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Positive solutions for the Riemann–Liouville-type fractional differential equation system with infinite-point boundary conditions on infinite intervals

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Abstract

In this paper, we study the existence and uniqueness of positive solutions for a class of a fractional differential equation system of Riemann–Liouville type on infinite intervals with infinite-point boundary conditions. First, the higher-order equation is reduced to the lower-order equation, and then it is transformed into the equivalent integral equation. Secondly, we obtain the existence and uniqueness of positive solutions for each fixed parameter $\lambda > 0$ by using the mixed monotone operators fixed-point theorem. The results obtained in this paper show that the unique positive solution has good properties: continuity, monotonicity, iteration, and approximation. Finally, an example is given to demonstrate the application of our main results.

Mathematics Subject Classification: 34A08; 34B16; 34B18

Keywords: Fractional differential equation system; Infinite intervals; Mixed monotone operators; Infinite point; Existence and uniqueness of positive solutions

1 Introduction

In recent years, fractional differential equations have developed rapidly in application and theory. They are widely used in a variety of fields, including fluid flow, signal and image processing, aerodynamics, and modeling of physical phenomena exhibiting anomalous diffusion. Some excellent research outcomes have also been obtained [1–14]. Solving the existence and uniqueness of positive solutions to boundary value problems of fractional differential equations has become an important research area.

In [15], employing a fixed-point theory in cones, the authors investigated the existence and multiplicity of positive solutions for multipoint boundary value problems of fractional differential equations on infinite intervals:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,+\infty), \\ u(0) = u'(0) = 0, \qquad D_{0^+}^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \end{cases} \tag{1}$$

where $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\beta_i \geq 0$, $i = 1,2,\ldots,m-2$ with $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$, and $D_{0^+}^{\alpha}$ denotes the Riemann–Liouville derivative.

In [16], by making use of the fixed-point theorem for generalized concave operators, the authors studied the existence and uniqueness of positive solutions for fractional differential equations with integral boundary conditions as follows:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \qquad u(1) = \beta \int_0^1 u(s)\,ds, \end{cases} \tag{2}$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$, $\lambda > 0$ is a parameter and $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative.

In [17], by using the fixed-point theorem of mixed monotone operators, the authors discussed the existence and uniqueness of positive solutions for the following fractional differential equation system:

$$\begin{cases} -D_t^{\alpha} x(t) = f(t,x(t),D_t^{\beta} x(t),y(t)), \\ -D_t^{\gamma} y(t) = g(t,x(t)), \quad 0 < t < 1, \\ D_t^{\beta} x(0) = 0, \qquad D_t^{\mu} x(1) = \sum_{j=1}^{p-2} a_j D_t^{\mu} x(\xi_j), \\ y(0) = 0, \qquad D_t^{\nu} y(1) = \sum_{j=1}^{p-2} b_j D_t^{\nu} y(\xi_j), \end{cases} \tag{3}$$

where $1 < \gamma < \alpha \leq 2$, $1 < \alpha - \beta < \gamma$, $0 < \beta \leq \mu < 1$, $0 < \nu < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $a_j, b_j \in [0,+\infty)$, and $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} < 1$, $\sum_{j=1}^{p-2} b_j \xi_j^{\gamma-1} < 1, D_t$ is the Riemann–Liouville fractional derivative.

The boundary value problems of fractional differential equations on infinite intervals are significant for the study of the unsteady flow of gases in semiinfinite porous media, the theory of drainage flow, and so on [15, 18–21]. In addition to having a larger practical application background, multipoint boundary value problems can more properly represent many significant physical phenomena, such as soil–water and wet-soil differentials, nonuniform electromagnetic field theory [15, 17, 20–25].

Through consulting the relevant literature, the existence and uniqueness of positive solutions to multipoint boundary value problems of fractional differential equations on infinite intervals have not been thoroughly investigated [20, 21, 26], most of these studies use fixed-point theorems on cones, but few papers use fixed-point theorems of mixed monotone operators, and there is even less literature on fractional differential equation systems.

Motivated by the studies above, in this work, we extend the multipoint boundary value problem to infinite point, and use the mixed monotone operators fixed-point theorem to investigate the existence and uniqueness of positive solutions to the boundary value problem of fractional differential equation system on infinite intervals as follows:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + \lambda p(t) f(t,x(t),D_{0^+}^{\gamma} x(t),y(t),D_{0^+}^{\gamma} y(t)) + \lambda q(t) m(t,D_{0^+}^{\gamma} x(t)) = 0, \\ D_{0^+}^{\beta} y(t) + r(t) g(t,x(t)) = 0, \quad t \in [0,+\infty), \\ D_{0^+}^{\gamma} x(0) = 0, \qquad D_{0^+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{\infty} a_i D_{0^+}^{\gamma} x(\xi_i), \\ I_{0^+}^{2-\beta} y(0) = 0, \qquad D_{0^+}^{\beta-1} y(+\infty) = \sum_{i=1}^{\infty} b_i I_{0^+}^{\sigma-\gamma} y(\eta_i), \end{cases} \tag{4}$$

where $2 < \alpha < 3$, $1 < \beta < 2$, $0 < \gamma < 1$, $1 < \alpha - \gamma < \beta$, $\beta - \gamma - 1 > 0$, $\sigma > \gamma$, $\lambda > 0$ is a parameter, $a_i, b_i \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < +\infty$, $0 < \eta_1 < \eta_2 < \cdots < \eta_i < \cdots < +\infty$, $i = 1, 2, \ldots, D_{0^+}^{\mu}$ is the Riemann–Liouville fractional derivative of order μ, $\mu \in \{\alpha, \beta, \gamma, \alpha - 1, \beta - 1\}$, $f : [0, +\infty)^5 \to [0, +\infty)$, $p, q, r : [0, +\infty) \to [0, +\infty)$ and $m, g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous.

In this paper, we study the existence and uniqueness of positive solutions to the boundary value problem (4), improving and generalizing the literature [17]. The main new features presented in this paper are as follows. First, we generalize the boundary conditions and intervals to give a more general form and more accurate results for the boundary value problem. Secondly, by using the fixed-point theorem of mixed monotone operators, the existence and uniqueness of positive solutions are obtained for every fixed parameter $\lambda > 0$. We also give some properties of positive solutions that depend on the parameter. In addition, the boundary value problem studied in this paper is a system, which is an extension of general fractional differential equations.

The rest of the paper is organized as follows. In Sect. 2, we introduce and derive several key definitions, lemmas, and properties. In Sect. 3, we obtain the existence and uniqueness of positive solutions for problem (4) and the unique positive solution has good properties such as continuity, monotonicity, iteration, and approximation. In Sect. 4, an example is presented to demonstrate the application of our main results. Finally, Sect. 5 presents a brief conclusion.

2 Preliminaries

In this section, we first present some definitions and lemmas to be used in the proof of our main results. They can also be found in the literature [16, 21, 26, 27].

Definition 1 ([26]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}^1$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2 ([26]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}^1$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha - n + 1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 1 ([26]) *Assume that $u \in C(0, 1) \cap L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L^1(0, 1)$. Then,*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \cdots + C_N t^{\alpha - N},$$

for some $C_i \in \mathbb{R}^1$ $(i = 1, 2, \ldots, N)$, where $N = [\alpha] + 1$.

Lemma 2 ([21]) *Let $\alpha,\beta > 0$, $f \in L^1[a,b]$. Then, $I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t) = I_{0^+}^{\beta} I_{0^+}^{\alpha} f(t)$ and $D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t)$, for all $t \in [a,b]$.*

Lemma 3 ([21]) *Let $\alpha,\beta > 0$ and $n = [\alpha]+1$, then the following relations hold*:

$$D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > n,$$
$$D_{0^+}^{\alpha} t^{k} = 0, \quad k = 0,1,2,\ldots,n-1.$$

To prove the main result of this paper we need the following lemmas.

Lemma 4 *Let $x(t) = I_{0^+}^{\gamma} u(t)$, $y(t) = I_{0^+}^{\gamma} v(t)$ and $u(t), v(t) \in C[0,+\infty)$, then the problem (4) can turn into the following modified problem*:

$$\begin{cases} D_{0^+}^{\alpha-\gamma} u(t) + \lambda p(t) f(t, I_{0^+}^{\gamma} u(t), u(t), I_{0^+}^{\gamma} v(t), v(t)) + \lambda q(t) m(t,u(t)) = 0, \\ D_{0^+}^{\beta-\gamma} v(t) + r(t) g(t, I_{0^+}^{\gamma} u(t)) = 0, \quad t \in [0,+\infty), \\ u(0) = 0, \qquad D_{0^+}^{\alpha-\gamma-1} u(+\infty) = \sum_{i=1}^{\infty} a_i u(\xi_i), \\ I_{0^+}^{2-\beta+\gamma} v(0) = 0, \qquad D_{0^+}^{\beta-\gamma-1} v(+\infty) = \sum_{i=1}^{\infty} b_i I_{0^+}^{\sigma} v(\eta_i). \end{cases} \tag{5}$$

Moreover, if $(u,v) \in C[0,+\infty) \times C[0,+\infty)$ is a positive solution of the problem (5), then $(I_{0^+}^{\gamma} u, I_{0^+}^{\gamma} v)$ is a positive solution of the problem (4).

