Choucha et al. *Boundary Value Problems* (2024) 2024:57
https://doi.org/10.1186/s13661-024-01864-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On a class of a coupled nonlinear viscoelastic Kirchhoff equations variable-exponents: global existence, blow up, growth and decay of solutions

Abdelbaki Choucha[1,2*], Mohamed Haiour[3] and Salah Boulaaras[4*]

*Correspondence:
abdelbaki.choucha@gmail.com;
abdel.choucha@lagh-univ.dz;
S.Boularas@qu.edu.sa
[1]Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Laghouat, Algeria
[4]Department of Mathematics, College of Science, Qassim University, 51452 Buraydah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this work, we consider a quasilinear system of viscoelastic equations with dispersion, source, and variable exponents. Under suitable assumptions on the initial data and the relaxation functions, we obtained that the solution of the system is global and bounded. Next, the blow-up is proved with negative initial energy. After that, the exponential growth of solutions is showed with positive initial energy, and by using an integral inequality due to Komornik, the general decay result is obtained in the case of absence of the source term.

Mathematics Subject Classification: 35B40; 35L70; 93D20

Keywords: Viscoelastic equation; Global existence; Blow up; Exponential growth; General decay; Variable exponents

1 Introduction

Numerous previous research works in the field of problems with variable exponents acknowledged the role of dampers which appear in many areas of applied sciences. The objective of this work is to provide further insight into the complex interactions that exist between dampers and variable exponents.

In the absence of a variable exponent, the reader can view the following papers related to the topic being studied: the general decay, blow-up, and growth of solutions [1, 3–5, 7–9, 11–15].

We present our current problem of a quasilinear system of viscoelastic equations with dispersion, source, and variable exponents, where we combine several damping terms into one, of course in the general case ($\eta \geq 0$).

In this work, we try to collect many studies in one paper, where the reader has in his hands four different proofs with the methods used under appropriate conditions, while comparing the differences.

First, we list some previous works similar to ours. Whereas in the absence of the source term with ($\eta = 0$), the study is found in [16], the authors studied the global existence and general decay of solutions for a quasilinear system with degenerate damping terms.

As for the presence of the source and degenerate damping terms in the absence of dispersion term, we mention the work done by the authors in [26], where they obtained global existence of solutions. Then, they proved the general decay result. Finally, they proved the finite time blow-up result of solutions with negative initial energy. For more information in this context, you can see the papers [6, 17, 18, 20, 22, 23].

On the other hand, there are many works that deal with the variable exponent. We mention, for example, our work [24]. In the presence of delay, the authors have demonstrated a nonlinear Kirchhoff-type equation with distributed delay and variable exponents. Under a suitable hypothesis they proved the blow-up of solutions, and by using an integral inequality due to Komornik, they obtained the general decay result. See also [2, 10, 21, 29], and [25], each of which examines a different problem with appropriate conditions.

In this work, we are examining the following problem:

$$\begin{cases} |\Phi_t|^\eta \Phi_{tt} - \mathcal{T}(\|\nabla\Phi\|_2^2)\Delta\Phi + \int_0^t h_1(t-\varsigma)\Delta\Phi(\varsigma)\,d\varsigma - \Delta\Phi_{tt} + g_1(\Phi_t) = f_1(\Phi,\Psi), \\ |\Psi_t|^\eta \Psi_{tt} - \mathcal{T}(\|\nabla\Psi\|_2^2)\Delta\Psi + \int_0^t h_2(t-\varsigma)\Delta\Psi(\varsigma)\,d\varsigma - \Delta\Psi_{tt} + g_2(\Psi_t) = f_2(\Phi,\Psi), \\ \Phi(x,t) = \Psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \Phi(x,0) = \Phi_0(x), \qquad \Phi_t(x,0) = \Phi_1(x), \quad x \in \Omega, \\ \Psi(x,0) = \Psi_0(x), \qquad \Psi_t(x,0) = \Psi_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where

$$g_1(\Phi_t) := \zeta_1 \big|\Phi_t(t)\big|^{m(x)-2}\Phi_t(t), \qquad g_2(\Psi_t) := \zeta_3 \big|\Psi_t(t)\big|^{s(x)-2}\Psi_t(t),$$

in which $\eta \geq 0$ for $N = 1,2$ and $0 < \eta \leq \frac{2}{N-2}$ for $N \geq 3$, and $h_i(\cdot) : R^+ \to R^+$ $(i = 1,2)$ symbolizes the relaxation function, $-\Delta_{tt}(\cdot)$ symbolizes the dispersion term, and $\mathcal{T}(\sigma)$ is a positive locally Lipschitz function for $\gamma, \sigma \geq 0$ such that $\mathcal{T}(\sigma) = \alpha_1 + \alpha_2\sigma^\gamma$, and

$$\begin{cases} f_1(\Phi,\Psi) = a_1|\Phi+\Psi|^{2(q(x)+1)}(\Phi+\Psi) + b_1|\Phi|^{q(x)}.\Phi.|\Psi|^{q(x)+2}, \\ f_2(\Phi,\Psi) = a_1|\Phi+\Psi|^{2(q(x)+1)}(\Phi+\Psi) + b_1|\Psi|^{q(x)}.\Psi.|\nu|^{q(x)+2}. \end{cases} \tag{1.2}$$

In this context, we consider $q(\cdot)$, $m(\cdot)$, and $s(\cdot)$ are variable exponents defined as measurable functions on $\overline{\Omega}$ in the following manner:

$$\begin{aligned} &1 \leq q^- \leq q(x) \leq q^+ \leq q^*, \\ &2 \leq m^- \leq m(x) \leq m^+ \leq m^*, \\ &2 \leq s^- \leq s(x) \leq s^+ \leq s^*, \end{aligned} \tag{1.3}$$

where

$$q^- = \inf_{x\in\overline{\Omega}} q(x), \qquad m^- = \inf_{x\in\overline{\Omega}} m(x), \qquad s^- = \inf_{x\in\overline{\Omega}} s(x),$$

$$q^+ = \sup_{x\in\overline{\Omega}} q(x), \qquad m^+ = \sup_{x\in\overline{\Omega}} m(x), \qquad s^+ = \sup_{x\in\overline{\Omega}} s(x), \tag{1.4}$$

with

$$\max\{m^+, s^+\} \le 2q^- + 1 \tag{1.5}$$

and

$$m^*, s^* = \frac{2(n-1)}{n-2} \quad \text{if } n \ge 3. \tag{1.6}$$

As for the division of the paper, it is as follows. In the following section, we present the hypotheses, concepts, and lemmas essential for our study. In Sect. 3, we obtain global existence of the solution of (1.1). Next, Sects. 4 and 5 are dedicated to proving the blow-up result, followed by the exponential growth of solutions. In Sect. 6, we establish the general decay when $f_1 = f_2 = 0$. Finally, we present the general conclusion in the last section.

2 Preliminaries

In this section, we give some related theory and put suitable hypotheses for the proof of our result.

(*H1*) Put a nonincreasing and differentiable function $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ where

$$h_i(t) \ge 0, \quad 1 - \int_0^\infty h_i(\varsigma)\, d\varsigma = l_i > 0, \quad i = 1, 2. \tag{2.1}$$

(*H2*) One can find $\xi_1, \xi_2 > 0$ in a way that

$$h_i'(t) \le -\xi_i h_i(t), \quad t \ge 0, i = 1, 2. \tag{2.2}$$

Lemma 2.1 *There exists $F(\Phi, \Psi)$ defined by*

$$\begin{aligned} F(\Phi, \Psi) &= \frac{1}{2(q(x)+2)}\big[\Phi f_1(\Phi, \Psi) + \Phi f_2(\Phi, \Psi)\big] \\ &= \frac{1}{2(q(x)+2)}\big[a_1|\Phi + \Psi|^{2(q(x)+2)} + 2b_1|\Phi.\Psi|^{q(x)+2}\big] \ge 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial \Phi} = f_1(\Phi, \Psi), \qquad \frac{\partial F}{\partial \Psi} = f_2(\Phi, \Psi).$$

For simplification, we put $\alpha_1 = \alpha_2 = 1$ and $a_1 = b_1 = 1$ for convenience.

Lemma 2.2 [6] *One can find $c_0 > 0$ and $c_1 > 0$ in a way that*

$$\begin{aligned} \frac{c_0}{2(q(x)+2)}\big(|\Phi|^{2(q(x)+2)} + |\Psi|^{2(q(x)+2)}\big) &\le F(\Phi, \Psi) \\ &\le \frac{c_1}{2(q(x)+2)}\big(|\Phi|^{2(q(x)+2)} + |\Psi|^{2(q(x)+2)}\big). \end{aligned} \tag{2.3}$$

Now, we consider $q : \Omega \to [1, \infty)$ is a measurable function.

After, introducing the Lebesgue space with a variable exponent $q(\cdot)$ as follows:

$$L^{q(\cdot)}(\Omega) = \left\{ \Phi : \Omega \to \mathbb{R};\ \text{measurable in } \Omega : \int_\Omega |\Phi|^{q(\cdot)}\, dx < \infty \right\},$$

we give the norm as follows:

$$\|\Phi\|_{q(\cdot)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{\Phi}{\lambda} \right|^{q(x)} dx \le 1 \right\}.$$

This space is fitted with the standard norm, $L^{q(\cdot)}(\Omega)$ is a Banach space. After that, we introduce the variable exponent Sobolev space $W^{1,q(\cdot)}(\Omega)$ as follows:

$$W^{1,q(\cdot)}(\Omega) = \left\{ \Phi \in L^{q(\cdot)}(\Omega); \nabla\Phi \text{ exists and } |\nabla\Phi| \in L^{q(\cdot)}(\Omega) \right\},$$

with the norm given by

$$\|\Phi\|_{1,q(\cdot)} = \|\Phi\|_{q(\cdot)} + \|\nabla\Phi\|_{q(\cdot)},$$

$W^{1,q(\cdot)}(\Omega)$ is a Banach space, and the closure of $C_0^\infty(\Omega)$ is given by $W_0^{1,q(\cdot)}(\Omega)$.

For $\Phi \in W_0^{1,q(\cdot)}(\Omega)$, we give the equivalent norm

$$\|\Phi\|_{1,q(\cdot)} = \|\nabla\Phi\|_{q(\cdot)}.$$

$W_0^{-1,q'(\cdot)}(\Omega)$ denotes the dual of $W_0^{1,q(\cdot)}(\Omega)$ in which $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.

Next, we offer the continuity condition of Log-Hölder:

$$\left|p(x) - p(y)\right| \le -\frac{M_1}{\log|x-y|} \quad \text{and} \quad \left|m(x) - m(y)\right| \le -\frac{M_2}{\log|x-y|} \tag{2.4}$$

for all $x, y \in \Omega$, where $M_1, M_2 > 0$ and $0 < \varrho < 1$ with $|x - y| < \varrho$.

Theorem 2.3 *Assume that* (2.1)–(2.2) *hold. Then, for any* $(\Phi_0, \Phi_1, \Psi_0, \Psi_1) \in \mathcal{H}$, (1.1) *has a unique solution for some* $T > 0$:

$$\begin{aligned}
&\Phi, \Psi \in C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big),\\
&\Phi_t \in C\big([0,T]; H_0^1(\Omega)\big) \cap L^{m(x)}\big(\Omega \times (0,T)\big),\\
&\Psi_t \in C\big([0,T]; H_0^1(\Omega)\big) \cap L^{s(x)}\big(\Omega \times (0,T)\big),
\end{aligned}$$

where

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

Now, we define the functional of energy.

