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Application of double Sumudu-generalized Laplace decomposition method and two-dimensional time-fractional coupled Burger's equation

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Abstract

The current paper concentrates on discovering the exact solutions of the time-fractional regular and singular coupled Burger's equations by involving a new technique known as the double Sumudu-generalized Laplace and Adomian decomposition method. Furthermore, some theorems of the double Sumudu-generalized Laplace properties are proved. Further, the offered method is a powerful tool for solving an enormous number of problems. The precision of the technique is evaluated with the aid of some examples, this method offers a solution precisely and successfully in a series form with smoothly calculated coefficients. The relation between both the approximate and exact solution is represented by a graph to display the high speed of this method's convergence.

Keywords: Double Sumudu-generalized Laplace; Double Sumudu transform; Inverse double Sumudu-generalized Laplace; Time-Fractional Coupled Burger's equation; Decomposition methods

1 Introduction

Burger's equation is one of the fundamental and essential nonlinear partial differential equations (PDE) containing diffusive properties and nonlinear expansion effects. Burger's equation was improved as a model of disorderly fluid movement. The fractional Burger's equation has received much interest and the solution to this problem becomes essential for mathematicians and physical phenomena. This problem has been found to demonstrate various types of events, for instance, a mathematical model of turbulence and an approximate theorem of flow through a trauma wave traveling in a viscous liquid [1, 2]. The authors in [3] introduced a semianalytical method that is called the local fractional Laplace homotopy analysis method to solve wave equations with local fractional derivatives and the authors used the same method to solve differential equations involving local fractional derivatives based on the local fractional calculus [4]. The Shehu transform and a semianalytical method have been used to solve multidimensional fractional diffusion equations [5]. The numerical solution of three-dimensional coupled Burger's Equations

has been studied by the Laplace decomposition method in [6, 7]. In recent years, substantial confirmation was offered on the Laplace decomposition method and its changes for discussing mathematical problems [8, 9]. In a previous study, the authors recommended various kinds of approximation and exact technique to solve fractional Burger's equation methods [10–12]. The researchers in [13] suggested the variational iteration method to gain Burger's equation. In [14], the Laplace decomposition method (LDM) was applied to determine the solution of two-dimensional nonlinear Burger's equations. The authors in [15] proposed a modification of the double Laplace decomposition method to obtain an analytical approximation solution of a coupled system of Burger's equation. There are many approaches where one can obtain a series of solutions, such as the Modified Laplace variational iteration method [16], and the He–Laplace method [17]. The approximate solution of wave problems in multidimensional orders was studied by applying the Aboodh homotopy integral transform method (AHITM), see [18]. The authors in [19] used the Yang Transform to obtain the approximation solution of nonlinear time-fractional Klein–Gordon equations. The authors in [20] employed the Fountain theorem and the symmetric Mountain-Pass theorem to study the novel trinonlocal Kirchhoff problem. The main aim of this paper is to offer a new hybrid of a double Sumudu-Generalized Laplace Transform to determine the exact solutions of the time-fractional regular and singular coupled Burger's equations. Finally, examples are given to clarify the proposed technique. Definitions will be recalled; the double Sumudu transform and the Generalized Laplace Transform that are useful in this article.

The Double Sumudu transform of the function $\psi(\chi,\sigma)$ is determined by $\Psi(\mu_1,\mu_2)$ in the following definition.

Definition 1 [21] let $\psi(\chi,\sigma)$ be a function we define as the double Sumudu Transform of function $\psi(\chi,\sigma)$, $\sigma,\chi\in\mathbb{R}^+$ is given by

$$\Psi(\mu_1,\mu_2)=S_2\big[\psi(\chi,\sigma)\big]=\int_0^\infty\int_0^\infty\frac{1}{u\nu}e^{-(\frac{\chi}{u}+\frac{\sigma}{\nu})}\psi(\chi,\sigma)\,d\chi\,d\sigma.$$

The generalized Laplace transform of the function $\psi(t)$ is given by G_α in the following definition.

Definition 2 If $\psi(t)$ is an integrable function defined for all $t\geq 0$, its generalized Laplace transform G_α is the integral of $\psi(t)$ times $s^\alpha e^{-\frac{t}{s}}$ from $t=0$ to ∞. It is a function of s, say $\Psi(s)$, and is denoted by $G_\alpha(\psi)$; thus

$$\Psi(s)=G_t(\psi)=s^\alpha\int_0^\infty\psi(t)e^{-\frac{t}{s}}\,dt,$$

where, $s\in\mathbb{C}$ and $\alpha\in Z$, for more details see [22].

Definition 3 [23–25]. The Caputo time-fractional derivative operator of order $\beta>0$ is presented by

$$D_t^\beta\psi(\chi,t)=\begin{cases}\frac{1}{\Gamma(m-\beta)}\int_0^t(t-\tau)^{m-\beta-1}\frac{\partial^m\psi(\chi,\tau)}{\partial\tau^m}\,d\tau, & m-1<\beta<m.\\ \frac{\partial^m\psi(\chi,t)}{\partial t^m}, \quad \text{for } m=\beta\in\mathbb{N}\end{cases}$$

2 Main results of double Sumudu-generalized Laplace transform

The definitions and existence condition of the double Sumudu-generalized Laplace transform are presented in this section. Here, we work with the double Sumudu-generalized Laplace transform, which is defined by

$$\begin{aligned} S_\chi S_\sigma G_t\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,s) \\ &= \frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\frac{t}{s})} f(\chi,\sigma,t)\,dt\,d\sigma\,d\chi \end{aligned} \tag{1}$$

and we note that the double Sumudu-generalized Laplace transform is a hybrid between the double Sumudu transform and the generalized Laplace transform. From the definition of the double Sumudu-generalized Laplace transform, we conclude the following:

1. if we put $\alpha = 0$ and $s = \frac{1}{s}$ we obtain the double Sumudu–Laplace transform

$$\begin{aligned} S_\chi S_\sigma L_t\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,s) \\ &= \frac{1}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\sigma,t)e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+st)}\,dt\,d\sigma\,d\chi; \end{aligned} \tag{2}$$

2. if we put $\alpha = 0$ and replacing s by ϖ we obtain the double Sumudu–Yang Transform

$$\begin{aligned} S_\chi S_\sigma Y\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,\varpi) \\ &= \frac{1}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\sigma,t)e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\frac{t}{\varpi})}\,dt\,d\sigma\,d\chi; \end{aligned} \tag{3}$$

3. At $\alpha = -1$ and replacing s by μ_3 we obtain the triple Sumudu Transform

$$\begin{aligned} S_\chi S_\sigma S_t\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,\mu_3) \\ &= \frac{1}{\mu_1\mu_2\mu_3}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\sigma,t)e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\frac{t}{\mu_3})}\,dt\,d\sigma\,d\chi; \end{aligned} \tag{4}$$

4. At $\alpha = 1$ we obtain the double Sumudu-Elzaki transform

$$\begin{aligned} S_\chi S_\sigma E_t\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,s) \\ &= \frac{s}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\sigma,t)e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\frac{t}{s})}\,dt\,d\sigma\,d\chi; \end{aligned} \tag{5}$$

5. At $\alpha = -1$ and $\frac{1}{\nu} = \frac{1}{s}$ we obtain the double Sumudu–Aboodh transform

$$\begin{aligned} S_\chi S_\sigma E_t\big(f(\chi,\sigma,t)\big) &= F(\mu_1,\mu_2,\nu) \\ &= \frac{1}{\mu_1\mu_2\nu}\int_0^\infty\int_0^\infty\int_0^\infty f(\chi,\sigma,t)e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\nu t)}\,dt\,d\sigma\,d\chi. \end{aligned} \tag{6}$$

From the analysis above concerning the double Sumudu-generalized Laplace transform, we note that the hybrid of the double Sumudu-generalized Laplace transform is more generic than the above transforms. Hence, the double Sumudu-generalized Laplace decomposition method is considered the most generic amongst other related methods.

2.1 Existence condition for the double Sumudu-generalized Laplace transform

In this section, the existence conditions and definitions of the double Sumudu-generalized Laplace transform are addressed as follows:

If $f(\chi,\sigma,t)$ is an exponential order a_1, a_2, and b as $\chi\to\infty$, $\sigma\to\infty$, $t\to\infty$, and if $\exists R>0$ thence $\forall\chi>\chi$, $\forall\sigma>\sigma$ and $\forall t>T$

$$\left|f(\chi,\sigma,t)\right|\le \mathrm{R}e^{a_1\chi+a_2\sigma+bt}, \tag{7}$$

for some χ, σ and T, we can write $f(\chi,\sigma,t)$ as follows:

$$f(\chi,\sigma,t)=O\left(e^{a_1\chi+a_2\sigma+bt}\right)\quad \text{as } \sigma\to\infty,\sigma\to\infty,t\to\infty,$$

equally,

$$\lim_{\substack{\chi\to\infty\\ \sigma\to\infty\\ t\to\infty}} e^{-\frac{1}{\mu}\chi-\frac{1}{\eta}\sigma-\frac{1}{\varepsilon}t}\left|f(\chi,\sigma,t)\right| = R\lim_{\substack{\chi\to\infty\\ \sigma\to\infty\\ t\to\infty}} e^{-(\frac{1}{\lambda_1}-a_1)\chi-(\frac{1}{\lambda_2}-a_2)\sigma-(\frac{1}{\eta}-c)t}=0, \tag{8}$$

whenever $\frac{1}{\lambda_1}>a$, $\frac{1}{\eta}>c$ and $\frac{1}{\lambda_2}>b$. The function $f(\chi,\sigma,t)$ does not expand quicker than $R(\chi,\sigma,t)e^{a_1\chi+a_2\sigma+bt}$ as $\chi\to\infty$, $\sigma\to\infty$, $t\to\infty$.