Proof The proof is similar to that for Lemma (2.5) in [17], hence, we omit it here. □

Lemma 5 *Let $\pi_1, \pi_2 \in L^1[0,+\infty)$. If $\Delta_1 = \Gamma(\alpha-\gamma) - \sum_{i=1}^{\infty} a_i \xi_i^{\alpha-\gamma-1} > 0$, $\Delta_2 = \Gamma(\beta+\sigma-\gamma) - \sum_{i=1}^{\infty} b_i \eta_i^{\beta+\sigma-\gamma-1} > 0$, then the following fractional differential equation boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha-\gamma} u(t) + \pi_1(t) = 0, \\ D_{0^+}^{\beta-\gamma} v(t) + \pi_2(t) = 0, \quad t \in [0,+\infty), \\ u(0) = 0, \qquad D_{0^+}^{\alpha-\gamma-1} u(+\infty) = \sum_{i=1}^{\infty} a_i u(\xi_i), \\ I_{0^+}^{2-\beta+\gamma} v(0) = 0, \qquad D_{0^+}^{\beta-\gamma-1} v(+\infty) = \sum_{i=1}^{\infty} b_i I_{0^+}^{\sigma} v(\eta_i) \end{cases} \tag{6}$$

has a unique solution

$$u(t) = \int_0^{+\infty} G(t,s)\pi_1(s)\,ds, \qquad v(t) = \int_0^{+\infty} K(t,s)\pi_2(s)\,ds, \tag{7}$$

where

$$G(t,s) = G_1(t,s) + \frac{1}{\Delta_1} \sum_{i=1}^{\infty} a_i t^{\alpha-\gamma-1} G_1(\xi_i,s) \tag{8}$$

and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha-\gamma)} \begin{cases} t^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}, & 0 \le s \le t < +\infty, \\ t^{\alpha-\gamma-1}, & 0 \le t \le s < +\infty, \end{cases}$$

$$K(t,s)=\frac{1}{\Gamma(\beta-\gamma)}\begin{cases}\frac{\chi(s)}{\chi(0)}t^{\beta-\gamma-1}-(t-s)^{\beta-\gamma-1}, & 0\le s\le t<+\infty,\\ \frac{\chi(s)}{\chi(0)}t^{\beta-\gamma-1}, & 0\le t\le s<+\infty,\end{cases}\tag{9}$$

$$\chi(s)=\Gamma(\beta+\sigma-\gamma)-\sum_{s\le\eta_i}b_i(\eta_i-s)^{\beta+\sigma-\gamma-1}.\tag{10}$$

Proof In view of Lemma 1, we know that the general solution of (6) can be written as

$$u(t)=-I_{0^+}^{\alpha-\gamma}\pi_1(t)+c_1t^{\alpha-\gamma-1}+c_2t^{\alpha-\gamma-2},\tag{11}$$

$$v(t)=-I_{0^+}^{\beta-\gamma}\pi_2(t)+d_1t^{\beta-\gamma-1}+d_2t^{\beta-\gamma-2},\tag{12}$$

for some $c_i,d_i\in\mathbb{R}$ $(i=1,2)$. By using the boundary conditions $u(0)=0$ and $I_{0^+}^{2-\beta+\gamma}v(0)=0$, we know that $c_2=0$, $d_2=0$. Therefore, by Lemma 3, we conclude

$$D_{0^+}^{\alpha-\gamma-1}u(t)=-\int_0^t\pi_1(s)\,ds+c_1\Gamma(\alpha-\gamma),$$

$$D_{0^+}^{\beta-\gamma-1}v(t)=-\int_0^t\pi_2(s)\,ds+d_1\Gamma(\beta-\gamma),$$

$$I_{0^+}^{\sigma}v(t)=-I_{0^+}^{\beta+\sigma-\gamma}\pi_2(t)+d_1\frac{\Gamma(\beta-\gamma)}{\Gamma(\beta+\sigma-\gamma)}t^{\beta+\sigma-\gamma-1}.$$

By the boundary condition $D_{0^+}^{\alpha-\gamma-1}u(+\infty)=\sum_{i=1}^{\infty}a_iu(\xi_i)$, we obtain

$$c_1=\frac{1}{\Delta_1}\int_0^{+\infty}\pi_1(s)\,ds-\sum_{i=1}^{\infty}\frac{a_i}{\Delta_1\Gamma(\alpha-\gamma)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\gamma-1}\pi_1(s)\,ds.\tag{13}$$

By the boundary condition $D_{0^+}^{\beta-\gamma-1}v(+\infty)=\sum_{i=1}^{\infty}b_iI_{0^+}^{\sigma}v(\eta_i)$, we have

$$d_1=\frac{\Gamma(\beta+\sigma-\gamma)}{\Delta_2\Gamma(\beta-\gamma)}\int_0^{+\infty}\pi_2(s)\,ds-\sum_{i=1}^{\infty}\frac{b_i}{\Delta_2\Gamma(\beta-\gamma)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\sigma-\gamma-1}\pi_2(s)\,ds.\tag{14}$$

Therefore, substituting (13) into (11), we have

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\pi_1(s)\,ds+\frac{t^{\alpha-\gamma-1}}{\Delta_1}\int_0^{+\infty}\pi_1(s)\,ds\\&\quad-\sum_{i=1}^{\infty}\frac{a_it^{\alpha-\gamma-1}}{\Delta_1\Gamma(\alpha-\gamma)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\gamma-1}\pi_1(s)\,ds\\&=\frac{1}{\Gamma(\alpha-\gamma)}\left(-\int_0^t(t-s)^{\alpha-\gamma-1}\pi_1(s)\,ds+\int_0^{+\infty}t^{\alpha-\gamma-1}\pi_1(s)\,ds\right)\\&\quad+\sum_{i=1}^{\infty}\frac{a_it^{\alpha-\gamma-1}}{\Delta_1\Gamma(\alpha-\gamma)}\left(\int_0^{+\infty}\xi_i^{\alpha-\gamma-1}\pi_1(s)\,ds-\int_0^{\xi_i}(\xi_i-s)^{\alpha-\gamma-1}\pi_1(s)\,ds\right)\\&=\int_0^{+\infty}\left(G_1(t,s)+\frac{1}{\Delta_1}\sum_{i=1}^{\infty}a_it^{\alpha-\gamma-1}G_1(\xi_i,s)\right)\pi_1(s)\,ds\\&=\int_0^{+\infty}G(t,s)\pi_1(s)\,ds.\end{aligned}$$

In a similar manner, substituting (14) into (12), we obtain

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\beta-\gamma)}\int_0^t (t-s)^{\beta-\gamma-1}\pi_2(s)\,ds + \frac{\Gamma(\beta+\sigma-\gamma)}{\Delta_2\Gamma(\beta-\gamma)}\int_0^{+\infty} t^{\beta-\gamma-1}\pi_2(s)\,ds \\ &\quad - \sum_{i=1}^{\infty}\frac{b_i t^{\beta-\gamma-1}}{\Delta_2\Gamma(\beta-\gamma)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\sigma-\gamma-1}\pi_2(s)\,ds \\ &= -\frac{1}{\Gamma(\beta-\gamma)}\int_0^t (t-s)^{\beta-\gamma-1}\pi_2(s)\,ds + \frac{1}{\Gamma(\beta-\gamma)}\int_0^{+\infty}\frac{\chi(s)}{\chi(0)}t^{\beta-\gamma-1}\pi_2(s)\,ds \\ &= \int_0^{+\infty} K(t,s)\pi_2(s)\,ds. \end{aligned}$$

Therefore, we obtain the expression (7) for the solution of problem (6). The proof is completed. □

Lemma 6 *Suppose that* $\chi(0) > 0$ *holds. Then,* $\chi(s) > 0$, $\frac{\chi(s)}{\chi(0)} \geq 1$, *for all* $s \in [0,+\infty)$.

Proof By (10), we have $\chi'(s) = (\beta+\sigma-\gamma-1)\sum_{s\leq\eta_i} b_i(\eta_i-s)^{\beta+\sigma-\gamma-2} > 0$. Then, $\chi(s)$ is a monotonically increasing function in $[0,+\infty)$. By $\chi(0) > 0$, for all $s \in [0,+\infty)$, we obtain $\chi(s) \geq \chi(0) > 0$ and $\frac{\chi(s)}{\chi(0)} \geq 1$. The proof is completed. □

The following properties of the Green functions play an important role in this paper.