Lemma 2.4 *Let* (2.1)–(2.2) *be satisfied and* (Φ,Ψ) *be a solution of* (1.1). *Then the functional* $\mathcal{E}(t)$ *is nonincreasing and defined as follows*:

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{\eta+2}\big[\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big] + \frac{1}{2}\big[\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big] \\ &\quad + \frac{1}{2(\gamma+1)}\big[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big] \\ &\quad + \frac{1}{2}\bigg[\bigg(1-\int_0^t h_1(\varsigma)\,d\varsigma\bigg)\|\nabla\Phi\|_2^2 + \bigg(1-\int_0^t h_2(\varsigma)\,d\varsigma\bigg)\|\nabla\Psi\|_2^2\bigg] \\ &\quad + \frac{1}{2}\big[(h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\big] - \int_\Omega F(\Phi,\Psi)\,dx \end{aligned} \tag{2.5}$$

fulfills

$$\begin{aligned}\mathcal{E}'(t) &= \frac{1}{2}\big[\big(h_1' o\nabla\Phi\big)(t) + \big(h_2' o\nabla\Psi\big)(t)\big] - \frac{1}{2}\big[h_1(t)\|\nabla\Phi\|_2^2 + h_2(t)\|\nabla\Psi\|_2^2\big] \\ &\quad - \zeta_1\int_\Omega \big|\Phi_t(t)\big|^{m(x)}\,dx - \zeta_3\int_\Omega \big|\Psi_t(t)\big|^{s(x)}\,dx \\ &\le 0. \end{aligned} \tag{2.6}$$

Proof By multiplying $(1.1)_1$, $(1.1)_2$ by Φ_t, Ψ_t and integrating over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\bigg\{\frac{1}{\eta+2}\|\Phi_t\|_{\eta+2}^{\eta+2} + \frac{1}{\eta+2}\|\Psi_t\|_{\eta+2}^{\eta+2} + \frac{1}{2}\|\nabla\Phi_t\|_2^2 + \frac{1}{2}\|\nabla\Psi_t\|_2^2 \\ &\quad + \frac{1}{2(\gamma+1)}\big[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big] \\ &\quad + \frac{1}{2}\bigg(1-\int_0^t h_1(\varsigma)\,d\varsigma\bigg)\|\nabla\Phi\|_2^2 + \frac{1}{2}\bigg(1-\int_0^t h_2(\varsigma)\,d\varsigma\bigg)\|\nabla\Psi\|_2^2 \\ &\quad + \frac{1}{2}(h_1 o\nabla\Phi)(t) + \frac{1}{2}(h_2 o\nabla\Psi)(t) - \int_\Omega F(\Phi,\Psi)\,dx\bigg\} \\ &= -\zeta_1\int_\Omega \big|\Phi_t(t)\big|^{m(x)}\,dx - \zeta_3\int_\Omega \big|\Psi_t(t)\big|^{s(x)}\,dx \\ &\quad + \frac{1}{2}\big(h_1' o\nabla\Phi\big) - \frac{1}{2}h_1(t)\|\nabla\Phi\|_2^2 + \frac{1}{2}\big(h_2' o\nabla\Psi\big) - \frac{1}{2}h_2(t)\|\nabla\Psi\|_2^2. \end{aligned} \tag{2.7}$$

Hence, we find (2.5) and (2.6). Then, we have $\mathcal{E}$ is a nonincreasing function. This ends the proof. □

3 Global existence

Now, we show that the solution of (1.1) is uniformly bounded and global in time. For this purpose, we set

$$\begin{aligned} I(t) &= \bigg[\bigg(1-\int_0^t h_1(\varsigma)\,d\varsigma\bigg)\|\nabla\Phi\|_2^2 + \bigg(1-\int_0^t h_2(\varsigma)\,d\varsigma\bigg)\|\nabla\Psi\|_2^2\bigg] \\ &\quad + \frac{1}{(\gamma+1)}\big[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big] \\ &\quad + \big[(h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\big] - 2\big(q^- + 2\big)\int_\Omega F(\Phi,\Psi)\,dx, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J(t) &= \frac{1}{2}\left[\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right] \\ &\quad + \frac{1}{2(\gamma+1)}\left[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right] \\ &\quad + \frac{1}{2}\left[(h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\right] - \int_\Omega F(\Phi,\Psi)\,dx. \end{aligned} \tag{3.2}$$

Hence,

$$\mathcal{E}(t) = J(t) + \frac{1}{\eta+2}\left[\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right] + \frac{1}{2}\left[\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right]. \tag{3.3}$$

Lemma 3.1 *Suppose that the initial data* $(\Phi_0,\Phi_1),(\Psi_0,\Psi_1) \in (H^1(\Omega)\times L^2(\Omega))^2$ *satisfy* $I(0) > 0$ *and*

$$\widehat{\xi} := \frac{c_1 C_*(p^+)}{l}\left(\frac{(2p^-+4)\mathcal{E}(0)}{l(p^-+1)}\right)^{p^++1} < 1. \tag{3.4}$$

Then $I(t) > 0$ *for any* $t \in [0,T]$.

Proof Since $I(0) > 0$, we deduce by continuity that there exists $0 < T^* \leq T$ such that $I(t) \geq 0$ for all $t \in [0,T^*]$.

This implies that $\forall t \in [0,T^*]$,

$$\begin{aligned} J(t) &\geq \frac{q^-+1}{2(q^-+2)}\left[\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right] \\ &\quad + \frac{q^-+1}{2(q^-+2)}\left\{\frac{1}{(\gamma+1)}\left[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right]\right\} \\ &\quad + \frac{q^-+1}{2(q^-+2)}\left[(h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\right] + \frac{1}{2(q^-+2)}I(t) \\ &\geq \frac{q^-+1}{2(q^-+2)}\left\{l_1\|\nabla\Phi\|_2^2 + l_2\|\nabla\Psi\|_2^2 + \frac{1}{(\gamma+1)}\left[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right]\right. \\ &\quad \left. + (h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\right\} \\ &\geq \frac{q^-+1}{2(q^-+2)}\left(l_1\|\nabla\Phi\|_2^2 + l_2\|\nabla\Psi\|_2^2\right). \end{aligned} \tag{3.5}$$

Hence, by (2.6) and (3.3), we get

$$\begin{aligned} \left(l_1\|\nabla\Phi\|_2^2 + l_2\|\nabla\Psi\|_2^2\right) &\leq \frac{2(q^-+2)}{q^-+1}J(t) \leq \frac{2(q^-+2)}{q^-+1}\mathcal{E}(t) \\ &\leq \frac{2(q^-+2)}{q^-+1}\mathcal{E}(0), \quad \forall t \in \left[0,T^*\right]. \end{aligned} \tag{3.6}$$

On the other hand, by using (2.3), we get

$$2\left(q^-+2\right)\int_\Omega F(\Phi,\Psi)\,dx < c_1\int_\Omega\left(|\Phi|^{2p^++4)} + |\Psi|^{2p^++4)}\right)dx. \tag{3.7}$$

Then the embedding $H_0^1(\Omega)) \hookrightarrow L^{2p^+ +4}(\Omega)$ yields

$$\begin{aligned}\int_\Omega \big(|\Phi|^{2p^+ +4} + |\Psi|^{2p^+ +4)}\big)\,dx &\le C_*\big(p^+\big)\big(\big\|\nabla\Phi(t)\big\|_2^{2p^+ +4} + \big\|\nabla\Psi(t)\big\|_2^{2p^+ +4}\big)\\ &= C_*\big(p^+\big)\big\{\big\|\nabla\Phi(t)\big\|_2^2\big\|\nabla\Phi(t)\big\|_2^{2p^+ +2}\\ &\quad + \big\|\nabla\Psi(t)\big\|_2^2\big\|\nabla\Psi(t)\big\|_2^{2p^+ +2}\big\}. \end{aligned} \tag{3.8}$$

By (3.6), we find

$$\begin{aligned} 2\big(q^- + 2\big)\int_\Omega F(\Phi,\Psi)\,dx &< c_1 C_*\big(p^+\big)\left(\frac{2(p^- + 2)\mathcal{E}(0)}{l_1(p^- + 1)}\right)^{p^+ +1}\big\|\nabla\Phi(t)\big\|_2^2\\ &\quad + c_1 C_*\big(p^+\big)\left(\frac{2(p^- + 2)\mathcal{E}(0)}{l_2(p^- + 1)}\right)^{p^+ +1}\big\|\nabla\Psi(t)\big\|_2^2\\ &< \widehat{\xi}\big(l_1\big\|\nabla\Phi(t)\big\|_2^2 + l_2\big\|\nabla\Psi(t)\big\|_2^2\big), \end{aligned} \tag{3.9}$$

where

$$\widehat{\xi} = c_1 C_*\big(p^+\big) l^{-(p^+ +2)}\left(\frac{(2p^- + 4)\mathcal{E}(0)}{(p^- + 1)}\right)^{p^+ +1},$$

the embedding constant $C_*(p^+)$ and $l = \min(l_1, l_2)$.

By (3.4) and (2.1), we obtain

$$\begin{aligned} 2\big(q^- + 2\big)\int_\Omega F(\Phi,\Psi)\,dx &< \xi\left(1 - \int_0^t h_1(\varsigma)\,d\varsigma\right)\big\|\nabla\Phi(t)\big\|_2^2\\ &\quad + \widehat{\xi}\left(1 - \int_0^t h_2(\varsigma)\,d\varsigma\right)\big\|\nabla\Psi(t)\big\|_2^2. \end{aligned} \tag{3.10}$$

According to (3.1),(3.6), and (3.10), we get

$$\begin{aligned} I(t) &> (1 - \widehat{\xi})\left\{\left(1 - \int_0^t h_1(\varsigma)\,d\varsigma\right)\big\|\nabla\Phi(t)\big\|_2^2 + \left(1 - \int_0^t h_2(\varsigma)\,d\varsigma\right)\big\|\nabla\Psi(t)\big\|_2^2\right\}\\ &> 0, \quad \forall t \in \big[0, T^*\big]. \end{aligned} \tag{3.11}$$

By repeating this procedure, T^* can be extended to T. This completes the proof. □

Remark 3.2 Under the conditions of Lemma 3.1, we have $J(t) \ge 0$, and consequently $\mathcal{E}(t) \ge 0$, $\forall t \in [0, T]$. Hence, by (3.3) and (3.5), we find

$$\begin{aligned} &\big\|\Phi_t(t)\big\|_{\eta+2}^{\eta+2} + \big\|\Psi_t(t)\big\|_{\eta+2}^{\eta+2} \le (\eta + 2)\mathcal{E}(0),\\ &\big\|\nabla\Phi_t(t)\big\|_2^2 + \big\|\nabla\Psi_t(t)\big\|_2^2 \le 2\mathcal{E}(0),\\ &\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} \le \frac{2q^- + 4}{q^- + 1}(\gamma + 1)\mathcal{E}(0). \end{aligned} \tag{3.12}$$

Theorem 3.3 *Suppose that the hypotheses of Lemma* 3.1 *hold, then the solution of* (1.1) *is global and bounded.*

Proof It suffices to show that

$$\begin{aligned}\left\|(\Phi,\Psi)\right\|_H &:= \|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}+\|\nabla\Phi\|_2^2+\|\nabla\Psi\|_2^2\\ &\quad+\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\end{aligned}$$

is bounded independently of t. To achieve this, we use (3.12) to get

$$\begin{aligned}\mathcal{E}(0) > \mathcal{E}(t) &= J(t)+\frac{1}{\eta+2}\left[\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\right]+\frac{1}{2}\left[\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\right]\\ &\geq \frac{q^-+1}{2(q^-+2)}\left(l_1\|\nabla\Phi\|_2^2+l_2\|\nabla\Psi\|_2^2\right)+\frac{1}{\eta+2}\left[\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\right]\\ &\quad+\frac{1}{2}\left[\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\right].\end{aligned}\tag{3.13}$$

Therefore,

$$\left\|(\Phi,\Psi)\right\|_H \leq C\mathcal{E}(0),$$

where $C(q^-,\eta,l_1,l_2)$ is a positive constant. □

4 Blow-up

Here, we establish the blow-up result for the solution of (1.1) with negative initial energy. Initially, we introduce the following functional:

$$\begin{aligned}\mathbb{H}(t) = -\mathcal{E}(t) &= -\frac{1}{\eta+2}\left[\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\right]-\frac{1}{2}\left[\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\right]\\ &\quad-\frac{1}{2(\gamma+1)}\left[\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\right]\\ &\quad-\frac{1}{2}\left[\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2+\left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right]\\ &\quad-\frac{1}{2}\left[(h_1o\nabla\Phi)(t)+(h_2o\nabla\Psi)(t)\right]+\int_\Omega F(\Phi,\Psi)\,dx.\end{aligned}\tag{4.1}$$

Theorem 4.1 *Assume that* (2.1)–(2.2) *and* $\mathcal{E}(0)<0$ *hold. Then the solution of* (1.1) *blows up in finite time.*

Proof From (2.5), the following can be written:

$$\mathcal{E}(t)\leq\mathcal{E}(0)\leq 0.\tag{4.2}$$

Therefore

$$\mathbb{H}'(t) = -\mathcal{E}'(t)\geq \zeta_1\int_\Omega\left|\Phi_t(t)\right|^{m(x)}dx-\zeta_3\int_\Omega\left|\Psi_t(t)\right|^{s(x)}dx,\tag{4.3}$$

hence

$$\mathbb{H}'(t)\geq\zeta_1\int_\Omega\left|\Phi_t(t)\right|^{m(x)}dx\geq 0$$

$$\mathbb{H}'(t) \geq \zeta_3 \int_\Omega \left|\Psi_t(t)\right|^{s(x)} dx \geq 0. \tag{4.4}$$

