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Solvability of a nonlinear second order m-point boundary value problem with p-Laplacian at resonance

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Abstract

We study the existence of solutions of the nonlinear second order m-point boundary value problem with p-Laplacian at resonance

$$\begin{cases} (\phi_p(x'))' = f(t,x,x'), & t \in [0,1], \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $a_i > 0$ ($i = 1, 2, \dots, m-2$) with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. Based on the topological transversality method together with the barrier strip technique and the cut-off technique, we obtain new existence results of solutions of the above problem. Meanwhile some examples are also given to illustrate our main results.

Mathematics Subject Classification: 34B10; 34B15; 34B18

Keywords: m-point boundary value problem; p-Laplacian; Resonance; Topological transversality method; Barrier strip technique; Cut-off technique

1 Introduction

In this paper, we consider the existence of solutions of the following m-point boundary value problem with p-Laplacian at resonance:

$$\begin{cases} (\phi_p(x'))' = f(t,x,x'), & t \in [0,1], \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $a_i > 0$ ($i = 1, 2, \dots, m-2$) with $\sum_{i=1}^{m-2} a_i = 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. By a solution to problem (1.1), we mean a function $x \in C^1[0,1]$ with $\phi_p(x') \in C^1[0,1]$ that satisfies problem (1.1).

It is well known that the research of m-point boundary value problems is significant in the theory of ordinary differential equations and practical applications, see [4, 13].

We specifically mention that the existence of solutions of problem (1.1) with $p = 2$ has been studied by Gupta [12], Feng and Webb [3], Infante and Webb [14], Ma [20], Infante and Zima [15], Liang and Lin [18], in which the Leray–Schauder continuation theorem, the Mawhin's continuation theorem, the theory of fixed point index, the nonlinear alternative of Leray–Schauder, the Leggett–Williams norm-type theorem due to O'Regan and Zima [21], and the fixed point theorem in cones are used, respectively. In the case that $p > 1$, problem (1.1) has been studied by García-Huidobro et al. [5–7] via some continuation lemma of Leray–Schauder type due to themselves, Zhu and Wang [24] via Mawhin's continuation theorem. For other types of works regarding the second order nonlocal boundary value problems involving p-Laplacian, we refer the reader to [1, 2, 8–10, 16, 23].

Inspired by the above works and [17, 22], the aim of this paper is to establish the new existence results of solutions to problem (1.1) by using the topological transversality method together with the barrier strip technique and the cut-off technique.

We would like to emphasize that the results of this paper are new even when $p = 2$, and the main research tool used is the topological transversality theorem, which is different from those in [5–7, 20, 24]. The advantage of the topological transversality theorem is that it can transform the resonance problem into a nonresonance problem by appropriately selecting a convex subset U of X. Besides, in [5–7], the order of growth of f with respect to the derivative term is less than or equal to p. However, in our results, we just impose some local sign conditions on the nonlinear term f and do not require the growth constraints, so the degree of the derivative term in f can exceed p.

This work is organized as follows: In Sect. 2, we first briefly introduce the topological transversality theory. And then, a modified boundary value problem is constructed by the cut-off technique. Finally, the barrier strip technique is used to estimate a prior bound of solutions of the modified boundary value problems in $C^1[0,1]$. In Sect. 3, the topological transversality method is used to establish some existence theorems of solutions of problem (1.1). As applications of our main results, some examples are given in the last section.

Throughout this paper, the following local conditions on f will be used.

(H$_1$) There exists $r_1 \leq 0 \leq r_2$ with $r_1^2 + r_2^2 > 0$ such that

$$f(t,r_1,0) \leq 0, \qquad f(t,r_2,0) \geq 0, \quad \forall t \in [0,1].$$

(H$_2$) There exists $R_1 \leq 0 \leq R_2$ with $R_1^2 + R_2^2 > 0$ such that

$$f(t,x,R_1) \geq 0, \qquad f(t,x,R_2) \leq 0, \quad \forall (t,x) \in [0,1] \times [r_1,r_2].$$

(H$_3$) There exists $R_1 \leq 0 \leq R_2$ with $R_1^2 + R_2^2 > 0$ such that

$$f(t,x,R_1) \geq 0, \qquad f(t,x,R_2) \geq 0, \quad \forall (t,x) \in [0,1] \times [r_1,r_2],$$

and

$$f(t,x,y) \leq \frac{\phi_p(R_2)}{1-\xi_1}, \quad \forall (t,x,y) \in [0,1] \times [r_1,r_2] \times [R_1,R_2].$$

(H$_4$) There exists $R_1 \leq 0 \leq R_2$ with $R_1^2 + R_2^2 > 0$ such that

$$f(t,x,R_1) \leq 0, \qquad f(t,x,R_2) \leq 0, \quad \forall (t,x) \in [0,1] \times [r_1,r_2],$$

and

$$f(t,x,y) \geq \frac{\phi_p(R_1)}{1-\xi_1}, \quad \forall (t,x,y) \in [0,1] \times [r_1,r_2] \times [R_1,R_2].$$

(H_5) There exists $R_1 \leq 0 \leq R_2$ with $R_1^2 + R_2^2 > 0$ such that

$$f(t,x,R_1) \leq 0, \qquad f(t,x,R_2) \geq 0, \quad \forall (t,x) \in [0,1] \times [r_1,r_2],$$

and

$$\frac{\phi_p(R_1)}{1-\xi_1} \leq f(t,x,y) \leq \frac{\phi_p(R_2)}{1-\xi_1}, \quad \forall (t,x,y) \in [0,1] \times [r_1,r_2] \times [R_1,R_2].$$

Here, the constants r_1 and r_2 in (H_2)–(H_5) are as in (H_1).