Theorem 1 *The function $f(\chi,\sigma,t)$ is defined on $(0,\chi)$, $(0,\sigma)$, and $(0,T)$ and of exponential order (χ,σ,t), then the double Sumudu-generalized Laplace transform of $f(\chi,\sigma,t)$ exists for all* $\mathrm{Re}\,\frac{1}{\mu_1}>\frac{1}{\lambda_1}$, $\mathrm{Re}\,\frac{1}{\mu_2}>\frac{1}{\lambda_2}$, $\mathrm{Re}\,\frac{1}{s}>\frac{1}{\eta}$.

Proof By utilizing Eq. (1) and Eq. (7), we obtain

$$\begin{aligned}\left|F(\mu_1,\mu_2,s)\right| &= \left|\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty e^{-(\frac{\chi}{\mu_1}+\frac{\sigma}{\mu_2}+\frac{t}{s})}f(\chi,\sigma,t)\,d\chi\,d\sigma\,dt\right|\\ &\le R\left|\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{(\frac{1}{\psi_1}-a)\chi-(\frac{1}{\psi_2}-b)\sigma-(\frac{1}{\nu}-c)t}\,d\chi\,d\sigma\,dt\right|\\ &= \frac{Rs^{\alpha+1}}{(1-a\mu_1)(1-c\mu_2)(1-bs)}.\end{aligned} \tag{9}$$

On using the condition $\mathrm{Re}\,\frac{1}{\mu_1}>\frac{1}{\lambda_1}$, $\mathrm{Re}\,\frac{1}{\mu_2}>\frac{1}{\lambda_2}$, $\mathrm{Re}\,\frac{1}{s}>\frac{1}{\eta}$, and Eq. (8), we obtain

$$\lim_{\substack{\chi\to\infty\\ \sigma\to\infty\\ t\to\infty}}\left|F(\psi_1,\psi_2,\nu)\right|=0\quad\text{or}\quad \lim_{\substack{\chi\to\infty\\ \sigma\to\infty\\ t\to\infty}}F(\psi_1,\psi_2,\nu)=0.$$

□

The inverse double Sumudu-generalized Laplace transform $S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}[S_\chi S_\sigma G_t(\psi(\chi,\sigma,t))]=\psi(\chi,\sigma,t)$ is denoted by the following formula

$$\begin{aligned}&\psi(\chi,\sigma,t)\\ &\quad=\frac{1}{(2\pi i)^3}\int_{\tau-i\infty}^{\tau-i\infty}\int_{\delta-i\infty}^{\delta-i\infty}\int_{\sigma-i\infty}^{\sigma-i\infty} e^{\frac{1}{\mu_1}\chi+\frac{1}{\mu_1}\sigma+\frac{1}{s}t}S_\chi S_\sigma G_t\left[\psi(\chi,\sigma,t)\right]ds\,d\mu_2\,d\mu_1.\end{aligned}$$

Theorem 2 *If the double Sumudu-generalized Laplace transform of the function $f(\chi,\sigma,t)$ is presented by $S_\chi S_\sigma G_t(f(\chi,\sigma,t))=F(\mu_1,\mu_2,s)$, then the double Sumudu-generalized*

Laplace transform of the functions

$$\chi\sigma f(\chi,\sigma,t),$$

is determined by

$$S_\chi S_\sigma G_t\big[\chi\sigma f(\chi,\sigma,t)\big] = \mu_1\mu_2\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 F(\mu_1,\mu_2,s)\big). \tag{10}$$

Proof By applying partial derivatives according to μ_1 for Eq. (10), we yield

$$\begin{aligned}\frac{\partial F(\mu_1,\mu_2,s)}{\partial\mu_1} &= \frac{\partial}{\partial\mu_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_1\mu_2}e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}f(\chi,\sigma,t)\,d\chi\,d\sigma\,dt,\\ &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\\ &\quad\times\left(\int_0^\infty \frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi\right)d\sigma\,dt,\end{aligned} \tag{11}$$

by handling the partial derivative inside the brackets, we obtain

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi &= \int_0^\infty\left(\frac{1}{\mu_1^3}\chi-\frac{1}{\mu_1^2}\right)e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi\\ &= \int_0^\infty \frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi\\ &\quad-\int_0^\infty \frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi,\end{aligned} \tag{12}$$

substituting Eq. (12) into Eq. (11), one can obtain the following equation

$$\begin{aligned}\frac{\partial F(\mu_1,\mu_2,s)}{\partial\mu_1} &= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi\right)d\sigma\,dt\\ &\quad-\int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}f(\chi,\sigma,t)\,d\chi\right)d\sigma\,dt\end{aligned} \tag{13}$$

and by taking derivatives according to μ_2 for Eq. (13), we achieve

$$\begin{aligned}\frac{\partial^2 F(\mu_1,\mu_2,s)}{\partial\mu_1\partial\mu_2} &= \frac{s^\alpha}{\mu_1^3}\int_0^\infty\int_0^\infty \chi e^{-(\frac{1}{\mu_1}\chi+\frac{1}{s}t)}\\ &\quad\times\left(\int_0^\infty e^{-\frac{1}{\mu_2}\sigma}\left(\frac{1}{\mu_2^3}\sigma-\frac{1}{\mu_2^2}\right)f(\chi,\sigma,t)\right)d\chi\,d\sigma\,dt\\ &\quad-\frac{s^\alpha}{\mu_1}\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{s}t)}\\ &\quad\times\left(\int_0^\infty e^{-\frac{1}{\mu_2}\sigma}\left(\frac{1}{\mu_2^3}\sigma-\frac{1}{\mu_2^2}\right)f(\chi,\sigma,t)\right)d\chi\,d\sigma\,dt.\end{aligned} \tag{14}$$

After the arrangement, Eq. (14), becomes

$$\frac{\partial^2 F(\mu_1,\mu_2,s)}{\partial\mu_1\partial\mu_2} = \frac{1}{\mu_1^2\mu_2^2}S_\chi S_\sigma G_t\big[\chi\sigma f(\chi,\sigma,t)\big]-\frac{1}{\mu_1^2\mu_2}S_\chi S_\sigma G_t\big[\chi f(\chi,\sigma,t)\big]$$

$$-\frac{1}{\mu_1\mu_2^2}S_\chi S_\sigma G_t\big[\sigma f(\chi,\sigma,t)\big]+\frac{1}{\mu_1\mu_2}S_\chi S_\sigma G_t\big[f(\chi,\sigma,t)\big], \tag{15}$$

by arranging the above equation, we obtain

$$\begin{aligned} S_\chi S_\sigma G_t\big[\chi\sigma(\chi,\sigma,t)\big] &= \mu_1^2\mu_2^2\frac{\partial^2 F(\mu_1,\mu_2,s)}{\partial\mu_1\partial\mu_2}+\mu_1^2\mu_2\frac{\partial F(\mu_1,\mu_2,s)}{\partial\mu_1} \\ &\quad+\mu_1\mu_2^2\frac{\partial F(\mu_1,\mu_2,s)}{\partial\mu_2}+\mu_1\mu_2F(\mu_1,\mu_2,s), \end{aligned}$$

thence,

$$S_\chi S_\sigma G_t\big[\chi\sigma f(\chi,\sigma,t)\big]=\mu_1\mu_2\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2F(\mu_1,\mu_2,s)\big).$$

The proof is completed. □

The double Sumudu-generalized Laplace transform of the function $\psi(\chi,\sigma,t)$ is determined by $S_\chi S_\sigma G_t[\psi(\chi,\sigma,t)]=\Psi(\mu_1,\mu_2,s)$ then, the double Sumudu-generalized Laplace transform of $\frac{\partial\psi}{\partial\chi}$, $\frac{\partial^2\psi}{\partial\chi^2}$, $D_t^\beta\psi$ is presented as

$$S_\chi S_\sigma G_t\left[\frac{\partial\psi}{\partial\chi}\right]=\frac{\Psi(\mu_1,\mu_2,s)-\Psi(0,\mu_2,s)}{\mu_1}, \tag{16}$$

$$S_\chi S_\sigma G_t\left(\frac{\partial^2\psi}{\partial\chi^2}\right)=\frac{\Psi(\mu_1,\mu_2,s)}{\mu_1^2}-\frac{\psi(0,\mu_2,s)}{\mu_1^2}-\frac{\psi_t(0,\mu_2,s)}{\mu_1}, \tag{17}$$

$$S_\chi S_\sigma G_t\left[\frac{\partial\psi}{\partial\sigma}\right]=\frac{\Psi(\mu_1,\mu_2,s)-\Psi(\mu_1,0,s)}{\mu_2},$$

$$S_\chi S_\sigma G_t\left(\frac{\partial^2\psi}{\partial\sigma^2}\right)=\frac{\Psi(\mu_1,\mu_2,s)}{\mu_2^2}-\frac{\Psi(\mu_1,0,s)}{\mu_2^2}-\frac{\psi_t(\mu_1,0,s)}{\mu_2} \tag{18}$$

and

$$S_\chi S_\sigma G_t\big[D_t^\beta\psi\big]=\frac{\Psi(\mu_1,\mu_2,s)}{s^\beta}-s^{\alpha-\beta+1}\Psi(\mu_1,\mu_2,0). \tag{19}$$

The next theorem offers the double Sumudu-generalized Laplace transform of the partial derivatives $\chi D_t^\beta\psi$ and $\sigma D_t^\beta\psi$.