By (4.1) and (2.3), we have

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \int_\Omega F(\Phi, \Psi)\, dx \\ &\leq \int_\Omega \frac{c_1}{2(q(x)+2)} \left(|\Phi|^{2(q(x)+2)} + |\Psi|^{2(q(x)+2)}\right) dx \\ &\leq \frac{c_1}{2(q^- + 2)} \left(\varrho(\Phi) + \varrho(\Psi)\right), \end{aligned} \tag{4.5}$$

where

$$\varrho(\varphi) = \varrho_{q(\cdot)}(\varphi) = \int_\Omega |\varphi|^{2(q(x)+2)}\, dx.$$

Lemma 4.2 *Let* $\exists c > 0$ *in a way that any solution of* (1.1) *satisfies*

$$\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} + \|\Psi\|_{2(q^-+2)}^{2(q^-+2)} \leq c\left(\varrho(\Phi) + \varrho(\Psi)\right). \tag{4.6}$$

Proof Let

$$\Omega_1 = \left\{x \in \Omega : \left|\Phi(x,t)\right| \geq 1\right\}, \qquad \Omega_2 = \left\{x \in \Omega : \left|\Phi(x,t)\right| < 1\right\}, \tag{4.7}$$

we have

$$\begin{aligned} \varrho(\Phi) &= \int_{\Omega_1} |\Phi|^{2(q(x)+2)}\, dx + \int_{\Omega_2} |\Phi|^{2(q(x)+2)}\, dx \\ &\geq \int_{\Omega_1} |\Phi|^{2(q^-+2)}\, dx + \int_{\Omega_2} |\Phi|^{2(q^++2)}\, dx \\ &\geq \int_{\Omega_1} |\Phi|^{2(q^-+2)}\, dx + c\left(\int_{\Omega_2} |\Phi|^{2(q^-+2)}\, dx\right)^{\frac{2(q^++2)}{2(q^-+2)}}, \end{aligned} \tag{4.8}$$

then

$$\begin{aligned} &\varrho(\Phi) \geq \int_{\Omega_1} |\Phi|^{2(q^-+2)}\, dx \\ &\left(\frac{\varrho(\Phi)}{c}\right)^{\frac{2(q^-+2)}{2(q^++2)}} \geq \int_{\Omega_1} |\Phi|^{2(q^-+2)}\, dx. \end{aligned} \tag{4.9}$$

Hence, we get

$$\begin{aligned} \|\Phi\|_{2(q^-+2)}^{2(q^-+2)} &\leq \varrho(\Phi) + c\left(\varrho(\Phi)\right)^{\frac{2(q^-+2)}{2(q^++2)}} \\ &\leq \left(\varrho(\Phi) + \varrho(\Psi)\right) + c\left(\varrho(\Phi) + \varrho(\Psi)\right)^{\frac{2(q^-+2)}{2(q^++2)}} \\ &\leq \left(\varrho(\Phi) + \varrho(\Psi)\right)\left[1 + c\left(\varrho(\Phi) + \varrho(\Psi)\right)^{\frac{2(q^-+2)}{2(q^++2)}-1}\right]. \end{aligned} \tag{4.10}$$

According to (4.5), we have

$$\frac{\mathbb{H}(0)}{c} \leq \big(\varrho(\Phi) + \varrho(\Psi)\big).$$

Therefore,

$$\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} \leq \big(\varrho(\Phi) + \varrho(\Psi)\big)\big[1 + c\big(\mathbb{H}(0)\big)^{\frac{2(q^-+2)}{2(q^++2)}-1}\big].$$

Hence

$$\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} \leq c\big(\varrho(\Phi) + \varrho(\Psi)\big). \tag{4.11}$$

Using the same method, we find

$$\|\Psi\|_{2(q^-+2)}^{2(q^-+2)} \leq c\big(\varrho(\Phi) + \varrho(\Psi)\big). \tag{4.12}$$

By combining the previous two inequalities (4.11) and (4.12), we get the result we want (4.6). □

Corollary 4.3

$$\begin{aligned}
&\int_\Omega |\Phi|^{m(x)}\,dx \leq c\Big(\big(\varrho(\Phi) + \varrho(\Psi)\big)^{m^-/2(q^-+2)} + \big(\varrho(\Phi) + \varrho(\Psi)\big)^{m^+/2(q^-+2)}\Big),\\
&\int_\Omega |\Psi|^{s(y)}\,dy \leq c\Big(\big(\varrho(\Phi) + \varrho(\Psi)\big)^{s^-/2(q^-+2)} + \big(\varrho(\Phi) + \varrho(\Psi)\big)^{s^+/2(q^-+2)}\Big).
\end{aligned} \tag{4.13}$$

Proof By (1.5), we get

$$\begin{aligned}
\int_\Omega |\Phi|^{m(x)}\,dx &\leq \int_{\Omega_1} |\Phi|^{m^+}\,dx + \int_{\Omega_2} |\Phi|^{m^-}\,dx\\
&\leq c\bigg(\int_{\Omega_1} |\Phi|^{2(q^-+2)}\,dx\bigg)^{\frac{m^+}{2(q^-+2)}} + c\bigg(\int_{\Omega_2} |\Phi|^{2(q^-+2)}\,dx\bigg)^{\frac{m^-}{2(q^-+2)}}\\
&\leq c\big(\|\Phi\|_{2(q^-+2)}^{m^+} + \|\Phi\|_{2(q^-+2)}^{m^-}\big).
\end{aligned} \tag{4.14}$$

Then, Lemma 4.2 gives $(4.13)_1$. Similarly, we get $(4.13)_2$. □

Here, we introduce the following new functional:

$$\begin{aligned}
\mathfrak{D}(t) = \mathbb{H}^{1-\alpha}(t) &+ \frac{\varepsilon}{\eta+1}\int_\Omega \big[\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\big]\,dx\\
&+ \varepsilon\int_\Omega [\nabla\Phi_t\nabla\Phi + \nabla\Psi_t\nabla\Psi]\,dx,
\end{aligned} \tag{4.15}$$

where $0 < \varepsilon$ will be considered later, and take

$$0 < \alpha < \min\bigg\{\bigg(1 - \frac{1}{2(q^-+2)} - \frac{1}{\eta+2}\bigg), \frac{1+2\gamma}{4(\gamma+1)}, \frac{2q^-+4-m^-}{(2q^-+4)(m^+-1)},$$

$$\left.\frac{2q^- + 4 - m^+}{(2q^- + 4)(m^+ - 1)}, \frac{2q^- + 4 - r^+}{(2q^- + 4)(s^- - 1)}, \frac{2q^- + 4 - s^-}{(2q^- + 4)(s^+ - 1)}\right\} < 1. \tag{4.16}$$

By multiplying $(1.1)_1$, $(1.1)_2$ by Φ, Ψ and with the help of (4.15), we obtain

$$\begin{aligned}\mathfrak{D}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right)\\ &\quad + \underbrace{\varepsilon\int_\Omega \nabla\Phi\int_0^t g(t-\varsigma)\nabla\Phi(\varsigma)\,d\varsigma\,dx}_{J_1} + \underbrace{\varepsilon\int_\Omega \nabla\Psi\int_0^t h(t-\varsigma)\nabla\Psi(\varsigma)\,d\varsigma\,dx}_{J_2}\\ &\quad \underbrace{-\varepsilon\zeta_1\int_\Omega \Phi\Phi_t|\Phi_t|^{m(x)-2}\,dx}_{J_3} \underbrace{-\varepsilon\zeta_3\int_\Omega \Psi\Psi_t|\Psi_t|^{s(x)-2}\,dx}_{J_4}\\ &\quad -\varepsilon\left(\|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2\right) - \varepsilon\left(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right)\\ &\quad + \underbrace{\varepsilon\int_\Omega \left(\Phi f_1(\Phi,\Psi) + \Psi f_2(\Phi,\Psi)\right)dx}_{J_5}.\end{aligned}$$

By (2.1), we obtain

$$\begin{aligned}\mathfrak{D}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right)\\ &\quad + \underbrace{\varepsilon\int_\Omega \nabla\Phi\int_0^t h_1(t-\varsigma)\nabla\Phi(\varsigma)\,d\varsigma\,dx}_{J_1} + \underbrace{\varepsilon\int_\Omega \nabla\Psi\int_0^t h_2(t-\varsigma)\nabla\Psi(\varsigma)\,d\varsigma\,dx}_{J_2}\\ &\quad \underbrace{-\varepsilon\zeta_1\int_\Omega \Phi\Phi_t|\Phi_t|^{m(x)-2}\,dx}_{J_3} \underbrace{-\varepsilon\zeta_3\int_\Omega \Psi\Psi_t|\Psi_t|^{s(x)-2}\,dx}_{J_4}\\ &\quad -\varepsilon\left(\|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2\right) - \varepsilon\left(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right)\\ &\quad + \underbrace{\varepsilon\left(2q^- + 4\right)\int_\Omega F(\Phi,\Psi)\,dx}_{J_5}.\end{aligned} \tag{4.17}$$

We have

$$\begin{aligned}J_1 &= \varepsilon\int_0^t h_1(t-\varsigma)\,d\varsigma\int_\Omega \nabla\Phi.\left(\nabla\Phi(\varsigma) - \nabla\Phi(t)\right)dx\,d\varsigma + \varepsilon\int_0^t h_1(\varsigma)\,d\varsigma\|\nabla\Phi\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t h_1(\varsigma)\,d\varsigma\|\nabla\Phi\|_2^2 - \frac{\varepsilon}{2}(h_1 o\nabla\Phi).\end{aligned} \tag{4.18}$$

$$\begin{aligned}J_2 &= \varepsilon\int_0^t h_2(t-\varsigma)\,d\varsigma\int_\Omega \nabla\Psi.\left(\nabla\Psi(\varsigma) - \nabla\Psi(t)\right)dx\,d\varsigma + \varepsilon\int_0^t h_2(\varsigma)\,d\varsigma\|\nabla\Psi\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t h_2(\varsigma)\,d\varsigma\|\nabla\Psi\|_2^2 - \frac{\varepsilon}{2}(h_2 o\nabla\Psi).\end{aligned} \tag{4.19}$$

From (4.17), we find

$$\mathfrak{D}'(t) \geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right) + \varepsilon\left(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right)$$

$$
\begin{aligned}
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right]\\
&-\frac{\varepsilon}{2}(h_1 o\nabla\Phi)-\frac{\varepsilon}{2}(h_2 o\nabla\Psi)-\varepsilon\left(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\right)\\
&+J_3+J_4+J_5. \qquad (4.20)
\end{aligned}
$$

At this stage we apply Young's inequality, which gives us for $\delta_1, \delta_2 > 0$ the following:

$$
J_3 \le \varepsilon\zeta_1\left\{\frac{1}{m^-}\int_\Omega \delta_1^{m(x)}|\Phi|^{m(x)}\,dx+\frac{m^+-1}{m^+}\int_\Omega \delta_1^{-\frac{m(x)}{m(x)-1}}|\Phi_t|^{m(x)}\,dx\right\}, \qquad (4.21)
$$

$$
J_4 \le \varepsilon\zeta_3\left\{\frac{1}{s^-}\int_\Omega \delta_2^{s(x)}|\Psi|^{s(x)}\,dx+\frac{s^+-1}{s^+}\int_\Omega \delta_2^{-\frac{s(x)}{s(x)-1}}|\Psi_t|^{s(x)}\,dx\right\}. \qquad (4.22)
$$

Therefore, by setting δ_1, δ_2 so that

$$
\delta_1^{-\frac{m(x)}{m(x)-1}}=\zeta_1\kappa\mathbb{H}^{-\alpha}(t), \qquad \delta_2^{-\frac{s(x)}{s(x)-1}}=\zeta_3\kappa\mathbb{H}^{-\alpha}(t), \qquad (4.23)
$$

substituting the previous two equalities into (4.20) gives us the following inequality:

$$
\begin{aligned}
\mathfrak{D}'(t) \ge{}& \left[(1-\alpha)-\varepsilon\kappa(\widehat{m}+\widehat{s})\right]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}\left(\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\right)\\
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right]\\
&+\varepsilon\left(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\right)-\frac{\varepsilon}{2}(h_1 o\nabla\Phi)-\frac{\varepsilon}{2}(h_2 o\nabla\Psi)\\
&-\varepsilon\frac{\zeta_1}{m^-}\int_\Omega(\zeta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|\Phi|^{m(x)}\,dx\\
&-\varepsilon\frac{\zeta_3}{s^-}\int_\Omega(\zeta_3\kappa)^{1-s(x)}\mathbb{H}^{\alpha(s(x)-1)}(t)|\Psi|^{s(x)}\,dx\\
&-\varepsilon\left(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\right)+J_5, \qquad (4.24)
\end{aligned}
$$

where $\widehat{m}=\frac{m^+-1}{m^-}$, $\widehat{s}=\frac{s^+-1}{s^-}$.