Lemma 7 *The Green functions* $G(t,s)$, $K(t,s)$ *defined by* (8) *and* (9) *have the following properties*:

(i) $G(t,s), K(t,s)$ *are continuous functions and* $G(t,s), K(t,s) \geq 0$,
$$\forall (t,s) \in [0,+\infty)\times[0,+\infty).$$

(ii) $$\frac{G(t,s)}{1+t^{\alpha-\gamma-1}} \leq \frac{1}{\Delta_1}, \quad \forall (t,s) \in [0,+\infty)\times[0,+\infty).$$

(iii) $$G(t,s) \geq \frac{1}{\Delta_1}\sum_{i=1}^{\infty} a_i t^{\alpha-\gamma-1}G_1(\xi_i,s), \quad \forall (t,s) \in [0,+\infty)\times[0,+\infty),$$
$$G(t,s) \leq \frac{t^{\alpha-\gamma-1}}{\Delta_1}, \quad \forall (t,s) \in [0,+\infty)\times[0,+\infty).$$

(iv) $$K(t,s) \leq \frac{\Gamma(\beta+\sigma-\gamma)t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)\chi(0)}, \quad \forall (t,s) \in [0,+\infty)\times[0,+\infty),$$
$$K(t,s) \geq \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\left(1-(1-s)^{\beta-\gamma-1}\right), \quad 0 \leq s,t \leq 1,$$

where $\Delta_1 = \Gamma(\alpha-\gamma) - \sum_{i=1}^{\infty} a_i\xi_i^{\alpha-\gamma-1} > 0$, $\chi(0) = \Gamma(\beta+\sigma-\gamma) - \sum_{i=1}^{\infty} b_i\eta_i^{\beta+\sigma-\gamma-1} > 0$.

Proof (i) According to the definition of $G(t,s)$, $K(t,s)$, it is clear that $G(t,s)$, $K(t,s)$ are continuous functions and $G(t,s), K(t,s) \geq 0$, for all $(t,s) \in [0,+\infty)\times[0,+\infty)$.

(ii) For all $(t,s) \in [0,+\infty) \times [0,+\infty)$, we obtain

$$\begin{aligned}\frac{G(t,s)}{1+t^{\alpha-\gamma-1}} &\leq \frac{1}{\Gamma(\alpha-\gamma)} + \sum_{i=1}^{\infty} \frac{a_i G_1(\xi_i,s)}{\Delta_1} \\ &\leq \frac{1}{\Gamma(\alpha-\gamma)} + \sum_{i=1}^{\infty} \frac{a_i \xi_i^{\alpha-\gamma-1}}{\Delta_1 \Gamma(\alpha-\gamma)} \\ &= \frac{1}{\Delta_1}.\end{aligned}$$

(iii) For all $(t,s) \in [0,+\infty) \times [0,+\infty)$, it is obvious from (8) that $G(t,s) \geq \frac{1}{\Delta_1}\sum_{i=1}^{\infty} a_i t^{\alpha-\gamma-1} G_1(\xi_i,s)$ and

$$\begin{aligned}G(t,s) &= G_1(t,s) + \frac{1}{\Delta_1}\sum_{i=1}^{\infty} a_i t^{\alpha-\gamma-1} G_1(\xi_i,s) \\ &\leq \left(\frac{1}{\Gamma(\alpha-\gamma)} + \sum_{i=1}^{\infty} \frac{a_i \xi_i^{\alpha-\gamma-1}}{\Delta_1 \Gamma(\alpha-\gamma)}\right) t^{\alpha-\gamma-1} \\ &= \frac{t^{\alpha-\gamma-1}}{\Delta_1}.\end{aligned}$$

(iv) For all $(t,s) \in [0,+\infty) \times [0,+\infty)$, by (9) and (10), we can obtain

$$K(t,s) \leq \frac{\chi(s) t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)\chi(0)} \leq \frac{\Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)} t^{\beta-\gamma-1}.$$

If $0 \leq s \leq t \leq +\infty$, then

$$\begin{aligned}K(t,s) &= \frac{1}{\Gamma(\beta-\gamma)}\left(\frac{\chi(s)}{\chi(0)} t^{\beta-\gamma-1} - (t-s)^{\beta-\gamma-1}\right) \\ &\geq \frac{1}{\Gamma(\beta-\gamma)}\left(t^{\beta-\gamma-1} - (t-s)^{\beta-\gamma-1}\right) \\ &= \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\left(1 - \left(1-\frac{s}{t}\right)^{\beta-\gamma-1}\right).\end{aligned}$$

If $0 \leq t \leq s \leq +\infty$, then

$$K(t,s) = \frac{\chi(s) t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)\chi(0)} \geq \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}.$$

In summary, for $0 \leq s,t \leq 1$, we have

$$K(t,s) \geq \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\left(1-(1-s)^{\beta-\gamma-1}\right).$$

The proof is completed. □

Suppose that $(E,\|\cdot\|)$ is a Banach space and θ is the zero element of E. A nonempty, closed, and convex set $P \subset E$ is a cone if it satisfies (1)$x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (2)$x \in P$,

$-x \in P \Rightarrow x = \theta$. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where the smallest N is called the normality constant of P. For $x, y \in E$, $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ ($h \geq \theta$ and $h \neq \theta$), if we define a set $P_h = \{u \in E \mid u \sim h\}$, it is easy to see that $P_h \subset P$.

Definition 3 ([16]) An operator $T : E \to E$ is said to be increasing if $u \leq v$ implies $Tu \leq Tv$.

Definition 4 ([26]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1,2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$.

Lemma 8 ([27]) *Let P be a normal cone, $A : P_h \to P_h$ be an increasing operator and $B : P_h \times P_h \to P_h$ be a mixed monotone operator. Assume that*:

(1) *for any $x \in P_h$, $t \in (0,1)$, there exists $\varphi_1(t) \in (t,1)$ such that*

$$A(tx) \geq \varphi_1(t)Ax;$$

(2) *for any $x, y \in P_h$, $t \in (0,1)$, there exists $\varphi_2(t) \in (t,1)$ such that*

$$B\big(tx, t^{-1}y\big) \geq \varphi_2(t)B(x,y).$$

Then:

(i) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$, such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq Au_0 + B(u_0, v_0) \leq Av_0 + B(v_0, u_0) \leq v_0;$$

(ii) *the operator equation $Ax + B(x,x) = x$ has a unique solution x^* in P_h;*

(iii) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = Ax_{n-1} + B(x_{n-1}, y_{n-1}), \qquad y_n = Ay_{n-1} + B(y_{n-1}, x_{n-1}), \qquad n = 1,2,\ldots,$$

we have $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Lemma 9 ([27]) *Assume that all the conditions of Lemma 8 hold. Let $x_\lambda (\lambda > 0)$ denote the unique solution of the operator equation $Ax + B(x,x) = \lambda x$. Then, we have the following conclusions*:

(1) *if $\varphi_i(t) > t^{\frac{1}{2}}$ $(i = 1,2)$ for $t \in (0,1)$, then x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*

(2) *if there exists $\beta \in (0,1)$, such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$ then x_λ is continuous in λ, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;*

(3) *if there exists $\beta \in (0, \frac{1}{2})$, such that $\varphi_i(t) \geq t^\beta$ $(i = 1,2)$ for $t \in (0,1)$, then $\lim_{\lambda \to +\infty} \|x_\lambda\| = 0$, $\lim_{\lambda \to 0^+} \|x_\lambda\| = \infty$.*

3 Main results

We are next concerned with problem (4) in the following space E defined by

$$E = \left\{ u \in C[0, +\infty) : \sup_{t \in [0,+\infty)} \frac{|u(t)|}{1 + t^{\alpha-\gamma-1}} < +\infty \right\}.$$

From [17], we know that E is a Banach space with the norm

$$\|u\| = \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^{\alpha-\gamma-1}}, \quad u \in E.$$

We define a cone $P \subset E$ by

$$P = \bigl\{u \in E : u(t) \geq 0, t \in [0,+\infty)\bigr\}.$$

For $u, v \in P$ with $u \leq v$, we have $0 \leq u(t) \leq v(t)$, $t \in [0,+\infty)$, and thus

$$\sup_{t\in[0,+\infty)} \frac{u(t)}{1+t^{\alpha-\gamma-1}} \leq \sup_{t\in[0,+\infty)} \frac{v(t)}{1+t^{\alpha-\gamma-1}}.$$

Hence, $\|u\| \leq \|v\|$. Hence, P is a normal cone. Let $h(t) = t^{\alpha-\gamma-1}$ and $\|h\| = 1$.