2 Preliminaries

Firstly, we briefly review some concepts and results of topological transversality theory. Let U be a convex subset of a Banach space X and $\mathcal{D} \subset U$ be an open set. Denote by $H_{\partial\mathcal{D}}(\overline{\mathcal{D}},U)$ the set of compact operators $F:\overline{\mathcal{D}} \to U$ that are fixed point free on $\partial\mathcal{D}$. We say that $F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}},U)$ is essential if every operator in $H_{\partial\mathcal{D}}(\overline{\mathcal{D}},U)$ that agrees with F on $\partial\mathcal{D}$ has a fixed point in $\mathcal{D}$.

The next two lemmas can be found in [11].

Lemma 2.1 *If $q \in \mathcal{D}$ and $F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}},U)$ is a constant operator, $F(x) = q$ for $x \in \overline{\mathcal{D}}$, then F is essential.*

Lemma 2.2 *Assume that*

(i) *$F \in H_{\partial\mathcal{D}}(\overline{\mathcal{D}},U)$ is essential;*

(ii) *$H:\overline{\mathcal{D}} \times [0,1] \to U$ is a compact homotopy, $H(\cdot,0) = F$ and $H(x,\lambda) \neq x$ for $x \in \partial\mathcal{D}$ and $\lambda \in [0,1]$.*

Then $H(\cdot,1)$ is essential and therefore it has a fixed point in $\mathcal{D}$.

According to the intermediate value property of continuous functions, it is not difficult to obtain the following.

Lemma 2.3 *Assume that*

(i) *$x \in C^1[0,1]$;*

(ii) *$x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, where $a_i > 0$ $(i = 1,2,\dots,m-2)$ with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.*

Then there exists $\eta \in [\xi_1, \xi_{m-2}]$ such that $x(1) = x(\eta)$.

Let the constants r_1, r_2, R_1, R_2 be such that

$$r_1 \leq 0 \leq r_2 \quad (r_1 \neq r_2), \qquad R_1 \leq 0 \leq R_2 \quad (R_1 \neq R_2).$$

We define the modification g of f as follows:

$$g(t,x,y)=\begin{cases} f(t,r_2,y)+x-r_2, & x>r_2; \\ f(t,x,y), & r_1\le x\le r_2; \\ f(t,r_1,y)+x-r_1, & x<r_1, \end{cases}$$

and define the modification h of g by setting

$$h(t,x,y)=\begin{cases} g(t,x,R_2), & y>R_2; \\ g(t,x,y), & R_1\le y\le R_2; \\ g(t,x,R_1), & y<R_1. \end{cases}$$

Obviously, both $g(t,x,y)$ and $h(t,x,y)$ are continuous on $[0,1]\times\mathbb{R}^2$.

We note here that in the following discussion, we agree that when (H_1) is true, the constants r_1 and r_2 in function g are the ones in (H_1), while for $i=2,3,4,5$, when (H_i) is true, the constants R_1 and R_2 in function h are the ones in (H_i).

Consider the family of the following modified boundary value problem:

$$\big(\phi_p\big(x'(t)\big)\big)'=\lambda h\big(t,x(t),x'(t)\big),\quad t\in[0,1], \tag{2.1}$$

$$x'(0)=0,\qquad x(1)=\sum_{i=1}^{m-2}a_i x(\xi_i), \tag{2.2}$$

where $\lambda\in(0,1]$.

The following lemma is a prior estimate of the possible solutions of the modified problem (2.1), (2.2) in $C^1[0,1]$.

Lemma 2.4 *Assume that* (H_1) *and* (H_2) *hold. Let x be a solution of the modified problem* (2.1), (2.2) *for some* $\lambda\in(0,1]$. *Then*

$$r_1\le x(t)\le r_2,\quad \forall t\in[0,1], \tag{2.3}$$

$$R_1\le x'(t)\le R_2,\quad \forall t\in[0,1]. \tag{2.4}$$

Proof We divided the proof into two steps.

Step 1. We prove that (2.3) holds.

Suppose on the contrary that there exists $t_0\in[0,1]$ such that $x(t_0)<r_1$ or $x(t_0)>r_2$. Without loss of generality, we may assume that $x(t_0)<r_1$. Let $t_1\in[0,1]$ be such that

$$x(t_1)=\min_{t\in[0,1]}x(t)<r_1. \tag{2.5}$$

By Lemma 2.3, we can assume that $t_1\in[0,1)$, and then from (2.2) we have $x'(t_1)=0$. It follows from (H_1) and the definition of h that

$$\begin{aligned}\big(\phi_p\big(x'(t)\big)\big)'|_{t=t_1}&=\lambda h\big(t_1,x(t_1),0\big)\\ &=\lambda g\big(t_1,x(t_1),0\big)\\ &=\lambda\big(f(t_1,r_1,0)+x(t_1)-r_1\big)<0,\end{aligned}$$

and thus there exists $\delta > 0$ such that $\phi_p(x'(t))$ is decreasing on $[t_1, t_1 + \delta) \subset [0,1)$. This together with the monotonicity of $\phi_p^{-1}(\cdot)$ implies that

$$x'(t) < x'(t_1) = 0, \quad \forall t \in (t_1, t_1 + \delta),$$

which contradicts (2.5). Therefore $x(t) \geq r_1$, $t \in [0,1]$. Similarly, we can obtain $x(t) \leq r_2$, $t \in [0,1]$. This implies that (2.3) holds.

Step 2. We use the barrier strip technique to show that (2.4) holds.

