Chorfi *Boundary Value Problems* (2024) 2024:60
https://doi.org/10.1186/s13661-024-01867-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The posteriori analysis of the spectral element discretization of the wave equation

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Abstract

This work focuses on discretizing a second-order linear wave equation using the implicit Euler scheme for time discretization and the spectral element method for spatial discretization. We prove that optimal adaptivity can be achieved by combining adaptive time steps with adaptive spectral mesh. We introduce two sets of error indicators for time and space, respectively, and derive optimal estimates.

Keywords: Second order wave equation; Implicit Euler method; Spectral element method; Posteriori analysis; Error indicators

1 Introduction

Over the past two decades, there has been significant attention given to a posteriori error analysis of partial differential equations. This theory has been extensively applied to elliptic and parabolic problems in the context of finite element approximation, as evidenced by various studies [1–13]. However, the a posteriori analysis of hyperbolic problems, whether using the finite element method or the spectral element method, remains relatively unexplored in the literature [14–24].

This study aims to advance the a posteriori error analysis for the initial-boundary-value problem associated with the second-order linear wave equation, which is discretized using the spectral element method. In this method, the solution of partial differential equations is approximated using higher-order polynomial functions over each element of the decomposition [25–27]. The discretization parameter comprises a K-tuple, determined by the maximum polynomial degree N_k on each element. Similar to the concept in the $h-p$ version of the finite element method (as discussed in [2, 7, 28]), this parameter also includes a quantity h_k representing the diameter of the element. Additionally, we demonstrate that converting the second-order wave equation into a first-order system involves time discretization equivalent to the backward Euler-time discretization of the corresponding first-order system.

This study extends the results obtained by Bernardi et al. [6] for the finite element method to the spectral element method. Specifically, we introduce two families of indicators, both of which are of residual type. The first family, as introduced in [11], is global concerning spatial variables but local concerning time discretization. The selection of the next time step relies on the time error indicator from this family. The second family serves as an efficient tool for mesh adaptivity. These indicators are local with respect to both temporal and spatial variables and can be explicitly computed as functions of the discrete solution and problem data. They are deemed optimal if their Hilbert sum is equivalent to the error, and the corresponding constant remains independent of the discretization parameter. The structure of this document is as follows:

In Sect. 2, we introduce the second-order linear wave equation and delve into the time-semi-discrete problem along with its spatial discretization.

Section 3 focuses on the construction of error indicators for the wave equation, accompanied by the establishment of upper and lower bounds derived from time and space indicators.

2 The discrete problems

We denote Ω as an open bounded connected domain in $\mathbb{R}^d$, where d takes on values of 1, 2, or 3. Let Γ represent its Lipschitz continuous boundary, and T denote a positive real number.

Considering $f \in L^1(0,T;H_0^1(\Omega))$, we examine the following initial-boundary-value problem for the second-order linear wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } \Omega \times]0,T[, \\ u = 0 & \text{on } \Gamma \times]0,T[, \\ u(\cdot,0) = u_0 & \text{in } \Omega, \\ \partial_t u(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{1}$$

Here, u represents the unknown function defined over $\Omega \times]0,T[$, while (u_0, v_0) denote the data functions defined over Ω.

Proposition 1 *For any given data $f \in L^1(0,T;H_0^1(\Omega))$ and $(u_0,v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, problem (1) possesses a unique solution u within the space $\mathcal{C}^1(0,T;L^2(\Omega)) \cap \mathcal{C}^0(0,T;H_0^1(\Omega))$, satisfying the following estimate for $0 \le t \le T$:*

$$\left(\|\partial_t u\|^2 + \|\nabla u\|^2\right)^{\frac{1}{2}} \le \left(\|v_0\|^2 + \|\nabla u_0\|^2\right)^{\frac{1}{2}} + \int_0^t \|f\|(s)\,ds. \tag{2}$$

The establishment of the well-posedness of system (1) relies on the Cauchy–Lipschitz theorem and the estimate (2). For a detailed proof, refer to ([29], Chap. 1, Th. 12.3). Additionally, for a broader examination of non-linear wave equations, consult [30–35].