Also, define a component of P by

$$P_h = \left\{x \in P : \frac{1}{M}h(t) \leq x(t) \leq Mh(t), t \in [0,+\infty)\right\},$$

where M is a constant and $M \geq 1$.

Lemma 10 *The vector* (u, v) *is a solution of system* (5) *if and only if* $(u, v) \in C[0,+\infty) \times C[0,+\infty)$ *is a solution of the following nonlinear integral equation system*:

$$\begin{cases} u(t) = \lambda \int_0^{+\infty} G(t,s)(p(s)f(s, I_{0^+}^{\gamma}u(s), u(s), I_{0^+}^{\gamma}v(s), v(s)) + q(s)m(s, u(s)))\,ds, \\ v(t) = \int_0^{+\infty} K(t,s)r(s)g(s, I_{0^+}^{\gamma}u(s))\,ds. \end{cases} \tag{15}$$

Obviously, system (15) is equivalent to the following integral equation

$$\begin{aligned} u(t) = \lambda \int_0^{+\infty} & G(t,s)p(s) \\ & \times f\biggl(s, I_{0^+}^{\gamma}u(s), u(s), I_{0^+}^{\gamma}\int_0^{+\infty} K(s,\tau)r(\tau)g\bigl(\tau, I_{0^+}^{\gamma}u(\tau)\bigr)\,d\tau, \\ & \int_0^{+\infty} K(s,\tau)r(\tau)g\bigl(\tau, I_{0^+}^{\gamma}u(\tau)\bigr)\,d\tau\biggr)\,ds \\ & + \lambda \int_0^{+\infty} G(t,s)q(s)m\bigl(s, u(s)\bigr)\,ds. \end{aligned} \tag{16}$$

To establish the existence and uniqueness of a solution to the boundary value problem (4), we need to make the following assumptions.

(H_1) $f(t, x_1, x_2, x_3, x_4) = \phi(t, x_1, x_2, x_3, x_4) + \varphi(t, x_1, x_2, x_3, x_4)$, where $\phi, \varphi : [0,+\infty)^5 \to [0,+\infty)$ are continuous, for any fixed $t \in [0,+\infty)$, $\phi(t, x_1, x_2, x_3, x_4)$ is increasing and $\varphi(t, x_1, x_2, x_3, x_4)$ is decreasing in $x_i \geq 0$ ($i = 1, 2, 3, 4$), respectively.

(H_2) $g(t,x) \in C([0,+\infty) \times [0,+\infty) \to [0,+\infty))$ is increasing in x, $g(t,0) \not\equiv 0$, and $\lim_{x\to+\infty} g(x, x^{\alpha-1}) = T_g \in \mathbb{R}$. Moreover, there exists $\omega \in (0,1)$, for all $t, x \in [0,+\infty)$, such that

$$g(t, lx) \geq l^{\omega} g(t,x), \quad l \in (0,1).$$

(H_3) If $x_i \geq 0$ $(i = 1,2,3,4)$ are bounded, then for all $t \in [0,+\infty)$, $\phi(t,(1+t^{\alpha-1})x_1,(1+t^{\alpha-1})x_2,(1+t^{\alpha-1})x_3,(1+t^{\alpha-1})x_4)$ and $\varphi(t,(1+t^{\alpha-1})x_1,(1+t^{\alpha-1})x_2,(1+t^{\alpha-1})x_3,(1+t^{\alpha-1})x_4)$ are bounded.

(H_4) $m(t,x) \in C([0,+\infty) \times [0,+\infty) \to [0,+\infty))$ is increasing in x, $m(t,0) \not\equiv 0$ and $\lim_{x\to+\infty} m(x,x^{\alpha-\gamma-1}) = T_m \in \mathbb{R}$. Moreover, there exists $\delta \in (0,1)$, for all $t,x \in [0,+\infty)$, such that

$$m(t,lx) \geq l^{\delta} m(t,x), \quad l \in (0,1).$$

(H_5) For any $l \in (0,1)$ and t,x_i $(i = 1,2,3,4) \in [0,+\infty)$, there exists $\nu \in (0,1)$, such that

$$\begin{aligned}&\phi(t,lx_1,lx_2,lx_3,lx_4) \geq l^{\nu}\phi(t,x_1,x_2,x_3,x_4),\\&\varphi\bigl(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3,l^{-1}x_4\bigr) \geq l^{\nu}\varphi(t,x_1,x_2,x_3,x_4).\end{aligned}$$

(H_6) The functions p, q, and r satisfy

$$0 < \int_0^{+\infty} p(s)\,ds < +\infty, \qquad 0 < \int_0^{+\infty} q(s)\,ds < +\infty, \qquad 0 < \int_0^{+\infty} r(s)\,ds < +\infty.$$

Remark 1 According to (H_2), (H_4), and (H_5), for all t,x,x_i $(i = 1,2,3,4) \in [0,+\infty)$, $\omega,\delta,\nu \in (0,1)$ and $l > 1$, we have

$$\begin{aligned}&g(t,lx) \leq l^{\omega} g(t,x), \qquad m(t,lx) \leq l^{\delta} m(t,x),\\&\phi(t,lx_1,lx_2,lx_3,lx_4) \leq l^{\nu}\phi(t,x_1,x_2,x_3,x_4),\\&\varphi\bigl(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3,l^{-1}x_4\bigr) \leq l^{\nu}\varphi(t,x_1,x_2,x_3,x_4).\end{aligned}$$

We define two operators $A : P \times P \to P$ and $B : P \to P$ by

$$\begin{aligned}A(u,v)(t) &= \int_0^{+\infty} \bigl[G(t,s)p(s)\phi\bigl(s,I_{0^+}^{\gamma}u(s),u(s),I_{0^+}^{\gamma}Cu(s),Cu(s)\bigr)\\&\quad + G(t,s)p(s)\varphi\bigl(s,I_{0^+}^{\gamma}v(s),v(s),I_{0^+}^{\gamma}Cv(s),Cv(s)\bigr)\bigr]\,ds,\\Bu(t) &= \int_0^{+\infty} G(t,s)q(s)m\bigl(s,u(s)\bigr)\,ds,\end{aligned}$$

where $Cu(t) = \int_0^{+\infty} K(t,s)r(s)g(s,I_{0^+}^{\gamma}u(s))\,ds, \forall u,\ v \in P,\ t \in [0,+\infty).G(t,s),K(t,s)$ are given in (8) and (9).

Theorem 1 *Suppose that* (H_1)–(H_6) *hold. Then*:

(a) *For any given* $\lambda > 0$, *problem* (4) *has a unique solution* $(x_\lambda^*, y_\lambda^*)$ *in* P_h, *where* $h(t) = t^{\alpha-\gamma-1}$, $t \in [0,+\infty)$. *Moreover, for any initial value* $x_0, y_0 \in P_h$, *defining the sequences*

$$\begin{aligned}x_n(t) = I_{0^+}^{\gamma}\biggl[&\int_0^{+\infty} \lambda G(t,s)p(s)f\bigl(s,x_{n-1}(s),D_{0^+}^{\gamma}x_{n-1}(s),y_{n-1}(s),D_{0^+}^{\gamma}y_{n-1}(s)\bigr)\,ds\\&+ \int_0^{+\infty} \lambda G(t,s)q(s)m\bigl(s,D_{0^+}^{\gamma}x_{n-1}(s)\bigr)\,ds\biggr],\end{aligned}$$

$$y_n(t) = I_{0^+}^{\gamma}\left[\int_0^{+\infty} K(t,s)r(s)g\big(s,x_{n-1}(s)\big)\,ds\right], \quad n = 1,2,\ldots,$$

we have $x_n(t) \to x_\lambda^*(t)$, $y_n(t) \to y_\lambda^*(t)$ *as* $n \to \infty$, *where* $G(t,s)$, $K(t,s)$ *are given as in Lemma* 5.

(b) *If* $\psi_i(l) > l^{\frac{1}{2}}$ $(i = 1,2)$ *for* $l \in (0,1)$, *then* x_λ^*, y_λ^* *is strictly increasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *can ensure* $x_{\lambda_1}^* < x_{\lambda_2}^*$, $y_{\lambda_1}^* < y_{\lambda_2}^*$. *If there exists* $\kappa \in (0,1)$, *such that* $\psi_i(l) > l^\kappa$ $(i = 1,2)$, $l \in (0,1)$, *then* x_λ^*, y_λ^* *is continuous in* λ, *that is,* $\lambda \to \lambda_0(\lambda_0 > 0)$ *ensures* $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\|y_\lambda^* - y_{\lambda_0}^*\| \to 0$. *If there exists* $\kappa \in (0,\frac{1}{2})$, *such that* $\psi_i(l) > l^\kappa$ $(i = 1,2)$, $l \in (0,1)$, *then* $\lim_{\lambda\to+\infty}\|x_\lambda^*\| = \infty$, $\lim_{\lambda\to+\infty}\|y_\lambda^*\| = \infty$, *and* $\lim_{\lambda\to0^+}\|x_\lambda^*\| = 0$, $\lim_{\lambda\to0^+}\|y_\lambda^*\| = 0$.

