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Existence of periodic solutions for a class of (ϕ_1, ϕ_2)-Laplacian difference system with asymptotically (p, q)-linear conditions

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Abstract

In this paper, we consider a (ϕ_1, ϕ_2)-Laplacian system as follows:

$$\begin{cases} \Delta\phi_1(\Delta u(t-1)) + \nabla_u F(t, u(t), v(t)) = 0, \\ \Delta\phi_2(\Delta v(t-1)) + \nabla_v F(t, u(t), v(t)) = 0, \end{cases}$$

where $F(t, u(t), v(t)) = -K(t, u(t), v(t)) + W(t, u(t), v(t))$ is T-periodic in t. By using the mountain pass theorem, we obtain that the (ϕ_1, ϕ_2)-Laplacian system has at least one periodic solution if W is asymptotically (p, q)-linear at infinity. Our results improve and extend some known works.

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Keywords: (ϕ_1, ϕ_2)-Laplacian system; Periodic solutions; Asymptotically (p, q)-linear condition; Mountain pass theorem

1 Introduction

Let N, $\mathbb{Z}$, and $\mathbb{R}$ represent the sets of all natural numbers, integers, and real numbers, respectively. In this paper, we investigate the following (ϕ_1, ϕ_2)-Laplacian difference system:

$$\begin{cases} \Delta\phi_1(\Delta u(t-1)) + \nabla_u F(t, u(t), v(t)) = 0 \\ \Delta\phi_2(\Delta v(t-1)) + \nabla_v F(t, u(t), v(t)) = 0, \end{cases} \tag{1.1}$$

where Δ is the forward difference operator, $t \in \mathbb{Z}$, $u, v \in \mathbb{R}^N$, $F(t, x_1, x_2) = -K(t, x_1, x_2) + W(t, x_1, x_2)$, $K, W : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are T-periodic in t, $\phi_i, i = 1, 2$ satisfy the following condition:

$(\mathcal{A}0)$ $\phi_i : \mathbb{R}^N \to \mathbb{R}^N$, *and* $\phi_i(0) = 0$, $\phi_i = \nabla\Phi_i$, $\Phi_i \in C^1(\mathbb{R}^N, [0, +\infty))$ *strictly convex*, $\Phi_i(0) = 0$.

Remark 1.1 Condition ($\mathcal{A}0$) is introduced in [1, 2] to depict the classical homeomorphism. If $\Phi_i(x) \to +\infty$ ($|x| \to \infty$), there exists $\delta_i > 0$ such that

$$\Phi_i(x) \geq \delta_i\big(|x| - 1\big), \quad x \in \mathbb{R}^N,$$

where $\delta_i = \min \Phi_i(x)$, ($|x| = 1, i = 1, 2$).

The variational method (see [3–5]) has become an important method to study periodic solutions, homoclinic solutions, ground state solutions, sign-changing solutions of differential equations ([6–9]), difference equations ([10–12]), Hamiltonian systems ([13–18]), poly-Laplacian system ([19, 20]), fractional problems ([21–23]), and so on. The nonlinear difference equations have become an important theoretical basis for computer science, ecology, engineering control, economics, etc. Mawhin ([1, 2]) considered the existence of periodic solutions for ϕ-Laplacian difference systems:

$$\Delta\phi\big[\Delta u(t-1)\big] = \nabla_u F\big[n, u(t)\big] + h(t) \quad (t \in \mathbb{Z}), \tag{1.2}$$

where $\phi = \nabla\Phi$, $\phi : \mathbb{R}^N \to B_a \subset \mathbb{R}^N$ or $\phi : B_a \to \mathbb{R}^N$. He studied three cases of ϕ: (1) $\phi : \mathbb{R}^N \to \mathbb{R}^N$; (2) $\phi : \mathbb{R}^N \to B_a$ ($a < +\infty$); (3) $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$.