In order to formulate the time semi-discrete problem, we partition the interval $[0,T]$ into sub-intervals $[t_i, t_{i+1}]$, where $1 \le i \le I$, with $0 = t_0 < t_1 < \cdots < t_I = T$. We define $\tau_i = t_{i+1} - t_i$, $\tau = (\tau_1, \dots, \tau_i)$, $|\tau| = \max_{1 \le i \le I} |\tau_i|$, and

$$\sigma_\tau = \max_{2 \le i \le I} \frac{\tau_i}{\tau_{i-1}}$$

as the regularity parameter. For any family $u^i = u(\cdot, t_i)$, $1 \le i \le I$, we define the function u_τ on the interval $[0,T]$ to be affine on each sub-interval $[t_{i-1}, t_i]$, where $1 \le i \le I$, such that $u_\tau(t_i) = u(\cdot, t_i)$. This function is given by:

$$\forall t \in [t_{i-1}, t_i], \quad u_\tau(t) = u^i - \frac{t_i - t}{\tau_i}\left(u^i - u^{i-1}\right).$$

We then employ the implicit Euler method to discretize the time derivative in problem (1), with the data $f = 0$ for simplification purposes. The time-discrete problem aims to find the sequence $u^i = u(x, t_i)_{0\leq i\leq I}$ in $H_0^1(\Omega)^{I+1}$ such that:

$$\begin{cases} \frac{u^{i+1}-u^i}{\tau_i} - \frac{u^i-u^{i-1}}{\tau_{i-1}} - \tau_i \Delta u^{i+1} = 0 & \text{in } \Omega, 1 \leq i \leq I, \\ u^{i+1} = 0 & \text{on } \Gamma, 1 \leq i \leq I, \\ u^0 = u_0 & \text{in } \Omega, \\ u^1 = u_0 + h_0 v_0 & \text{in } \Omega, \end{cases} \tag{3}$$

Given that (u_0, v_0) is an element of $H_0^1(\Omega) \times H_0^1(\Omega)$, when the values of u^0 and v^0 are known, we establish that u^{i+1}, for $i \geq 1$, serves as a solution to the following variational formulation:

Find u^{i+1} in $H_0^1(\Omega)$ such that for any $v \in H_0^1(\Omega)$ we have:

$$\begin{aligned} &\int_\Omega u^{i+1}(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} + \tau_i^2 \int_\Omega \nabla u^{i+1}(\mathbf{x}) \nabla v(\mathbf{x})\, d\mathbf{x} \\ &\quad = \int_\Omega \left(u^i + \frac{\tau_i}{\tau_{i-1}} \left(u^i - u^{i-1} \right) \right)(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{4}$$

Proposition 2 *If (u_0, v_0) lies in $H_0^1(\Omega) \times H_0^1(\Omega)$, then problem (4) possesses a unique solution u^{i+1} for $i \geq 1$ within $H_0^1(\Omega)$, satisfying the following stability conditions:*

$$\left\| \frac{u^{i+1} - u^i}{\tau_i} \right\|^2 + \left\| \nabla u^{i+1} \right\|^2 \leq \|v_0\|^2 + 2\|\nabla u_0\|^2 + 2\tau_0^2 \|\nabla v_0\|^2. \tag{5}$$

and

$$\left\| \frac{u^{i+1} - u^i}{\tau_i} \right\|^2 + \left\| \nabla u^{i+1} \right\|^2 \leq 2\left(\left\| v^1 \right\|^2 + \left\| \nabla u^1 \right\|^2 \right). \tag{6}$$

Proof We employ the Lax–Milgram theorem to readily demonstrate the uniqueness of the solution to the variational formulation (4). Refer to [21] for the verification of stability conditions (5) and (6). □

Next, we present the a priori time error estimate in the following theorem.

Theorem 1 *For the solution u of problem (1) and $(u^i)_{1\leq i\leq I}$, the solution of problem (3), the a priori error estimate holds for $0 \leq i \leq I$:*

$$\begin{aligned} &\left\| \frac{u(t_{i+1}) - u(t_i)}{\tau_i} - \partial_t u(t_{i+1}) \right\|^2 + \left\| \nabla \left(u(t_i) - u^i \right) \right\|^2 \\ &\quad \leq C\tau^2 \left(\int_0^{t_i} \left(\left\| \partial_t^3 u \right\| + \left\| \partial_t^2 \nabla u \right\| \right)(s)\, ds \right)^2, \end{aligned} \tag{7}$$

where C is a positive constant that remains independent of the step τ.

Refer to [21] for the proof of Theorem 1. The estimate (7) is of order 1, as the time discretization relies on the implicit Euler scheme.

In the subsequent discussion, we will concentrate on the a posteriori analysis of the spectral element method in one dimension, given that the polynomial inverse inequalities are not optimal for the spectral method in dimensions $d \geq 2$. Now, we delineate the discrete space. Let Λ denote the interval $]-1,1[$. For each discrete time $t_i, 0 \leq i \leq I$, we introduce a partition P_i of the interval Λ such that

$$-1 = a_0 \leq a_1 \leq \cdots \leq a_{K-1} \leq a_K = 1,$$

and $\Lambda_k =]a_{k-1}, a_k[$, $1 \leq k \leq K$. Let h_k the length of the sub-interval Λ_k, and $h = \max_{1 \leq k \leq K} h_k$. The discrete parameter δ is a K-tuple of couples $(h_k, N_k), 1 \leq k \leq K$, where a integer $N_k \geq 2$.