Proof We first consider the existence of a positive solution to problem (16). The proof process is divided into four steps as follows.

First, we show that $A : P_h \times P_h \to P$, $B : P_h \to P$ are well defined. For $u \in P_h$, the constant $M \geq 1$, $t \in [0,+\infty)$, we have

$$\frac{1}{M}t^{\alpha-\gamma-1} \leq u(t) \leq Mt^{\alpha-\gamma-1}. \tag{17}$$

Then,

$$\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}t^{\alpha-1} \leq I_{0^+}^{\gamma}u(t) \leq \frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1}. \tag{18}$$

By (18) and (H_2), for all $t \in [0,+\infty)$, $\omega \in (0,1)$, we have

$$\begin{aligned} g\big(t,I_{0^+}^{\gamma}u(t)\big) &\leq g\left(t,\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1}\right) \\ &\leq \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} g\big(t,t^{\alpha-1}\big) \\ &\leq \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} T_g \end{aligned} \tag{19}$$

and

$$\begin{aligned} g\big(t,I_{0^+}^{\gamma}u(t)\big) &\geq g\left(t,\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}t^{\alpha-1}\right) \\ &\geq \left(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\right)^{\omega} g\big(t,t^{\alpha-1}\big) \\ &\geq \left(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\right)^{\omega} g(t,0). \end{aligned} \tag{20}$$

Thus, from (19), (20), and (iv) in Lemma 7, we have

$$\begin{aligned} Cu(t) &= \int_0^{+\infty} K(t,s)r(s)g\big(s,I_{0^+}^{\gamma}u(s)\big)\,ds \\ &\leq \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{\Gamma(\beta+\sigma-\gamma)t^{\beta-\gamma-1}T_g}{\Gamma(\beta-\gamma)\chi(0)} \int_0^{+\infty} r(s)\,ds, \end{aligned} \tag{21}$$

for any $0 < \zeta \le 1$, $t \in [0, +\infty)$, we have

$$Cu(t) \ge \left(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\right)^{\omega} \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \left(1-(1-s)^{\beta-\gamma-1}\right) r(s) g(s,0)\, ds. \tag{22}$$

Hence, by (21) and (22), we obtain

$$\begin{aligned} I_{0^+}^{\gamma} Cu(t) &\le \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \\ &\quad \times \left[\left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{s^{\beta-\gamma-1} T_g \Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)} \int_0^{+\infty} r(\tau)\, d\tau\right] ds, \\ I_{0^+}^{\gamma} Cu(t) &\ge \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \\ &\quad \times \left[\left(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\right)^{\omega} \frac{s^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \left(1-(1-\tau)^{\beta-\gamma-1}\right) r(\tau) g(\tau,0)\, d\tau\right] ds. \end{aligned} \tag{23}$$

At the same time, for any $t \in [0, +\infty)$, we also obtain

$$\begin{aligned} &0 \le \frac{M\Gamma(\alpha-\gamma)t^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})} < +\infty, \qquad 0 \le \frac{Mt^{\alpha-\gamma-1}}{1+t^{\alpha-1}} < +\infty, \\ &0 \le \frac{1}{\Gamma(\gamma)(1+t^{\alpha-1})} \int_0^t (t-s)^{\gamma-1} \\ &\quad \times \left[\left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{s^{\beta-\gamma-1} T_g \Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)} \int_0^{\infty} r(\tau)\, d\tau\right] ds \\ &= \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{T_g \Gamma(\beta+\sigma-\gamma) t^{\beta-1} \int_0^{+\infty} r(\tau)\, d\tau}{\chi(0)\Gamma(\beta)(1+t^{\alpha-1})} < +\infty, \\ &0 \le \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{t^{\beta-\gamma-1} T_g \Gamma(\beta+\sigma-\gamma)}{(1+t^{\alpha-1})\Gamma(\beta-\gamma)\chi(0)} \int_0^{+\infty} r(s)\, ds < +\infty. \end{aligned}$$

Therefore, by (17), (18), (19), (21), (23), and (H_1), (H_3), there exists a positive constant Q_ϕ such that

$$\begin{aligned} &\phi\left(s, I_{0^+}^{\gamma} u(s), u(s), I_{0^+}^{\gamma} Cu(s), Cu(s)\right) \\ &\quad \le \phi\Bigg(s, \frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} s^{\alpha-1}, Ms^{\alpha-\gamma-1}, \\ &\qquad \frac{1}{\Gamma(\gamma)} \int_0^s (s-t)^{\gamma-1} \\ &\qquad \times \left[\left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{t^{\beta-\gamma-1} T_g \Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)} \int_0^{+\infty} r(\tau)\, d\tau\right] dt, \\ &\qquad \left(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\right)^{\omega} \frac{s^{\beta-\gamma-1} T_g \Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)} \int_0^{+\infty} r(\tau)\, d\tau\Bigg) \\ &\quad \le Q_\phi. \end{aligned} \tag{24}$$

In a similar manner, there exists a positive constant Q_φ such that

$$\begin{aligned}
&\varphi\bigl(s, I_{0^+}^{\gamma} v(s), v(s), I_{0^+}^{\gamma} Cv(s), Cv(s)\bigr) \\
&\quad \leq \varphi\biggl(s, \frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)} s^{\alpha-1}, \frac{1}{M} s^{\alpha-\gamma-1}, \\
&\qquad \frac{1}{\Gamma(\gamma)} \int_0^s (s-t)^{\gamma-1} \\
&\qquad \times \biggl[\biggl(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\biggr)^{\omega} \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \bigl(1-(1-\tau)^{\beta-\gamma-1}\bigr) r(\tau) g(\tau,0)\, d\tau\biggr] dt, \\
&\qquad \biggl(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\biggr)^{\omega} \frac{s^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \bigl(1-(1-\tau)^{\beta-\gamma-1}\bigr) r(\tau) g(\tau,0)\, d\tau\biggr) \\
&\quad \leq Q_\varphi.
\end{aligned} \tag{25}$$

By (24), (25), and (ii) in Lemma 7, we have

$$\begin{aligned}
\frac{|A(u,v)(t)|}{1+t^{\alpha-\gamma-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-\gamma-1}} p(s) \bigl[\phi\bigl(s, I_{0^+}^{\gamma} u(s), u(s), I_{0^+}^{\gamma} Cu(s), Cu(s)\bigr) \\
&\quad + \varphi\bigl(s, I_{0^+}^{\gamma} v(s), v(s), I_{0^+}^{\gamma} Cv(s), Cv(s)\bigr)\bigr] ds \\
&\leq \frac{(Q_\phi + Q_\varphi)}{\Delta_1} \int_0^{+\infty} p(s)\, ds < +\infty.
\end{aligned}$$

For the operator $Bu(t) = \int_0^{+\infty} G(t,s) q(s) m(s,u(s))\, ds$, by (H_4), we obtain that

$$\begin{aligned}
&m\bigl(t,u(t)\bigr) \leq m\bigl(t, Mt^{\alpha-\gamma-1}\bigr) \leq M^{\delta} m\bigl(t, t^{\alpha-\gamma-1}\bigr) \leq M^{\delta} T_m, \\
&m\bigl(t,u(t)\bigr) \geq m\biggl(t, \frac{1}{M} t^{\alpha-\gamma-1}\biggr) \geq \frac{1}{M^{\delta}} m\bigl(t, t^{\alpha-\gamma-1}\bigr) \geq \frac{1}{M^{\delta}} m(t,0),
\end{aligned} \tag{26}$$

where $M \geq 1$, $\delta \in (0,1)$, $t \in [0,+\infty)$. Thus, by (26) and (ii) in Lemma 7, we obtain

$$\frac{|Bu(t)|}{1+t^{\alpha-\gamma-1}} = \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-\gamma-1}} q(s) m\bigl(s,u(s)\bigr)\, ds \leq \frac{M^{\delta} T_m}{\Delta_1} \int_0^{+\infty} q(s)\, ds < +\infty.$$

Hence, we see that $A : P_h \times P_h \to P$ and $B : P_h \to P$ are well defined.