Let

$$\begin{aligned} S_1 &= \{ t \in [0,1] : R_1 - 1 \leq x'(t) < R_1 \}, \\ S_2 &= \{ t \in [0,1] : R_2 < x'(t) \leq R_2 + 1 \}. \end{aligned} \tag{2.6}$$

We now assert that the sets S_1 and S_2 are empty. We only prove that $S_1 = \emptyset$. Similarly, it can be shown that $S_2 = \emptyset$. Suppose on the contrary that $S_1 \neq \emptyset$. Choose $t_0 \in S_1$. Then $R_1 - 1 \leq x'(t_0) < R_1$, and so it follows from (2.2) that $0 < t_0 \leq 1$. By the continuity of $x'(t)$ on $[0,1]$ and the fact of $x'(0) = 0$, there exist $0 < t_1 < t_2 \leq t_0$ such that

$$R_1 - 1 \leq x'(t_0) \leq x'(t_2) < x'(t_1) < R_1 \tag{2.7}$$

and

$$x'(t_2) \leq x'(t) \leq x'(t_1), \quad \forall t \in [t_1, t_2].$$

Consequently, $[t_1, t_2] \subset S_1$; whereas from (2.3) and (H_2) we obtain

$$\begin{aligned} \big(\phi_p\big(x'(t)\big)\big)' &= \lambda h\big(t, x(t), x'(t)\big) \\ &= \lambda g\big(t, x(t), R_1\big) \\ &= \lambda f\big(t, x(t), R_1\big) \geq 0, \quad \forall t \in [t_1, t_2]. \end{aligned}$$

Hence $\phi_p(x'(t))$ is nondecreasing on $[t_1, t_2]$. Furthermore, by the monotonicity of $\phi_p^{-1}(\cdot)$, we have that $x'(t)$ is nondecreasing on $[t_1, t_2]$. Consequently,

$$x'(t_1) \leq x'(t_2),$$

which contradicts (2.7). This implies that $S_1 = \emptyset$. Notice that $x'(0) = 0$ and $x' \in C[0,1]$, it follows that (2.4) holds. This completes the proof of the lemma. □

Lemma 2.5 *Assume that* (H_1) *and* (H_3) *hold. Let x be a solution of the modified problem* (2.1), (2.2) *for some* $\lambda \in (0,1]$. *Then $x(t)$ satisfies* (2.3) *and* (2.4).

Proof From the proof of Step 1 of Lemma 2.4, it is easy to see that $x(t)$ satisfies (2.3). We now show that (2.4) holds. To do this, we let S_i $(i = 1,2)$ be as in (2.6). Then $S_1 = \emptyset$ by the proof of Step 2 of Lemma 2.4. Therefore, it follows from (2.2) that

$$x'(t) \geq R_1, \quad \forall t \in [0,1]. \tag{2.8}$$

We now prove that $S_2 = \emptyset$. Indeed, by contradiction, we assume that there exists $t_0 \in S_2$ such that $R_2 < x'(t_0) \le R_2 + 1$. According to (2.2), we have $0 < t_0 \le 1$. In view of Lemma 2.3 and the mean value theorem of differentials, there exists $\zeta \in (\eta, 1) \subset [\xi_1, 1]$ such that $x'(\zeta) = 0$. Integrating (2.1) from ζ to 1, applying (2.3) and (H_3), we obtain

$$\begin{aligned}\phi_p\big(x'(1)\big) &= \lambda \int_\zeta^1 h\big(t, x(t), x'(t)\big)\,\mathrm{d}t \\ &= \lambda \int_\zeta^1 g\big(t, x(t), \theta\big(x'(t)\big)\big)\,\mathrm{d}t \\ &= \lambda \int_\zeta^1 f\big(t, x(t), \theta\big(x'(t)\big)\big)\,\mathrm{d}t \\ &\le \lambda \int_\zeta^1 \frac{\phi_p(R_2)}{1-\xi_1}\,\mathrm{d}t \\ &< \phi_p(R_2),\end{aligned}$$

where $\theta(y) := \max\{R_1, \min\{y, R_2\}\}$ for $y \in \mathbb{R}$. Consequently, $x'(1) < R_2$, and so $0 < t_0 < 1$. From the continuity of $x'(t)$ on $[0,1]$, there exist $t_0 \le t_1 < t_2 < 1$ such that

$$R_2 < x'(t_2) < x'(t_1) \le x'(t_0) \le R_2 + 1, \tag{2.9}$$

and

$$x'(t_2) \le x'(t) \le x'(t_1), \quad \forall t \in [t_1, t_2].$$

Consequently, $[t_1, t_2] \subset S_2$. It follows from the definition of h, (2.3), and (H_3) that

$$\begin{aligned}\big(\phi_p\big(x'(t)\big)\big)' &= \lambda h\big(t, x(t), x'(t)\big) \\ &= \lambda g\big(t, x(t), R_2\big) \\ &= \lambda f\big(t, x(t), R_2\big) \ge 0, \quad \forall t \in [t_1, t_2].\end{aligned}$$

This implies that

$$\phi_p\big(x'(t_1)\big) \le \phi_p\big(x'(t_2)\big),$$

and thus, by the monotonicity of $\phi_p^{-1}(\cdot)$, we obtain

$$x'(t_1) \le x'(t_2),$$

which contradicts (2.9). This shows that $S_2 = \emptyset$. Notice that $x'(0) = 0$ and $x' \in C[0,1]$, we have

$$x'(t) \le R_2, \quad \forall t \in [0,1].$$

This together with (2.8) implies that (2.4) holds. This completes the proof of the lemma. □

Similar to the proof of Lemma 2.5, we can easily show the following two results.