Let's begin by revisiting the following formulas, which we will utilize later on. Consider $\xi_0 < \cdots < \xi_N$ as the zeros of the polynomial $(1-x^2)L_N'$, and ρ_j as their corresponding weights, where L_N represents the Legendre polynomial defined on Λ. The Gauss–Lobatto quadrature formula on the interval $\Lambda =]-1,1[$ can be expressed as:

$$\forall \phi \in \mathbb{P}_{2N-1}(\Lambda); \quad \int_{-1}^{1} \phi(x)\,dx = \sum_{j=0}^{N} \phi\big(\xi_j^N\big)\rho_j^N, \tag{8}$$

where $\mathbb{P}_N(\Lambda)$ is the space of polynomials, defined on Λ, with degree $\leq N$.

We define a discrete scalar product for any continuous functions u and v over $\overline{\Lambda}$ as follows:

$$(u,v)_\delta = \sum_{k=1}^{K} \sum_{j=0}^{N_k} u\big(\xi_j^{N_k}\big) v\big(\xi_j^{N_k}\big) \rho_j^{N_k}, \tag{9}$$

where $\xi_j^{N_k} = F_k^{-1}(\xi_j^N)$ and $\rho_j^{N_k} = (a_k - a_{k-1})\rho_j^N$, $0 \leq j \leq N$, such that F_k is the bijection from Λ_k into Λ.

Let i_δ denote the Lagrange interpolation operator on the set of nodes $\xi_j^{N_k}$, taking values in

$$Y_\delta = \big\{ v_\delta \in H^1(\Lambda); v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K \big\}.$$

For every function φ continuous over $\overline{\Lambda}_k$, $i_\delta(\varphi)|\Lambda_k$ belongs to $\mathbb{P}_{N_k}(\Lambda_k)$, and we confirm

$$i_\delta(\varphi)_{|\Lambda_k}\big(\xi_j^{N_k}\big) = \varphi_{|\Lambda_k}\big(\xi_j^{N_k}\big).$$

We consider the following property, which will be widely used in the following:

$$\forall u_\delta \in Y_\delta, \quad \|u_\delta\|^2_{L^2(\Lambda)} \leq (u_\delta, u_\delta)_\delta \leq 3\|u_\delta\|^2_{L^2(\Lambda)}. \tag{10}$$

We define the discrete space as

$$X_\delta^i = \big\{ v_\delta \in H_0^1(\Lambda); \forall \Lambda_k \in P_i \quad v_{\delta|\Lambda_k} \in \mathbb{P}_{N_k}(\Lambda_k), 1 \leq k \leq K \big\}. \tag{11}$$

We introduce the orthogonal projection operator Π_δ^i defined on $H_0^1(\Omega)$ into X_δ^i. If $w \in H_0^1(\Omega)$, $\Pi_\delta^i w$ belongs to X_δ^i such that:

$$\forall t_\delta \in X_\delta^i, \quad \left(\frac{\partial (w - \Pi_\delta^i w)}{\partial x}, \frac{\partial t_\delta}{\partial x} \right) = 0. \tag{12}$$

By employing the Galerkin method along with numerical integration, we formulate the discrete problem derived from problem (3) as follows: Given that u_0 and v_0 are continuous over $\overline{\Lambda}$, the task is to find $(u_\delta^i, 0 \leq i \leq I)$ in $\prod_{i=0}^I X_\delta^i$ such that:

$$u_\delta^0 = i_\delta u_0 \quad \text{and} \quad u_\delta^1 = i_\delta u_0 + \tau_0 i_\delta v_0, \tag{13}$$

$$\forall v_\delta \in X_\delta^{i+1}, \quad \left(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v_\delta \right)_\delta + \tau_i \left(\frac{\partial u_\delta^{i+1}}{\partial x} \frac{\partial v_\delta}{\partial x} \right)_\delta = 0. \tag{14}$$

As in the problem (4), we prove that u_δ^{i+1}, $1 \leq i \leq I$ is the solution of the following discrete variational problem:

Find u_δ^{i+1} in X_δ^{i+1} such that:

$$\forall v_\delta \in X_\delta^{i+1}, \quad \left(u_\delta^{i+1}, v_\delta \right)_\delta + \tau_i^2 \left(\frac{\partial u_\delta^{i+1}}{\partial x}, \frac{\partial v_\delta}{\partial x} \right)_\delta = \left(\Pi_\delta^{i+1} u_\delta^i + \frac{\tau_i}{\tau_{i-1}} \left(u_\delta^i - \Pi_\delta^i u_\delta^{i-1} \right), v_\delta \right)_\delta. \tag{15}$$

Hence, utilizing the Lax–Milgram theorem, we readily establish that the problem (13)-(14) possesses a unique solution.