Secondly, we prove that $A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$. Similar to the proof in the first step, by (H_1) and (H_3), for $s \in [0,+\infty)$, there exists a positive constant N_ϕ that satisfies $0 < N_\phi < Q_\phi$, such that

$$\begin{aligned}
&\phi\bigl(s, I_{0^+}^{\gamma} u(s), u(s), I_{0^+}^{\gamma} Cu(s), Cu(s)\bigr) \\
&\quad \geq \phi\biggl(s, \frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)} s^{\alpha-1}, \frac{1}{M} s^{\alpha-\gamma-1}, \\
&\qquad \frac{1}{\Gamma(\gamma)} \int_0^s (s-t)^{\gamma-1} \\
&\qquad \times \biggl[\biggl(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\biggr)^{\omega} \frac{t^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \bigl(1-(1-\tau)^{\beta-\gamma-1}\bigr) r(\tau) g(\tau,0)\, d\tau\biggr] dt, \\
&\qquad \biggl(\frac{\Gamma(\alpha-\gamma)}{M\Gamma(\alpha)}\biggr)^{\omega} \frac{s^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} \int_0^{\zeta} \bigl(1-(1-\tau)^{\beta-\gamma-1}\bigr) r(\tau) g(\tau,0)\, d\tau\biggr) \\
&\quad \geq N_\phi.
\end{aligned} \tag{27}$$

In a similar way, there exists a positive constant N_φ that satisfies $0 < N_\varphi < Q_\varphi$, such that

$$\begin{aligned}
&\varphi\big(s, I_{0^+}^\gamma v(s), v(s), I_{0^+}^\gamma Cv(s), Cv(s)\big)\\
&\quad\geq \varphi\bigg(s, \frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}s^{\alpha-1}, Ms^{\alpha-\gamma-1},\\
&\quad\frac{1}{\Gamma(\gamma)}\int_0^s (s-t)^{\gamma-1}\\
&\quad\times\bigg[\bigg(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\bigg)^\omega \frac{t^{\beta-\gamma-1}T_g\Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)}\int_0^\infty r(\tau)\,d\tau\bigg]dt,\\
&\quad\bigg(\frac{M\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}+1\bigg)^\omega \frac{s^{\beta-\gamma-1}T_g\Gamma(\beta+\sigma-\gamma)}{\Gamma(\beta-\gamma)\chi(0)}\int_0^{+\infty} r(\tau)\,d\tau\bigg)\\
&\quad\geq N_\varphi.
\end{aligned}\tag{28}$$

Let

$$l_1 = \frac{1}{\Delta_1}\sum_{i=1}^\infty a_i(N_\phi + N_\varphi)\int_0^\zeta p(s)G_1(\xi_i, s)\,ds, \quad l_2 = \frac{1}{\Delta_1}(Q_\phi + Q_\varphi)\int_0^{+\infty} p(s)\,ds,$$

where $\Delta_1 = \Gamma(\alpha-\gamma) - \sum_{i=1}^\infty a_i\xi_i^{\alpha-\gamma-1} > 0$ and $0 < \zeta \leq 1$.

In view of $G_1(\xi_i, s) \leq \frac{\xi_i^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}$, $\Gamma(\alpha-\gamma) > \sum_{i=1}^\infty a_i\xi_i^{\alpha-\gamma-1} > 0$ and $0 < N_\varphi < Q_\varphi$, $0 < N_\phi < Q_\phi$, we obtain that

$$\begin{aligned}
0 < l_1 &= \frac{1}{\Delta_1}\sum_{i=1}^\infty a_i(N_\phi + N_\varphi)\int_0^\zeta p(s)G_1(\xi_i, s)\,ds\\
&\leq \frac{1}{\Delta_1\Gamma(\alpha-\gamma)}\sum_{i=1}^\infty a_i\xi_i^{\alpha-\gamma-1}(N_\phi + N_\varphi)\int_0^\zeta p(s)\,ds\\
&\leq \frac{\Gamma(\alpha-\gamma)(Q_\phi + Q_\varphi)}{\Delta_1\Gamma(\alpha-\gamma)}\int_0^{+\infty} p(s)\,ds\\
&= \frac{(Q_\phi + Q_\varphi)}{\Delta_1}\int_0^{+\infty} p(s)\,ds = l_2.
\end{aligned}$$

It follows that $0 < l_1 \leq l_2$. From (27), (28), and (iii) in Lemma 7, we have

$$\begin{aligned}
A(u,v)(t) &\leq \frac{t^{\alpha-\gamma-1}}{\Delta_1}(Q_\phi + Q_\varphi)\int_0^{+\infty} p(s)\,ds\\
&= l_2 t^{\alpha-\gamma-1} = l_2 h(t)
\end{aligned}$$

and

$$\begin{aligned}
A(u,v)(t) &\geq \frac{t^{\alpha-\gamma-1}}{\Delta_1}\sum_{i=1}^\infty a_i(N_\phi + N_\varphi)\int_0^\zeta p(s)G_1(\xi_i, s)\,ds\\
&= l_1 t^{\alpha-\gamma-1} = l_1 h(t).
\end{aligned}$$

Hence, $l_1 h(t) \leq A(u,v)(t) \leq l_2 h(t)$, $\forall t \in [0, +\infty)$. Therefore, $A: P_h \times P_h \to P_h$.

For the operator $Bu(t) = \int_0^{+\infty} G(t,s)q(s)m(s,u(s))\,ds$, let

$$l_3 = \frac{1}{M^\delta \Delta_1} \sum_{i=1}^{\infty} a_i \int_0^\zeta G_1(\xi_i,s)q(s)m(s,0)\,ds, \qquad l_4 = \frac{M^\delta T_m}{\Delta_1} \int_0^{+\infty} q(s)\,ds,$$

where $\Delta_1 = \Gamma(\alpha-\gamma) - \sum_{i=1}^{\infty} a_i \xi_i^{\alpha-\gamma-1} > 0$, $M \geq 1$, $0 < \zeta \leq 1$, $\delta \in (0,1)$ and in a similar way as before, we can obtain $0 < l_3 \leq l_4$.

By (26) and (iii) in Lemma 7, we have

$$\begin{aligned} Bu(t) &\leq \frac{M^\delta T_m t^{\alpha-\gamma-1}}{\Delta_1} \int_0^{+\infty} q(s)\,ds \\ &= l_4 t^{\alpha-\gamma-1} = l_4 h(t) \end{aligned}$$

and

$$\begin{aligned} Bu(t) &\geq \frac{1}{M^\delta \Delta_1} \sum_{i=1}^{\infty} a_i t^{\alpha-\gamma-1} \int_0^\zeta G_1(\xi_i,s)q(s)m(s,0)\,ds \\ &= l_3 t^{\alpha-\gamma-1} = l_3 h(t). \end{aligned}$$

Hence, $l_3 h(t) \leq Bu(t) \leq l_4 h(t)$, $\forall t \in [0,+\infty)$. Therefore, $A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$.

Next, we prove that $A : P_h \times P_h \to P_h$ is a mixed monotone operator and $B : P_h \to P_h$ is an increasing operator. For any $u_i, v_i \in P_h$ $(i = 1,2)$ and $u_1 \leq u_2$, $v_1 \geq v_2$, we have $u_1(t) \leq u_2(t)$, $v_1(t) \geq v_2(t)$ for all $t \in [0,+\infty)$.

By the monotonicity of $I_{0^+}^\gamma$, g, h, ϕ, φ, we conclude

$$\begin{aligned} A(u_1,v_1)(t) &= \int_0^{+\infty} G(t,s)p(s)\bigl[\phi\bigl(s, I_{0^+}^\gamma u_1(s), u_1(s), I_{0^+}^\gamma Cu_1(s), Cu_1(s)\bigr) \\ &\quad + \varphi\bigl(s, I_{0^+}^\gamma v_1(s), v_1(s), I_{0^+}^\gamma Cv_1(s), Cv_1(s)\bigr)\bigr]\,ds \\ &\leq \int_0^{+\infty} G(t,s)p(s)\bigl[\phi\bigl(s, I_{0^+}^\gamma u_2(s), u_2(s), I_{0^+}^\gamma Cu_2(s), Cu_2(s)\bigr) \\ &\quad + \varphi\bigl(s, I_{0^+}^\gamma v_2(s), v_2(s), I_{0^+}^\gamma Cv_2(s), Cv_2(s)\bigr)\bigr]\,ds \\ &= A(u_2,v_2)(t) \end{aligned}$$

and

$$\begin{aligned} Bu_1(t) &= \int_0^{+\infty} G(t,s)q(s)m\bigl(s,u_1(s)\bigr)\,ds \\ &\leq \int_0^{+\infty} G(t,s)q(s)m\bigl(s,u_2(s)\bigr)\,ds = Bu_2(t). \end{aligned}$$

Hence, $A : P_h \times P_h \to P_h$ is a mixed monotone operator and $B : P_h \to P_h$ is an increasing operator.