Theorem 3 *The double Sumudu-generalized Laplace transform of the fractional partial derivatives $\chi D_t^\beta\psi$ and $\sigma D_t^\beta\psi$ is achieved by*

$$\begin{aligned} S_\chi S_\sigma G_t\big[\chi D_t^\beta\psi\big] &= \frac{\mu_1\mu_2}{s^\beta}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Psi(\mu_1,\mu_2,s)\big) \\ &\quad-\mu_1\mu_2s^{\alpha-\beta+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Psi(\mu_1,\mu_2,0)\big) \end{aligned} \tag{20}$$

and

$$\begin{aligned} S_\chi S_\sigma G_t\big[\sigma D_t^\beta\psi\big] &= \frac{\mu_2}{s^\beta}\frac{\partial}{\partial\mu_1}\big(\mu_2\Psi(\mu_1,\mu_2,s)\big) \\ &\quad-\mu_2s^{\alpha-\beta+1}\frac{\partial}{\partial\mu_2}\big(\mu_1\Psi(\mu_1,\mu_2,0)\big). \end{aligned} \tag{21}$$

Proof By employing partial derivatives according to μ_1 for Eq. (1), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\mu_1}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)\\
&\quad=\frac{\partial}{\partial\mu_1}\int_0^\infty\int_0^\infty\int_0^\infty\frac{s^\alpha}{\mu_1\mu_2}e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}D_t^\beta\psi\,d\chi\,d\sigma\,dt,\\
&\quad=\int_0^\infty\int_0^\infty\frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty\frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt
\end{aligned}\tag{22}$$

and the partial derivative within the brackets can be calculated as follows:

$$\begin{aligned}
\int_0^\infty\frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi&=\int_0^\infty\left(\frac{1}{\mu_1^3}\chi-\frac{1}{\mu_1^2}\right)e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\\
&=\int_0^\infty\frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\\
&\quad-\int_0^\infty\frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi,
\end{aligned}\tag{23}$$

by putting Eq. (23) into Eq. (22), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\mu_1}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)\\
&\quad=\int_0^\infty\int_0^\infty\frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty\frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt\\
&\qquad-\int_0^\infty\int_0^\infty\frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty\frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt,
\end{aligned}\tag{24}$$

therefore, Eq. (24) becomes

$$\begin{aligned}
&\frac{\partial}{\partial\mu_1}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)\\
&\quad=\frac{1}{\mu_1^2}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\chi D_t^\beta\psi\,d\chi\,d\sigma\,dt\right)\\
&\qquad-\frac{1}{\mu_1}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}D_t^\beta\psi\,d\chi\,d\sigma\,dt\right),
\end{aligned}\tag{25}$$

hence,

$$\frac{\partial}{\partial\mu_1}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)=\frac{1}{\mu_1^2}S_\chi S_\sigma G_t\left[\chi D_t^\beta\psi\right]-\frac{1}{\mu_1}S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\tag{26}$$

and by arranging the above equation, we will obtain the proof of Eq. (20) as follows

$$\begin{aligned}
S_\chi S_\sigma G_t\left[\chi D_t^\beta\psi\right]&=\frac{\mu_1}{s^\beta}\frac{\partial}{\partial\mu_1}\left(\mu_1\Psi(\mu_1,\mu_2,s)\right)\\
&\quad-\mu_1 s^{\alpha-\beta+1}\frac{\partial}{\partial\mu_1}\left(\mu_1\Psi(\mu_1,\mu_2,0)\right).
\end{aligned}$$

Similarly, we can prove Eq. (21). □

The double Sumudu-generalized Laplace transform of the partial derivatives is presented in the upcoming theorem:

Theorem 4 *The double Sumudu-generalized Laplace transform of the fractional partial derivatives $\chi\sigma D_t^\beta \psi$ is determined by*

$$S_\chi S_\sigma G_t\big[\chi\sigma D_t^\beta \psi\big] = \frac{\mu_1\mu_2}{s^\beta}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Psi(\mu_1,\mu_2,s)\big) - \mu_1\mu_2 s^{\alpha-\beta+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Psi(\mu_1,\mu_2,0)\big). \tag{27}$$

Proof By taking partial derivatives according to μ_1 for Eq. (1), we have

$$\begin{aligned}
&\frac{\partial}{\partial\mu_1}\big(S_\chi S_\sigma G_t\big[D_t^\beta\psi\big]\big)\\
&\quad= \frac{\partial}{\partial\mu_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_1\mu_2}e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}D_t^\beta\psi\,d\chi\,d\sigma\,dt,\\
&\quad= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt
\end{aligned} \tag{28}$$

and we calculate the partial derivative inside brackets as follows:

$$\begin{aligned}
\int_0^\infty \frac{\partial}{\partial\mu_1}\frac{1}{\mu_1}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi &= \int_0^\infty\left(\frac{1}{\mu_1^3}\chi-\frac{1}{\mu_1^2}\right)e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\\
&= \int_0^\infty \frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\\
&\quad- \int_0^\infty \frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi.
\end{aligned} \tag{29}$$

Putting Eq. (29) into Eq. (28), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\mu_1}\big(S_\chi S_\sigma G_t\big[D_t^\beta\psi\big]\big)\\
&\quad= \int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt\\
&\qquad- \int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt,
\end{aligned} \tag{30}$$

the partial derivative with respect to μ_2 for Eq. (30) is calculated as the following:

$$\begin{aligned}
&\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(S_\chi S_\sigma G_t\big[D_t^\beta\psi\big]\big)\\
&\quad= \frac{\partial}{\partial\mu_2}\left(\int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^3}\chi e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt\right)\\
&\qquad- \frac{\partial}{\partial\mu_2}\left(\int_0^\infty\int_0^\infty \frac{s^\alpha}{\mu_2}e^{-(\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\left(\int_0^\infty \frac{1}{\mu_1^2}e^{-\frac{1}{\mu_1}\chi}D_t^\beta\psi\,d\chi\right)d\sigma\,dt\right),
\end{aligned} \tag{31}$$

therefore, Eq. (31) becomes

$$\begin{aligned}
&\frac{\partial^2}{\partial\mu_1\partial\mu_2}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)\\
&\quad=\frac{1}{\mu_1^2\mu_2^2}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\chi\sigma D_t^\beta\psi\,d\chi\,d\sigma\,dt\right)\\
&\qquad+\frac{1}{\mu_1\mu_2}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}D_t^\beta\psi\,d\chi\,d\sigma\,dt\right)\\
&\qquad-\frac{1}{\mu_1\mu_2^2}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\sigma D_t^\beta\psi\,d\chi\,d\sigma\,dt\right)\\
&\qquad-\frac{1}{\mu_1^2\mu_2}\left(\frac{s^\alpha}{\mu_1\mu_2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{\mu_1}\chi+\frac{1}{\mu_2}\sigma+\frac{1}{s}t)}\chi D_t^\beta\psi\,d\chi\,d\sigma\,dt\right),
\end{aligned}\tag{32}$$

thence

$$\begin{aligned}
&\frac{\partial^2}{\partial\mu_1\partial\mu_2}\left(S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\right)\\
&\quad=\frac{1}{\mu_1^2\mu_2^2}S_\chi S_\sigma G_t\left[\chi\sigma D_t^\beta\psi\right]+\frac{1}{\mu_1\mu_2}S_\chi S_\sigma G_t\left[D_t^\beta\psi\right]\\
&\qquad-\frac{1}{\mu_1\mu_2^2}S_\chi S_\sigma G_t\left[\sigma D_t^\beta\psi\right]-\frac{1}{\mu_1^2\mu_2}S_\chi S_\sigma G_t\left[\chi D_t^\beta\psi\right]
\end{aligned}\tag{33}$$

and one can rearrange Eq. (33), to prove Eq. (27)

$$\begin{aligned}
S_\chi S_\sigma G_t\left[\chi\sigma D_t^\beta\psi\right]&=\frac{\mu_1\mu_2}{s^\beta}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\left(\mu_1\mu_2\Psi(\mu_1,\mu_2,s)\right)\\
&\quad-\mu_1\mu_2 s^{\alpha-\beta+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\left(\mu_1\mu_2\Psi(\mu_1,\mu_2,0)\right).
\end{aligned}$$

□

3 Double Sumudu-generalized Laplace decomposition method and two-dimensional time-fractional coupled Burger's equation

This section aims to make use of the double Sumudu-generalized Laplace decomposition method (DSGLTDM) to solve the two-dimensional time-fractional coupled Burger's equation. In the upcoming analysis, we deem the two-dimensional fractional coupled Burger's equation to be:

$$\begin{aligned}
&D_t^\beta\psi+\psi\psi_\chi+\phi\psi_\sigma=\frac{1}{\Re}(\psi_{\chi\chi}+\psi_{\sigma\sigma})\\
&D_t^\beta\phi+\psi\phi_\chi+\phi\phi_\sigma=\frac{1}{\Re}(\phi_{\chi\chi}+\phi_{\sigma\sigma})\\
&n-1<\beta<n;
\end{aligned}\tag{34}$$

with the following conditions

$$\psi(\chi,\sigma,0)=f_1(\chi,\sigma),\qquad \phi(\chi,\sigma,0)=g_1(\chi,\sigma),\tag{35}$$

where $D_t^\beta=\frac{\partial^\beta}{\partial t^\beta}$ stands for the fractional Caputo derivative, $\Re$ is the Reynolds number, and the velocity components are determined by $\psi(\chi,\sigma,t)$ and $\phi(\chi,\sigma,t)$ in the χ and σ

directions, respectively. The two-dimensional coupled Burger's equations are the same as the incompressible Navier–Stokes equations with the pressure-gradient terms removed. With the purpose to gain the solution of Eq. (34), first, operating the double Sumudu-generalized Laplace for Eq. (34) and using the double Sumudu transform for Eq. (35) we gain

$$\begin{aligned} S_\chi S_\sigma G_t\left[D_t^\beta \psi\right] &= S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\psi_{\chi\chi} + \psi_{\sigma\sigma})\right] \\ &\quad - S_\chi S_\sigma G_t[\psi\psi_\chi + \phi\psi_\sigma] \end{aligned} \tag{36}$$

and

$$\begin{aligned} S_\chi S_\sigma G_t\left[D_t^\beta \phi\right] &= S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\phi_{\chi\chi} + \phi_{\sigma\sigma})\right] \\ &\quad - S_\chi S_\sigma G_t[\psi\phi_\chi + \phi\phi_\sigma], \end{aligned} \tag{37}$$

by putting Eq. (19) into Eq. (36) and Eq. (37), we obtain

$$\begin{aligned} \frac{\Psi(\mu_1,\mu_2,s)}{s^\beta} &= s^{\alpha-\beta+1}F_1(\mu_1,\mu_2) \\ &\quad + S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\psi_{\chi\chi} + \psi_{\sigma\sigma})\right] \\ &\quad - S_\chi S_\sigma G_t[\psi\psi_\chi + \phi\psi_\sigma] \end{aligned} \tag{38}$$

