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Finite-time blowup for the 3-D viscous primitive equations of oceanic and atmospheric dynamics

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Abstract

In this paper, we prove that for certain class of initial data, the corresponding solutions to the 3-D viscous primitive equations blow up in finite time. Specifically, we find a special solution to simplify the 3-D systems, assuming that the pressure function $p(x, y, t)$ is a concave function. We also consider the equations on the line $x = 0$, $y = 0$.

Keywords: Blow-up; Primitive equations; Viscous; Special solution

1 Introduction

To the best of our knowledge, many researchers devote their studies to some nonlinear partial differential equations using mathematical physics methods [1–10]. Many works have investigated the primitive equations of ocean and atmospheric dynamics since the 1990s. In [7–13], the authors have shown the global well-posedness of strong solutions to primitive systems. In [14], it was shown that for certain class of smooth initial data, the solutions of the 2-D inviscid primitive equations are blown up in finite time by looking for a self-similar solution. In [15], the blow-up for the 2-D Prandtl equation in the maximum norm of u_x or u_{xy} was introduced.

It is worth mentioning that it has been proved that for certain class of initial data, the corresponding solutions of the 3-D primitive equations without viscosity blow up in finite time by looking for a self-similar solution in [16]. However, we cannot find the self-similar solution to the viscous system.

In 2023, the corresponding solutions of the 2-D viscous primitive equations were proven in [17]; the main problem was to solve the viscosity term. In this paper, we study the corresponding solutions to the 3-D primitive equations with viscosity. We need not only to solve the viscosity similarly to the method described in [17] but also to consider the effect of dimensional changes.

The three-dimensional primitive equations for large-scale oceanic and atmospheric dynamics are given by the system of partial differential equations:

$$u_t + uu_x + vu_y + wu_z + p_x - Rv = \nu_H \Delta_H u + \nu_3 u_{zz}, \tag{1.1}$$

$$v_t + uv_x + vv_y + wv_z + p_y + Ru = \nu_H \Delta_H v + \nu_3 v_{zz}, \tag{1.2}$$

$$p_z + T = 0, \tag{1.3}$$

$$T_t + uT_x + vT_y + wT_z = Q + \kappa_H \Delta_H T + \kappa_3 T_{zz}, \tag{1.4}$$

$$u_x + v_y + w_z = 0, \tag{1.5}$$

with the initial value (u_0, v_0, T_0) and the relevant geophysical boundary conditions. Here, the horizontal velocity (u, v), the vertical velocity w, the temperature T, and the pressure p are the unknowns. R is the rotation parameter, ν_H is the horizontal viscosity parameter, ν_3 is the vertical viscosity parameter, κ_H is the horizontal diffusion parameter, and κ_3 is the vertical diffusion parameter. Here, $\Delta_H = \partial_{xx} + \partial_{yy}$ denotes the horizontal Laplacian operator.

2 Derivation of the reduced equations

In this section, we try to construct a special solution to the 3-D primitive equations and get the expression of the term of pressure to simplify the system. Then, we restrict the reduced equation on the surface $x = 0$, $y = 0$.

For simplicity, we take $T(x, y, z, t) = 0$ and consider the simplified primitive equations without the Coriolis force

$$u_t + uu_x + vu_y + wu_z + p_x = u_{zz}, \tag{2.1}$$

$$v_t + uv_x + vv_y + wv_z + p_y = v_{zz}, \tag{2.2}$$

$$p_z = 0, \tag{2.3}$$

$$u_x + v_y + w_z = 0, \tag{2.4}$$

in the horizontal channel $\Omega = \{(x, y, z) : 0 \leq z \leq H, (x, y) \in D \subset R^2\}$ and $t \in [0, T)$, with initial and boundary conditions:

$$(u, v)|_{t=0} = (u_0, v_0)(x, y, z), \tag{2.5}$$

$$(u, v, w)|_{z=0} = (u, v, w)|_{z=H} = 0, \tag{2.6}$$

$$(u, v)|_{\partial D} = (u_1, v_1). \tag{2.7}$$

Specifically, we assume that $p(x, y, t)$ is a concave function, with respect to the variable x. Then, we have