Lemma 2.6 *Assume that* (H_1) *and* (H_4) *hold. Let* x *be a solution of the modified problem* (2.1), (2.2) *for some* $\lambda \in (0,1]$. *Then* $x(t)$ *satisfies* (2.3) *and* (2.4).

Lemma 2.7 *Assume that* (H_1) *and* (H_5) *hold. Let* x *be a solution of the modified problem* (2.1), (2.2) *for some* $\lambda \in (0,1]$. *Then* $x(t)$ *satisfies* (2.3) *and* (2.4).

Now, we denote by $X = C^1[0,1] \times \mathbb{R}$ the Banach space equipped with the norm

$$\|(x,r)\| = \|x\|_\infty + \|x'\|_\infty + |r|, \quad (x,r) \in X.$$

Set

$$U = \{(x,r) \in X : x(0) = 0, r \in \mathbb{R}\}$$

and

$$\mathcal{D} = \left\{ (x,r) \in U \,\middle|\, \begin{array}{l} r_1 - r_2 - 1 < x(t) < r_2 - r_1 + 1 \text{ on } [0,1], \\ R_1 - 1 < x'(t) < R_2 + 1 \text{ on } [0,1], \\ r_1 - 1 < r < r_2 + 1 \end{array} \right\}.$$

Obviously, U is a convex subset of X and $\mathcal{D}$ is an open subset of U.

Lemma 2.8 *Let the operator* $H : \overline{\mathcal{D}} \times [0,1] \to U$ *be defined by*

$$H(x,r,\lambda) = \left(0, \lambda r - \lambda \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right). \tag{2.10}$$

Then $H(x,r,\lambda)$ *is a compact operator.*

Proof Notice that to prove $H(x,r,\lambda)$ is a compact operator, it is sufficient to show the second component of $H(x,r,\lambda)$, that is,

$$H_2(x,r,\lambda) := \lambda r - \lambda \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s$$

is compact since the first component $H_1(x,r,\lambda) \equiv 0$ of $H(x,r,\lambda)$ is compact.

We now divided the proof into two steps.

Step 1. We prove that $H_2(x,r,\lambda)$ is continuous. To do this, we let $\{(x_n,r_n,\lambda_n)\}_{n=1}^\infty \subset \overline{\mathcal{D}} \times [0,1]$ be such that $(x_n,r_n,\lambda_n) \to (x_0,r_0,\lambda_0) \in \overline{\mathcal{D}} \times [0,1] (n \to \infty)$. Then

$$\|x_n - x_0\|_\infty \to 0, \qquad \|x_n' - x_0'\|_\infty \to 0, \qquad |r_n - r_0| \to 0, \qquad |\lambda_n - \lambda_0| \to 0, \quad n \to \infty.$$

From the continuity of h and ϕ_p^{-1}, we have

$$\begin{aligned} H_2(x_n,r_n,\lambda_n) &= \lambda_n r_n - \lambda_n \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x_n(\tau)+r_n, x_n'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\to \lambda_0 r_0 - \lambda_0 \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x_0(\tau)+r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= H_2(x_0,r_0,\lambda_0) \quad (n\to\infty). \end{aligned}$$

Hence $H_2(x,r,\lambda)$ is continuous on $\overline{\mathcal{D}}\times[0,1]$.

Step 2. We show that $H_2(\overline{\mathcal{D}}\times[0,1])$ is a relatively compact set in $\mathbb{R}$. Notice that for all $(x,r,\lambda)\in\overline{\mathcal{D}}\times[0,1]$, we have

$$\begin{aligned} \big|H_2(x,r,\lambda)\big| &= \left|\lambda r - \lambda \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau)+r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\ &\le |r| + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^1 \big|h\big(\tau, x(\tau)+r, x'(\tau)\big)\big|\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\le \max\{-r_1+1, r_2+1\} + \phi_p^{-1}(M) =: C > 0, \end{aligned}$$

where

$$M = \max\left\{ \big|h(t,x,y)\big| \,\middle|\, \begin{array}{l} 0\le t\le 1, 2r_1-r_2-2\le x\le 2r_2-r_1+2, \\ R_1-1\le y\le R_2+1 \end{array} \right\}. \tag{2.11}$$

Thus, $H_2(\overline{\mathcal{D}}\times[0,1])$ is a relatively compact set in $\mathbb{R}$. This completes the proof of the lemma. □

Lemma 2.9 *Assume that* (H_1) *holds. Let the operator* $F:\overline{\mathcal{D}}\to U$ *be defined by*

$$F(x,r) = \left(0, r - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau)+r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right).$$

Then F is essential.

Proof It follows from (2.10) that $H(\cdot,\cdot,1)=F(\cdot,\cdot)$, $H(x,r,0)=(0,0)\in\mathcal{D}$ for $(x,r)\in\overline{\mathcal{D}}$, and thus from Lemma 2.1 we know that $H(x,r,0)$ is essential. Besides, $H(x,r,\lambda)$ is compact by Lemma 2.8.