Zhang and Wang in [24] investigated the existence of homoclinic solutions for the following (ϕ_1, ϕ_2)-Laplacian systems:

$$\begin{cases} \Delta\phi_1(\Delta u_1(t-1)) + \nabla_{u_1} V(t, u_1(t), u_2(t)) = f_1(t) \\ \Delta\phi_2(\Delta u_2(t-1)) + \nabla_{u_2} V(t, u_1(t), u_2(t)) = f_2(t), \end{cases} \tag{1.3}$$

where $\phi : \mathbb{R}^N \to \mathbb{R}^N$, when $V = -K + W$, K possess p-sublinear, W possess p-superlinear growth, by using a linking theorem, they obtained the existence of homoclinic solutions for system (1.3). In [25], Deng et al. studied the existence of periodic solution for system (1.3) with classical or bounded homeomorphism $f_1 = f_2 = 0$. Using the saddle point theorem and the least action principle, they obtained that system has at least one periodic solution under (p, q)-sublinear condition and Lipschitz condition. In [26], Zhang et al. studied the (ϕ_1, ϕ_2)-Laplacian difference system with a parameter. Using the Clark's theorem, they obtained system has multiplicity results of homoclinic solutions under sub(p, q)-linear growth or (p, q)-linear growth. In [27], by using the genus theory, Wang et al. considered the existence and multiplicity of weak solution for (ϕ_1, ϕ_2)-Laplacian elliptic system, under sub-linear growth condition and symmetric conditions. However, few people investigated the existence and multiplicity of solution for system (1.1) under asymptotically linear growth.

Inspired by the results above, in this paper, we study the existence of periodic solutions for (ϕ_1, ϕ_2)-Laplacian system (1.1) with classical homeomorphism, when W satisfies asymptotically (p, q)-linear condition at infinity.

Theorem 1.1 *Suppose that* ($\mathcal{A}0$) *holds*, K *and* W *satisfy the following conditions*:

($\mathcal{A}1$) *there exist constants* $c_1, c_2 > 0$, $p, q > 1$ *such that*

$$\Phi_1(x) \geq c_1|x|^p, \qquad \Phi_2(y) \geq c_2|y|^q,$$

and

$$\big(\phi_1(x),x\big) + \big(\phi_2(y),y\big) \le \max\{p,q\}\big[\Phi_1(x) + \Phi_2(y)\big];$$

$(K1)$ *there exist constants* $b_1, b_2 > 0$, $\lambda_1 \in (1,p]$, $\lambda_2 \in (1,q]$ *such that*

$$K(t,0,0) = 0, \qquad K(t,u,v) \ge b_1|u|^{\lambda_1} + b_2|v|^{\lambda_2}, \quad \forall (t,u,v) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \times \mathbb{R}^N;$$

$(K2)$

$$\big(\nabla_u K(t,u,v),u\big) + \big(\nabla_v K(t,u,v),v\big) \le \max\{p,q\}K(t,u,v),$$
$$\forall (t,u,v) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \times \mathbb{R}^N;$$

$(W1)$

$$\limsup_{|(u,v)|\to 0} \frac{W(t,u,v)}{|u|^p + |v|^q} < \min\{b_1,b_2\}, \quad \textit{uniformly for } t \in \mathbb{Z}[1,T];$$

$(W2)$ *there exists* $g \in L^1(\mathbb{Z}[1,T],\mathbb{R})$ *such that*

$$\big(\nabla_u W(t,u,v),u\big) + \big(\nabla_v W(t,u,v),v\big) - \max\{p,q\}W(t,u,v) \ge g(t)$$

and

$$\lim_{|(u,v)|\to\infty} \big[\big(\nabla_u W(t,u,v),u\big) + \big(\nabla_v W(t,u,v),v\big) - \max\{p,q\}W(t,u,v)\big] = +\infty$$
$$\forall t \in \mathbb{Z}[1,T];$$

$(W3)$ *there exist constants* $a_1, a_2 > 0$, $d > 0$ *such that*

$$W(t,u,v) \le a_1|u|^p + a_2|v|^q + d, \quad \forall (t,u,v) \in \mathbb{Z}[1,T] \times \mathbb{R}^N \times \mathbb{R}^N;$$

$(W4)$ *there exists* $(u_0,v_0) \in \mathbb{R}^N \times \mathbb{R}^N$ *such that*

$$\sum_{t=1}^{T} \left[K(t,u_0,v_0) - W(t,u_0,v_0) - \frac{g(t)}{\max\{p,q\}}\right] < 0.$$

Then system (1.1) *possesses at least one nontrivial periodic solution.*

Remark 1.2 There are many examples that satisfy $(\mathcal{A}0)$ and $(\mathcal{A}1)$, such as

$$\phi_1(x) = a_0 p|x|^{p-1} + b_0|x|^{\alpha-1}, \qquad \phi_2(y) = c_0 q|y|^{q-1} + d_0|y|^{\beta-1},$$

for some $a_0, b_0, c_0, d_0 > 0$, where $1 \le \alpha \le p$, $1 \le \beta \le q$.