and

$$\begin{aligned} \frac{\Phi(\mu_1,\mu_2,s)}{s^\beta} &= s^{\alpha-\beta+1}G_1(\mu_1,\mu_2) \\ &\quad + S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\phi_{\chi\chi} + \phi_{\sigma\sigma})\right] \\ &\quad - S_\chi S_\sigma G_t[\psi\phi_\chi + \phi\phi_\sigma], \end{aligned} \tag{39}$$

therefore, by rearranging Eq. (38) and Eq. (39) we obtain

$$\begin{aligned} \Psi(\mu_1,\mu_2,s) &= s^{\alpha+1}F_1(\mu_1,\mu_2) \\ &\quad + s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\psi_{\chi\chi} + \psi_{\sigma\sigma})\right] \\ &\quad - s^\beta S_\chi S_\sigma G_t[\psi\psi_\chi + \phi\psi_\sigma], \end{aligned} \tag{40}$$

and

$$\begin{aligned} \Phi(\mu_1,\mu_2,s) &= s^{\alpha+1}G_1(\mu_1,\mu_2) \\ &\quad + s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\phi_{\chi\chi} + \phi_{\sigma\sigma})\right] \\ &\quad - s^\beta S_\chi S_\sigma G_t[\psi\phi_\chi + \phi\phi_\sigma] \end{aligned} \tag{41}$$

and by employing the inverse double Sumudu-generalized Laplace for Eq. (38) and Eq. (39), we yield

$$\begin{aligned}\psi(\chi,\sigma,t) &= f_1(\chi,\sigma)\\ &\quad + S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^{\beta}S_{\chi}S_{\sigma}G_t\left[\frac{1}{\Re}(\psi_{\chi\chi}+\psi_{\sigma\sigma})\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^{\beta}S_{\chi}S_{\sigma}G_t[A_n+B_n]\right]\end{aligned} \tag{42}$$

and

$$\begin{aligned}\phi(\chi,\sigma,t) &= g_1(\chi,\sigma)\\ &\quad + S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^{\beta}S_{\chi}S_{\sigma}G_t\left[\frac{1}{\Re}(\phi_{\chi\chi}+\phi_{\sigma\sigma})\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^{\beta}S_{\chi}S_{\sigma}G_t[C_n+D_n]\right],\end{aligned} \tag{43}$$

where some terms of the Adomian polynomials A_n B_n, C_n, and D_n are determined by

$$\begin{aligned}&A_0 = \psi_0\psi_{0\chi}, \qquad A_1 = \psi_0\psi_{1\chi}+\psi_1\psi_{0\chi},\\ &A_2 = \psi_0\psi_{2\chi}+\psi_1\psi_{1\chi}+\psi_2\psi_{0\chi},\\ &A_3 = \psi_0\psi_{3\chi}+\psi_1\psi_{2\chi}+\psi_2\psi_{1\chi}+\psi_3\psi_{0\chi},\end{aligned} \tag{44}$$

$$\begin{aligned}&B_0 = \phi_0\psi_{0\sigma}, \qquad B_1 = \phi_0\psi_{1\sigma}+\phi_1\psi_{0\sigma},\\ &B_2 = \phi_0\psi_{2\sigma}+\phi_1\psi_{1\sigma}+\phi_2\psi_{0\sigma},\\ &B_3 = \phi_0\psi_{3\sigma}+\phi_1\psi_{2\sigma}+\phi_2\psi_{1\sigma}+\phi_3\psi_{0\sigma},\end{aligned} \tag{45}$$

$$\begin{aligned}&C_0 = \psi_0\phi_{0\chi}, \qquad C_1 = \psi_0\phi_{1\chi}+\psi_1\phi_{0\chi},\\ &C_2 = \psi_0\phi_{2\chi}+\psi_1\phi_{1\chi}+\psi_2\phi_{0\chi},\\ &C_3 = \psi_0\phi_{3\chi}+\psi_1\phi_{2\chi}+\psi_2\phi_{1\chi}+\psi_3\phi_{0\chi}.\end{aligned} \tag{46}$$

$$\begin{aligned}&D_0 = \phi_0\phi_{0\sigma}, \qquad D_1 = \phi_0\phi_{1\sigma}+\phi_1\phi_{0\sigma},\\ &D_2 = \phi_0\phi_{2\sigma}+\phi_1\phi_{1\sigma}+\phi_2\phi_{0\sigma},\\ &D_3 = \phi_0\phi_{3\sigma}+\phi_1\phi_{2\sigma}+\phi_2\phi_{1\sigma}+\phi_3\phi_{0\sigma}\end{aligned} \tag{47}$$

and $S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}$ denotes the inverse double Sumudu-generalized Laplace. The double Sumudu-generalized Laplace decomposition method (DSGLTDM) defines the solutions $\psi(\chi,\sigma,t)$ and $\psi(\chi,\sigma,t)$ and is represented by the following infinite series.

$$\psi(\chi,\sigma,t) = \sum_{n=0}^{\infty}\psi_n(\chi,\sigma,t) \tag{48}$$

$$\phi(\chi,\sigma,t) = \sum_{n=0}^{\infty}\phi_n(\chi,\sigma,t). \tag{49}$$

Moreover, the nonlinear terms $\psi\psi_\chi$, $\phi\frac{\partial\psi}{\partial\sigma}$, $\psi\frac{\partial\phi}{\partial\chi}$ and $\phi\frac{\partial\phi}{\partial\sigma}$ are presented by:

$$\psi\psi_\chi = \sum_{n=0}^{\infty} A_n, \qquad \phi\frac{\partial\psi}{\partial\sigma} = \sum_{n=0}^{\infty} B_n, \qquad \psi\frac{\partial\phi}{\partial\chi} = \sum_{n=0}^{\infty} C_n, \qquad \phi\frac{\partial\phi}{\partial\sigma} = \sum_{n=0}^{\infty} D_n. \tag{50}$$

By substituting Eq. (48) and Eq. (49) into Eq. (42) and Eq. (43), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(\chi,\sigma,t) &= f_1(\chi,\sigma) + S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}\left(\sum_{n=0}^{\infty}(\psi_{n\chi\chi}+\psi_{n\sigma\sigma})\right)\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\sum_{n=0}^{\infty}(A_n+B_n)\right]\right]\end{aligned} \tag{51}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n(\chi,\sigma,t) &= g_1(\chi,\sigma) + S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}\left(\sum_{n=0}^{\infty}(\phi_{n\chi\chi}+\phi_{n\sigma\sigma})\right)\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\sum_{n=0}^{\infty}(C_n+D_n)\right]\right].\end{aligned} \tag{52}$$

By comparing both sides of Eq. (51) and Eq. (52), we obtain

$$\begin{aligned}\psi_0(\chi,\sigma,t) &= f_1(\chi,\sigma),\\ \phi_0(\chi,\sigma,t) &= g_1(\chi,\sigma).\end{aligned} \tag{53}$$

Generally, the remnant terms are presented by

$$\begin{aligned}\psi_{n+1}(\chi,\sigma,t) &= S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}\left((\psi_{n\chi\chi}+\psi_{n\sigma\sigma})\right)\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(A_n+B_n)\right]\right]\end{aligned} \tag{54}$$

and

$$\begin{aligned}\phi_{n+1}(\chi,\sigma,t) &= S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}\left((\phi_{n\chi\chi}+\phi_{n\sigma\sigma})\right)\right]\right]\\ &\quad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(C_n+D_n)\right]\right],\end{aligned} \tag{55}$$

where the inverse double Sumudu-generalized Laplace transform is determined by $S_{\zeta_1}^{-1}S_{\zeta_2}^{-1}G_s^{-1}$. We assume that the inverse exists for Eqs. (54) and (55). For the goal of illustrating the advantages and the reliability of the (DSGLTDM) we solve the two-dimensional fractional coupled Burger's equations.

Example 1 [13, 26, 27] Consider the singular two-dimensional time-fractional coupled Burger's equations to be determined by

$$D_t^\alpha\psi + \psi\psi_\chi + \phi\psi_\sigma = \psi_{\chi\chi} + \psi_{\sigma\sigma}, \qquad \chi,\sigma,t > 0,$$

$$D_t^\alpha \phi + \psi\phi_\chi + \phi\phi_\sigma = \phi_{\chi\chi} + \phi_{\sigma\sigma}, \qquad \chi,\sigma,t > 0,$$
$$n-1 < \alpha < n, \tag{56}$$

with the following conditions

$$\psi(\chi,\sigma,0) = \chi + \sigma, \qquad \phi(\chi,\sigma,0) = \chi - \sigma. \tag{57}$$

Utilizing the previous steps one can obtain

$$\begin{aligned}
\psi(\chi,\sigma,t) &= \chi + \sigma + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\psi_{\chi\chi} + \psi_{\sigma\sigma})\right]\right] \\
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t[A_n + B_n]\right], \\
\phi(\chi,\sigma,t) &= \chi - \sigma + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\frac{1}{\Re}(\phi_{\chi\chi} + \phi_{\sigma\sigma})\right]\right] \\
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t[C_n + D_n]\right].
\end{aligned} \tag{58}$$

The zeroth components ψ_0 and ϕ_0 are proposed by the Adomian method; combining the initial conditions and the sources terms as follows:

$$\begin{aligned}
\psi_0(\chi,\sigma,t) &= \chi + \sigma \\
\phi_0(\chi,\sigma,t) &= \chi - \sigma.
\end{aligned} \tag{59}$$