We now show that

$$H(x,r,\lambda)\neq(x,r), \quad \forall (x,r)\in\partial\mathcal{D}, \lambda\in[0,1]. \tag{2.12}$$

Obviously, $H(x,r,0)\neq(x,r)$ for all $(x,r)\in\partial\mathcal{D}$. Suppose that $H(x_0,r_0,\lambda_0)=(x_0,r_0)$ for some $(x_0,r_0)\in\partial\mathcal{D}$ and $\lambda_0\in(0,1]$. Then $x_0=0$ and

$$\lambda_0 r_0 - \lambda_0 \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x_0(\tau)+r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s = r_0.$$

Consequently,

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_0^s h(\tau, r_0, 0)\,\mathrm{d}\tau\right)\mathrm{d}s = r_0\left(1 - \frac{1}{\lambda_0}\right).$$

If $r_0 > r_2$, from (H_1) we have

$$h(\tau, r_0, 0) = g(\tau, r_0, 0) = f(\tau, r_2, 0) + r_0 - r_2 > 0, \quad \forall \tau \in [0,1].$$

The monotonicity of $\phi_p^{-1}(\cdot)$ and $\phi_p^{-1}(0) = 0$ imply that

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_0^s h(\tau, r_0, 0)\,\mathrm{d}\tau\right)\mathrm{d}s > \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}(0)\,\mathrm{d}s = 0,$$

which contradicts $r_0(1 - 1/\lambda_0) \le 0$. Hence, $r_0 \le r_2$. Similarly, we can obtain $r_0 \ge r_1$. Consequently, $(0, r_0) \in \mathcal{D}$, which contradicts $(0, r_0) \in \partial\mathcal{D}$. This shows that (2.12) holds. Therefore, according to Lemma 2.2, $F(\cdot,\cdot) = H(\cdot,\cdot,1)$ is essential. This completes the proof of the lemma. □

Lemma 2.10 *Let the operator $G : \overline{\mathcal{D}} \times [0,1] \to U$ be defined by*

$$G(x, r, \lambda) = \begin{pmatrix} \int_0^t \phi_p^{-1}\left(\lambda \int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ r - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \end{pmatrix}^*,$$

where the symbol "∗" denotes the transpose of vector. Then G is a compact operator.

Proof Firstly, we define operators $G_1 : \overline{\mathcal{D}} \times [0,1] \to C^1[0,1]$ and $G_2 : \overline{\mathcal{D}} \times [0,1] \to \mathbb{R}$ by

$$G_1(x, r, \lambda) = \int_0^t \phi_p^{-1}\left(\lambda \int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s$$

and

$$G_2(x, r, \lambda) = r - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Next, we divided the proof into three steps.

Step 1. We prove that $G_1 : \overline{\mathcal{D}} \times [0,1] \to C^1[0,1]$ is continuous.

Let $\{(x_n, r_n, \lambda_n)\}_{n=1}^{\infty} \subset \overline{\mathcal{D}} \times [0,1]$ be such that $(x_n, r_n, \lambda_n) \to (x_0, r_0, \lambda_0) \in \overline{\mathcal{D}} \times [0,1] (n \to \infty)$. Then

$$\|x_n - x_0\|_\infty \to 0, \qquad \left\|x_n' - x_0'\right\|_\infty \to 0,$$
$$|r_n - r_0| \to 0, \qquad |\lambda_n - \lambda_0| \to 0 \quad (n \to \infty).$$

The continuity of h and ϕ_p^{-1} imply that

$$\phi_p^{-1}\left(\lambda_n \int_0^t h\big(\tau, x_n(\tau) + r_n, x_n'(\tau)\big)\,\mathrm{d}\tau\right) \to \phi_p^{-1}\left(\lambda_0 \int_0^t h\big(\tau, x_0(\tau) + r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)$$

uniformly on $[0,1]$ as $n \to \infty$, and

$$\int_0^t \phi_p^{-1}\left(\lambda_n \int_0^s h\big(\tau, x_n(\tau) + r_n, x_n'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ \to \int_0^t \phi_p^{-1}\left(\lambda_0 \int_0^s h\big(\tau, x_0(\tau) + r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s$$

uniformly on $[0,1]$ as $n \to \infty$. That is,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} G_1(x_n, r_n, \lambda_n) - \frac{\mathrm{d}}{\mathrm{d}t} G_1(x_0, r_0, \lambda_0) \right\|_\infty \to 0 \quad (n \to \infty)$$

and

$$\left\| G_1(x_n, r_n, \lambda_n) - G_1(x_0, r_0, \lambda_0) \right\|_\infty \to 0 \quad (n \to \infty).$$

This shows that $G_1 : \overline{\mathcal{D}} \times [0,1] \to C^1[0,1]$ is continuous.

Step 2. We show that the set $G_1(\overline{\mathcal{D}} \times [0,1])$ is relatively compact in $C^1[0,1]$. Obviously, for all $(x,r) \in \overline{\mathcal{D}}$, $\lambda \in [0,1]$, we have

$$\big|G_1(x,r,\lambda)\big| = \left|\int_0^t \phi_p^{-1}\left(\lambda \int_0^s h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right| \le \phi_p^{-1}(M),$$
$$\left|\frac{\mathrm{d}}{\mathrm{d}t} G_1(x,r,\lambda)\right| = \left|\phi_p^{-1}\left(\lambda \int_0^t h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\right| \le \phi_p^{-1}(M),$$

where M is defined by (2.11). Since $\phi_p^{-1}(\cdot)$ is uniformly continuous on $[-M, M]$, for any $\varepsilon > 0$, there exists $\mu > 0$ such that when $|s_2 - s_1| < \mu$ ($\forall s_1, s_2 \in [-M, M]$),

$$\big|\phi_p^{-1}(s_2) - \phi_p^{-1}(s_1)\big| < \varepsilon.$$

Now, we choose $\delta = \mu/M$. Then, for all $t_1, t_2 \in [0,1]$ with $|t_2 - t_1| < \delta$ and for all $(x,r) \in \overline{\mathcal{D}}$, we have

$$\left|\int_{t_1}^{t_2} h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right| \le M|t_2 - t_1| < \mu.$$