Now, by using (4.5) and $(4.13)_1$, we have

$$
\begin{aligned}
&\frac{\zeta_1}{m^-}\int_\Omega(\zeta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|\Phi|^{m(x)}\,dx\\
&\quad\le\frac{\zeta_1}{m^-}\int_\Omega(\zeta_1\kappa)^{1-m^-}\mathbb{H}^{\alpha(m^+-1)}(t)|\Phi|^{m(x)}\,dx\\
&\quad=C_1\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega|\Phi|^{m(x)}\,dx\\
&\quad\le C_2\left\{\left(\varrho(\Phi)+\varrho(\Psi)\right)^{\frac{m^-}{2(q^-+2)}+\alpha(m^+-1)}\right.\\
&\qquad\left.\times\left(\varrho(\Phi)+\varrho(\Psi)\right)^{\frac{m^+}{2(q^-+2)}+\alpha(m^+-1)}\right\}. \qquad (4.25)
\end{aligned}
$$

By (4.16), we find

$$
r=m^-+\alpha\left(2q^-+4\right)\left(m^+-1\right)\le\left(2q^-+4\right),
$$

$$r = m^+ + \alpha\big(2q^- + 4\big)\big(m^+ - 1\big) \le \big(2q^- + 4\big).$$

We use the following inequality to advance the proof:

$$Z^\gamma \le Z + 1 \le \left(1 + \frac{1}{\nu}\right)(Z + \nu), \quad \forall Z \ge 0, 0 < \gamma \le 1, \nu > 0, \tag{4.26}$$

with $\nu = \frac{1}{\mathbb{H}(0)}$. Then we have

$$\begin{aligned} \big(\varrho(\Phi) + \varrho(\Psi)\big)^{\frac{m^-}{2(q^-+2)}+\alpha(m^+-1)} &\le \left(1 + \frac{1}{\mathbb{H}(0)}\right)\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(0)\big) \\ &\le C_3\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(t)\big) \end{aligned} \tag{4.27}$$

and

$$\big(\varrho(\Phi) + \varrho(\Psi)\big)^{\frac{m^+}{2(q^-+2)}+\alpha(m^+-1)} \le C_3\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(t)\big), \tag{4.28}$$

where $C_3 = 1 + \frac{1}{\mathbb{H}(0)}$. Substituting (4.27) and (4.28) into (4.25), we get

$$\begin{aligned} &\frac{\zeta_1}{m^-}\int_\Omega (\zeta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|\Phi|^{m(x)}\,dx \\ &\quad \le C_4\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(t)\big). \end{aligned} \tag{4.29}$$

Similarly, we get

$$\begin{aligned} &\frac{\zeta_3}{s^-}\int_\Omega (\zeta_3\kappa)^{1-s(x)}\mathbb{H}^{\alpha(s(x)-1)}(t)|\Psi|^{s(x)}\,dx \\ &\quad \le C_5\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(t)\big), \end{aligned} \tag{4.30}$$

where $C_4 = C_4(\kappa) = C_3\frac{\zeta_1}{m^-}(\zeta_1\kappa)^{1-m^-}$, $C_5 = C_5(\kappa) = C_3\frac{\zeta_3)}{s^-}(\zeta_3\kappa)^{1-s^-}$.

At this stage, combining (4.29), (4.30), and (4.24), and by (2.3) we find

$$\begin{aligned} \mathfrak{D}'(t) \ge{}& \big[(1-\alpha) - \varepsilon\kappa(\widehat{m} + \widehat{s})\big]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\big(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big) \\ &- \varepsilon\left[\left(1 - \frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1 - \frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right] \\ &+ \varepsilon\big(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big) - \frac{\varepsilon}{2}(h_1 o\nabla\Phi) - \frac{\varepsilon}{2}(h_2 o\nabla\Psi) + J_5 \\ &- \varepsilon(C_4 + C_5)\big(\big(\varrho(\Phi) + \varrho(\Psi)\big) + \mathbb{H}(t)\big) - \varepsilon\big(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big). \end{aligned} \tag{4.31}$$

Now, for $0 < a < 1$, from (4.1) and (2.3) we have

$$\begin{aligned} J_5 &= \varepsilon\big(2q^- + 4\big)\int_\Omega F(\Phi,\Psi)\,dx \\ &= \varepsilon a\big(2q^- + 4\big)\int_\Omega F(\Phi,\Psi)\,dx + \varepsilon(1-a)\big(2q^- + 4\big)\mathbb{H}(t) \\ &\quad + \frac{\varepsilon(1-a)(2q^- + 4)}{\eta+2}\big(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon(1-a)\big(q^-+2\big)\big(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\big)\\
&+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2\\
&+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\\
&+\varepsilon(1-a)\big(q^-+2\big)\big((h_1o\nabla\Phi)+(h_2o\nabla\Psi)\big)\\
&+\frac{\varepsilon(1-a)(q^-+2)}{(\gamma+1)}\big(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\big).
\end{aligned}
\tag{4.32}
$$

Substituting (4.32) in (4.31) and applying (2.3) gives

$$
\begin{aligned}
\mathfrak{D}'(t)\geq{}&\big\{(1-\alpha)-\varepsilon\kappa(\widehat{m}+\widehat{s})\big\}\mathbb{H}^{-\alpha}\mathbb{H}'(t)\\
&+\varepsilon\big\{(1-a)\big(q^-+2\big)+1\big\}\big(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\big)\\
&+\varepsilon\left\{\frac{\varepsilon(1-a)(2q^-+4)}{\eta+2}+\frac{1}{\eta+1}\right\}\big(\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\big)\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)-\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\right\}\|\nabla\Phi\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)\left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)-\left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\right\}\|\nabla\Psi\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^-+2\big)-\frac{1}{2}\right\}(h_1o\nabla\Phi+h_2o\nabla\Psi)\\
&+\varepsilon\left\{\frac{(1-a)(q^-+2)}{\gamma+1}-1\right\}\big(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\big)\\
&+\varepsilon\big\{c_0a-\big(C_4(\kappa)+C_5(\kappa)\big)\big\}\big(\varrho(\Phi)+\varrho(\Psi)\big)\\
&+\varepsilon\big\{(1-a)\big(2q^-+4\big)-\big(C_4(\kappa)+C_5(\kappa)\big)\big\}\mathbb{H}(t).
\end{aligned}
\tag{4.33}
$$

By choosing $0<a$ so small that

$$
\big(q^-+2\big)(1-a)>1+\gamma,
$$

we have

$$
\begin{aligned}
\lambda_1&:=\big(q^-+2\big)(1-a)-1>0,\\
\lambda_2&:=\big(q^-+2\big)(1-a)-\frac{1}{2}>0,\\
\lambda_3&:=\frac{(q^-+2)(1-a)}{\gamma+1}-1>0.
\end{aligned}
$$

At this moment we present this supposition

$$
\max\left\{\int_0^\infty h_1(\varsigma)\,d\varsigma,\int_0^\infty h_2(\varsigma)\,d\varsigma\right\}<\frac{(q^-+2)(1-a)-1}{((q^-+2)(1-a)-\frac{1}{2})}=\frac{2\lambda_1}{2\lambda_1+1}
\tag{4.34}
$$

gives

$$\lambda_4 = \left\{\left(\left(q^- + 2\right)(1-a) - 1\right) - \int_0^t h_1(\varsigma)\,d\varsigma\left(\left(q^- + 2\right)(1-a) - \frac{1}{2}\right)\right\} > 0,$$
$$\lambda_5 = \left\{\left(\left(q^- + 2\right)(1-a) - 1\right) - \int_0^t h_2(\varsigma)\,d\varsigma\left(\left(q^- + 2\right)(1-a) - \frac{1}{2}\right)\right\} > 0.$$

Next, we choose κ large enough such that

$$\lambda_6 = ac_0 - \left(C_4(\kappa) + C_5(\kappa)\right) > 0,$$
$$\lambda_7 = 2\left(q^- + 2\right)(1-a) - \left(C_4(\kappa) + C_5(\kappa)\right) > 0.$$

At this point, take κ, a, and we pick ε in a way that

$$\lambda_8 = (1-\alpha) - \varepsilon\kappa(\widehat{m} + \widehat{s}) > 0$$

and

$$\mathfrak{D}(0) = \mathbb{H}^{1-\alpha}(0) + \frac{\varepsilon}{\eta+1}\int_\Omega \left[\Phi_0|\Phi_1|^\eta\Phi_1 + \Psi_0|\Psi_1|^\eta\Psi_1\right]dx$$
$$+ \varepsilon\int_\Omega [\nabla\Phi_1\nabla\Phi_0 + \nabla\Psi_1\nabla\Psi_0]\,dx > 0. \tag{4.35}$$

Hence, from (4.33) we deduce for some $\mu > 0$

$$\mathfrak{D}'(t) \geq \mu\left\{\mathbb{H}(t) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right.$$
$$+ \|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2 + \|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2 + (h_1 o\nabla\Phi) + (h_2 o\nabla\Psi)$$
$$\left. + \varrho(\Phi) + \varrho(\Psi)\right\} \tag{4.36}$$

and

$$\mathfrak{D}(t) \geq \mathfrak{D}(0) > 0, \quad t > 0. \tag{4.37}$$

Next, by Hölder's and Young's inequalities, we find

$$\left|\int_\Omega \left(\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\right)dx\right|^{\frac{1}{1-\alpha}} \leq C\left[\|\Phi\|_{2(q^-+2)}^{\frac{\theta}{1-\alpha}} + \|\Phi_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}}\right.$$
$$\left. + \|\Psi\|_{2(q^-+2)}^{\frac{\theta}{1-\alpha}} + \|\Psi_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}}\right], \tag{4.38}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

Pick $\mu = (\eta+2)(1-\alpha)$ to get the below

$$\frac{\theta}{1-\alpha} = \frac{\eta+2}{(1-\alpha)(\eta+2) - 1} \leq 2\left(q^- + 2\right).$$

Consequently, by the application of (4.5), (4.16), and (4.26), we have

$$\|\Phi\|_{2(q^-+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} \leq d\big(\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{H}(t)\big)$$
$$\|\Psi\|_{2(q^-+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} \leq d\big(\|\Psi\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{H}(t)\big), \quad \forall t \geq 0.$$

Then we have

$$\left|\int_\Omega \big(\Phi|\Phi_t|^\eta \Phi_t + \Psi|\Psi_t|^\eta \Psi_t\big)\, dx\right|^{\frac{1}{1-\alpha}} \leq c\big\{\varrho(\Phi) + \varrho(\Psi) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \mathbb{H}(t)\big\}. \tag{4.39}$$

In the same way, we have

$$\left|\int_\Omega (\nabla\Phi\nabla\Phi_t + \nabla\Psi\nabla\Psi_t)\, dx\right|^{\frac{1}{1-\alpha}} \leq C\big[\|\nabla\Phi\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla\Phi_t\|_2^{\frac{\mu}{1-\alpha}} + \|\nabla\Psi\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla\Psi_t\|_2^{\frac{\mu}{1-\alpha}}\big],$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

In this, by assuming $\theta = 2(\gamma+1)(1-\alpha)$, we get

$$\frac{\mu}{1-\alpha} = \frac{2(\gamma+1)}{2(1-\alpha)(\gamma+1)-1} \leq 2$$
$$\left|\int_\Omega (\nabla\Phi\nabla\Phi_t + \nabla\Psi\nabla\Psi_t)\, dx\right|^{\frac{1}{1-\alpha}} \leq c\big\{\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} + \|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big\}. \tag{4.40}$$

Thus, by (4.39) and (4.40), we have

$$\begin{aligned}
\mathfrak{D}^{\frac{1}{1-\alpha}}(t) &= \bigg(\mathbb{H}^{1-\alpha}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega \big(\Phi|\Phi_t|^\eta \Phi_t + \Psi|\Psi_t|^\eta \Psi_t\big)\, dx \\
&\quad + \varepsilon\int_\Omega (\nabla\Phi_t\nabla\Phi + \nabla\Psi_t\nabla\Psi)\, dx\bigg)^{\frac{1}{1-\alpha}} \\
&\leq c\bigg(\mathbb{H}(t) + \left|\int_\Omega \big(\Phi|\Phi_t|^\eta \Phi_t + \Psi|\Psi_t|^\eta \Psi_t\big)\, dx\right|^{\frac{1}{1-\alpha}} + \|\nabla\Phi\|_2^{\frac{2}{1-\alpha}} + \|\nabla\Psi\|_2^{\frac{2}{1-\alpha}} \\
&\quad + \|\nabla\Phi_t\|_2^{\frac{2}{1-\alpha}} + \|\nabla\Psi_t\|_2^{\frac{2}{1-\alpha}}\bigg) \\
&\leq c\big(\mathbb{H}(t) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} + \|\nabla\Phi_t\|_2^2 \\
&\quad + \|\nabla\Psi_t\|_2^2 + (h_1 o\nabla\Phi) + (h_2 o\nabla\Psi) + \varrho(\Phi) + \varrho(\Psi)\big) \\
&\leq c\big\{\mathbb{H}(t) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} \\
&\quad + \|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2 + \|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2 + (h_1 o\nabla\Phi) + (h_2 o\nabla\Psi) \\
&\quad + \varrho(\Phi) + \varrho(\Psi)\big\}.
\end{aligned} \tag{4.41}$$

Now, (4.36) and (4.41) imply

$$\mathfrak{D}'(t) \geq \lambda \mathfrak{D}^{\frac{1}{1-\alpha}}(t), \tag{4.42}$$

where $0 < \lambda$, this relies only on β and c.