Remark 1.3 Theorem 1.1 extend the results in [12] (p-Laplacian discrete system) and [14] (second-order Hamiltonian system). We consider the more general (ϕ_1,ϕ_2)-Laplacian system (1.1). Even if $\phi_1 = \phi_2 =: \phi$, $u = v$, $F(t,u,v) \equiv F(t,v,u)$, system (1.1) is still different from [12, 28], and [10].

Remark 1.4 From $(W1)$–$(W3)$, it can be concluded that W satisfies asymptotically (p,q)-linear condition at infinity. Moreover, there are many examples that satisfy Theorem 1.1, which we will illustrate in the fourth part.

2 Preliminaries

We use $(\cdot,\cdot)$ and $|\cdot|$ to represent the inner product and the Euclidean norm in $\mathbb{R}^N$. Define

$$E_T = \{u := \{u(t)\} \big| u(t+T) = u(t), u(t) \in \mathbb{R}^N, t \in \mathbb{Z}\},$$

and

$$\|u\|_s = \left(\sum_{t=1}^{T}\left(\left|\Delta u(t)\right|^s + \left|u(t)\right|^s\right)\right)^{1/s}, \quad u \in E_T.$$

Let $E = E_T \times E_T$, for $\varpi = (u,v)^\tau \in E$, define

$$\|\omega\| = \|(u,v)\| = \|u\|_p + \|v\|_q.$$

Then E is separable and reflexive Banach space. Moreover, define

$$\|v\|_{[r]} = \left(\sum_{t=1}^{T}\left|v(t)\right|^r\right)^{1/r}, \quad r > 1, \qquad \|v\|_\infty = \max_{t\in\mathbb{Z}[1,T]}\left|v(t)\right|.$$

For $u, v \in E_T$, it is easy to obtain that there exists $C_0 > 0$ such that

$$\|u\|_\infty \le C_0\|u\|_p, \qquad \|v\|_\infty \le C_0\|v\|_q. \tag{2.1}$$

Define $\mathcal{J} : E \to \mathbb{R}$, as

$$\mathcal{J}(u,v) = \sum_{t=1}^{T}\left[\Phi_1\left(\Delta u(t)\right) + \Phi_2\left(\Delta v(t)\right) - F\left(t,u(t),v(t)\right)\right]. \tag{2.2}$$

Then $\mathcal{J} \in C^1(E,\mathbb{R})$, for each $\varpi = (u,v)^\tau$, $\psi = (\psi_1,\psi_2)^\tau \in E$, one can easily check that

$$\begin{aligned}\left\langle \mathcal{J}'(\varpi),\psi\right\rangle &= \left\langle \mathcal{J}'(u,v),(\psi_1,\psi_2)\right\rangle = \left\langle \mathcal{J}_u(u,v),\psi_1\right\rangle + \left\langle \mathcal{J}_v(u,v),\psi_2\right\rangle \\ &= \sum_{t=1}^{T}\Big[\left(\phi_1\left(\Delta u(t)\right),\Delta\psi_1(t)\right) + \left(\nabla_u K\left(t,u(t),v(t)\right),\psi_1(t)\right) \\ &\quad - \left(\nabla_u W\left(t,u(t),v(t)\right),\psi_1(t)\right)\Big] \\ &\quad + \sum_{t=1}^{T}\Big[\left(\phi_2\left(\Delta v(t)\right),\Delta\psi_2(t)\right) + \left(\nabla_v K\left(t,u(t),v(t)\right),\psi_2(t)\right) \\ &\quad - \left(\nabla_v W\left(t,u(t),v(t)\right),\psi_2(t)\right)\Big].\end{aligned} \tag{2.3}$$

Lemma 2.1 (*see* [25]) *For any* $\varpi = (u,v)^\tau$, $\psi = (\psi_1,\psi_2)^\tau \in E$, *we have*:

$$-\sum_{t=1}^{T}\big(\Delta\phi_1\big(\Delta u(t-1)\big),\psi_1(t)\big) = \sum_{t=1}^{T}\big(\phi_1\big(\Delta u(t)\big),\Delta\psi_1(t)\big),$$
$$-\sum_{t=1}^{T}\big(\Delta\phi_2\big(\Delta v(t-1)\big),\psi_2(t)\big) = \sum_{t=1}^{T}\big(\phi_2\big(\Delta v(t)\big),\Delta\psi_2(t)\big).$$