Remark 1 Opting to utilize various spectral meshes at each time step led us to employ the Π_δ^i operators, diverging from the conventional approach of fixed-grid spectral discretization for the wave equation (as outlined in [21]).

3 A posteriori analysis of the discretizations

In this section, we commence by introducing two sets of error indicators. The first pertains to the temporal discretization, while the second addresses the spectral discretization. We establish upper and lower bounds on the error, initially concentrating on the temporal discretization, followed by an examination of the spatial discretization.

3.1 A posteriori analysis of the time discretization

We define the time indicators for each $1 \leq i \leq I$,

$$\kappa_i = \tau_i \left\| \frac{\partial (u_\delta^{i+1} - u_\delta^i)}{\partial x} \right\| + \tau_i \left\| \frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}} \right\|. \tag{16}$$

This type of time indicators was initially introduced in [11]. Additionally, their utilization in the a posteriori analysis of the finite element discretization of certain parabolic problems (such as the heat equation) can be found in [6]. It is worth noting that given the knowledge of the discrete solutions u_δ^{i+1}, u_δ^i, and u_δ^{i-1}, the time indicator κ_i can be readily computed.

Let $v^i = \frac{u^i - u^{i-1}}{\tau_{i-1}}$ for $1 \le i \le I$. Thus, the residual problem if $U = \binom{u}{v}$, and $U_\tau = \binom{u_\tau}{v_\tau}$ is:

$$\begin{cases} \partial_t (U - U_\tau) - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} (U - U_\tau) = \binom{D_u}{D_v} & \text{in } \Omega \times]0, T[, \\ u - u_\tau = 0 & \text{on } \Gamma \times]0, T[, \\ (U - U_\tau)(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{17}$$

where $D_u(x,t) = v - v_\tau$, for $t_i \le t \le t_{i+1}$, $1 \le i \le I-1$, and $D_u(x,t) = 0$, for $0 \le t \le t_1$ likewise $D_v(x,t) = \frac{\partial^2 (u - u_\tau)}{\partial x^2}$, for $t_i \le t \le t_{i+1}$, $1 \le i \le I-1$ and $D_v(x,t) = \frac{\partial^2 u_\tau}{\partial x^2}$, for $0 \le t \le t_1$.

Proposition 3 *The a posteriori error estimate between the solution u of problem* (1) *with* $f = 0$, *and the solution* $(u^i)_{0 \le i \le I}$ *of problem* (3), *remains valid for* $0 \le i \le I$:

$$\begin{aligned} & \left\| (\partial_t u)(t_{i+1}) - \frac{u^{i+1} - u^i}{\tau_i} \right\|_{H^{-1}(\Lambda)} + \left\| u(t_{i+1}) - u^{i+1} \right\| \\ & \quad \le C \Bigg(\sum_{j=1}^{i} \tau_j \bigg(\left\| \frac{\partial (u^{j+1} - u_\delta^{j+1})}{\partial x} \right\| + \left\| \frac{\partial (u^j - u_\delta^j)}{\partial x} \right\| \bigg) \\ & \qquad + \left\| \left(u^{j+1} - u_\delta^{j+1}\right) - \left(u^j - \Pi_\delta^{j+1} u_\delta^j\right) \right\| \\ & \qquad + \left(\frac{\tau_j}{\tau_{j-1}} \right) \left\| \left(u^j - u_\delta^j\right) - \left(u^{j-1} - \Pi_\delta^j u_\delta^{j-1}\right) \right\| + \kappa_j + \tau_0 \|\nabla u_0\| + \tau_0^2 \|\nabla v_0\| \Bigg). \end{aligned} \tag{18}$$

Proof Taking the inner product of (17) with $\binom{u - u_\delta}{\Delta^{-1}(v - v_\delta)}$ yields:

$$\Sigma(t) = \left(\|u - u_\delta\|^2 + \|v - v_\delta\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Thus,

$$\frac{1}{2} \frac{d^2 \Sigma}{dt} = (D_u, u - u_\delta) + \left(D_v, \Delta^{-1}(v - v_\delta)\right) \le \left(\|D_u\|^2 + \|D_v\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}} \Sigma.$$