$$p_{xx} \geq 0. \tag{2.8}$$

We construct the solution to the system (2.1)–(2.4) with the structure

$$\big(u(x, y, z, t), k(x, y, t)u(x, y, z, t), w(x, y, z, t)\big), \tag{2.9}$$

with $u(x, y, z, t)$ being strictly increasing in $0 \leq z \leq H$,

$$u_z > 0. \tag{2.10}$$

After the above assumptions, we have

$$v_0 = ku_0, \qquad v_1 = ku_1. \tag{2.11}$$

Plugging the form (2.9) into (2.2) and using (2.1), we get

$$uk_t + u^2 k_x + ku^2 k_y - kp_x + p_y = 0. \tag{2.12}$$

By differentiating (2.12) with respect to z and using (2.10), we have

$$k_t + 2u(k_x + kk_y) = 0. \tag{2.13}$$

By differentiating (2.13) with respect to z, we get

$$(k_x + kk_y) = 0. \tag{2.14}$$

Combing (2.13) and (2.14), we get $k_t = 0$. Plugging these equations into (2.12) gives the following

$$p_y - kp_x = 0. \tag{2.15}$$

The systems to be studied in this paper can be formulated as follows

$$u_t + uu_x + vu_y + wu_z + p_x = u_{zz}, \tag{2.16}$$

$$v_t + uv_x + vv_y + wv_z + p_y = v_{zz}, \tag{2.17}$$

$$u_x + v_y + w_z = 0, \tag{2.18}$$

with initial and boundary conditions:

$$(u,v)|_{t=0} = (u_0, ku_0)(x,y,z), \tag{2.19}$$

$$(u,v,w)|_{z=0} = (u,v,w)|_{z=H} = 0, \tag{2.20}$$

$$(u,v)|_{\partial D} = (u_1, ku_1), \tag{2.21}$$

there $u_z(x,y)$, $k(x,y)$ and $p(x,y,t)$ satisfy (2.10), (2.14), and (2.15).

Lemma 2.1 *If* $(u,v,w)(x,y,z,t)$ *is the classical solution to* (2.16)–(2.21), *then* $v(x,y,z,t) = k(x,y)u(x,y,z,t)$. *See Lemma* 2.1 *in* [16].

Therefore, studying the problem (2.16)–(2.21) is equivalent to studying the following reduced problem for only two unknown functions u and w

$$u_t + uu_x + ku \cdot u_y + wu_z + p_x = u_{zz}, \tag{2.22}$$

$$u_x + (ku)_y + w_z = 0, \tag{2.23}$$

with initial and boundary condition

$$(u,w)|_{z=0} = (u,w)|_{z=H} = 0, \tag{2.24}$$

$$u|_{\partial D} = u_1, \tag{2.25}$$

$$u|_{t=0} = u_0(x,y,z). \tag{2.26}$$

In addition, we impose the following condition

$$\begin{aligned} &u(x,y,z,t) = -u(-x,y,z,t); \qquad w(x,y,z,t) = w(-x,y,z,t);\\ &p(x,y,t) = p(-x,y,z,t). \end{aligned} \tag{2.27}$$

By differentiating equation (2.22) with respect to x, we obtain

$$u_{tx} + u_x^2 + uu_{xx} + (ku)_x u_y + ku \cdot u_{xy} + w_x u_z + wu_{xz} + p_{xx} = u_{zzx}. \tag{2.28}$$

By averaging (2.28) with respect to the z variable over $[0,H]$ and multiplying $\frac{1}{H}$, we obtain

$$\frac{1}{H}\int_o^H \big[u_{tx} + 2u_x^2 + 2uu_{xx} + (ku\cdot u)_{xy} - u_{zzx}\big]\,dz + p_{xx} = 0. \tag{2.29}$$

Thus,

$$\frac{1}{H}\int_o^H \big[u_t + 2uu_x + (ku\cdot u)_y - u_{zz}\big]\,dz + p_x = C(y,t) \tag{2.30}$$

for some function $C(y,t)$. Due to (2.28), we know that p_x and u are odd functions, with respect to the variable x, then

$$C(y,t) = 0, \tag{2.31}$$

and consequently

$$p_x = -\frac{1}{H}\int_0^H \big[u_t + 2uu_x + (ku\cdot u)_y - u_{zz}\big]\,dz. \tag{2.32}$$