Thus, when $|t_2 - t_1| < \delta$ ($t_1, t_2 \in [0,1]$), we have for all $(x,r,\lambda) \in \overline{\mathcal{D}} \times [0,1]$,

$$\left|\phi_p^{-1}\left(\lambda \int_0^{t_2} h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right) - \phi_p^{-1}\left(\lambda \int_0^{t_1} h\big(\tau, x(\tau) + r, x'(\tau)\big)\,\mathrm{d}\tau\right)\right| < \varepsilon.$$

This shows that $\{\frac{\mathrm{d}}{\mathrm{d}t} G_1(x,r,\lambda) : (x,r,\lambda) \in \overline{\mathcal{D}} \times [0,1]\}$ is equicontinuous on $[0,1]$. Therefore, $G_1(\overline{\mathcal{D}} \times [0,1])$ is a relatively compact set in $C^1[0,1]$ by Arzelà-Ascoli theorem.

Step 3. We prove that $G_2 : \overline{\mathcal{D}} \times [0,1] \to \mathbb{R}$ is a compact operator.

The compactness of the operator G_2 is clear, the proof is similar to the one of H_2 in Lemma 2.8.

In summary, $G : \overline{\mathcal{D}} \times [0,1] \to U$ is a compact operator. This completes the proof of the lemma. □

3 Main results

With the preparatory work in Sect. 2, we can now establish the existence results of solutions of problem (1.1).

Theorem 3.1 *Assume that* (H_1) *and* (H_2) *hold. Then boundary value problem* (1.1) *has at least one solution* $x = x(t)$ *satisfying* (2.3) *and* (2.4).

Proof We note that to obtain the existence of solutions satisfying (2.3) and (2.4) of problem (1.1), it is sufficient to prove that the modified problem (2.1), (2.2) has a solution $x = x(t)$ satisfying (2.3) and (2.4).

Below, we will prove in two steps.

Step 1. We prove that if the operator $G(\cdot,\cdot,1)$, which is defined in Lemma 2.10, has a fixed point, then the modified problem (2.1), (2.2) has a solution satisfying (2.3) and (2.4).

Suppose that (x_1, r_1) is a fixed point of $G(\cdot,\cdot,1)$. It follows from the definition of the operator G that

$$x_1(t) = \int_0^t \phi_p^{-1}\left(\int_0^s h\bigl(\tau, x_1(\tau) + r_1, x_1'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad \forall t \in [0,1]$$

and

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \phi_p^{-1}\left(\int_0^s h\bigl(\tau, x_1(\tau) + r_1, x_1'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s = 0.$$

Furthermore, we have

$$x_1'(t) = \phi_p^{-1}\left(\int_0^t h\bigl(\tau, x_1(\tau) + r_1, x_1'(\tau)\bigr)\,\mathrm{d}\tau\right), \quad \forall t \in [0,1],$$

and thus

$$x_1'(0) = 0, \qquad x_1(1) = \sum_{i=1}^{m-2} a_i x_1(\xi_i).$$

Setting $x_2(t) = x_1(t) + r_1$ for $t \in [0,1]$, it is easy to see that x_2 is a solution of the modified problem (2.1), (2.2) with $\lambda = 1$, and the validity of (2.3) and (2.4) follows from Lemma 2.4.

Step 2. We show that the operator $G(\cdot,\cdot,1)$ has a fixed point.

Notice that $G(\cdot,\cdot,0) = F(\cdot,\cdot)$, F is essential by Lemma 2.9, and G is a compact operator by Lemma 2.10. For the existence of a fixed point of $G(\cdot,\cdot,1)$, it is sufficient to verify that

$$G(x,r,\lambda) \neq (x,r), \quad \forall (x,r) \in \partial\mathcal{D}, \lambda \in [0,1].$$

Suppose on the contrary that $G(x_0, r_0, \lambda_0) = (x_0, r_0)$ for some $(x_0, r_0) \in \partial\mathcal{D}$ and $\lambda_0 \in [0,1]$. If $\lambda_0 = 0$, then from the proof of Lemma 2.10, we have $(x_0, r_0) \notin \partial\mathcal{D}$, which contradicts $(x_0, r_0) \in \partial\mathcal{D}$. If $\lambda_0 \in (0,1]$, then we have

$$x_0(t) = \int_0^t \phi_p^{-1}\left(\lambda_0 \int_0^s h\bigl(\tau, x_0(\tau) + r_0, x_0'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad \forall t \in [0,1]$$

and

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_0^s h\big(\tau, x_0(\tau)+r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s = 0. \tag{3.1}$$

Hence

$$x_0'(t) = \phi_p^{-1}\left(\lambda_0 \int_0^t h\big(\tau, x_0(\tau)+r_0, x_0'(t)\big)\,\mathrm{d}\tau\right), \quad \forall t \in [0,1].$$

In particular,

$$x_0'(0) = 0.$$

Notice that $\phi_p^{-1}(s_1 s_2) = \phi_p^{-1}(s_1)\phi_p^{-1}(s_2)$ for all $s_1, s_2 \in \mathbb{R}$ and (3.1), we have

$$\begin{aligned} x_0(1) - \sum_{i=1}^{m-2} a_i x_0(\xi_i) &= \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\lambda_0 \int_0^s h\big(\tau, x_0(\tau)+r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \phi_p^{-1}(\lambda_0) \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \phi_p^{-1}\left(\int_0^s h\big(\tau, x_0(\tau)+r_0, x_0'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= 0, \end{aligned}$$

that is,

$$x_0(1) = \sum_{i=1}^{m-2} a_i x_0(\xi_i).$$

Setting $x(t) = x_0(t) + r_0$ for $t \in [0,1]$. Then we can see that x is a solution of the modified problem (2.1), (2.2) with $\lambda = \lambda_0$. It follows from Lemma 2.4 that

$$r_1 \le x(t) = x_0(t) + r_0 \le r_2, \quad \forall t \in [0,1] \tag{3.2}$$

and

$$R_1 - 1 < R_1 \le x'(t) = x_0'(t) \le R_2 < R_2 + 1, \quad \forall t \in [0,1].$$