So,

$$\frac{d\Sigma}{dt} \le \left(\|D_u\|^2 + \|D_v\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}} \le \|D_u\| + \|D_v\|_{H^{-1}(\Omega)}. \tag{19}$$

Since $\Sigma(0) = 0$, then by integration (19) between 0 and t_{i+1}, we have

$$\Sigma(t_{i+1}) \le \int_0^{t_{i+1}} \left(\|D_u\| + \|D_v\|_{H^{-1}(\Omega)} \right) dt.$$

Since

$$\forall t \in [t_j, t_{j+1}], \quad u_\tau(t) = u^{j+1} - \frac{t_{j+1} - t}{\tau_j} \left(u^{j+1} - u^j\right),$$

then

$$\int_{t_j}^{t_{j+1}} \|D_v\|_{H^{-1}(\Omega)}\,dt = \frac{\partial^2(u^{j+1}-u^j)}{\partial x^2}\int_{t_j}^{t_{j+1}}\left(\frac{t_{j+1}-t}{\tau_j}\right)dt = \left(\frac{\tau_j}{2}\right)\left(\frac{\partial^2(u^{j+1}-u^j)}{\partial x^2}\right).$$

And we conclude by the triangular inequality

$$\left\|\frac{\partial^2(u^{j+1}-u^j)}{\partial x^2}\right\| \le \left\|\frac{\partial^2(u_\delta^{j+1}-u_\delta^j)}{\partial x^2}\right\| + \left\|\frac{\partial^2(u^{j+1}-u_\delta^{j+1})}{\partial x^2}\right\| + \left\|\frac{\partial^2(u^j-u_\delta^j)}{\partial x^2}\right\|.$$

Employing identical arguments, we evaluate $\int_{t_j}^{t_{j+1}} \|D_u\|\,dt$. The combination of all these inequalities yields the desired result (18). □

In the forthcoming proposition, we establish an upper bound for the error indicators κ_i for each $0 \le i \le I$.

Proposition 4 *For indicators κ_i, $0 \le i \le I$, the following estimate is applicable*:

$$\begin{aligned}\kappa_i \le{}& \left\|\int_{t_i}^{t_{i+1}} \frac{\partial^2(u-u_\delta)}{\partial x^2}\,dt\right\| + \left\|\int_{t_i}^{t_{i+1}}(v-v_\delta)\,dt\right\| \\ &+ \sum_{k=0}^{1}\left\|(\partial_t u)(t_{i+1-k}) - \frac{u^{i+1-k}-u^{i-k}}{\tau_{i-k}}\right\|_{H^{-1}(\Omega)} + \left\|u(t_{i+1-k})-u^{i+1-k}\right\| \\ &+ \tau_i\sum_{k=0}^{1}\left\|\frac{\partial^2(u^{i+1-k}-u_\delta^{i+1-k})}{\partial x^2}\right\| \\ &+ \left\|\frac{(u^{i+1-k}-u_\delta^{i+1-k})-(u^{i-k}-\Pi_\delta^{i+1-k}u_\delta^{i-k})}{\tau_{i-k}}\right\|.\end{aligned} \tag{20}$$

Proof Applying the triangle inequality, we only need to bound the following terms:

$$\tau_i\left\|\frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right\|, \quad \tau_i\left\|v^{i+1}-v^i\right\|. \tag{21}$$

i) To bound the first term in (21), we take the inner product of the second line of (17) with $(u^{i+1}-u^i)$ and integrate over the time interval $[t_i,t_{i+1}]$. Thus, we obtain:

$$\frac{\tau_i}{2}\left\|\frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right\|^2 \le \int_{t_i}^{t_{i+1}}\left(\partial_t(v-v_\tau), u^{i+1}-u^i\right)dt + \left(\int_{t_i}^{t_{i+1}}\frac{\partial^2(u-u_\tau)}{\partial x^2}\,dt, \frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right).$$

Then, through integration by parts, we deduce that

$$\frac{\tau_i}{2}\left\|\frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right\|^2 \leq \left((\partial_t u)(t_{i+1}) - \frac{u^{i+1}-u^i}{\tau_i}, u^{i+1}-u^i\right) - \left((\partial_t u)(t_i) - \frac{u^i-u^{i-1}}{\tau_{i-1}}, u^{i+1}-u^i\right) + \left(\int_{t_i}^{t_{i+1}} \frac{\partial^2(u-u_\tau)}{\partial x^2}\,dt, \frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right).$$