Through Lemma 2.1, we obtain

$$\begin{aligned}
&\sum_{t=1}^{T}\Big[\big(\phi_1\big(\Delta u(t)\big),\Delta\psi_1(t)\big) + \big(\nabla_u K\big(t,u(t),v(t)\big),\psi_1(t)\big) - \big(\nabla_u W\big(t,u(t),v(t)\big),\psi_1(t)\big)\Big] \\
&\quad + \sum_{t=1}^{T}\Big[\big(\phi_2\big(\Delta v(t)\big),\Delta\psi_2(t)\big) + \big(\nabla_v K\big(t,u(t),v(t)\big),\psi_2(t)\big) \\
&\quad - \big(\nabla_v W\big(t,u(t),v(t)\big),\psi_2(t)\big)\Big] \\
&= \sum_{t=1}^{T}\Big[-\big(\Delta\phi_1\big(\Delta u(t-1)\big),\psi_1(t)\big) + \big(\nabla_u K\big(t,u(t),v(t)\big),\psi_1(t)\big) \\
&\quad - \big(\nabla_u W\big(t,u(t),v(t)\big),\psi_1(t)\big)\Big] \\
&\quad + \sum_{t=1}^{T}\Big[-\big(\Delta\phi_2\big(\Delta v(t-1)\big),\psi_2(t)\big) + \big(\nabla_v K\big(t,u(t),v(t)\big),\psi_2(t)\big) \\
&\quad - \big(\nabla_v W\big(t,u(t),v(t)\big),\psi_2(t)\big)\Big] \\
&= \sum_{t=1}^{T}\Big[\big(-\Delta\phi_1\big(\Delta u(t-1)\big) + \nabla_u K\big(t,u(t),v(t)\big) - \nabla_u W\big(t,u(t),v(t)\big),\psi_1(t)\big)\Big] \\
&\quad + \sum_{t=1}^{T}\Big[\big(-\Delta\phi_2\big(\Delta v(t-1)\big) + \nabla_v K\big(t,u(t),v(t)\big) - \nabla_v W\big(t,u(t),v(t)\big),\psi_2(t)\big)\Big].
\end{aligned} \tag{2.4}$$

From the above equation, we can easily obtain that the critical points of $\mathcal{J}$ in E are periodic solutions of system (1.1).

Let X be a real Banach space. For $J \in C^1(X,\mathbb{R})$, we say that J satisfies the (PS)-condition, if any sequence $\{\varpi_m\} \in X$, $J(\varpi_m)$ is bounded, and $J'(\varpi_m) \to 0$ $(m \to \infty)$ possesses a convergent subsequence.

Lemma 2.2 (*see* [4]) *Suppose* X *is a real Banach space*, $J \in C^1(X,\mathbb{R})$ *satisfies the* (*PS*)-*condition and the following conditions*:

(*i*) $J(0) = 0$;

(*ii*) *there exist constants* $\rho,\alpha > 0$ *such that* $J|_{\partial B_\rho(0)} \geq \alpha$;

(*iii*) *there exists* $e \in X \setminus \overline{B}_\rho(0)$ *such that* $J(e) \leq 0$,

then J *possesses a critical value* $c \geq \alpha$ *given by*

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} J\big(g(s)\big),$$

where $B_\rho(0)$ is an open ball in X of radius ρ at 0, *and*

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}.$$

Remark 2.1 Under the weaker Cerami condition than (PS), the mountain pass theorem still holds. A sequence $\{\omega_n\}$ is called Cerami-sequence (henceforth denoted by (C)-sequence), if $\mathcal{J}\{\omega_n\}$ is bounded and $(1+\|u_n\|)\|\mathcal{J}'(u_n)\| \to 0$ $(n \to \infty)$, if any (C)-sequence for $\mathcal{J}$ has a convergent subsequence, we call the functional $\mathcal{J}$ satisfies (C)-condition.

3 Proofs

Lemma 3.1 *Assume that* $(\mathcal{A}0)$, $(\mathcal{A}1)$, $(K1)$, $(K2)$, $(W2)$, *and* $(W3)$ *hold. Then* J *satisfies* (C)*-condition.*