Further simplification of (4.42) leads us to

$$\mathfrak{D}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathfrak{D}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda \frac{\alpha}{(1-\alpha)} t}.$$

Hence, $\mathfrak{D}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda \alpha \mathfrak{D}^{\alpha/(1-\alpha)}(0)}.$$

This ends the proof. □

5 Growth of solution

Here, the exponential growth of solution of problem (1.1) is established with positive initial energy.

First, based on Theorem 3.3, we have a solution that is global in time. To achieve the objectives of our results in this section, we first introduce the following function:

$$\begin{aligned} \Upsilon(t) := {} & \left(1 - \int_0^t h_1(\varsigma)\, d\varsigma\right) \|\nabla \Phi\|_2^2 + \left(1 - \int_0^t h_2(\varsigma)\, d\varsigma\right) \|\nabla \Psi\|_2^2 + (h_1 \circ \nabla \Phi) \\ & + (h_2 \circ \nabla \Psi). \end{aligned} \tag{5.1}$$

Then, we present the following lemma, which is similar to the one presented first by Vitillaro [28] to study a class of single wave equations, also see [27].

Lemma 5.1 *Suppose that* (2.1) *and* (2.2) *hold. Let* (u, v, z, y) *be a solution of* (1.1). *Assume further that*

$$\|\nabla \Phi_0\|_2^2 + \|\nabla \Psi_0\|_2^2 > \alpha_1^2, \qquad \mathcal{E}(0) < d_1. \tag{5.2}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\Upsilon(t) > \alpha_2^2 \tag{5.3}$$

and

$$2\big(p^- + 2\big) \int_0^L F(\Phi, \Psi)\, dx \geq (B\alpha_2)^{2(p^+ + 2)} \tag{5.4}$$

with

$$B = \left(\frac{c_1 C_*(p^+)}{l^{(p^+ + 2)}}\right)^{\frac{1}{2(p^+ + 2)}}, \quad \alpha_1 = B^{-\frac{p^+ + 2}{p^+ + 1}},$$

$$d_1 = \left[\frac{1}{2(p^- + 2)(1-a)} - \frac{1}{2(p^- + 2)}\right]\alpha_1^2. \tag{5.5}$$

We also know which number $d = [\frac{1}{2} - \frac{1}{2(p^-+2)}]\alpha_1^2 > d_1 > \mathcal{E}(0)$.

Moreover, one can easily see that, from (5.5), the condition $\mathcal{E}(0) < d_1$ is equivalent to inequality (3.4).

Since $0 < a < 1$, we will appoint it later, we have $2 < 2(p^- + 2)(1 - a) < 2(p^- + 2)$.

For this purpose, we set the functional

$$\begin{aligned}
\mathbb{T}(t) = d_1 - \mathcal{E}(t) = d_1 &- \frac{1}{\eta+2}\left[\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right] - \frac{1}{2}\left[\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right] \\
&- \frac{1}{2(\gamma+1)}\left[\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right] \\
&- \frac{1}{2}\left[\left(1 - \int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1 - \int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right] \\
&- \frac{1}{2}\left[(h_1 o\nabla\Phi)(t) + (h_2 o\nabla\Psi)(t)\right] + \int_\Omega F(\Phi,\Psi)\,dx.
\end{aligned} \tag{5.6}$$

Theorem 5.2 *Assume that (2.1)–(2.2) are satisfied and $\mathcal{E}(0) < d_1$, then*

$$2\left(q^- + 2\right) > \frac{\eta+2}{\eta+1}. \tag{5.7}$$

Then the solution of problem (1.1) grows exponentially.

Proof To achieve our goal, by (2.5) we first deduce

$$\mathcal{E}(t) \le \mathcal{E}(0) < d_1, \tag{5.8}$$

with the help of (4.3) and (4.4) and (5.2), (2.3), we have

$$\begin{aligned}
0 \le \mathbb{T}(0) \le \mathbb{T}(t) &\le d_1 - \frac{1}{2}\alpha_1^2 + \frac{1}{2(p^- + 2)}\left[\|\Phi + \Psi\|_{2(p^-+2)}^{2(p^-+2)} + 2\|\Phi\Psi\|_{(p^-+2)}^{p^-+2}\right] \\
&\le d - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2(p^- + 2)}\left[\|\Phi\|_{2(p^-+2)}^{2(p^-+2)} + \|\Psi\|_{2(p^-+2)}^{2(p^-+2)}\right] \\
&\le -\frac{1}{2(p^- + 2)}\alpha_1^2 + \frac{c_1}{2(p^- + 2)}\left[\|\Phi\|_{2(p^-+2)}^{2(p^-+2)} + \|\Psi\|_{2(p^-+2)}^{2(p^-+2)}\right] \\
&\le \frac{c_1}{2(p^- + 2)}\left[\|\Phi\|_{2(p^-+2)}^{2(p^-+2)} + \|\Psi\|_{2(p^-+2)}^{2(p^-+2)}\right].
\end{aligned} \tag{5.9}$$

In this, we introduce the functional

$$\begin{aligned}
\mathfrak{R}(t) = \mathbb{T}(t) &+ \frac{\varepsilon}{\eta+1}\int_\Omega \left[\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\right]dx \\
&+ \varepsilon\int_\Omega [\nabla\Phi_t\nabla v + \nabla\Psi_t\nabla\Psi]\,dx,
\end{aligned} \tag{5.10}$$

where $\varepsilon > 0$.

From $(1.1)_1$, $(1.1)_2$, and (5.10), we get

$$\begin{aligned}\mathfrak{R}'(t) &= \mathbb{T}'(t) + \frac{\varepsilon}{\eta+1}\big(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big) + \varepsilon\big(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big)\\
&\quad + \varepsilon\int_\Omega \nabla\Phi\int_0^t h_1(t-\varsigma)\nabla\Phi(\varsigma)\,d\varsigma\,dx + \varepsilon\int_\Omega \nabla\Psi\int_0^t h_2(t-\varsigma)\nabla\Psi(\varsigma)\,d\varsigma\,dx\\
&\quad - \varepsilon\zeta_1\int_\Omega \Phi\Phi_t|\Phi_t|^{m(x)-2}\,dx - \varepsilon\zeta_3\int_\Omega \Psi\Psi_t|\Psi_t|^{s(x)-2}\,dx\\
&\quad - \varepsilon\big(\|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2\big) - \varepsilon\big(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big)\\
&\quad + \varepsilon\int_\Omega \big(\Phi f_1(\Phi,\Psi) + \Psi f_2(\Phi,\Psi)\big)\,dx.\end{aligned}$$

By (2.1), we find

$$\begin{aligned}\mathfrak{R}'(t) &\geq \mathbb{T}'(t) + \frac{\varepsilon}{\eta+1}\big(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big) + \varepsilon\big(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big)\\
&\quad \underbrace{+ \varepsilon\int_\Omega \nabla\Phi\int_0^t h_1(t-\varsigma)\nabla\Phi(\varsigma)\,d\varsigma\,dx}_{I_1} \underbrace{+ \varepsilon\int_\Omega \nabla\Psi\int_0^t h_2(t-\varsigma)\nabla\Psi(\varsigma)\,d\varsigma\,dx}_{I_2}\\
&\quad \underbrace{- \varepsilon\zeta_1\int_\Omega \Phi\Phi_t|\Phi_t|^{m(x)-2}\,dx}_{I_3} \underbrace{- \varepsilon\zeta_3\int_\Omega \Psi\Psi_t|\Psi_t|^{s(x)-2}\,dx}_{I_4}\\
&\quad - \varepsilon\big(\|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2\big) - \varepsilon\big(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big)\\
&\quad \underbrace{+ \varepsilon\big(2q^- + 4\big)\int_\Omega F(\Phi,\Psi)\,dx}_{I_5}.\end{aligned} \tag{5.11}$$

Similarly to J_1, J_2 in (4.18) and (4.19), we estimate I_1, I_2:

$$I_1 = J_1 \geq \frac{\varepsilon}{2}\int_0^t h_1(\varsigma)\,d\varsigma\,\|\nabla\Phi\|_2^2 - \frac{\varepsilon}{2}(h_1 o\nabla\Phi), \tag{5.12}$$

$$I_2 = J_2 \geq \frac{\varepsilon}{2}\int_0^t h_2(\varsigma)\,d\varsigma\,\|\nabla\Psi\|_2^2 - \frac{\varepsilon}{2}(h_2 o\nabla\Psi). \tag{5.13}$$

From (5.11), we find

$$\begin{aligned}\mathfrak{R}'(t) &\geq \mathbb{T}'(t) + \frac{\varepsilon}{\eta+1}\big(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\big) + \varepsilon\big(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\big)\\
&\quad - \varepsilon\left[\left(1 - \frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1 - \frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right]\\
&\quad - \frac{\varepsilon}{2}(h_1 o\nabla\Phi) - \frac{\varepsilon}{2}(h_2 o\nabla\Psi) - \varepsilon\big(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\big)\\
&\quad + I_3 + I_4 + I_5.\end{aligned} \tag{5.14}$$

Similar to J_3 and J_4 in (4.21)–(4.22), we estimate I_3 and I_4. By Young's inequality, we find for $\delta_1, \delta_2 > 0$

$$I_3 \leq \varepsilon\zeta_1\left\{\frac{1}{m^-}\int_\Omega \delta_1^{m(x)}|\Phi|^{m(x)}\,dx + \frac{m^+-1}{m^+}\int_\Omega \delta_1^{-\frac{m(x)}{m(x)-1}}|\Phi_t|^{m(x)}\,dx\right\} \tag{5.15}$$

and

$$I_4 \leq \varepsilon\zeta_3\left\{\frac{1}{s^-}\int_\Omega \delta_2^{s(x)}|\Psi|^{s(x)}\,dx + \frac{s^+-1}{s^+}\int_\Omega \delta_2^{-\frac{s(x)}{s(x)-1}}|\Psi_t|^{s(x)}\,dx\right\}. \tag{5.16}$$