The remainder components ψ_{n+1}, ϕ_{n+1}, $n \geq 0$ are determined by utilizing the relation

$$\begin{aligned}
\psi_{n+1}(\chi,\sigma,t) &= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(\psi_{n\chi\chi} + \psi_{n\sigma\sigma})\right]\right] \\
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(A_n + B_n)\right]\right]
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
\phi_{n+1}(\chi,\sigma,t) &= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(\phi_{n\chi\chi} + \phi_{n\sigma\sigma})\right]\right] \\
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(C_n + D_n)\right]\right],
\end{aligned} \tag{61}$$

for $n = 0, 1, 2, \ldots$, so, at $n = 0$

$$\begin{aligned}
\psi_1(\chi,\sigma,t) &= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\left((\psi_{0\chi\chi} + \psi_{0\sigma\sigma})\right)\right]\right] \\
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(A_0 + B_0)\right]\right] \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[(2\chi)\right]\right] \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[2s^{\alpha+\beta+1}\mu_1\right] \\
\psi_1(\chi,\sigma,t) &= \frac{-2\chi t^\beta}{\Gamma(\alpha+1)}
\end{aligned}$$

and

$$\phi_1(\chi,\sigma,t) = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^\beta S_\chi S_\sigma G_t\left[\left((\phi_{0\chi\chi} + \phi_{0\sigma\sigma})\right)\right]\right]$$

$$
\begin{aligned}
&\quad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^{\beta} S_{\chi} S_{\sigma} G_t\left[(C_0 + D_0)\right]\right] \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[s^{\beta} S_{\chi} S_{\sigma} G_t\left[(2\chi)\right]\right] \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[2 s^{\alpha+\beta+1} \mu_2\right] \\
\phi_1(\chi, \sigma, t) &= \frac{-2\sigma t^{\beta}}{\Gamma(\alpha+1)},
\end{aligned}
$$

likely, at $n = 1$, we have

$$
\begin{aligned}
\psi_2 &= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left((\psi_0 \psi_{1\chi} + \psi_1 \psi_{0\chi} + \phi_0 \psi_{1\sigma} + \phi_1 \psi_{0\sigma})\right)\right) \\
&\quad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left((\psi_{1\chi\chi} + \psi_{1\sigma\sigma})\right)\right), \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left(\frac{-t^{\beta}}{\Gamma(\alpha+1)}(4\chi + 4\sigma)\right)\right) \\
&= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left((4\mu_1 + 4\mu_2) s^{\alpha+2\beta+1}\right) \\
&= \frac{4(\chi+\sigma) t^{2\beta}}{\Gamma(2\alpha+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_2 &= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t (D_1 + E_1)\right) \\
&\quad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t (\phi_{1\chi\chi} + \phi_{1\sigma\sigma})\right) \\
&= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left(\frac{-t^{\beta}}{\Gamma(\beta+1)}(-4\chi + 4\sigma)\right)\right) \\
&= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left((-4\mu_1 + 4\mu_2) s^{\alpha+2\beta+1}\right) \\
&= \frac{4(\chi-\sigma) t^{2\beta}}{\Gamma(2\alpha+1)},
\end{aligned}
$$

at $n = 2$, we produce

$$
\begin{aligned}
\psi_3 &= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t (\psi_{0\chi} \psi_2 + \psi_{1\chi} \psi_1 + \psi_{\chi 2} \psi_0 + \phi_0 \psi_{\sigma 2} + \phi_1 \psi_{\sigma 1} + \phi_2 \psi_{\sigma 0})\right) \\
&\quad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left((\psi_{\chi\chi 2} + \psi_{\sigma\sigma 2})\right)\right), \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left(\left(16\chi + \frac{4\chi\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right)\right) \\
&= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(16\mu_1 s^{\alpha+3\beta+1} + \frac{4\Gamma(2\beta+1)\mu_1 s^{\alpha+3\beta+1}}{(\Gamma(\beta+1))^2}\right) \\
&= \left(-16\chi - \frac{4\Gamma(2\beta+1)\chi}{(\Gamma(\beta+1))^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

similar to

$$
\begin{aligned}
\phi_3 &= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t (D_2 + E_2)\right) \\
&\quad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left((\phi_{2\chi\chi} + \phi_{2\sigma\sigma})\right)\right)
\end{aligned}
$$

$$= -S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(s^{\beta} S_{\chi} S_{\sigma} G_t\left(\left(16\sigma + \frac{4\sigma\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right)\right)$$
$$= \left(-16\sigma - \frac{4\sigma\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)}.$$

Thus, the solution of Eq. (56) is

$$\psi(\chi,\sigma,t) = \sum_{n=0}^{\infty} u_n = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots$$
$$\psi(\chi,\sigma,t) = \chi + \sigma - \frac{2\chi t^{\beta}}{\Gamma(\beta+1)} + \frac{4(\chi+\sigma)t^{2\beta}}{\Gamma(2\beta+1)} - \left(16\chi + \frac{4\chi\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots$$

and

$$\phi(\chi,\sigma,t) = \sum_{n=0}^{\infty} v_n = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\phi(\chi,\sigma,t) = \chi - \sigma - \frac{2\sigma t^{\beta}}{\Gamma(\beta+1)} + \frac{4(\chi+\sigma)t^{2\beta}}{\Gamma(2\beta+1)} - \left(16\sigma + \frac{4\sigma\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots,$$

at $\beta = 1$ the solution of the above equation becomes

$$\begin{aligned}\psi(\chi,\sigma,t) &= \chi + \sigma - 2\chi t + 2(\chi+\sigma)t^2 - 4\chi t^3 + 4(\chi+\sigma)t^4 - 8\chi t^5 \\ &\quad + 8(\chi+\sigma)t^6 - 16\chi t^7 + 16(\chi+\sigma)t^8 \cdots \\ &= \chi\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right) + \sigma\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right) \\ &\quad - 2\chi t\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right)\end{aligned}$$
$$\psi(\chi,\sigma,t) = \frac{(\chi + \sigma - 2\chi t)}{1 - 2t^2}$$

and

$$\begin{aligned}\phi(\chi,\sigma,t) &= \chi - \sigma - \frac{2\sigma t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4(\chi-\sigma)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \left(-16\sigma - \frac{4\sigma\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \\ &= (\chi - \sigma) - 2\sigma t + 2(\chi-\sigma)t^2 - 4\sigma t^3 + 4(\chi-\sigma)t^4 - 8\sigma t^5 + 8(\chi-\sigma)t^6 \\ &\quad - 16\sigma t^7 + 16(\chi-\sigma)t^8 \cdots \\ &= \chi\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right) - \sigma\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right) \\ &\quad - 2\sigma t\left(1 + 2t^2 + 4t^4 + 8t^6 + \cdots\right)\end{aligned}$$
$$\phi(\chi,\sigma,t) = \frac{(\chi - \sigma - 2\sigma t)}{1 - 2t^2}.$$

We achieved the same results that were presented in [13, 26, 27].

Table 1 Comparison between the exact and approximation solutions for $\psi(\chi,\sigma,t)$

Exact $\beta = 1$	The method $\beta = 0.95$	Error	The method $\beta = 0.99$	Error
0	0	0	0	0
0.8000	0.8777	0.0777	0.8148	0.0148
1.6000	1.7554	0.1554	1.6295	0.0295
2.4000	2.6331	0.2331	2.4443	0.0443
3.2000	3.5108	0.3108	3.2591	0.0591
4.0000	4.3886	0.3886	4.0739	0.0739
4.8000	5.2663	0.4663	4.8886	0.0886
5.6000	6.1440	0.5440	5.7034	0.1034
6.4000	7.0217	0.6217	6.5182	0.1182
7.2000	7.8994	0.6994	7.3330	0.1330
8.0000	8.7771	0.7771	8.1477	0.1477

Table 2 Comparison between the exact and approximation solutions for $\phi(\chi,\sigma,t)$

Exact $\beta = 1$	The method $\beta = 0.95$	Error	The method $\beta = 0.99$	Error
0	0	0	0	0
−0.6000	−0.6777	0.0777	−0.6148	0.0148
−1.2000	−1.3554	0.1554	−1.2295	0.0295
−1.8000	−2.0331	0.2331	−1.8443	0.0443
−2.4000	−2.7108	0.3108	−2.4591	0.0591
−3.0000	−3.3886	0.3886	−3.0739	0.0739
−3.6000	−4.0663	0.4663	−3.6886	0.0886
−4.2000	−4.7440	0.5440	−4.3034	0.1034
−4.8000	−5.4217	0.6217	−4.9182	0.1182
−5.4000	−6.0994	0.6994	−5.5330	0.1330
−6.0000	−6.7771	0.7771	−6.1477	0.1477

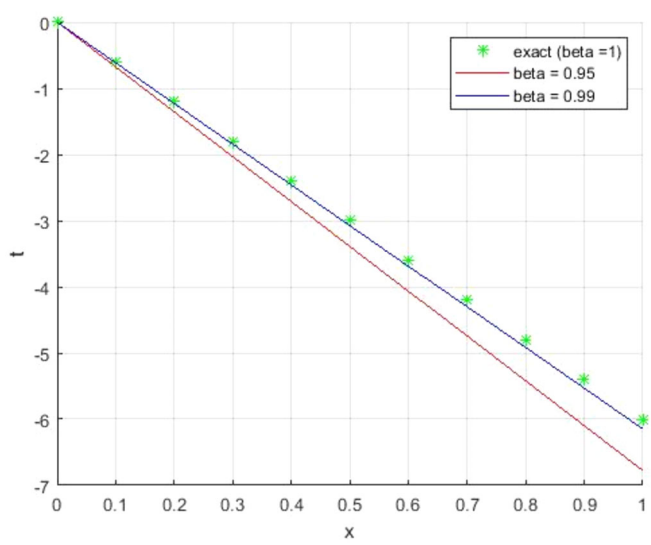

Figure 1 The comparison between the exact and numerical solutions for $\psi(\chi,\sigma,t)$

Table 1 and Table 2 above show the comparison between exact and approximate solutions of Example 1.