Finally, we prove that $A(lu, l^{-1}v) \geq \psi_1(l)A(u,v)$ and $B(lu) \geq \psi_2(l)Bu$, for any $u, v \in P_h$, $l \in (0,1)$. By (H_1) and (H_5), for all $\gamma, l, \omega \in (0,1)$, $s \in [0,+\infty)$ and $u, v \in P_h$, we have

$$\begin{aligned} &\phi\bigl(s, I_{0^+}^\gamma lu(s), lu(s), I_{0^+}^\gamma Clu(s), Clu(s)\bigr) \\ &\quad \geq \phi\bigl(s, lI_{0^+}^\gamma u(s), lu(s), l^\omega I_{0^+}^\gamma Cu(s), l^\omega Cu(s)\bigr) \end{aligned} \tag{29}$$

$$\begin{aligned}&\geq \phi\big(s, lI_{0^+}^{\gamma}u(s), lu(s), lI_{0^+}^{\gamma}Cu(s), lCu(s)\big)\\&\geq l^{\nu}\phi\big(s, I_{0^+}^{\gamma}u(s), u(s), I_{0^+}^{\gamma}Cu(s), Cu(s)\big)\end{aligned}$$

and

$$\begin{aligned}&\varphi\big(s, I_{0^+}^{\gamma}l^{-1}v(s), l^{-1}v(s), I_{0^+}^{\gamma}Cl^{-1}v(s), Cl^{-1}v(s)\big)\\&\quad\geq \varphi\big(s, l^{-1}I_{0^+}^{\gamma}v(s), l^{-1}v(s), l^{-\omega}I_{0^+}^{\gamma}Cv(s), l^{-\omega}Cv(s)\big)\\&\quad\geq \varphi\big(s, l^{-1}I_{0^+}^{\gamma}v(s), l^{-1}v(s), l^{-1}I_{0^+}^{\gamma}Cv(s), l^{-1}Cv(s)\big)\\&\quad\geq l^{\nu}\varphi\big(s, I_{0^+}^{\gamma}v(s), v(s), I_{0^+}^{\gamma}Cv(s), Cv(s)\big).\end{aligned}\tag{30}$$

Let

$$\psi_1(l) = l^{\nu} \in (l,1), \qquad \psi_2(l) = l^{\delta} \in (l,1),$$

where $l, \nu, \delta \in (0,1)$. By (29), (30), and (H_1), we can obtain

$$\begin{aligned}A\big(lu, l^{-1}v\big)(t) &= \int_0^{+\infty} G(t,s)p(s)\big[\phi\big(s, I_{0^+}^{\gamma}lu(s), lu(s), I_{0^+}^{\gamma}Clu(s), Clu(s)\big)\\&\quad + \varphi\big(s, l^{-1}I_{0^+}^{\gamma}v(s), l^{-1}v(s), I_{0^+}^{\gamma}Cl^{-1}v(s), Cl^{-1}v(s)\big)\big]\,ds\\&\geq \int_0^{+\infty} G(t,s)p(s)l^{\nu}\big[\phi\big(s, I_{0^+}^{\gamma}u(s), u(s), I_{0^+}^{\gamma}Cu(s), Cu(s)\big)\\&\quad + \varphi\big(s, I_{0^+}^{\gamma}v(s), v(s), I_{0^+}^{\gamma}Cv(s), Cv(s)\big)\big]\,ds\\&= l^{\nu}A(u,v)(t) = \psi_1(l)A(u,v)(t)\end{aligned}$$

and by (H_4), we have

$$\begin{aligned}B(lu)(t) &= \int_0^{+\infty} G(t,s)q(s)m\big(s, lu(s)\big)\,ds\\&\geq \int_0^{+\infty} G(t,s)q(s)l^{\delta}m\big(s, u(s)\big)\,ds\\&= l^{\delta}Bu(t) = \psi_2(l)Bu(t).\end{aligned}$$

Hence, for any $u, v \in P_h$, $l \in (0,1)$, there exist $\psi_1(l), \psi_2(l) \in (l,1)$ such that $A(lu, l^{-1}v) \geq \psi_1(l)A(u,v)$ and $B(lu) \geq \psi_2(l)Bu$.

Therefore, two operators A and B satisfy the conditions of Lemma 8. Hence, there exists a unique positive solution of integral equation (16) in P_h. By Lemma 10, there exists a unique positive solution $(u_{\lambda}^*, v_{\lambda}^*)$ of (5) in P_h.

Let $x_{\lambda}^* = I_{0^+}^{\gamma}u_{\lambda}^*$, $y_{\lambda}^* = I_{0^+}^{\gamma}v_{\lambda}^*$. By Lemma 4, the monotonicity and continuity of $I_{0^+}^{\gamma}$, for $\lambda > 0$, there exists a unique $z_{\lambda}^* = (x_{\lambda}^*, y_{\lambda}^*) \in P_h$ such that $A(z_{\lambda}^*, z_{\lambda}^*) + Bz_{\lambda}^* = \frac{1}{\lambda}z_{\lambda}^*$. It follows that $\lambda(A(z_{\lambda}^*, z_{\lambda}^*) + Bz_{\lambda}^*) = z_{\lambda}^*$, and

$$\begin{aligned}x_{\lambda}^*(t) = I_{0^+}^{\gamma}\bigg[&\int_0^{+\infty} \lambda G(t,s)p(s)f\big(s, x(s), D_{0^+}^{\gamma}x(s), y(s), D_{0^+}^{\gamma}y(s)\big)\,ds\\&+ \int_0^{+\infty} \lambda G(t,s)q(s)m\big(s, D_{0^+}^{\gamma}x(s)\big)\,ds\bigg],\end{aligned}$$

$$y_\lambda^*(t) = I_{0^+}^\gamma\left[\int_0^{+\infty} K(t,s)r(s)g\big(s,x(s)\big)\,ds\right], \quad n = 1,2,\ldots.$$

Consequently, for given $\lambda > 0$, $(x_\lambda^*, y_\lambda^*)$ is a unique positive solution of problem (4) in P_h. For any initial value $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned} x_n(t) = I_{0^+}^\gamma\bigg[&\int_0^{+\infty} \lambda G(t,s)p(s)f\big(s, x_{n-1}(s), D_{0^+}^\gamma x_{n-1}(s), y_{n-1}(s), D_{0^+}^\gamma y_{n-1}(s)\big)\,ds \\ &+ \int_0^{+\infty} \lambda G(t,s)q(s)m\big(s, D_{0^+}^\gamma x_{n-1}(s)\big)\,ds\bigg], \\ y_n(t) = I_{0^+}^\gamma\bigg[&\int_0^{+\infty} K(t,s)r(s)g\big(s,x_{n-1}(s)\big)\,ds\bigg], \quad n = 1,2,\ldots. \end{aligned}$$

By Lemma 8, we have $x_n(t) \to x_\lambda^*(t)$, $y_n(t) \to y_\lambda^*(t)$ as $n \to \infty$, where $G(t,s), K(t,s)$ are given as in Lemma 5.

Further, for all $l \in (0,1)$, by Lemma 9, if $\psi_i(l) > l^{\frac{1}{2}}$ $(i = 1,2)$, then Lemma 9 (1) ensures that x_λ^*, y_λ^* are strictly increasing in λ, that is $0 < \lambda_1 < \lambda_2$ can guarantee $x_{\lambda_1}^* < x_{\lambda_2}^*$, $y_{\lambda_1}^* < y_{\lambda_2}^*$. If there exists $\kappa \in (0,1)$, such that $\psi_i(l) > l^\kappa$ $(i = 1,2)$, then Lemma 9 (2) tells us that x_λ^*, y_λ^* are continuous in λ, that is, $\lambda \to \lambda_0 (\lambda_0 > 0)$ can ensure $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\|y_\lambda^* - y_{\lambda_0}^*\| \to 0$. If there exists $\kappa \in (0, \frac{1}{2})$, such that $\psi_i(l) > l^\kappa$ $(i = 1,2)$, then Lemma 9 (3) tells us that $\lim_{\lambda\to+\infty} \|x_\lambda^*\| = \infty$, $\lim_{\lambda\to+\infty} \|y_\lambda^*\| = \infty$, and $\lim_{\lambda\to 0^+} \|x_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|y_\lambda^*\| = 0$.

Therefore, the proof of Theorem 1 is completed. □

4 Example

In this section, we present a simple example to explain the main results.