Proof Presume that $\{\varpi_n = (u_n, v_n)^\tau\} \subset E$ is a (C) sequence for $\mathcal{J}$, then $\mathcal{J}(\varpi_n)$ is bounded, $(1+\|\varpi_n\|)\|\mathcal{J}'(\varpi_n)\| \to 0$ $(n \to \infty)$. Hence, there exists $M > 0$ such that $|\mathcal{J}(\varpi_n)| \leq M$, $(1+\|\varpi_n\|)\|\mathcal{J}'(\varpi_n)\| \leq M$. Then, by (2.2), (2.3), $(\mathcal{A}1)$, and $(K2)$, we have

$$\begin{aligned}(1+\max\{p,q\})M &\geq \max\{p,q\}\mathcal{J}(u_n,v_n) - \langle \mathcal{J}_{u_n}(u_n,v_n), u_n\rangle - \langle \mathcal{J}_{v_n}(u_n,v_n), v_n\rangle \\ &\geq \sum_{t=1}^{T}\Big[\big(\nabla_{u_n} W(t,u_n(t),v_n(t)), u_n(t)\big) + \big(\nabla_{v_n} W(t,u_n(t),v_n(t)), v_n(t)\big) \\ &\quad - \max\{p,q\} W(t,u_n(t),v_n(t))\Big]. \end{aligned} \tag{3.1}$$

Now, we demonstrate that $\{\varpi_n = (u_n, v_n)^\tau\}$ is bounded, through contradiction. If $\{\varpi_n\}$ is unbounded, then $\{\varpi_n\}$ has a subsequence, still remember $\{\varpi_n = (u_n, v_n)^\tau\}$, and $\|u_n\|_p + \|v_n\|_q \to +\infty$, $(n \to \infty)$. Therefore, we can suppose that $\|u_n\|_p \to +\infty$. Then there are two situations.

(i): $\|v_n\|_q \to +\infty$

Let $z_1^{(n)} = \frac{u_n}{\|u_n\|_p}$, $z_2^{(n)} = \frac{v_n}{\|v_n\|_q}$, then $\|z_1^{(n)}\|_p = 1$ and $\|z_2^{(n)}\|_q = 1$. Hence, $\{z_i^{(n)}\}(i=1,2)$ has a convergent subsequence, still remember $\{z_i^{(n)}\}(i=1,2)$, such that $z_i^{(n)} \to z_i (i=1,2)$, $(n \to \infty)$, for some $(z_1, z_2) \in E$. Then

$$z_1^{(n)}(t) \to z_1(t), \qquad z_2^{(n)}(t) \to z_2(t), \quad \text{for all } t \in \mathbb{Z}, \text{ as } n \to \infty. \tag{3.2}$$

By $(\mathcal{A}1)$, (K1), and (W3), we have

$$\begin{aligned}\mathcal{J}(u_n,v_n) &= \sum_{t=1}^{T}\big[\Phi_1(\Delta u_n(t)) + \Phi_2(\Delta v_n(t)) + K(t,u_n(t),v_n(t)) - W(t,u_n(t),v_n(t))\big] \\ &\geq \min\{c_1,c_2\}\sum_{t=1}^{T}\big[|\Delta u_n(t)|^p + |\Delta v_n(t)|^q\big] - \sum_{t=1}^{T}\big[a_1|u_n(t)|^p + a_2|v_n(t)|^q\big] - dT \\ &\geq C_1\big(\|u_n\|_p^p + \|v_n\|_q^q\big) - C_2\left[\sum_{t=1}^{T}|u_n(t)|^p + \sum_{t=1}^{T}|v_n(t)|^q\right] - dT, \end{aligned} \tag{3.3}$$

where $C_1 = \min\{c_1, c_2\}$, $C_2 = \min\{c_1, c_2\} + \max\{a_1, a_2\}$. Then, we have

$$\frac{\mathcal{J}(u_n,v_n)}{\|u_n\|_p^p + \|v_n\|_q^q} \geq C_1 - C_2\left[\frac{\sum_{t=1}^{T}|u_n(t)|^p}{\|u_n\|_p^p + \|v_n\|_q^q} + \frac{\sum_{t=1}^{T}|v_n(t)|^q}{\|u_n\|_p^p + \|v_n\|_q^q}\right] - \frac{dT}{\|u_n\|_p^p + \|v_n\|_q^q}$$

$$\geq C_1 - C_2\left[\sum_{t=1}^{T}\frac{|u_n(t)|^p}{\|u_n\|_p^p} + \sum_{t=1}^{T}\frac{|v_n(t)|^q}{\|v_n\|_q^q}\right] - \frac{dT}{\|u_n\|_p^p + \|v_n\|_q^q}$$
$$\geq C_1 - C_2\left[\sum_{t=1}^{T}\left|z_1^{(n)}(t)\right|^p + \sum_{t=1}^{T}\left|z_2^{(n)}(t)\right|^q\right] - \frac{dT}{\|u_n\|_p^p + \|v_n\|_q^q}. \tag{3.4}$$