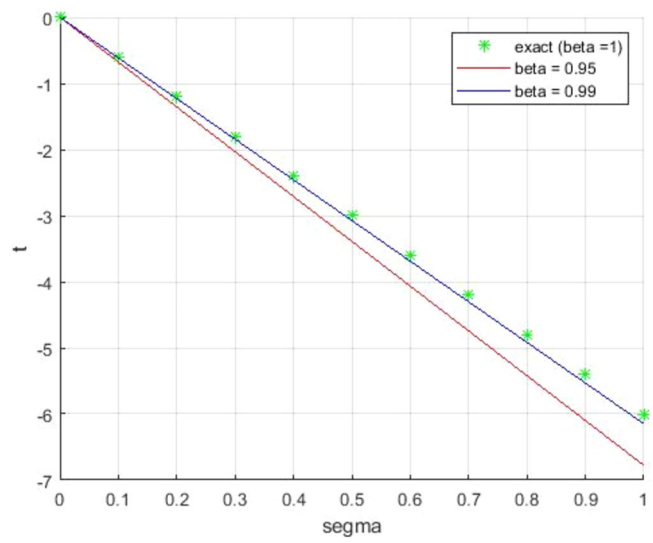

Figure 2 The comparison between the exact and numerical solutions for $\phi(\chi,\sigma,t)$

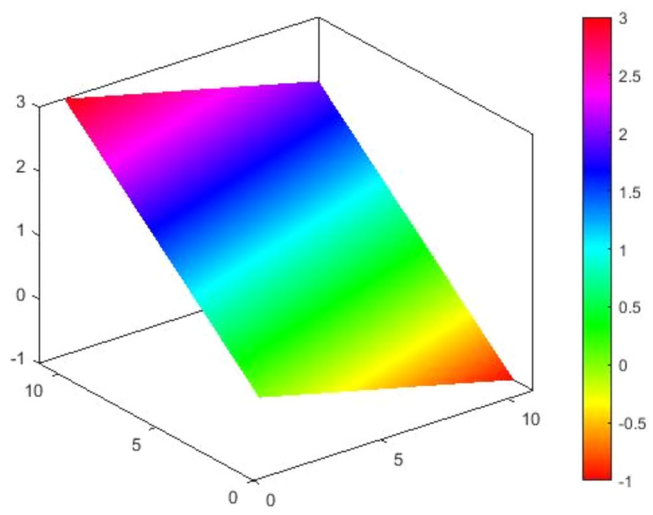

Figure 3 The surface shows the function $\psi(\chi,\sigma,t)$

The comparison between the exact and numerical solutions for the Eq. (56) is shown in Figs. 1 and 2. We obtain the exact solution at $\beta = 1$ and the different values of β such as ($\beta = 0.95$, $\beta = 0.99$) shows the approximate solution. The surfaces in Figs. 3 and 4 show the exact solution of the functions $\psi(\chi,\sigma,t)$ and $\phi(\chi,\sigma,t)$ at $\chi = 0$, respectively.

4 Double Sumudu-generalized Laplace decomposition method and singular two-dimensional time-fractional coupled Burger's equation

The objective of this section is to interpret the utilization of the double Sumudu-generalized Laplace decomposition method for solving the singular two-dimensional

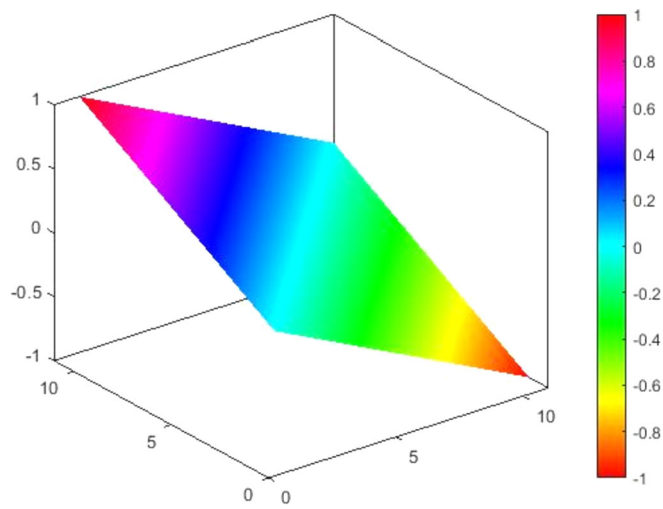

Figure 4 The surface shows the function $\phi(\chi,\sigma,t)$

time-fractional coupled Burger's equations in the following form

$$\begin{aligned}
&D_t^\alpha \psi + \frac{1}{\chi}\psi\psi_\chi + \frac{1}{\sigma}\phi\phi_\sigma - \frac{1}{\chi}(\chi\psi_\chi)_\chi - \frac{1}{\sigma}(\sigma\psi_\sigma)_\sigma = f(\chi,\sigma,t),\\
&D_t^\alpha \phi + \frac{1}{\chi}\psi\phi_\chi + \frac{1}{\sigma}\phi\phi_\sigma - \frac{1}{\chi}(\chi\phi_\chi)_\chi - \frac{1}{\sigma}(\sigma\phi_\sigma)_\sigma = g(\chi,\sigma,t),\\
&\chi,\sigma,t > 0,
\end{aligned} \tag{62}$$

with the initial condition

$$\psi(\chi,\sigma,0) = f_1(\chi,\sigma), \qquad \phi(\chi,\sigma,0) = g_1(\chi,\sigma), \tag{63}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional Caputo derivative and $\frac{1}{\chi}(\chi\psi_\chi)_\chi$, $\frac{1}{\sigma}(\sigma\psi_\sigma)_\sigma$, $\frac{1}{\chi}(\chi\phi_\chi)_\chi$, $\frac{1}{\sigma}(\sigma\phi_\sigma)_\sigma$ are the so-called Bessel operators, $\psi(\chi,\sigma,t)$ and $\phi(\chi,\sigma,t)$ are the velocity components to be presented, $f(\chi,\sigma,t)$, $g(\chi,\sigma,t)$, $f_1(\chi,\sigma)$, and $g_1(\chi,\sigma)$ are given functions. For the purpose of obtaining the solution of Eq. (62), we apply the next steps:

Step 1: Multiply both sides of Eq. (62) by $\chi\sigma$ to yield

$$\begin{aligned}
&\chi\sigma D_t^\alpha \psi + \sigma\psi\psi_\chi + \chi\phi\psi_\sigma - \sigma(\chi\psi_\chi)_\chi - \chi(\sigma\psi_\sigma)_\sigma = \chi\sigma f(\chi,\sigma,t),\\
&\chi\sigma D_t^\alpha \phi + \sigma\psi\phi_\chi + \chi\phi\phi_\sigma - \sigma(\chi\phi_\chi)_\chi - \chi(\sigma\phi_\sigma)_\sigma = \chi\sigma g(\chi,\sigma,t),\\
&\chi,\sigma,t > 0.
\end{aligned} \tag{64}$$

Step 2: Taking the double Sumudu-generalized Laplace transform for either side of Eq. (64) we gain

$$\begin{aligned}
\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Psi(\mu_1,\mu_2,s)\big) &= s^{\alpha+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 F_1(\mu_1,\mu_2)\big)\\
&\quad + \frac{s^\beta}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\sigma(\chi\psi_\chi)_\chi + \chi(\sigma\psi_\sigma)_\sigma\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma\psi\psi_\chi + \chi\phi\psi_\sigma) \\
&+ \frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\chi\sigma f(\chi,\sigma,t)\big)
\end{aligned} \tag{65}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2\Phi(\mu_1,\mu_2,s)\big) &= s^{\alpha+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 G_1(\mu_1,\mu_2)\big) \\
&+ \frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\sigma(\chi\phi_\chi)_\chi + \chi(\sigma\phi_\sigma)_\sigma\big) \\
&- \frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma\psi\phi_\chi + \chi\phi\phi_\sigma) \\
&+ \frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\chi\sigma g(\chi,\sigma,t)\big).
\end{aligned} \tag{66}
$$

Step 3: By taking the double integral for both sides of Eq. (65) and Eq. (66) from 0 to μ_1 and 0 to μ_2 according to μ_1 and μ_2, respectively, we obtain

$$
\begin{aligned}
\Psi(\mu_1,\mu_2,s) &= \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(s^{\alpha+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 F_1(\mu_1,\mu_2)\big)\right)d\mu_1\,d\mu_2 \\
&+ \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\sigma(\chi\psi_\chi)_\chi + \chi(\sigma\psi_\sigma)_\sigma\big)\right)d\mu_1\,d\mu_2 \\
&- \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma\psi\psi_\chi + \chi\phi\psi_\sigma)\right)d\mu_1\,d\mu_2 \\
&+ \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\frac{s^{\beta}}{\mu_1\mu_2}\big(S_\chi S_\sigma G_t\big(\chi\sigma f(\chi,\sigma,t)\big)\big)\,d\mu_1\,d\mu_2
\end{aligned} \tag{67}
$$

and

$$
\begin{aligned}
\Phi(\mu_1,\mu_2,s) &= \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(s^{\alpha+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 G_1(\mu_1,\mu_2)\big)\right)d\mu_1\,d\mu_2 \\
&+ \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\sigma(\chi\phi_\chi)_\chi + \chi(\sigma\phi_\sigma)_\sigma\big)\right)d\mu_1\,d\mu_2 \\
&- \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma\psi\phi_\chi + \chi\phi\phi_\sigma)\right)d\mu_1\,d\mu_2 \\
&+ \frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\frac{s^{\beta}}{\mu_1\mu_2}\big(S_\chi S_\sigma G_t\big(\chi\sigma g(\chi,\sigma,t)\big)\big)\,d\mu_1\,d\mu.
\end{aligned} \tag{68}
$$

