,dx > Nd_n^2 \quad \forall n > \nu_N.$$

Hence

$$I_\lambda(w_n) \le \frac{M^2 d_n^2}{b-a} - \lambda \min_{x\in[a,b]} e^{-\Gamma(x)} N d_n^2 = d_n^2\left(\frac{M^2}{b-a} - \lambda \min_{x\in[a,b]} e^{-\Gamma(x)} N\right).$$

Taking the choice of N into account, also in this case one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Finally, we observe that

$$]\lambda_1, \lambda_2[\subseteq \left]0, \frac{1}{\alpha}\right[.$$

Indeed, $0 \le \lambda_1 < \lambda_2 = \frac{m^2}{2A(b-a)\max_{x\in[a,b]} e^{-\Gamma(x)}} \le \frac{1}{\alpha}$, from (3.2).

Then, from (b) of Theorem 2.5, for each $\lambda \in]\lambda_1, \lambda_2[$, the functional $I_\lambda = \Phi - \lambda\Psi$ admits a sequence $\{u_n\}$ of critical points such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, which are generalized solutions of problem (M_λ). □

Now, we deal with the autonomous case. To this end, let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function and consider the autonomous problem

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda g(u) & \text{in }]a, b[, \\ u(a) = u'(b) = 0. \end{cases} \tag{AM_λ}$$

Hence, put

$$G(x) = \int_0^x g(\xi)\,d\xi \quad \forall x \in \mathbb{R},$$

and

$$\Psi(u) = \int_a^b e^{-\Gamma(x)} G\big(u(x)\big)\,dx.$$

Moreover, put

$$\bar{\lambda}_1 = \frac{2M^2}{(b-a)^2 \min_{x\in[a,b]} e^{-\Gamma(x)} \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}},$$

$$\bar{\lambda}_2 = \frac{m^2}{2(b-a)^2 \max_{x\in[a,b]} e^{-\Gamma(x)} \liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}}.$$

Here we point out a consequence of Theorem 3.1, also assuming in addition that γ and δ are continuous functions.

Corollary 3.1 *Assume that*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} < \frac{1}{2} K \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2}.$$

Then, for each $\lambda \in]\bar{\lambda}_1, \bar{\lambda}_2[$, *problem* ($AM_\lambda$) *admits a sequence of pairwise distinct positive classical solutions.*

Proof Put

$$g^+(t) = \begin{cases} g(0) & \text{if } t \leq 0, \\ g(t) & \text{if } t > 0. \end{cases}$$

Theorem 3.1 ensures that problem

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda g^+(u) & \text{in }]a,b[, \\ u(a) = u'(b) = 0 \end{cases}$$

admits a sequence of pairwise distinct solutions for all $\lambda \in]\bar{\lambda}_1, \bar{\lambda}_2[$, which are classical since γ and δ are continuous. Moreover, Proposition 2.6 guarantees that such solutions are all (except at most one) strictly positive and they are also solutions of problem (AM_λ), hence the conclusion. □

We provide an explicit example.

Example 3.2 Consider the autonomous problem (AM_λ) in $[0,1]$, with

$$\gamma(x) = -1, \qquad \delta(x) = -\frac{1}{2}.$$

Put

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!} \quad \forall n \in \mathbb{N}$$

and let $g : \mathbb{R} \to \mathbb{R}$ the nonnegative continuous function defined by

$$g(\xi) = \begin{cases} \frac{32(n+1)!^2[(n+1)!^2 - n!^2]}{\pi} \sqrt{\frac{1}{16(n+1)!^2} - (\xi - \frac{n!(n+2)}{2})^2} & \text{if } \xi \in \bigcup_{n\in\mathbb{N}} [a_n, b_n] \\ 0 & \text{otherwise.} \end{cases}$$

By simple calculations, one has that

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = 0, \qquad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^2} = 4,$$

$$\min_{x\in[0,1]} e^{-\Gamma(x)} = 1, \qquad \max_{x\in[0,1]} e^{-\Gamma(x)} = e,$$
$$m = \sqrt{1-\frac{8}{\pi^2}}, \qquad M = \sqrt{e},$$
$$K = \frac{1-\frac{8}{\pi^2}}{2e^2}.$$

Hence, from Corollary 3.1, for each $\lambda > \frac{e}{2}$ the given problem admits a sequence of pairwise distinct classical positive solutions.

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