Taking the limit of the above inequality, by (3.2), we have

$$\sum_{t=1}^{T}\left|z_1(t)\right|^p + \sum_{t=1}^{T}\left|z_2(t)\right|^q \geq \frac{C_1}{C_2} > 0, \tag{3.5}$$

then, there exists a set $\Omega_1 = \{t \in \mathbb{Z} : z_1(t) \neq 0 \text{ or } z_2(t) \neq 0\} \neq \phi \subset \mathbb{Z}$ such that

$$\left|z_1(t)\right| + \left|z_2(t)\right| > 0, \quad \forall t \in \Omega_1.$$

Thus

$$\lim_{n\to\infty}\left|u_n(t)\right| + \left|v_n(t)\right| = +\infty, \quad \text{for all } t \in \Omega_1. \tag{3.6}$$

Then, by (3.6) and ($W2$), for $t \in \Omega_1$, we have

$$\lim_{n\to\infty}\Big[\big(\nabla_{u_n}W\big(t,u_n(t),v_n(t)\big),u_n(t)\big) + \big(\nabla_{v_n}W\big(t,u_n(t),v_n(t)\big),v_n(t)\big)$$
$$- \max\{p,q\}W\big(t,u_n(t),v_n(t)\big)\Big]$$
$$= +\infty. \tag{3.7}$$

By ($\mathcal{A}2$), (3.6), and (3.7), we get

$$\lim_{n\to\infty}\inf\sum_{t=1}^{T}\Big[\big(\nabla_{u_n}W\big(t,u_n(t),v_n(t)\big),u_n(t)\big) + \big(\nabla_{v_n}W\big(t,u_n(t),v_n(t)\big),v_n(t)\big)$$
$$- \max\{p,q\}W\big(t,u_n(t),v_n(t)\big)\Big]$$
$$\geq \lim_{n\to\infty}\inf\sum_{t\in\Omega_1}\Big[\big(\nabla_{u_n}W\big(t,u_n(t),v_n(t)\big),u_n(t)\big) + \big(\nabla_{v_n}W\big(t,u_n(t),v_n(t)\big),v_n(t)\big)$$
$$- \max\{p,q\}W\big(t,u_n(t),v_n(t)\big)\Big] + \sum_{t\in\mathbb{Z}[1,T]\setminus\Omega_1} g(t)$$
$$\geq \sum_{t\in\Omega_1}\lim_{n\to\infty}\inf\Big[\big(\nabla_{u_n}W\big(t,u_n(t),v_n(t)\big),u_n(t)\big) + \big(\nabla_{v_n}W\big(t,u_n(t),v_n(t)\big),v_n(t)\big)$$
$$- \max\{p,q\}W\big(t,u_n(t),v_n(t)\big)\Big] + \sum_{t\in\mathbb{Z}[1,T]\setminus\Omega_1} g(t)$$
$$= +\infty, \tag{3.8}$$

which contradicts (3.1). Hence $\{u_n\}$ is bounded.

(ii): $\|v_n\|_q$ is boundness

For this case, there exists $C_3 > 0$ such that

$$\|u_n\|_p \to +\infty, \quad \text{as } n \to \infty, \quad \text{and} \quad \|v_n\|_q \leq C_3.$$

Let $z_1^{(n)} = \frac{u_n}{\|u_n\|_p}$, then $\|z_1^{(n)}\|_p = 1$. Hence, $\{z_1^{(n)}\}$ has a convergent subsequence, still remember $\{z_1^{(n)}\}$, such that $z_1^{(n)} \to z_1$, $(n \to \infty)$, for some $z_1 \in E_T$. Then

$$z_1^{(n)}(t) \to z_1(t), \quad \text{for all } t \in \mathbb{Z}, \text{ as } n \to \infty. \tag{3.9}$$

Hence, we have

$$\begin{aligned}
&\frac{\mathcal{J}(u_n, v_n)}{\|u_n\|_p^p + \|v_n\|_q^q} \\
&\quad \geq C_1 - C_2\left[\frac{\sum_{t=1}^T |u_n(t)|^p}{\|u_n\|_p^p + \|v_n\|_q^q} + \frac{\sum_{t=1}^T |v_n(t)|^q}{\|u_n\|_p^p + \|v_n\|_q^q}\right] - \frac{dT}{\|u_n\|_p^p + \|v_n\|_q^q} \\
&\quad \geq C_1 - C_2\left[\sum_{t=1}^T \frac{|u_n(t)|^p}{\|u_n\|_p^p} + \sum_{t=1}^T \frac{|v_n(t)|^q}{\|u_n\|_p^p}\right] - \frac{dT}{\|u_n\|_p^p} \\
&\quad \geq C_1 - C_2 \sum_{t=1}^T \left|z_1^{(n)}(t)\right|^p - C_2 \frac{C_3^q}{\|u_n\|_p^p} - \frac{dT}{\|u_n\|_p^p}.
\end{aligned} \tag{3.10}$$