Notice that $x_0(0) = 0$, and so (3.2) yields $r_1 - 1 < r_1 \le r_0 \le r_2 < r_2 + 1$. Thus

$$r_1 - r_2 - 1 < r_1 - r_0 \le x_0(t) \le r_2 - r_0 < r_2 - r_1 + 1, \quad \forall t \in [0,1].$$

Consequently, $(x_0, r_0) \in \mathcal{D}$, which contradicts $(x_0, r_0) \in \partial\mathcal{D}$.

In summary, the conclusion of Theorem 3.1 holds. This completes the proof of the theorem. □

The following conclusion can be obtained by Theorem 3.1 immediately.

Corollary 3.1 *Assume that all the conditions of Theorem 3.1 with $r_1 = 0$ hold. Then problem (1.1) has at least one nonnegative solution $x = x(t)$ satisfying (2.3) and (2.4). Furthermore, the solution $x = x(t)$ is nondecreasing on $[0,1]$ provided $R_1 = 0$.*

The following theorems can be proved by using similar arguments to those of Theorem 3.1.

Theorem 3.2 *Assume that (H_1) and (H_3) hold. Then the boundary value problem (1.1) has at least one solution $x = x(t)$ satisfying (2.3) and (2.4).*

Corollary 3.2 *Assume that all the conditions of Theorem 3.2 with $r_1 = 0$ hold. Then problem (1.1) has at least one nonnegative solution $x = x(t)$ satisfying (2.3) and (2.4). Furthermore, the solution $x = x(t)$ is nondecreasing on $[0,1]$ provided $R_1 = 0$.*

Theorem 3.3 *Assume that (H_1) and (H_4) hold. Then the boundary value problem (1.1) has at least one solution $x = x(t)$ satisfying (2.3) and (2.4).*

Corollary 3.3 *Assume that all the conditions of Theorem 3.3 with $r_1 = 0$ hold. Then problem (1.1) has at least one nonnegative solution $x = x(t)$ satisfying (2.3) and (2.4). Furthermore, the solution $x = x(t)$ is nondecreasing on $[0,1]$ provided $R_1 = 0$.*

Theorem 3.4 *Assume that (H_1) and (H_5) hold. Then the boundary value problem (1.1) has at least one solution $x = x(t)$ satisfying (2.3) and (2.4).*

Remark 3.1 Theorem 3.4 takes Theorem 2.1 of [19, 20] as a special case.

Corollary 3.4 *Assume that all the conditions of Theorem 3.4 with $r_1 = 0$ hold. Then problem (1.1) has at least one nonnegative solution $x = x(t)$ satisfying (2.3) and (2.4). Furthermore, the solution $x = x(t)$ is nondecreasing on $[0,1]$ provided $R_1 = 0$.*

Remark 3.2 If the ϕ_p in problem (1.1) is replaced by ϕ, which is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and satisfies $\phi(s_1 s_2) = \phi(s_1)\phi(s_2)$ for all $s_1, s_2 \in \mathbb{R}$, Theorems 3.1 – 3.4 and Corollaries 3.1 – 3.4 are still true.

4 Some examples

As applications of our results, this section will provide four illustrative examples.

Example 4.1 Consider the m-point boundary value problem with p-Laplacian at resonance

$$\begin{cases} (\phi_p(x'(t)))' = \sum_{i=0}^{n} c_i (x(t))^i + P_l(x'(t)), & t \in [0,1], \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{4.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$; n is an even number, $c_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n)$; the polynomial $\sum_{i=0}^{n} c_i x^i$ has at least two distinct real roots with different signs; $P_l(y)$ is a polynomial of degree $l \in \mathbb{N}$, $P_l(0) = 0$; and $a_i > 0$ $(i = 1, 2, \ldots, m-2)$ with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Let

$$f(t,x,y) = \sum_{i=0}^{n} c_i x^i + P_l(y), \quad (t,x,y) \in [0,1] \times \mathbb{R}^2.$$

Obviously, $f \in C([0,1] \times \mathbb{R}^2)$. Let r_1 and r_2 be the minimum and maximum real root of the polynomial $\sum_{i=0}^{n} c_i x^i$, respectively. Then from the assumption it follows that $r_1 \le 0 \le r_2$, $r_1^2 + r_2^2 > 0$ and

$$f(t,r_1,0) = 0, \qquad f(t,r_2,0) = 0, \quad \forall t \in [0,1].$$

Hence, condition (H_1) of Theorem 3.1 is satisfied.

Let

$$F = \max_{r_1 \le x \le r_2} \left| \sum_{i=0}^{n} c_i x^i \right|.$$

Then, for $x \in [r_1, r_2]$, $y \in \mathbb{R}$, we have

$$P_l(y) - F \le f(t,x,y) \le P_l(y) + F.$$

Thus, if there exists $y_1 < 0$ such that $P_l(y_1) - F > 0$ and there exists $y_2 > 0$ such that $P_l(y_2) + F < 0$, then condition (H_2) of Theorem 3.1 is satisfied.