Therefore, by setting δ_1, δ_2 so that

$$\delta_1^{-\frac{m(x)}{m(x)-1}} = \frac{\zeta_1}{2}\kappa, \qquad \delta_2^{-\frac{s(x)}{s(x)-1}} = \frac{\zeta_3}{2}\kappa, \tag{5.17}$$

substituting in (5.14), we obtain

$$\begin{aligned}
\mathfrak{R}'(t) \geq{}& \left[1-\varepsilon\kappa(\widehat{m}+\widehat{s})\right]\mathbb{T}'(t) + \frac{\varepsilon}{\eta+1}\left(\|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2}\right) \\
&- \varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right] \\
&+ \varepsilon\left(\|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2\right) - \frac{\varepsilon}{2}(h_1 o\nabla\Phi) - \frac{\varepsilon}{2}(h_2 o\nabla\Psi) \\
&- \varepsilon\frac{\zeta_1}{m^-}\int_\Omega\left(\frac{\zeta_1\kappa}{2}\right)^{1-m(x)}|\Phi|^{m(x)}\,dx - \varepsilon\frac{\zeta_3}{s^-}\int_\Omega\left(\frac{\zeta_3\kappa}{2}\right)^{1-s(x)}|\Psi|^{s(x)}\,dx \\
&- \varepsilon\left(\|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)}\right) + I_5,
\end{aligned} \tag{5.18}$$

where $\widehat{m} = \frac{m^+-1}{m^-}$, $\widehat{s} = \frac{s^+-1}{s^-}$. By using (4.5) and (4.13), we have

$$\begin{aligned}
\frac{\zeta_1}{m^-}\int_\Omega\left(\frac{\zeta_1\kappa}{2}\right)^{1-m(x)}|\Phi|^{m(x)}\,dx &\leq \frac{\zeta_1}{m^-}\int_\Omega\left(\frac{\zeta_1\kappa}{2}\right)^{1-m^-}|\Phi|^{m(x)}\,dx \\
&= C_8\int_\Omega |\Phi|^{m(x)}\,dx \\
&\leq C_9\left\{\left(\varrho(\Phi)+\varrho(\Psi)\right)^{\frac{m^-}{2(q^-+2)}} \right. \\
&\quad \left. + \left(\varrho(\Phi)+\varrho(\Psi)\right)^{\frac{m^+}{2(q^-+2)}}\right\}.
\end{aligned} \tag{5.19}$$

By (1.5), we get

$$r = m^- \leq \left(2q^- + 4\right), \qquad r = m^+ \leq \left(2q^- + 4\right)$$

and by (4.26) with $\nu = \frac{1}{\mathbb{T}(0)}$. Then we have

$$\begin{aligned}
\left(\varrho(\Phi)+\varrho(\Psi)\right)^{\frac{m^-}{2(q^-+2)}} &\leq \left(1+\frac{1}{\mathbb{T}(0)}\right)\left(\left(\varrho(\Phi)+\varrho(\Psi)\right)+\mathbb{T}(0)\right) \\
&\leq C_{10}\left(\left(\varrho(\Phi)+\varrho(\Psi)\right)+\mathbb{T}(t)\right)
\end{aligned} \tag{5.20}$$

and

$$\big(\varrho(\Phi)+\varrho(\Psi)\big)^{\frac{m^+}{2(q^-+2)}} \le C_{10}\Big(\big(\varrho(\Phi)+\varrho(\Psi)\big)+\mathbb{T}(t)\Big), \tag{5.21}$$

where $C_{10}=1+\frac{1}{\mathbb{T}(0)}$. Substituting (5.20) and (5.21) into (5.19), we get

$$\frac{\zeta_1}{m^-}\int_\Omega\left(\frac{\zeta_1\kappa}{2}\right)^{1-m(x)}|\Phi|^{m(x)}\,dx \le C_{11}\Big(\big(\varrho(\Phi)+\varrho(\Psi)\big)+\mathbb{T}(t)\Big). \tag{5.22}$$

Similarly, we find

$$\frac{\zeta_3}{s^-}\int_\Omega\left(\frac{\zeta_3\kappa}{2}\right)^{1-s(x)}|\Psi|^{s(x)}\,dx \le C_{12}\Big(\big(\varrho(\Phi)+\varrho(\Psi)\big)+\mathbb{T}(t)\Big), \tag{5.23}$$

where $C_{11}=C_{11}(\kappa)=C_9\frac{\zeta_1}{m^-}(\frac{\zeta_1\kappa}{2})^{1-m^-}$, $C_{12}=C_{12}(\kappa)=C_9\frac{\zeta_3}{s^-}(\frac{\zeta_3\kappa}{2})^{1-s^-}$.

Combining (5.22), (5.23), and (5.18), we have

$$\begin{aligned}
\mathfrak{R}'(t) \ge{}& \big[1-\varepsilon\kappa(\widehat{m}+\widehat{s})\big]\mathbb{T}'(t)+\frac{\varepsilon}{\eta+1}\big(\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\big)\\
&-\varepsilon\left[\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2+\left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\right]\\
&+\varepsilon\big(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\big)-\frac{\varepsilon}{2}(h_1o\nabla\Phi)-\frac{\varepsilon}{2}(h_2o\nabla\Psi)+I_5\\
&-\varepsilon(C_{11}+C_{12})\Big(\big(\varrho(\Phi)+\varrho(\Psi)\big)+\mathbb{T}(t)\Big)-\varepsilon\big(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\big).
\end{aligned} \tag{5.24}$$

Here, for $0<a<1$, from (5.6) and (2.3) we have

$$\begin{aligned}
J_7 ={}& \varepsilon\big(2q^-+4\big)\int_\Omega F(\Phi,\Psi)\,dx\\
={}& \varepsilon a\big(2q^-+4\big)\int_\Omega F(\Phi,\Psi)\,dx+\varepsilon(1-a)\big(2q^-+4\big)\big(\mathbb{T}(t)-d_1\big)\\
&+\frac{\varepsilon(1-a)(2q^-+4)}{\eta+2}\big(\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\big)\\
&+\varepsilon(1-a)\big(q^-+2\big)\big(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\big)\\
&+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2\\
&+\varepsilon(1-a)\big(q^-+2\big)\left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)\|\nabla\Psi\|_2^2\\
&+\varepsilon(1-a)\big(q^-+2\big)\big((h_1o\nabla\Phi)+(h_2o\nabla\Psi)\big)\\
&+\frac{\varepsilon(1-a)(q^-+2)}{(\gamma+1)}\big(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\big).
\end{aligned} \tag{5.25}$$

Substituting (5.25) in (5.24) and applying (2.3) and (5.4), we get

$$\begin{aligned}
\mathfrak{R}'(t) \ge{}& \big\{1-\varepsilon\kappa(\widehat{m}+\widehat{s})\big\}\mathbb{T}'(t)\\
&+\varepsilon\big\{(1-a)\big(q^-+2\big)+1\big\}\big(\|\nabla\Phi_t\|_2^2+\|\nabla\Psi_t\|_2^2\big)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\left\{\frac{\varepsilon(1-a)(2q^{-}+4)}{\eta+2}+\frac{1}{\eta+1}\right\}\left(\|\Phi_t\|_{\eta+2}^{\eta+2}+\|\Psi_t\|_{\eta+2}^{\eta+2}\right)\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)-\left(1-\frac{1}{2}\int_0^t h_1(\varsigma)\,d\varsigma\right)\right\}\|\nabla\Phi\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)\left(1-\int_0^t h_2(\varsigma)\,d\varsigma\right)-\left(1-\frac{1}{2}\int_0^t h_2(\varsigma)\,d\varsigma\right)\right\}\|\nabla\Psi\|_2^2\\
&+\varepsilon\left\{(1-a)\big(q^{-}+2\big)-\frac{1}{2}\right\}(h_1o\nabla\Phi+h_2o\nabla\Psi)\\
&+\varepsilon\left\{\frac{(1-a)(q^{-}+2)}{\gamma+1}-1\right\}\left(\|\nabla\Phi\|_2^{2(\gamma+1)}+\|\nabla\Psi\|_2^{2(\gamma+1)}\right)\\
&+\varepsilon\Big\{c_0\underbrace{\Big(a-2\big(p^{-}+2\big)(1-a)\,d_1(B\alpha_2)^{-2(p^{+}+2)}\Big)}_{\widehat{c}}-C_{13}(\kappa)\Big\}\big(\varrho(\Phi)+\varrho(\Psi)\big)\\
&+\varepsilon\big\{(1-a)\big(2q^{-}+4\big)-C_{13}(\kappa)\big\}\mathbb{T}(t),
\end{aligned}
\tag{5.26}
$$

where $C_{13}(\kappa)=C_{11}(\kappa)+C_{12}(\kappa)$, by (5.5),(2.3), and (5.4), one can check that $\widehat{c}>0$.

Here, assume $0<a$ so small that

$$\big(q^{-}+2\big)(1-a)>1+\gamma,$$

we have

$$
\begin{aligned}
&\lambda_1:=\big(q^{-}+2\big)(1-a)-1>0,\\
&\lambda_2:=\big(q^{-}+2\big)(1-a)-\frac{1}{2}>0,\\
&\lambda_3:=\frac{(q^{-}+2)(1-a)}{\gamma+1}-1>0,
\end{aligned}
$$

and we assume

$$\max\left\{\int_0^\infty h_1(\varsigma)\,d\varsigma,\int_0^\infty h_2(\varsigma)\,d\varsigma\right\}<\frac{(q^{-}+2)(1-a)-1}{((q^{-}+2)(1-a)-\frac{1}{2})}=\frac{2\lambda_1}{2\lambda_1+1},\tag{5.27}$$

which gives

$$
\begin{aligned}
&\lambda_4=\left\{\Big(\big(q^{-}+2\big)(1-a)-1\Big)-\int_0^t h_1(\varsigma)\,d\varsigma\left(\big(q^{-}+2\big)(1-a)-\frac{1}{2}\right)\right\}>0,\\
&\lambda_5=\left\{\Big(\big(q^{-}+2\big)(1-a)-1\Big)-\int_0^t h_2(\varsigma)\,d\varsigma\left(\big(q^{-}+2\big)(1-a)-\frac{1}{2}\right)\right\}>0.
\end{aligned}
$$

Next, we pick κ large enough such that

$$
\begin{aligned}
&\lambda_6=c_0\widehat{c}-C_{13}(\kappa)>0,\\
&\lambda_7=2\big(q^{-}+2\big)(1-a)-C_{13}(\kappa)>0.
\end{aligned}
$$

At this point, we fix κ, a and select ε so small that

$$\lambda_8=1-\varepsilon\kappa(\widehat{m}+\widehat{s})>0$$

and

$$\begin{aligned}\Re(0) = \mathbb{T}(0) + \frac{\varepsilon}{\eta+1}\int_\Omega \big[\Phi_0|\Phi_1|^\eta\Phi_1 + \Psi_0|\Psi_1|^\eta\Psi_1\big]\,dx \\ + \varepsilon\int_\Omega [\nabla\Phi_1\nabla\Phi_0 + \nabla\Psi_1\nabla\Psi_0]\,dx > 0,\end{aligned} \tag{5.28}$$

and from (5.9) and (5.10)

$$\Re(t) \le c\big[\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} + \|\Psi\|_{2(q^-+2)}^{2(q^-+2)}\big]. \tag{5.29}$$

Thus, for some $\mu_1 > 0$, (5.26) implies

$$\begin{aligned}\Re'(t) \ge \mu_1\big\{&\mathbb{T}(t) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|w_t\|_{\eta+2}^{\eta+2} + \|\nabla v\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} \\ &+ \|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2 + \|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2 + (h_1 o\nabla\Phi) + (h_2 o\nabla\Psi) \\ &+ \varrho(\Phi) + \varrho(\Psi)\big\}\end{aligned} \tag{5.30}$$

and

$$\Re(t) \ge \Re(0) > 0, \quad t > 0. \tag{5.31}$$

After, by Hölder's and Young's inequalities, we find

$$\begin{aligned}\left|\int_\Omega \big(\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\big)\,dx\right| \le C\big[&\|\Phi\|_{2(q^-+2)}^\theta + \|\Phi_t\|_{\eta+2}^\mu \\ &+ \|\Psi\|_{2(q^-+2)}^\theta + \|\Psi_t\|_{\eta+2}^\mu\big],\end{aligned} \tag{5.32}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Next, assume $\mu = (\eta+2)$ to reach

$$\theta = \frac{(\eta+2)}{(\eta+1)} \le 2\big(q^- + 2\big).$$

By using (5.7) and (4.26), we find

$$\begin{aligned}&\|\Phi\|_{2(q^-+2)}^{\frac{\eta+2}{(\eta+1)}} \le K\big(\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{T}(t)\big) \\ &\|\Psi\|_{2(q^-+2)}^{\frac{\eta+2}{(\eta+1)}} \le K\big(\|\Psi\|_{2(q^-+2)}^{2(q^-+2)} + \mathbb{T}(t)\big), \quad \forall t \ge 0.\end{aligned}$$

Then

$$\begin{aligned}&\left|\int_\Omega \big(\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\big)\,dx\right| \\ &\quad \le c\big\{\big(\varrho(\Phi) + \varrho(\Psi)\big) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \mathbb{T}(t)\big\}.\end{aligned} \tag{5.33}$$