Example 1 We consider the following fractional differential equation boundary value problem

$$\begin{cases} D_{0^+}^{2.25}x(t) + \lambda e^{-t}f(t, x(t), D_{0^+}^{0.5}x(t), y(t), D_{0^+}^{0.5}y(t)) + \lambda e^{-2t}m(t, D_{0^+}^{0.5}x(t)) = 0, \\ D_{0^+}^{1.8}y(t) + e^{-3t}g(t, x(t)) = 0, \quad t \in [0,+\infty), \\ D_{0^+}^{0.5}x(0) = 0, \qquad D_{0^+}^{1.25}x(+\infty) = \sum_{i=1}^\infty a_i D_{0^+}^{0.5}x(\xi_i), \\ I_{0^+}^{0.2}y(0) = 0, \qquad D_{0^+}^{0.8}y(+\infty) = \sum_{i=1}^\infty b_i I_{0^+}^{0.1}y(\eta_i), \end{cases} \tag{31}$$

where $\alpha = 2.25$, $\beta = 1.8$, $\gamma = 0.5$, $\sigma = 0.6$, $a_i = \frac{1}{4^i}$, $b_i = \frac{1}{5^i}$, $\xi_i = 1 - \frac{1}{2i+2}$, $\eta_i = 1 - \frac{1}{2i+3}$, $i = 1,2,\ldots$, and

$$\begin{aligned} &p(t) = e^{-t}, \qquad q(t) = e^{-2t}, \qquad r(t) = e^{-3t}, \\ &f(t,x_1,x_2,x_3,x_4) = \phi(t,x_1,x_2,x_3,x_4) + \varphi(t,x_1,x_2,x_3,x_4), \\ &\phi(t,x_1,x_2,x_3,x_4) = \frac{x_1^{0.11} + x_2^{0.125} + x_3^{0.056} + x_4^{0.04}}{1+t^{1.25}}, \\ &\varphi(t,x_1,x_2,x_3,x_4) = \frac{x_1^{-0.066} + x_2^{-0.11} + x_3^{-0.125} + x_4^{-0.083}}{1+t^{1.25}}, \\ &g(t,x) = \frac{15x^{0.076}}{1+x^{0.076}} + 10, \qquad m(t,x) = \frac{23x^{0.22}}{1+x^{0.22}} + 1. \end{aligned}$$

Then, we have

$$\begin{aligned}\Delta_1 &= \Gamma(\alpha-\gamma) - \sum_{i=1}^{\infty} a_i \xi_i^{\alpha-\gamma-1} \\ &\geq \Gamma(1.75) - \left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^i} + \cdots\right) \approx 0.5857 > 0\end{aligned}$$

and

$$\begin{aligned}\Delta_2 &= \Gamma(\beta+\sigma-\gamma) - \sum_{i=1}^{\infty} b_i \eta_i^{\beta+\sigma-\gamma-1} \\ &\geq \Gamma(1.9) - \left(\frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^i} + \cdots\right) \approx 0.7118 > 0.\end{aligned}$$

Let us check that all the required conditions of Theorem 1 are satisfied.

(1) It is obvious that $\phi, \varphi : [0,+\infty) \times [0,+\infty)^4 \to [0,+\infty)$ are continuous. For any fixed $t \in [0,+\infty)$, $\phi(t,x_1,x_2,x_3,x_4)$ is increasing and $\varphi(t,x_1,x_2,x_3,x_4)$ is decreasing in $x_i \geq 0$ $(i = 1,2,3,4)$.

(2) Clearly, the function $g(t,x) \in C([0,+\infty) \times [0,+\infty) \to [0,+\infty))$ is increasing in x, $g(t,0) \not\equiv 0$ and $\lim_{x\to+\infty} g(x,x^{1.25}) = 25$. Moreover, there exists $\omega = 0.076 \in (0,1)$, for all $t,x \in [0,+\infty)$, we have $g(t,lx) \geq l^{0.076} g(t,x)$, $l \in (0,1)$.

(3) We observe easily that if $x_i \geq 0$ $(i = 1,2,3,4)$ are bounded, then for all $t \in [0,+\infty)$, $\phi(t,(1+t^{1.25})x_1,(1+t^{1.25})x_2,(1+t^{1.25})x_3,(1+t^{1.25})x_4)$ and $\varphi(t,(1+t^{1.25})x_1,(1+t^{1.25})x_2,(1+t^{1.25})x_3,(1+t^{1.25})x_4)$ are bounded.

(4) Clearly, the function $m(t,x) \in C([0,+\infty) \times [0,+\infty) \to [0,+\infty))$ is increasing in x, $m(t,0) \not\equiv 0$ and $\lim_{x\to+\infty} m(x,x^{0.75}) = 24$. Moreover, there exists $\delta = 0.22 \in (0,1)$, for all $t,x \in [0,+\infty)$, we have $m(t,lx) \geq l^{0.22} m(t,x)$, $l \in (0,1)$.

(5) For any $l \in (0,1)$ and t, x_i $(i = 1,2,3,4) \in [0,+\infty)$, taking $\nu = 0.125 \in (0,1)$, such that

$$\begin{aligned}&\phi(t,lx_1,lx_2,lx_3,lx_4) \geq l^{0.125}\phi(t,x_1,x_2,x_3,x_4), \\ &\varphi\left(t,l^{-1}x_1,l^{-1}x_2,l^{-1}x_3,l^{-1}x_4\right) \geq l^{0.125}\varphi(t,x_1,x_2,x_3,x_4).\end{aligned}$$

(6) The functions p, q, and r satisfy

$$\begin{aligned}&0 < \int_0^{+\infty} e^{-s}\,ds = 1 < +\infty, \qquad 0 < \int_0^{+\infty} e^{-2s}\,ds = \frac{1}{2} < +\infty, \\ &0 < \int_0^{+\infty} e^{-3s}\,ds = \frac{1}{3} < +\infty.\end{aligned}$$

Hence, all the conditions of Theorem 1 are satisfied. Hence, we can claim that for $\lambda > 0$, there exists a unique positive solution $(x_\lambda^*, y_\lambda^*)$ of problem (31) in P_h, and for any initial value $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned}x_n(t) = I_{0^+}^{0.5}\bigg[&\int_0^{+\infty} \lambda G(t,s)e^{-s} f\left(s,x_{n-1}(s),D_{0^+}^{0.5}x_{n-1}(s),y_{n-1}(s),D_{0^+}^{0.5}y_{n-1}(s)\right) ds \\ &+ \int_0^{+\infty} \lambda G(t,s)e^{-2s} m\left(s,D_{0^+}^{0.5}x_{n-1}(s)\right) ds\bigg],\end{aligned}$$

$$y_n(t) = I_{0^+}^{0.5}\left[\int_0^{+\infty} K(t,s)e^{-3s}g\big(s,x_{n-1}(s)\big)\,ds\right], \quad n = 1,2,\ldots,$$

by Theorem 1, we have $x_n(t) \to x_\lambda^*(t)$, $y_n(t) \to y_\lambda^*(t)$ as $n \to \infty$.

Furthermore, since $\psi_1(l) = l^{0.125} > l^{\frac{1}{2}}$, $\psi_2(l) = l^{0.22} > l^{\frac{1}{2}}$, $l \in (0,1)$. We find from Theorem 1 that x_λ^*, y_λ^* are strictly increasing in λ, that is $0 < \lambda_1 < \lambda_2$ ensures $x_{\lambda_1}^* < x_{\lambda_2}^*$, $y_{\lambda_1}^* < y_{\lambda_2}^*$. Taking $\kappa \in (0.23,1)$, and $\psi_i(l) > l^\kappa$ $(i = 1,2)$, $l \in (0,1)$. Using Theorem 1, we know that x_λ^*, y_λ^* are continuous in λ, that is, $\lambda \to \lambda_0(\lambda_0 > 0)$ ensures $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$, $\|y_\lambda^* - y_{\lambda_0}^*\| \to 0$. Taking $\kappa \in (0.23,0.5)$ and $\psi_i(l) > l^\kappa$ $(i = 1,2)$, $l \in (0,1)$, we find from Theorem 1 that $\lim_{\lambda\to+\infty}\|x_\lambda^*\| = \infty$, $\lim_{\lambda\to+\infty}\|y_\lambda^*\| = \infty$ and $\lim_{\lambda\to0^+}\|x_\lambda^*\| = 0$, $\lim_{\lambda\to0^+}\|y_\lambda^*\| = 0$.

5 Conclusion

In this paper, by using the fixed-point theorem of mixed monotone operators, we study the existence and uniqueness of positive solutions to the boundary value problem of the fractional differential equation system on infinite intervals with infinite-point boundary conditions. The results obtained in this paper show that the unique positive solution has good properties: continuity, monotonicity, iteration, and approximation. It is worth pointing out that this paper generalizes the boundary conditions and intervals. Compared with the existing literature, this paper has a more general form and more accurate results and can be widely used in physics, chemistry, electrical networks, economics, rheology, and other fields.

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Author contributions
Yang Yu and Qi Ge wrote the main manuscript text. All authors have equal contributions. All authors reviewed the manuscript.

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