Applying Cauchy–Schwarz inequality then

$$\frac{\tau_i}{2}\left\|\frac{\partial^2(u^{i+1}-u^i)}{\partial x^2}\right\| \leq \left\|(\partial_t u)(t_{i+1}) - \frac{u^{i+1}-u^i}{\tau_i}\right\|_{H^{-1}(\Omega)} + \left\|(\partial_t u)(t_i) - \frac{u^i-u^{i-1}}{\tau_{i-1}}\right\|_{H^{-1}(\Omega)} + \left\|\int_{t_i}^{t_{i+1}} \frac{\partial^2(u-u_\tau)}{\partial x^2}\,dt\right\|.$$

ii) Similarly to estimating the first term in (21), for bounding the second term of (21), we take the inner product of the first equation in (17) with $v^{i+1}-v^i$ and integrate over the interval $[t_i, t_{i+1}]$. This yields:

$$\frac{\tau_i}{2}\left\|v^{i+1}-v^i\right\|^2 \leq \left(u(t_{i+1})-u^{i+1}, v^{i+1}-v^i\right) - \left(u(t_i)-u^i, v^{i+1}-v^i\right) - \left(\int_{t_i}^{t_{i+1}}(v-v_\tau)\,dt, v^{i+1}-v^i\right).$$

This allows us to derive the estimate (20). □

3.2 A posteriori analysis of the spectral discretization

For each $1 \leq i \leq I$ and each $\Lambda_k, 1 \leq k \leq K$, we define the spectral indicators

$$\beta_i^k = \left\|u_\delta^i - \Pi_\delta^{i+1}u_\delta^i\right\| + N_k^{-1}\left\|\frac{u_\delta^{i+1}-\Pi_\delta^{i+1}u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}\right\|. \tag{22}$$

These indicators are local and respect both the time and spatial variables, depending on the local discrete solution. Consequently, they can be explicitly computed for each time iteration. It is noteworthy that the first term in (22) arises due to the utilization of different spatial meshes across various time levels, while the other term is consistent with standard residual-based error bounds for the elliptic equation (refer to [1]). Subsequently, the residual problem is derived from the system (13)–(14). For each $1 \leq i \leq I$, we define:

$$v^i = \frac{u^i - u^{i-1}}{\tau_{i-1}}, \qquad v_\delta^i = \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, \qquad eu_\delta^i = u^i - u_\delta^i, \qquad ev_\delta^i = v^i - v_\delta^i. \tag{23}$$

Thus, from problems (3) and (13)–(14), we deduce that the error vector $E_\delta^i = \binom{eu_\delta^i}{ev_\delta^i}$ constitutes the solution to the following residual problem:

$$\begin{cases} \frac{E_\delta^{i+1}-E_\delta^i}{\tau_i} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} E_\delta^{i+1} = \begin{pmatrix} \xi u_\delta^i \\ \xi v_\delta^i \end{pmatrix} & \text{in } \Omega, 0 \le i \le I, \\ eu_\delta^{i+1} = 0 & \text{on } \Gamma, 0 \le i \le I, \\ E_\delta^1 = \begin{pmatrix} u^0 - u_\delta^0 + \tau_0(v^0 - v_\delta^0) \\ v^0 - v_\delta^0 \end{pmatrix} & \text{in } \Omega. \end{cases} \tag{24}$$

The two terms ξu_δ^i and ξv_δ^i belongs to $H^{-1}(\Omega)$ and are defined as

$$\begin{aligned} \langle \xi u_\delta^i, v \rangle &= \left(\frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i}, v \right) \\ \langle \xi v_\delta^i, v \rangle &= -\frac{1}{\tau_i} \left(\frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}}, v \right) - \left(\frac{\partial u_\delta^{i+1}}{\partial x}, \frac{\partial v}{\partial x} \right), \end{aligned} \tag{25}$$

where $\langle \cdot, \cdot \rangle$ is the duality product between $H^{-1}(\Omega)$, and $H_0^1(\Omega)$. The proposition following this deals with bounding the error estimate using the refinement spectral indicators.