Hence

$$\Re(t) = \left(\mathbb{T}(t) + \frac{\varepsilon}{\eta+1}\int_\Omega \big(\Phi|\Phi_t|^\eta\Phi_t + \Psi|\Psi_t|^\eta\Psi_t\big)\,dx\right.$$

$$
\begin{aligned}
&+\varepsilon \int_\Omega (\nabla\Phi_t\nabla\Phi + \nabla\Psi_t\nabla\Psi)\,dx\Big) \\
&\le c\Big(\mathbb{T}(t) + \|\Phi_t\|_{\eta+2}^{\eta+2} + \|\Psi_t\|_{\eta+2}^{\eta+2} + \|\nabla\Phi\|_2^2 + \|\nabla\Psi\|_2^2 \\
&\quad + \|\nabla\Phi_t\|_2^2 + \|\nabla\Psi_t\|_2^2 + \|\nabla\Phi\|_2^{2(\gamma+1)} + \|\nabla\Psi\|_2^{2(\gamma+1)} \\
&\quad + (h_1 o\nabla\Phi) + (h_2 o\nabla\Psi) + \big(\varrho(\Phi) + \varrho(\Psi)\big)\Big). \qquad (5.34)
\end{aligned}
$$

From (5.30) and (5.34), we have

$$\mathfrak{R}'(t) \ge \lambda_1 \mathfrak{R}(t), \qquad (5.35)$$

where $\lambda_1 > 0$, this relies on μ_1 and c. Hence, (5.35) gives

$$\mathfrak{R}(t) \ge \mathfrak{R}(0)e^{(\lambda_1 t)} \quad \forall t > 0. \qquad (5.36)$$

Then (5.29) and (5.36) imply

$$\|\Phi\|_{2(q^-+2)}^{2(q^-+2)} + \|\Psi\|_{2(q^-+2)}^{2(q^-+2)} \ge Ce^{(\lambda_1 t)}, \quad \forall t > 0.$$

This implies that the solution grows exponentially with $L^{2(p^-+2)}$-norm. This ends the proof. □

6 General decay

In this section, we state and prove the general decay of system (1.1) in the case $f_1 = f_2 = 0$. For this goal, problem (1.1) can be written as

$$
\begin{cases}
|\Phi_t|^\eta \Phi_{tt} - \mathcal{T}(\|\nabla\Phi\|_2^2)\Delta\Phi + \int_0^t h_1(t-\varsigma)\Delta\Phi(\varsigma)\,d\varsigma - \Delta\Phi_{tt} + g_1(\Phi_t) = 0, \\
\Phi(x,0) = \Phi_0(x), \qquad \Phi_t(x,0) = \Phi_1(x), \quad \text{in } \Omega \\
\Phi(x,t) = 0, \quad \text{in } \partial\Omega \times (0,T),
\end{cases} \qquad (6.1)
$$

where

$$g_1(\Phi_t) = \zeta_1 \big|\Phi_t(t)\big|^{m(x)-2}\Phi_t(t).$$

We introduce the modified functional of energy $\mathfrak{E}$ of (6.1) as follows:

$$
\begin{aligned}
\mathfrak{E}(t) &= \frac{1}{\eta+2}\|\Phi_t\|_{\eta+2}^{\eta+2} + \frac{1}{2}\|\nabla\Phi_t\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla\Phi\|_2^{2(\gamma+1)} \\
&\quad + \frac{1}{2}\left(1 - \int_0^t h_1(\varsigma)\,d\varsigma\right)\|\nabla\Phi\|_2^2 + \frac{1}{2}(h_1 o\nabla\Phi)(t). \qquad (6.2)
\end{aligned}
$$

From Lemma 2.4, the functional of energy satisfies

$$\mathfrak{E}'(t) \le -\zeta_1 \int_\Omega \big|\Phi_t(t)\big|^{m(x)}\,dx + \frac{1}{2}\big(h_1' o\nabla\Phi\big)(t) - \frac{1}{2}h_1(t)\|\nabla\Phi\|_2^2 \le 0. \qquad (6.3)$$

Lemma 6.1 (*Komornik*, [19]) *Assume a nonincreasing function* $E:\mathbb{R}_+\to\mathbb{R}_+$ *and suppose that* $\exists\sigma,\omega>0$ *in a manner that*

$$\int_{\Im}^{\infty} E^{1+\aleph}(t)\,dt \le \frac{1}{\omega}E^{\aleph}(0)E(\Im)=cE(\Im),\quad \forall\Im>0. \tag{6.4}$$

Then we have $\forall t\ge 0$

$$\begin{cases} E(t)\le cE(0)/(1+t)^{\frac{1}{\aleph}}, & \text{if } \aleph>0,\\ E(t)\le cE(0)e^{-\omega t}, & \text{if } \aleph=0. \end{cases} \tag{6.5}$$

Theorem 6.2 *Suppose that* (1.3), (2.1)–(2.2), *and* (2.4) *hold. Then there exist* $c,\lambda>0$ *so that the solution of* (6.1) *satisfies*

$$\begin{cases} \mathfrak{E}(t)\le c\mathfrak{E}(0)/(1+t)^{\frac{2}{m^+-2}}, & \text{if } m^+>2,\\ \mathfrak{E}(t)\le c\mathfrak{E}(0)e^{-\lambda t}, & \text{if } m(x)=2. \end{cases} \tag{6.6}$$

Proof Multiplying $(6.1)_1$ by $\Phi\mathfrak{E}^p(t)$ for $p>0$,
then integrating over $\Omega\times(\Im,T)$, where $\Im<T$, gives

$$\begin{aligned} &\int_{\Im}^{T}\mathfrak{E}^p(t)\int_{\Omega}\biggl\{\Phi|\Phi_t|^{\eta}\Phi_{tt}-\mathcal{T}\bigl(\|\nabla\Phi\|_2^2\bigr)\Phi\Delta\Phi+\int_0^t h_1(t-\varsigma)\Phi\Delta\Phi(\varsigma)\,d\varsigma\\ &\quad-\Phi\Delta\Phi_{tt}+\zeta_1\Phi\Phi_t\bigl|\Phi_t(t)\bigr|^{m(x)-2}\biggr\}\,dx\,dt=0, \end{aligned} \tag{6.7}$$

we deduce that

$$\begin{aligned} &\int_{\Im}^{T}\mathfrak{E}^p(t)\int_{\Omega}\biggl\{\frac{d}{dt}\frac{1}{\eta+1}\bigl(\Phi|\Phi_t|^{\eta}\Phi_t\bigr)-\frac{1}{\eta+1}|\Phi_t|^{\eta+2}+\frac{d}{dt}(\nabla\Phi\nabla\Phi_t)-|\nabla\Phi_t|^2\\ &\qquad+\mathcal{T}\bigl(\|\nabla\Phi\|_2^2\bigr)|\nabla\Phi|^2-\int_0^t h_1(t-\varsigma)\nabla\Phi\nabla\Phi(\varsigma)\,d\varsigma+\zeta_1\Phi\Phi_t\bigl|\Phi_t(t)\bigr|^{m(x)-2}\biggr\}\,dx\,dt\\ &\quad=0. \end{aligned} \tag{6.8}$$

By (6.2) and the relation

$$\begin{aligned} &\frac{d}{dt}\biggl(\mathfrak{E}^p(t)\int_{\Omega}\bigl(\Phi|\Phi_t|^{\eta}\Phi_t+\nabla\Phi\nabla\Phi_t\bigr)\,dx\biggr)\\ &\quad=p\mathfrak{E}^{p-1}(t)\mathfrak{E}'(t)\biggl(\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t\,dx+\int_{\Omega}\nabla\Phi\nabla\Phi_t\,dx\biggr)\\ &\qquad+\mathfrak{E}^p(t)\frac{d}{dt}\biggl(\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t\,dx+\int_{\Omega}\nabla\Phi\nabla\Phi_t\,dx\biggr), \end{aligned}$$

we deduce

$$(\eta+2)\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt$$

$$
\begin{aligned}
&= \underbrace{\int_{\Im}^{T} \frac{d}{dt}\left(\mathfrak{E}^p(t)\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t\,dx\right)dt}_{I_1} - \underbrace{p\int_{\Im}^{T}\left(\mathfrak{E}^{p-1}(t)\mathfrak{E}'(t)\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t\,dx\right)dt}_{I_2} \\
&\quad + (\eta+1)\underbrace{\int_{\Im}^{T}\frac{d}{dt}\left(\mathfrak{E}^p(t)\int_{\Omega}\nabla\Phi\nabla\Phi_t\,dx\right)dt}_{I_3} \\
&\quad \underbrace{-(\eta+1)p\int_{\Im}^{T}\left(\mathfrak{E}^{p-1}(t)\mathfrak{E}'(t)\int_{\Omega}\nabla\Phi\nabla\Phi_t\,dx\right)dt}_{I_4} - \underbrace{\frac{\eta}{2}\int_{\Im}^{T}\left(\mathfrak{E}^p(t)\int_{\Omega}|\nabla\Phi_t|^2\,dx\right)dt}_{I_5} \\
&\quad + \underbrace{\frac{\eta+2}{2}\int_{\Im}^{T}\left(\mathfrak{E}^p(t)\left(1-\int_0^t h_1(\varsigma)\,d\varsigma\right)\int_{\Omega}|\nabla\Phi|^2\,dx\right)dt}_{I_6} \\
&\quad + \underbrace{\left((\eta+1)+\frac{\eta+2}{2(\gamma+1)}\right)\int_{\Im}^{T}\left(\mathfrak{E}^p(t)\int_{\Omega}\|\nabla\Phi\|_2^{2\gamma}|\nabla\Phi|^2\,dx\right)dt}_{I_7} \\
&\quad + \underbrace{(\eta+1)\zeta_1\int_{\Im}^{T}\left(\mathfrak{E}^p(t)\int_{\Omega}\Phi\Phi_t\big|\Phi_t(t)\big|^{m(x)-2}\,dx\right)dt}_{I_8} \\
&\quad + \underbrace{\frac{\eta+2}{2}\int_{\Im}^{T}\big(\mathfrak{E}^p(t)(h_1\circ\nabla\Phi)(t)\big)\,dt}_{I_9} \\
&\quad \underbrace{-(\eta+1)\int_{\Im}^{T}\left(\mathfrak{E}^p(t)\int_0^t h_1(t-\varsigma)\int_{\Omega}\nabla\Phi\nabla\Phi(\varsigma)\,dx\,d\varsigma\right)dt}_{I_{10}}.
\end{aligned}
\tag{6.9}
$$

Now, we estimate $I_j, j = 1,\dots,10$, of the RHS in (6.9), we have

$$
\begin{aligned}
I_1 &= \mathfrak{E}^p(T)\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t(x,T)\,dx - \mathfrak{E}^p(\Im)\int_{\Omega}\Phi|\Phi_t|^{\eta}\Phi_t(x,\Im)\,dx \\
&\le c\mathfrak{E}^p(T)\left\{\left\|\Phi(x,T)\right\|_2^2 + \left\|\Phi_t(x,T)\right\|_{\eta+2}^{\eta+2}\right\} \\
&\quad + c\mathfrak{E}^p(\Im)\left\{\left\|\Phi(x,\Im)\right\|_2^2 + \left\|\Phi_t(x,\Im)\right\|_{\eta+2}^{\eta+2}\right\} \\
&\le c\mathfrak{E}^p(T)\left\{c_*\left\|\nabla\Phi(T)\right\|_2^2 + \mathfrak{E}(T)\right\} \\
&\quad + c\mathfrak{E}^p(\Im)\left\{c_*\left\|\nabla\Phi(\Im)\right\|_2^2 + \mathfrak{E}(\Im)\right\} \\
&\le c_1\left(\mathfrak{E}^{p+1}(T) + \mathfrak{E}^{p+1}(\Im)\right).
\end{aligned}
\tag{6.10}
$$

Because $\mathfrak{E}$ is a nonincreasing function, we find

$$
I_1 \le c\mathfrak{E}^{p+1}(\Im) \le \mathfrak{E}^p(0)\mathfrak{E}(\Im) \le c\mathfrak{E}(\Im). \tag{6.11}
$$

Similarly, we find

$$
I_2 \le -p\int_{\Im}^{T}\mathfrak{E}^{p-1}(t)\mathfrak{E}'(t)\big(c_*\mathfrak{E}(t) + \mathfrak{E}(t)\big)\,dt
$$

$$\leq -c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}'(t)\,dt \leq c\mathfrak{E}^{p+1}(\Im) \leq c\mathcal{E}(\Im), \tag{6.12}$$

$$\begin{aligned} I_3 &\leq c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\big(\|\nabla\Phi\|_2^2 + \|\nabla\Phi_t\|_2^2\big)\,dt \\ &\leq c\mathfrak{E}^{p+1}(\Im) \leq \mathfrak{E}^{p}(0)\mathfrak{E}(\Im) \leq c\mathfrak{E}(\Im), \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} I_4 &\leq -(\eta+1)p\int_{\Im}^{T}\mathfrak{E}^{p-1}(t)\mathfrak{E}'(t)\big(c\mathfrak{E}(t)\big)\,dt \\ &\leq -c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}'(t)\,dt \leq c\mathfrak{E}^{p+1}(\Im) \leq c\mathfrak{E}(\Im). \end{aligned} \tag{6.14}$$