Let $n \to \infty$. (3.10) implies that

$$\sum_{t=1}^T \left|z_1^n(t)\right|^p > 0,$$

then, there exists a set $\Omega_2 = \{t \in \mathbb{Z} : z_1(t) \neq 0\} \neq \phi \subset \mathbb{Z}$ such that

$$\left|z_1(t)\right| > 0, \quad \forall t \in \Omega_2.$$

Thus

$$\lim_{n\to\infty} \left|u_n(t)\right| + \left|v_n(t)\right| = +\infty, \quad \text{for all } t \in \Omega_2. \tag{3.11}$$

The remaining proof is similar to the case (i), we can also get that $\{u_n\}$ is bounded. □

Likewise, we can show that $\{v_n\}$ is bounded. So $\{\varpi_n = (u_n, v_n)^\tau\}$ is bounded in E. Meanwhile, E is finite dimensional, thus there exists a convergent subsequence. Then J satisfies (C)-condition.

Lemma 3.2 *Assume that* $(\mathcal{A}0)$, $(\mathcal{A}1)$, $(K1)$, $(W1)$, *and* $(W3)$ *hold, there exist* $\rho > 0$, $\alpha > 0$ *such that* $J|_{\partial B_\rho(0)} \geq \alpha$.

Proof From $(W1)$ and $(W3)$, there exist constants $0 < \varepsilon < \min\{b_1, b_2, c_1, c_2\}$, $\tau_1 > p$, $\tau_2 > q$ and $C_4, C_5 > 0$ such that

$$W(t,u,v) \leq \left(\min\{b_1,b_2\} - \varepsilon\right)\left(|u|^p + |v|^q\right) + C_4|u|^{\tau_1} + C_5|v|^{\tau_2}. \tag{3.12}$$

Let

$$\rho = \min\left\{\frac{1}{\max\{p,q\}C_0}, \left(\frac{\varepsilon}{pC_4TC_0^{\tau_1}}\right)^{\frac{1}{\tau_1-p}}, \left(\frac{\varepsilon}{qC_5TC_0^{\tau_2}}\right)^{\frac{1}{\tau_2-q}}\right\}.$$

$\|u\|_p \le \|(u,v)\| = \rho$, $\|v\|_q \le \|(u,v)\| = \rho$, then $|u(t)| \le \|u\|_\infty \le C_0\|u\|_p \le C_0\rho < 1$, $|v(t)| \le \|v\|_\infty \le C_0\|v\|_q \le C_0\rho < 1$. Then, from (2.1), (2.2), (3.12), ($\mathcal{A}1$), and ($K1$), we have