Proposition 5 *The a posteriori error estimate between the solution* (u^i) *of problem* (3) *and the solution* (u_δ^i) *of problem* (13)-(14) *holds for all* $1 \le i \le I-1$,

$$\begin{aligned} &\left\| \frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i} \right\|_{H^{-1}(\Omega)} + \left\| u^{i+1} - u_\delta^{i+1} \right\| \\ &\quad \le C \left(\sum_{j=1}^{i} \left(\sum_{k=1}^{K} (\beta_j^k)^2 \right)^{\frac{1}{2}} + \left\| u^0 - u_\delta^0 \right\| + \tau_0 \left\| v^0 - v_\delta^0 \right\| \right). \end{aligned} \tag{26}$$

Proof Applying inequality (6) to the residual problem (24), and noting that for any $a \ge 0$, $b \ge 0$, $\frac{1}{\sqrt{2}}(a+b) \le \sqrt{a^2+b^2} \le a+b$, we derive:

$$\begin{aligned} \left\| ev_\delta^{i+1} \right\|_{H^{-1}(\Omega)} + \left\| eu_\delta^{i+1} \right\| \le C \Bigg(&\left\| ev_\delta^1 \right\|_{H^{-1}(\Omega)} + \left\| eu_\delta^1 \right\| \\ &+ \sum_{j=1}^{i} \tau_j \left(\left\| \xi v_\delta^{j+1} \right\|_{H^{-1}(\Omega)} + \left\| \xi u_\delta^{j+1} \right\| \right) \Bigg). \end{aligned} \tag{27}$$

Next, we need to constrain the terms on the right-hand side of inequality (27). The upper bounds of $\|eu_\delta^1\|_{H^{-1}(\Omega)}$ and $\|ev_\delta^1\|$ are established using the final equation of the system (24). By utilizing the definition of ξu_δ^i, we demonstrate that

$$\left\| \xi u_\delta^{j+1} \right\| = \left\| \frac{u_\delta^j - \Pi_\delta^{j+1} u_\delta^j}{\tau_j} \right\| = \frac{1}{\tau_j} \left(\sum_{k=1}^{K} \left\| u_\delta^j - \Pi_\delta^{j+1} u_\delta^j \right\|_{L^2(\Lambda_k)}^2 \right)^{\frac{1}{2}}.$$

Since,

$$\left\| \xi v_\delta^{j+1} \right\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle \xi v_\delta^{j+1}, v \rangle}{\|v\|},$$

and using the equality (14), we have for any $v \in H_0^1(\Omega)$ and $v_\delta \in X_\delta^i$

$$\|\xi v_\delta^{j+1}\|_{H^{-1}(\Omega)} = -\frac{1}{\tau_j}\left(\frac{u_\delta^{j+1} - \Pi_\delta^{j+1} u_\delta^j}{\tau_j} - \frac{u_\delta^j - \Pi_\delta^j u_\delta^{j-1}}{\tau_{j-1}}, v - v_\delta\right) - \left(\frac{\partial u_\delta^j}{\partial x}, \frac{\partial (v - v_\delta)}{\partial x}\right).$$

We consider for any function $v \in H_0^1(\Omega)$ the function

$$v_\delta = \sum_{k=1}^{K} \pi_{N_k-1}^{1,0}\big(v - v(a_{k-1})\psi_{k-1} - v(a_k)\psi_k\big) + \sum_{k=1}^{K} v(a_k)\psi_k,$$

Here, ψ_k represents affine functions on Λ_k, which are equal to 1 at node a_k and 0 at other nodes a_l for $l \neq k$. $\pi_{N_k-1}^{1,0}$ denotes the orthogonal projection operator from $H_0^1(\Lambda_k)$ onto $\mathbb{P}_{N_k}(\Lambda_k) \cap H_0^1(\Lambda_k)$. For properties of this operator, we refer the reader to [25]. Since $v \in H_0^1(\Omega)$, the function v_δ belongs to the space X_δ. Subsequently, through integration by parts, we derive:

$$\|\xi v_\delta^{j+1}\|_{H^{-1}(\Omega)} = -\frac{1}{\tau_j}\left(\frac{u_\delta^{j+1} - \Pi_\delta^{j+1} u_\delta^j}{\tau_j} - \frac{u_\delta^i - \Pi_\delta^j u_\delta^{j-1}}{\tau_{j-1}}, v - v_\delta\right).$$

Therefore, we establish result (26) by employing the Cauchy–Schwarz inequality. □

The following proposition concerns the upper bound estimate of the spectral indicators.

Proposition 6 *The subsequent estimate is valid for the indicators β_i^k, where $1 \le i \le I$.*

$$\begin{aligned} \beta_i^k \le C\Bigg(&\sum_{j=0}^{1}\Bigg(\bigg\|\frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}}\bigg\|_{H^{-1}(\Lambda_k)} \\ &+ \|u^{i+1-j} - u_\delta^{i+1-j}\|_{L^2(\Lambda_k)}\Bigg) + \tau_i\Bigg(\bigg\|\frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i}\bigg\|_{L^2(\Lambda_k)} \\ &+ \bigg\|\frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x}\bigg\|_{L^2(\Lambda_k)}\Bigg)\Bigg), \end{aligned} \tag{28}$$

where C is a positive constant independent of τ and δ.