Substituting (2.32) into system (2.22), we obtain the closed system

$$u_t + uu_x + ku\cdot u_y + wu_z - u_{zz} = -\frac{1}{H}\int_0^H \big[u_t + 2uu_x + (ku\cdot u)_y - u_{zz}\big]\,dz. \tag{2.33}$$

By differentiating with respect to x, we have

$$\begin{aligned} &u_{tx} + u_x^2 + uu_x + (ku)_x u_y + ku\cdot u_{xy} + w_x u_z + wu_{xz} - u_{zzx}\\ &\quad = \frac{1}{H}\int_0^H \big[u_{tx} + 2u_x^2 + 2uu_{xx} + (ku\cdot u)_{yx} - u_{xzz}\big]\,dz. \end{aligned} \tag{2.34}$$

Let us consider the restriction of the evolution of equation (2.34) on the surface $x = 0$, $y = 0$. Since u is an odd function, and w is an even function, with respect to the variable x, we have

$$u(0,0,z,t) = 0; \qquad w_x(0,0,z,t) = 0. \tag{2.35}$$

This, together with (2.34), implies

$$\begin{aligned} &u_{tx}(0,0,z,t) + \big(u_x(0,0,z,t)\big)^2 - \int_0^z u_x\,dz \cdot u_{xz}(0,0,z,t) \\ &\quad = \frac{1}{H}\int_0^H \big[u_{tx}(0,0,z,t) + 2\big(u_x(0,0,z,t)\big)^2 - u_{xzz}\big]\,dz. \end{aligned} \tag{2.36}$$

Denoting by

$$a(z,t) = -u_x(0,0,z,t), \tag{2.37}$$

we obtain

$$a_t = a^2 + a_{zz} - \int_0^z a\,dz \cdot a_z - \frac{1}{H}\int_0^H \big(-a_t + 2a^2 + a_{zz}\big)\,dz, \tag{2.38}$$

with the initial and boundary conditions

$$a(z,0) = a_0(z), \qquad a(0,t) = 0, \qquad a(H,t) = 0. \tag{2.39}$$

In particular, due to equations (2.8) and (2.32), we have

$$f(t) = \frac{1}{H}\int_0^H \big(-a_t + 2a^2 + a_{zz}\big)\,dz \le 0. \tag{2.40}$$

3 Main result and the proof

In this section, we will give Lemma 3.1 and the main result.

Lemma 3.1 *Define*

$$F(a) = \int_0^H a^2\,dz, \qquad E(a) = \int_0^H \left(\frac{1}{2}a_z^2 - \frac{1}{4}a^3\right)dz, \qquad G = -\frac{E}{F^\beta}, \tag{3.1}$$

where $\beta \in (1, \frac{5}{4})$. *Let* a_0 *be nonnegative, compactly supported initial data such that* $E(a_0) < 0$, $a_{zz}(0,t) = 0$ *and* $a_{zz}(H,t) = 0$. *Then, there exists a finite time* T *such that either*

$$\lim_{t\to T} \max_z a = +\infty, \tag{3.2}$$

or at least one of the following results

$$\lim_{t\to T} a_z(0,t) = +\infty, \tag{3.3}$$

$$\lim_{t\to T} a_z(H,t) = +\infty. \tag{3.4}$$

Remark 3.1 Using Lemma 2.1, we reduce the system (2.1)–(2.7) to the system (2.22)–(2.26) with two unknown functions and three variables. Then, we consider the problem on the surface $x = 0$ and $y = 0$. So, the proof of Lemma 3.1 takes the same method as in [17].

Theorem 3.1 *Assuming that* $p_{xx} \geq 0$ *and the conditions* (2.10) *and* (2.27) *are satisfied, we can get Lemma* 3.1. *Then, smooth solutions to* (2.16)–(2.21) *do not exist globally in time.*

Remark 3.2 In the Lemma 3.1, $a(z,t) = -u_x(0,0,z,t)$, so the blowup is in the norm of u_x or u_{xy}.

4 Conclusions

This study has shown that the 3-D primitive equations with viscosity blow up on the surface $x = 0$ and $y = 0$. In this paper, we consider the simplified equations. We plan to solve unreduced systems in the future.

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Lin Zheng wrote the whole manuscript and reviewed it.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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