Next, we get

$$\begin{aligned} I_5 &= -\frac{\eta}{2}c\int_{\Im}^{T}\big(\mathfrak{E}^{p}(t)\|\nabla\Phi_t\|_2^2\big)\,dt \\ &\leq c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}(t)\,dt \leq c\mathfrak{E}^{p+1}(\Im) \leq c\mathfrak{E}(\Im). \end{aligned} \tag{6.15}$$

After that, we get

$$I_6 \leq (\eta+2)\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}(t)\,dt \leq c\mathfrak{E}^{p+1}(\Im) \leq c\mathfrak{E}(\Im). \tag{6.16}$$

For the next term, we have

$$\begin{aligned} I_7 &= \big(2(\gamma+1)(\eta+1)+(\eta+2)\big)\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\frac{\|\nabla\Phi\|_2^{2(\gamma+1)}}{2(\gamma+1)}\right)dt \\ &\leq c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}(t)\,dt \leq c\mathfrak{E}^{p+1}(\Im) \leq c\mathfrak{E}(\Im), \end{aligned} \tag{6.17}$$

by Young's inequality, we find

$$\begin{aligned} I_8 &= (\eta+1)\zeta_1\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}\Phi\Phi_t\big|\Phi_t(t)\big|^{m(x)-2}\,dx\right)dt \\ &\leq \varepsilon\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}\big|\Phi(t)\big|^{m(x)}\,dx\right)dt \\ &\quad + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\big|\Phi_t(t)\big|^{m(x)}\,dx\right)dt \\ &\leq \varepsilon\int_{\Im}^{T}\mathfrak{E}^{p}(t)\left[\int_{\Omega_+}\big|\Phi(t)\big|^{m^+}\,dx + \int_{\Omega_-}\big|\Phi(t)\big|^{m^-}\,dx\right]dt \\ &\quad + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\big|\Phi_t(t)\big|^{m(x)}\,dx\right)dt. \end{aligned}$$

Here, utilizing $H_0^1(\Omega)\hookrightarrow L^{m^-}(\Omega)$ and $H_0^1(\Omega)\hookrightarrow L^{m^+}(\Omega)$, we get

$$I_8 \leq \varepsilon\int_{\Im}^{T}\mathfrak{E}^{p}(t)\big[c\big\|\nabla\Phi(t)\big\|_2^{m^+} + c\big\|\nabla\Phi(t)\big\|_2^{m^-}\big]\,dt$$

$$
\begin{aligned}
&+ c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\left|\Phi_t(t)\right|^{m(x)}dx\right)dt\\
&\le \varepsilon\int_{\Im}^{T}\mathfrak{E}^{p}(t)\left[c\mathfrak{E}^{\frac{m^+-2}{2}}(0)\mathfrak{E}(t)+c\mathfrak{E}^{\frac{m^--2}{2}}(0)\mathfrak{E}(t)\right]dt\\
&\quad + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\left|\Phi_t(t)\right|^{m(x)}dx\right)dt\\
&\le c\varepsilon\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\left|\Phi_t(t)\right|^{m(x)}dx\right)dt
\end{aligned}
\tag{6.18}
$$

and

$$
I_9 \le (\eta+2)\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}(t)\,dt \le c\mathfrak{E}^{p+1}(\Im)\le c\mathfrak{E}(\Im).
\tag{6.19}
$$

By Young's inequality, we get

$$
\begin{aligned}
I_{10} &\le (\eta+1)\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\big(c\|\nabla\Phi\|_2^2 + c(h_1\circ\nabla\Phi)(t)\big)\,dt\right.\\
&\le c\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}(t)\,dt \le c\mathfrak{E}^{p+1}(\Im)\le c\mathfrak{E}(\Im).
\end{aligned}
\tag{6.20}
$$

By substituting (6.11)–(6.20) into (6.9), we find

$$
\begin{aligned}
\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt &\le c\varepsilon\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt + c\mathfrak{E}(\Im)\\
&\quad + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\left|\Phi_t(t)\right|^{m(x)}dx\right)dt.
\end{aligned}
\tag{6.21}
$$

Now, we choose ε so small that

$$
\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt \le c\mathfrak{E}(\Im) + c\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}c_{\varepsilon}(x)\left|\Phi_t(t)\right|^{m(x)}dx\right)dt.
\tag{6.22}
$$

After, we fix ε, $c_{\varepsilon}(x)\le M$ since $m(x)$ is bounded.

Then, by (6.3), we have

$$
\begin{aligned}
\int_{\Im}^{T}\mathfrak{E}^{p+1}(t)\,dt &\le c\mathfrak{E}(\Im) + cM\int_{\Im}^{T}\left(\mathfrak{E}^{p}(t)\int_{\Omega}\left|\Phi_t(t)\right|^{m(x)}dx\right)dt\\
&\le c\mathfrak{E}(\Im) - \frac{cM}{\zeta_1}\int_{\Im}^{T}\mathfrak{E}^{p}(t)\mathfrak{E}'(t)\,dt\\
&\le c\mathfrak{E}(\Im) + \frac{cM}{\zeta_1(p+1)}\left[\mathfrak{E}^{p+1}(\Im)-\mathfrak{E}^{p+1}(T)\right]\le c\mathfrak{E}(\Im).
\end{aligned}
\tag{6.23}
$$

Taking $T\to\infty$, we get

$$
\int_{\Im}^{\infty}\mathfrak{E}^{p+1}(t)\,dt \le c\mathfrak{E}(\Im).
\tag{6.24}
$$

Hence, Komornik's Lemma 6.1 (with $\aleph = p = \frac{m^+-2}{2}$) gives (6.6). This ends the proof. □

7 Conclusion

In this paper, we investigated a coupled nonlinear viscoelastic Kirchhoff-type system with sources and variable exponents. Firstly, we showed the global existence of the solution. Next, we proved the blow-up result with negative initial energy. After that, we established the exponential growth of solution but with positive initial energy. At the end of this study we obtained the general decay by Komornik's lemma in the case of absence of the source terms.

As for the future vision, we will apply the same method to study other systems, but with the addition of some damping terms.

Author contributions

All the authors contributed to the study. All authors read and approve the final manuscript.

Funding

No funding is associated with the current research work.

Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethical approval

There is no ethical issue in this work. All the authors actively participated in this research and approved it for publication.

Competing interests

The authors declare no competing interests.

Author details

[1]Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Laghouat, Algeria. [2]Laboratory of Mathematics and Applied Sciences, Ghardaia University, Ghardaia, Algeria. [3]Department of Mathematics, Faculty of Sciences, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria. [4]Department of Mathematics, College of Science, Qassim University, 51452 Buraydah, Saudi Arabia.

Received: 7 March 2024 Accepted: 21 April 2024 Published online: 10 May 2024

References

1. Agre, K., Rammaha, M.A.: Systems of nonlinear wave equations with damping and source terms. Differ. Integral Equ. **19**, 1235–1270 (2007)
2. Al-Mahdi, A.: The coupling system of Kirchhoff and Euler-Bernoulli plates with logarithmic source terms: strong damping versus weak damping of variable-exponent type. AIMS Math. **8**(11), 27439–27459 (2023). https://doi.org/10.3934/math.20231404
3. Ball, J.: Remarks on blow-up and nonexistence theorems for nonlinear evolutions equation. Q. J. Math. **28**, 473–486 (1977)
4. Ben Aissa, A., Ouchenane, D., Zennir, K.: Blow up of positive initial-energy solutions to systems of nonlinear wave equations with degenerate damping and source terms. Nonlinear Stud. **19**(4), 523–535 (2012)
5. Bland, D.R.: The Theory of Linear Viscoelasticity. Courier Dover Publications, Mineola (2016)
6. Boulaaras, S., Choucha, A., Ouchenane, D., et al.: Blow up, growth, and decay of solutions for a class of coupled nonlinear viscoelastic Kirchhoff equations with distributed delay and variable exponents. J. Inequal. Appl. **2024**, 55 (2024). https://doi.org/10.1186/s13660-024-03132-2
7. Boulaaras, S., Choucha, A., Ouchenane, D., Cherif, B.: Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms. Adv. Differ. Equ. **2020**, 310 (2020)
8. Boulaaras, S., Choucha, A., Ouchenane, D., Jan, R.: Blow up, growth, and decay of solutions for a class of coupled nonlinear viscoelastic Kirchhoff equations with distributed delay and variable exponents. J. Inequal. Appl. **2024**, 55 (2024). https://doi.org/10.1186/s13660-024-03132-2.
9. Cavalcanti, M.M., Cavalcanti, D., Ferreira, J.: Existence and uniform decay for nonlinear viscoelastic equation with strong damping. Math. Methods Appl. Sci. **24**, 1043–1053 (2001)
10. Choucha, A., Boulaaras, S., Jan, R., Alharbi, R.: Blow-up and decay of solutions for a viscoelastic Kirchhoff-type equation with distributed delay and variable exponents. Math. Methods Appl. Sci., 1–18 (2024). https://doi.org/10.1002/mma.9950
11. Choucha, A., Boulaaras, S., Ouchenane, D., Beloul, S.: General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms. Math. Methods Appl. Sci. **44**. 1–22 (2020). https://doi.org/10.1002/mma.7121
12. Choucha, A., Boulaaras, S.M.: Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan-Taylor damping. Bound. Value Probl. (2021). https://doi.org/10.1186/s13661-021-01555-0

13. Choucha, A., Ouchenane, D., Boulaaras, S.: Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms. J. Nonlinear Funct. Anal. (2020). https://doi.org/10.23952/jnfa.2020.31
14. Choucha, A., Ouchenane, D., Zennir, K.: Exponential growth of solution with L_p-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data. Open J. Math. Anal. **4**(1), 76–83 (2020)
15. Coleman, B.D., Noll, W.: Foundations of linear viscoelasticity. Rev. Mod. Phys. **33**(2), 239 (1961)
16. Ekinci, F., Piskin, E., Boulaaras, S.M., Mekawy, I.: Global existence and general decay of solutions for a quasilinear system with degenerate damping terms. J. Funct. Spaces (2021)
17. Georgiev, V., Todorova, G.: Existence of a solution of the wave equation with nonlinear damping and source term. J. Differ. Equ. **109**, 295–308 (1994)
18. Kirchhoff, G.: Vorlesungen uber Mechanik. Tauber, Leipzig (1883)
19. Komornik, V.: Exact controlability and stabilisation. The multiplier method, Masson and Wiley
20. Liu, W.: General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source. Nonlinear Anal. **73**, 1890–1904 (2010)
21. Mesaoudi, S., Kafini, M.: On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay. Ann. Pol. Math. **122**(1), 49–70 (2019)
22. Mesloub, F., Boulaaras, S.: General decay for a viscoelastic problem with not necessarily decreasing kernel. J. Appl. Math. Comput. **58**, 647–665 (2018). https://doi.org/10.1007/S12190-017-1161-9
23. Messaoudi, S.A., Said-Houari, B.: Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. J. Math. Anal. Appl. **365**, 277–287 (2010)
24. Ouchenane, D., Boulaaras, S., Choucha, A., Alngga, M.: Blow-up and general decay of solutions for a Kirchhoff-type equation with distributed delay and variable-exponents. Quaest. Math. (2023). https://doi.org/10.2989/16073606.2023.2183156
25. Pişkin, E.: Blow up of solutions for a nonlinear viscoelastic wave equations with variable exponents. Middle East J. Sci. **5**(2), 134–145 (2019)
26. Pişkin, E., Ekinci, F.: General decay and blowup of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms. Math. Methods Appl. Sci. **42**(16), 5468–5488 (2019)
27. Said-Houari, B.: Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms. Differ. Integral Equ. **23**, 79–92 (2010)
28. Vitillaro, E.: Global existence theorems for a class of evolution equations with dissipation. Arch. Ration. Mech. Anal. **149**, 155–182 (1999)
29. Yilmaz, N., Piskin, E., Boulaaras, S.: Viscoelastic plate equation with variable exponents: existence and blow-up. J. Anal. (2024). https://doi.org/10.1007/s41478-024-00765-w

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.