Step 4: On using the inverse double Sumudu-generalized Laplace decomposition method for Eqs. (67) and (68), we obtain

$$
\begin{aligned}
\psi(\chi,\sigma,t) &= S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(s^{\alpha+1}\frac{\partial^2}{\partial\mu_1\partial\mu_2}\big(\mu_1\mu_2 F_1(\mu_1,\mu_2)\big)\right)d\mu_1\,d\mu_2\right] \\
&+ S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1} \\
&\times\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^{\beta}}{\mu_1\mu_2} S_\chi S_\sigma G_t\big(\sigma(\chi\psi_\chi)_\chi + \chi(\sigma\psi_\sigma)_\sigma\big)\right)d\mu_1\,d\mu_2\right]
\end{aligned}
$$

$$
\begin{aligned}
&- S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
&\times \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t (\sigma \psi \psi_\chi + \chi \phi \psi_\sigma) \right) d\mu_1 \, d\mu_2 \right] \\
&+ S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
&\times \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma f(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right]
\end{aligned} \tag{69}
$$

and

$$
\begin{aligned}
\phi(\chi, \sigma, t) = {} & S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(s^{\alpha+1} \frac{\partial^2}{\partial \mu_1 \partial \mu_2} \big(\mu_1 \mu_2 G_1(\mu_1, \mu_2) \big) \right) d\mu_1 \, d\mu_2 \right] \\
& + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
& \times \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \big(\sigma (\chi \phi_\chi)_\chi + \chi (\sigma \phi_\sigma)_\sigma \big) \right) d\mu_1 \, d\mu_2 \right] \\
& - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t (\sigma \psi \phi_\chi + \chi \phi \phi_\sigma) \right) d\mu_1 \, d\mu_2 \right] \\
& + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
& \times \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma g(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right].
\end{aligned} \tag{70}
$$

Step 5: Substituting Eqs. (48), (50), and Eq. (49) into Eqs. (69) and (70), we have

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \psi_n(\chi, \sigma, t) \\
& \quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(s^{\alpha+1} \frac{\partial^2}{\partial \mu_1 \partial \mu_2} \big(\mu_1 \mu_2 F_1(\mu_1, \mu_2) \big) \right) d\mu_1 \, d\mu_2 \right] \\
& \qquad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \left(\sigma \left(\chi \sum_{n=0}^{\infty} \psi_{n\chi} \right)_\chi \right. \right. \right. \\
& \qquad \left. \left. \left. + \chi \left(\sigma \sum_{n=0}^{\infty} \psi_{n\sigma} \right)_\sigma \right) \right) d\mu_1 \, d\mu_2 \right] \\
& \qquad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
& \qquad \times \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \left(\sigma \sum_{n=0}^{\infty} A_n + \chi \sum_{n=0}^{\infty} B_n \right) \right) d\mu_1 \, d\mu_2 \right] \\
& \qquad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma f(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \phi_n(\chi, \sigma, t) \\
& \quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(s^{\alpha+1} \frac{\partial^2}{\partial \mu_1 \partial \mu_2} \big(\mu_1 \mu_2 G_1(\mu_1, \mu_2) \big) \right) d\mu_1 \, d\mu_2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \left(\sigma \left(\chi \sum_{n=0}^{\infty} \phi_{n\chi} \right)_\chi \right.\right.\right. \\
&\left.\left.\left. + \chi \left(\sigma \sum_{n=0}^{\infty} \phi_{n\sigma} \right)_\sigma \right) \right) d\mu_1 \, d\mu_2 \right] \\
&- S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \\
&\times \left(\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \left(\sigma \sum_{n=0}^{\infty} C_n + \chi \sum_{n=0}^{\infty} D_n \right) \right) d\mu_1 \, d\mu_2 \right) \\
&+ S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma g(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right].
\end{aligned}
$$

Step 6: On utilizing the double Sumudu-generalized Laplace decomposition method, we present the recursive relations to obtain:

$$
\begin{aligned}
&\psi_0(\chi, \sigma, t) \\
&\quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(s^{\alpha+1} \frac{\partial^2}{\partial \mu_1 \partial \mu_2} \big(\mu_1 \mu_2 F_1(\mu_1, \mu_2) \big) \right) d\mu_1 \, d\mu_2 \right] \\
&\qquad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma f(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right]
\end{aligned} \tag{71}
$$

and

$$
\begin{aligned}
&\phi_0(\chi, \sigma, t) \\
&\quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(s^{\alpha+1} \frac{\partial^2}{\partial \mu_1 \partial \mu_2} \big(\mu_1 \mu_2 G_1(\mu_1, \mu_2) \big) \right) d\mu_1 \, d\mu_2 \right] \\
&\qquad + S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \frac{s^\beta}{\mu_1 \mu_2} \big(S_\chi S_\sigma G_t \big(\chi \sigma g(\chi, \sigma, t) \big) \big) \, d\mu_1 \, d\mu_2 \right].
\end{aligned} \tag{72}
$$

The remainder components ψ_{n+1} and ϕ_{n+1}, $n \geq 0$ are determined by

$$
\begin{aligned}
&\psi_{n+1}(\chi, \sigma, t) \\
&\quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \big(\sigma (\chi \psi_{n\chi})_\chi + \chi (\sigma \psi_{n\sigma})_\sigma \big) \right) d\mu_1 \, d\mu_2 \right] \\
&\qquad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t (\sigma A_n + \chi B_n) \right) d\mu_1 \, d\mu_2 \right]
\end{aligned} \tag{73}
$$

and

$$
\begin{aligned}
&\phi_{n+1}(\chi, \sigma, t) \\
&\quad = S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left[\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t \big(\sigma (\chi \phi_{n\chi})_\chi + \chi (\sigma \phi_{n\sigma})_\sigma \big) \right) d\mu_1 \, d\mu_2 \right] \\
&\qquad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1} \left(\frac{1}{\mu_1 \mu_2} \int_0^{\mu_1} \int_0^{\mu_2} \left(\frac{s^\beta}{\mu_1 \mu_2} S_\chi S_\sigma G_t (\sigma C_n + \chi D_n) \right) d\mu_1 \, d\mu_2 \right)
\end{aligned} \tag{74}
$$

and $S_\chi S_\sigma G_t$ is the double Sumudu-generalized Laplace transform with respect to χ, σ, t and the inverse double Sumudu-generalized Laplace transform is denoted by $S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}$ according to μ_1, μ_2, s. We assumed that the inverse double Sumudu-generalized Laplace transform with respect to μ_1, μ_2, and s exists for Eqs. (71), (72), (73), and (74). In the following example, we use the double Sumudu-generalized Laplace transform Adomain decomposition method to solve singular two-dimensional time-fractional coupled Burger's equations.

Example 2 [26] Consider that the singular two-dimensional time-fractional coupled Burger's equations are presented by

$$\begin{aligned}
&D_t^\alpha \psi + \frac{1}{\chi}\psi\psi_\chi + \frac{1}{\sigma}\phi\psi_\sigma - \frac{1}{\chi}(\chi\psi_\chi)_\chi - \frac{1}{\sigma}(\sigma\psi_\sigma)_\sigma = \left(\chi^2 - \sigma^2\right)e^t,\\
&D_t^\alpha \phi + \frac{1}{\chi}\psi\phi_\chi + \frac{1}{\sigma}\nu\phi\phi_\sigma - \frac{1}{\chi}(\chi\phi_\chi)_\chi - \frac{1}{\sigma}(\sigma\phi_\sigma)_\sigma = \left(\chi^2 - \sigma^2\right)e^t,\\
&\chi, \sigma, t > 0,
\end{aligned} \tag{75}$$

with the initial condition

$$\psi(\chi,\sigma,0) = \chi^2 - \sigma^2, \qquad \phi(\chi,\sigma,0) = \chi^2 - \sigma^2.$$

By using our method above, we successfully obtain

$$\begin{aligned}
\psi_0(\chi,\sigma,t) &= \chi^2 - \sigma^2 + \left(\chi^2 - \sigma^2\right)\left(\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \cdots\right),\\
\phi_0(\chi,\sigma,t) &= \chi^2 - \sigma^2 + \left(\chi^2 - \sigma^2\right)\left(\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \cdots\right),
\end{aligned}$$

and the remainder components ψ_{n+1} and ϕ_{n+1}, $n \geq 0$ are given by

$$\begin{aligned}
&\psi_{n+1}(\chi,\sigma,t)\\
&\quad= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2} S_\chi S_\sigma G_t\left(\sigma(\chi\psi_{n\chi})_\chi + \chi(\sigma\psi_{n\sigma})_\sigma\right)\right) d\mu_1\, d\mu_2\right]\\
&\qquad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma A_n + \chi B_n)\right) d\mu_1\, d\mu_2\right]
\end{aligned} \tag{76}$$

and

$$\begin{aligned}
&\phi_{n+1}(\chi,\sigma,t)\\
&\quad= S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2} S_\chi S_\sigma G_t\left(\sigma(\chi\phi_{n\chi})_\chi + \chi(\sigma\phi_{n\sigma})_\sigma\right)\right) d\mu_1\, d\mu_2\right]\\
&\qquad - S_{\mu_1}^{-1} S_{\mu_2}^{-1} G_s^{-1}\left(\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2} S_\chi S_\sigma G_t(\sigma C_n + \chi D_n)\right) d\mu_1\, d\mu_2\right).
\end{aligned} \tag{77}$$

By substituting $n = 0$, into Eqs. (76) and (77) we have

$$\begin{aligned}
&\psi_1(\chi,\sigma,t)\\
&\quad = S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2}S_\chi S_\sigma G_t\big(\sigma(\chi\psi_{0\chi})_\chi + \chi(\sigma\psi_{0\sigma})_\sigma\big)\right)d\mu_1\,d\mu_2\right]\\
&\qquad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2}S_\chi S_\sigma G_t(\sigma A_0 + \chi B_0)\right)d\mu_1\,d\mu_2\right],\\
&\psi_1(\chi,\sigma,t) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\phi_1(\chi,\sigma,t)\\
&\quad = S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left[\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2}S_\chi S_\sigma G_t\big(\sigma(\chi\phi_{0\chi})_\chi + \chi(\sigma\phi_{0\sigma})_\sigma\big)\right)d\mu_1\,d\mu_2\right]\\
&\qquad - S_{\mu_1}^{-1}S_{\mu_2}^{-1}G_s^{-1}\left(\frac{1}{\mu_1\mu_2}\int_0^{\mu_1}\int_0^{\mu_2}\left(\frac{s^\beta}{\mu_1\mu_2}S_\chi S_\sigma G_t(\sigma C_0 + \chi D_0)\right)d\mu_1\,d\mu_2\right),\\
&\phi_1(\chi,\sigma,t) = 0.
\end{aligned}$$