In summary, problem (4.1) has at least one solution $x = x(t)$ provided

$$\inf_{y>0} P_l(y) + F < 0 < \sup_{y<0} P_l(y) - F.$$

Note that if l is an odd number and the coefficient of y^l in $P_l(y)$ is negative, then the above inequality holds.

Example 4.2 Consider the m-point boundary value problem with p-Laplacian at resonance

$$\begin{cases} (\phi_p(x'(t)))' = x^2(t) - \frac{1}{4} - \frac{1}{2^n}(x'(t))^n \cos(\frac{\pi}{2}x'(t)), \quad t \in [0,1], \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{4.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, n is an even number, $a_i > 0$ $(i = 1,2,\dots,m-2)$ with $\sum_{i=1}^{m-2} a_i = 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Let

$$f(t,x,y) = x^2 - \frac{1}{4} - \frac{1}{2^n} y^n \cos\left(\frac{\pi}{2} y\right), \quad (t,x,y) \in [0,1] \times \mathbb{R}^2.$$

Obviously, $f \in C([0,1] \times \mathbb{R}^2)$. We choose $r_1 = -1/2$, $r_2 = 1/2$, $R_1 = -2$, $R_2 = 2$. Then

$$f(t,r_1,0) = f(t,r_2,0) = 0, \quad \forall t \in [0,1].$$

This implies that (H_1) holds. On the other hand, we have

$$f(t,x,R_1) = f(t,x,R_2) > 0, \quad \forall (t,x) \in [0,1] \times [r_1,r_2].$$

Moreover, we have

$$f(t,x,y) \le 1 < \frac{\phi_p(R_2)}{1-\xi_1}, \quad \forall (t,x,y) \in [0,1] \times \left[-\frac{1}{2},\frac{1}{2}\right] \times [-2,2].$$

Hence, (H_3) holds. From Theorem 3.2, problem (4.2) has at least one solution.

Example 4.3 Consider the m-point boundary value problem with p-Laplacian at resonance

$$\begin{cases} (\phi_p(x'(t)))' = x^2(t) - \frac{1}{4} + (x'(t))^3 \sin(\frac{\pi}{2}x'(t)), & t \in [0,1], \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{4.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $a_i > 0$ $(i = 1,2,\dots,m-2)$ with $\sum_{i=1}^{m-2} a_i = 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Let

$$f(t,x,y) = x^2 - \frac{1}{4} + y^3 \sin\left(\frac{\pi}{2}y\right), \quad (t,x,y) \in [0,1] \times \mathbb{R}^2.$$

Clearly, $f \in C([0,1] \times \mathbb{R}^2)$. We select $r_1 = -1/2$, $r_2 = 1/2$, $R_1 = -2$, $R_2 = 2$. Then

$$f(t,r_1,0) = f(t,r_2,0) = 0, \quad \forall t \in [0,1].$$

This implies that (H_1) holds. On the other hand, we have

$$f(t,x,R_1) = f(t,x,R_2) \le 0, \quad \forall t \in [0,1] \times [r_1,r_2]$$

and

$$f(t,x,y) > -1 > \frac{\phi_p(R_1)}{1-\xi_1}, \quad \forall (t,x,y) \in [0,1] \times \left[-\frac{1}{2},\frac{1}{2}\right] \times [-2,2].$$

Hence, (H_4) is also true. From Theorem 3.3, problem (4.3) has at least one solution.

Example 4.4 Consider the m-point boundary value problem with p-Laplacian at resonance

$$\begin{cases} (\phi_p(x'(t)))' = \frac{1}{4} - \frac{1}{2}x(t)\cos(2\pi x(t)) + \frac{1}{2^{n+1}}(x'(t))^n \sin(\frac{\pi}{4}x'(t)), & t \in [0,1], \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases} \tag{4.4}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, n is an even number, $a_i > 0$ $(i = 1,2,\dots,m-2)$ with $\sum_{i=1}^{m-2} a_i = 1$, and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Let

$$f(t,x,y)=\frac{1}{4}-\frac{1}{2}x\cos(2\pi x)+\frac{1}{2^{n+1}}y^n\sin\left(\frac{\pi}{4}y\right),\quad (t,x,y)\in[0,1]\times\mathbb{R}^2.$$

Evidently, $f\in C([0,1]\times\mathbb{R}^2)$. We choose $r_1=-1/2$, $r_2=1/2$, $R_1=-2$, $R_2=2$. Then, for all $t\in[0,1]$, we have

$$f(t,r_1,0)=0,\qquad f(t,r_2,0)>0.$$

This implies that (H_1) holds. On the other hand, for all $(t,x)\in[0,1]\times[r_1,r_2]$, it follows that

$$f(t,x,R_1)\le 0,\qquad f(t,x,R_2)>0.$$

Moreover, when $(t,x,y)\in[0,1]\times[-1/2,1/2]\times[-2,2]$, we have

$$\frac{\phi_p(R_1)}{1-\xi_1}<f(t,x,y)<\frac{\phi_p(R_2)}{1-\xi_1}.$$

This shows that (H_5) is satisfied. From Theorem 3.4, problem (4.4) has at least one solution.

Abbreviations
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Author contributions
This work was carried out in collaboration between the three authors. MP designed the study and guided the research. ML and LW performed the analysis and wrote the first draft of the manuscript. ML, MP, and LW managed the analysis of the study. The three authors read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
We certify that this manuscript is original and has not been published and has not been submitted elsewhere for publication while being considered by Boundary Value Problems. In addition, the study is not split up into several parts to increase the number of submissions and to be submitted to various journals or to one journal over time.

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