Proof We successively bound the two terms in β_i^k, labeled as ${\beta_1}_i^k$ and ${\beta_2}_i^k$. From the first equation of system (24), we have:

$$\frac{eu_\delta^{i+1} - eu_\delta^i}{\tau_i} - ev_\delta^{i+1} = \xi u_\delta^{i+1} = \frac{u_\delta^i - \Pi_\delta^{i+1} u_\delta^i}{\tau_i}.$$

Next, we take the L^2 norm of this equation and multiply by τ_i, resulting in:

$${\beta_1}_i^k \le \sum_{j=0}^{1}\|u^{i+1-j} - u_\delta^{i+1-j}\|_{L^2(\Lambda_k)} + \tau_i\bigg\|\frac{(u^{i+1} - u_\delta^{i+1}) - (u^i - \Pi_\delta^{i+1} u_\delta^i)}{\tau_i}\bigg\|_{L^2(\Lambda_k)}. \tag{29}$$

Let v_δ be the function defined as $(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}})\psi_k$ on the interval Λ_k, and 0 elsewhere on $\Lambda \backslash \Lambda_k$, where ψ_k represents affine functions on Λ_k equal to 1 at the node a_k and

0 at other nodes a_l for $l \neq k$. Then, we demonstrate through integration by parts that:

$$\langle \xi v_\delta^{i+1}, v_\delta \rangle = -\frac{1}{\tau_i} \left\| \left(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}} \right) \psi_k^{\frac{1}{2}} \right\|_{L^2(\Lambda_k)}^2 .$$

Therefore, taking the inner product of the second equation of system (24) with $-\tau_i v_\delta$ yields:

$$\begin{aligned} -\frac{1}{\tau_i} & \left\| \left(\frac{u_\delta^{i+1} - \Pi_\delta^{i+1} u_\delta^i}{\tau_i} - \frac{u_\delta^i - \Pi_\delta^i u_\delta^{i-1}}{\tau_{i-1}} \right) \psi_k^{\frac{1}{2}} \right\|_{L^2(\Lambda_k)}^2 \\ & \leq \left(\sum_{j=0}^{1} \left\| \frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}} \right\|_{H^{-1}(\Lambda_k)} \right. \\ & \quad \left. + \left\| \frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x} \right\|_{L^2(\Lambda_k)} \right) \left\| \frac{\partial v_\delta}{\partial x} \right\|_{L^2(\Lambda_k)} . \end{aligned}$$

We now employ the following two inverse inequalities (refer to [25] and [26] for the proof). For any $\varphi_N \in \mathbb{P}_N(\Lambda)$, we have:

$$\int_{-1}^{1} (\varphi_N')^2(\zeta)(1-\zeta^2)^2 \, d\zeta \leq cN^2 \int_{-1}^{1} \varphi_N^2(\zeta)(1-\zeta^2) \, d\zeta ,$$

and

$$\int_{-1}^{1} \varphi_N^2(\zeta) \, d\zeta \leq c\, N^2 \int_{-1}^{1} \varphi_N^2(\zeta)(1-\zeta^2) \, d\zeta .$$

Combining all these inequalities, we deduce the existence of a constant C such that:

$$\begin{aligned} \beta_{2_i}^{\,k} \leq C & \left(\sum_{j=0}^{1} \left\| \frac{(u^{i+1-j} - u_\delta^{i+1-j}) - (u^{i-j} - \Pi_\delta^{i+1-j} u_\delta^{i-j})}{\tau_{i-j}} \right\|_{H^{-1}(\Lambda_k)} \right. \\ & \quad \left. + \left\| \frac{\partial (u^{i+1} - u_\delta^{i+1})}{\partial x} \right\|_{L^2(\Lambda_k)} \right). \end{aligned} \tag{30}$$

Ultimately, by considering (29) and (30), we obtain the desired result (28). □

4 Conclusion

In the discretization of partial differential equations, a posteriori analysis proves highly effective for mesh adaptivity. Our focus in this work lies in applying a posteriori analysis to the discretization of the second-order wave equation using the spectral element method. We have developed two types of residual indicators and established their optimal upper and lower error bounds. The resolution algorithm and implementation of these findings will be detailed in our forthcoming paper.

Acknowledgements
Researchers Supporting Project number (RSP2024R153), King Saud University, Riyadh, Saudi Arabia.

Author contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Funding
Not applicable.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Received: 27 March 2024 Accepted: 25 April 2024 Published online: 15 May 2024

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