In a similar way, at $n = 1$, we have

$$\psi_2(\chi,\sigma,t) = 0, \qquad \phi_2(\chi,\sigma,t) = 0.$$

The solution of Eq. (75) is determined by

$$\begin{aligned}
&\psi(\chi,\sigma,t) = \psi_0 + \psi_1 + \psi_2 + \cdots,\\
&\phi(\chi,\sigma,t) = \phi_0 + \phi_1 + \phi_2 + \cdots + .
\end{aligned}$$

Thence, the exact solution is denoted by

$$\begin{aligned}
\psi(\chi,\sigma,t) &= \chi^2 - \sigma^2 + \big(\chi^2 - \sigma^2\big)\left(\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \cdots\right),\\
\phi(\chi,\sigma,t) &= \chi^2 - \sigma^2 + \big(\chi^2 - \sigma^2\big)\left(\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \cdots\right),
\end{aligned}$$

when we put $\alpha = 1$, we obtain the exact solution of Eq. (75) as follows:

$$\begin{aligned}
\psi(\chi,\sigma,t) &= \big(\chi^2 - \sigma^2\big)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\cdots\right) = \big(\chi^2 - \sigma^2\big)e^t,\\
\phi(\chi,\sigma,t) &= \big(\chi^2 - \sigma^2\big)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\cdots\right) = \big(\chi^2 - \sigma^2\big)e^t.
\end{aligned}$$

Table 3 and Table 4 below shows the comparison between exact and approximate solutions of Example 2.

Table 3 Comparison between the exact and approximation solutions for $\psi(\chi,\sigma,t)$

Exact $\beta = 1$	The method $\beta = 0.95$	Error	The method $\beta = 0.99$	Error
−1.0000	−1.0000	0	−1.0000	0
−1.0941	−1.1094	0.0153	−1.0970	0.0029
−1.1725	−1.1956	0.0231	−1.1769	0.0045
−1.2280	−1.2560	0.0280	−1.2335	0.0054
−1.2522	−1.2826	0.0304	−1.2581	0.0060
−1.2344	−1.2650	0.0306	−1.2404	0.0060
−1.1622	−1.1909	0.0286	−1.1679	0.0057
−1.0211	−1.0456	0.0245	−1.0260	0.0049
−0.7939	−0.8122	0.0183	−0.7976	0.0036
−0.4610	−0.4711	0.0101	−0.4630	0.0020
0	0	0	0	0

Table 4 Comparison between the exact and approximation solutions for $\phi(\chi,\sigma,t)$

Exact $\beta = 1$	The method $\beta = 0.95$	Error	The method $\beta = 0.99$	Error
−1.0000	−1.0000	0	−1.0000	0
−1.0941	−1.1094	0.0153	−1.0970	0.0029
−1.1725	−1.1956	0.0231	−1.1769	0.0045
−1.2280	−1.2560	0.0280	−1.2335	0.0054
−1.2522	−1.2826	0.0304	−1.2581	0.0060
−1.2344	−1.2650	0.0306	−1.2404	0.0060
−1.1622	−1.1909	0.0286	−1.1679	0.0057
−1.0211	−1.0456	0.0245	−1.0260	0.0049
−0.7939	−0.8122	0.0183	−0.7976	0.0036
−0.4610	−0.4711	0.0101	−0.4630	0.0020
0	0	0	0	0

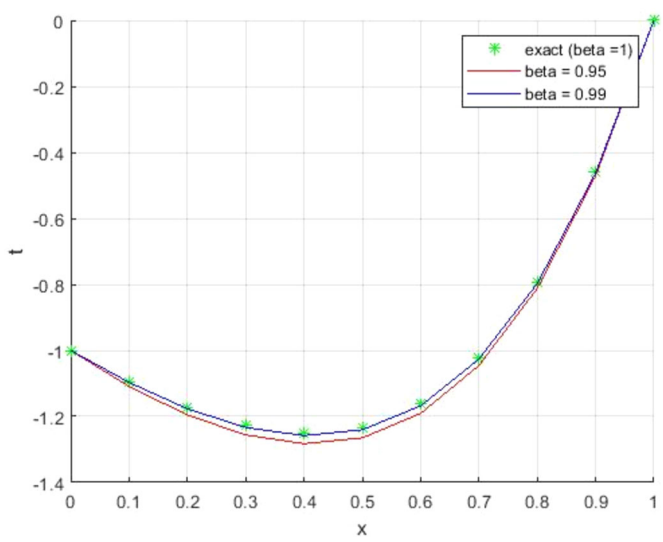

Figure 5 The comparison between the exact and numerical solutions for $\psi(\chi,\sigma,t)$

The comparison between the exact and numerical solutions for the Eq. (75) is shown in Figs. 5 and 6. We obtain the exact solution at $\beta = 1$ and the different values of β such as ($\beta = 0.95$, $\beta = 0.99$) shows the approximate solution. The surfaces in Figs. 7 and 8 show the exact solution of the functions $\psi(\chi,\sigma,t)$ and $\phi(\chi,\sigma,t)$ at $\chi = 0$, respectively.

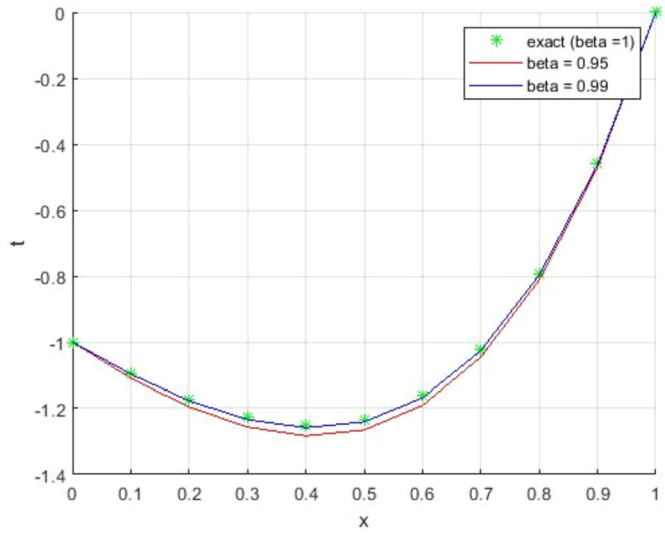

Figure 6 The comparison between the exact and numerical solutions for $\phi(\chi,\sigma,t)$

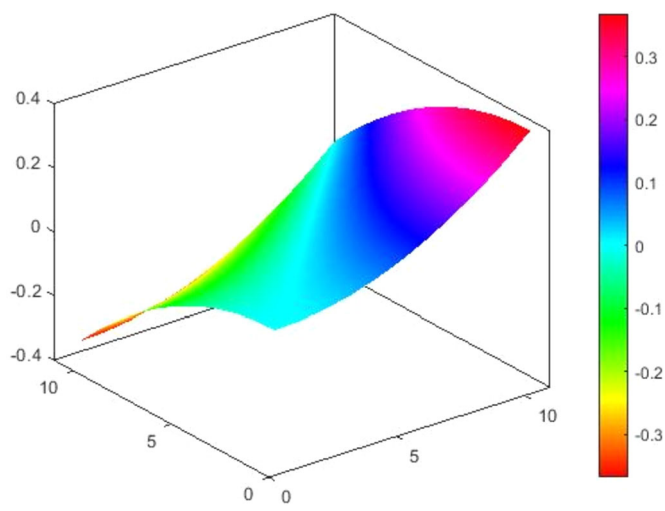

Figure 7 The surface shows the function $\psi(\chi,\sigma,t)$

5 Conclusions

In this research paper, double Sumudu-generalized Laplace transforms and Adomian decomposition have been profitably joined to obtain a new potent method called the double Sumudu-generalized Laplace Adomian decomposition method (DSGLTDM). This technique has been employed to solve regular and singular two-dimensional time-fractional coupled Burger's equations. By involving this approach in some examples we have obtained new effective relations to solve our problems. Our method shows that the series solution can converge very quickly to the solutions. In this study, the technique utilized to obtain exact and approximation solutions can also be expanded to solve other nonlinear partial differential equations of physical interest. We see that the results of Examples 1 and 2 are the same as those of applying the Laplace–Adomian decomposition method, vari-

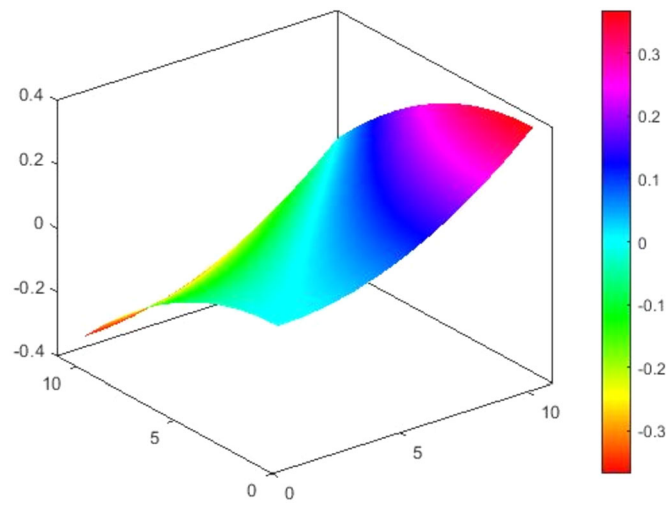

Figure 8 The surface shows the function $\phi(\chi, \sigma, t)$

ational iteration method (VIM), and Triple Laplace–Adomian Decomposition Method, [13, 26, 27]. The advantage of DSGLTDM is that it generates other methods, such as the double Sumudu–Laplace transform decomposition method, see Eq. (2), the double Sumudu–Yang Transform decomposition method, see Eq. (3), the triple Sumudu Transform decomposition method, see Eq. (4), the double Sumudu–Elzaki transform decomposition method, see Eq. (5), and double Sumudu–Aboodh transform, see Eq. (6).

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Author contributions
I all parts of this paper I did by myself no body help me methodology, data collection, and analysis.

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Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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