$$\begin{aligned}
\mathcal{J}(u,v) &= \sum_{t=1}^{T}\left[\Phi_1\big(\Delta u(t)\big) + \Phi_2\big(\Delta v(t)\big) + K\big(t,u(t),v(t)\big) - W\big(t,u(t),v(t)\big)\right] \\
&\ge c_1\sum_{t=1}^{T}\left|\Delta u(t)\right|^p + c_2\sum_{t=1}^{T}\left|\Delta v(t)\right|^q + b_1\sum_{t=1}^{T}\left|u(t)\right|^{\lambda_1} + b_2\sum_{t=1}^{T}\left|v(t)\right|^{\lambda_2} \\
&\quad - \big(\min\{b_1,b_2\} - \varepsilon\big)\sum_{t=1}^{T}\left|u(t)\right|^p \\
&\quad - \big(\min\{b_1,b_2\} - \varepsilon\big)\sum_{t=1}^{T}\left|v(t)\right|^q - C_4\sum_{t=1}^{T}\left|u(t)\right|^{\tau_1} - C_5\sum_{t=1}^{T}\left|v(t)\right|^{\tau_2} \\
&\ge \varepsilon\left(\sum_{t=1}^{T}\left|\Delta u(t)\right|^p + \sum_{t=1}^{T}\left|u(t)\right|^p\right) + \varepsilon\left(\sum_{t=1}^{T}\left|\Delta v(t)\right|^q + \sum_{t=1}^{T}\left|v(t)\right|^q\right) \\
&\quad - C_4\sum_{t=1}^{T}\left|u(t)\right|^{\tau_1} - C_5\sum_{t=1}^{T}\left|v(t)\right|^{\tau_2} \\
&\ge \varepsilon\|u\|_p^p + \varepsilon\|v\|_q^q - C_4T\|u\|_\infty^{\tau_1} - C_5T\|v\|_\infty^{\tau_2} \\
&\ge \varepsilon\|u\|_p^p + \varepsilon\|v\|_q^q - C_4TC_0^{\tau_1}\|u\|_p^{\tau_1} - C_5TC_0^{\tau_2}\|v\|_q^{\tau_2} \\
&\ge \big(\varepsilon - C_4TC_0^{\tau_1}\|u\|_p^{\tau_1-p}\big)\|u\|_p^p + \big(\varepsilon - C_5TC_0^{\tau_2}\|v\|_q^{\tau_2-q}\big)\|v\|_q^q \\
&\ge \min\big\{\varepsilon - C_4TC_0^{\tau_1}\rho^{\tau_1-p}, \varepsilon - C_5TC_0^{\tau_2}\rho^{\tau_2-q}\big\}\big(\|u\|_p^p + \|v\|_q^q\big) \\
&\ge \varepsilon\min\left\{1-\frac{1}{p}, 1-\frac{1}{q}\right\}\frac{1}{\max\{2^{p-1},2^{q-1}\}}\big(\|u\|_p + \|v\|_q\big)^{\min\{p,q\}} \\
&= \varepsilon\min\left\{1-\frac{1}{p}, 1-\frac{1}{q}\right\}\frac{1}{\max\{2^{p-1},2^{q-1}\}}\big\|(u,v)\big\|^{\min\{p,q\}}.
\end{aligned} \tag{3.13}$$

Let $\alpha := \varepsilon\min\{1-\frac{1}{p}, 1-\frac{1}{q}\}\frac{1}{\max\{2^{p-1},2^{q-1}\}}\rho^{\min\{p,q\}} > 0$. So (3.13) shows that $\|(u,v)\| = \rho$ implies that $\mathcal{J}(u,v) \ge \alpha$. □

Lemma 3.3 *Assume that* ($\mathcal{A}0$), ($K2$), ($W2$), *and* ($W4$) *hold. Then* $J(u^*,v^*) \le 0$, *where* $(u^*,v^*) \in E \setminus \overline{B}_\rho(0)$.

Proof Let $\psi(s) = s^{-\max\{p,q\}}W(t,su_0,sv_0)$ $(s > 0)$. By ($W2$), we obtain

$$\begin{aligned}
\psi'(s) &= s^{-\max\{p,q\}-1}\big[-\max\{p,q\}W(t,su_0,sv_0) + \big(\nabla_{su_0}W(t,su_0,sv_0), su_0\big) \\
&\quad + \big(\nabla_{sv_0}W(t,su_0,sv_0), sv_0\big)\big] \\
&\ge s^{-\max\{p,q\}-1}g(t),
\end{aligned}$$

Then

$$\int_1^\zeta \psi'(s)\,ds \ge \int_1^\zeta s^{-\max\{p,q\}-1}g(t)\,ds$$

when $\zeta > 1$, that is

$$W(t,\zeta u_0,\zeta v_0) \geq \zeta^{\max\{p,q\}} W(t,u_0,v_0) + \frac{g(t)}{\max\{p,q\}}\left(\zeta^{\max\{p,q\}} - 1\right). \tag{3.14}$$

By $(K2)$, we have

$$K(t,\zeta u_0,\zeta v_0) \leq \zeta^{\max\{p,q\}} K(t,u_0,v_0). \tag{3.15}$$

Combining with (3.14), (3.15), and $(W4)$, we have

$$\begin{aligned}
\mathcal{J}(\zeta u_0,\zeta v_0) &= \sum_{t=1}^{T}\left[K(t,\zeta u_0,\zeta v_0) - W(t,\zeta u_0,\zeta v_0)\right] \\
&\leq \zeta^{\max\{p,q\}} \sum_{t=1}^{T}\left[K(t,u_0,v_0) - W(t,u_0,v_0) - \frac{g(t)}{\max\{p,q\}}\right] \\
&\quad + \frac{1}{\max\{p,q\}} \sum_{t=1}^{T} g(t) \\
&\to -\infty, \quad \text{as } \zeta \to \infty.
\end{aligned} \tag{3.16}$$

Hence, there exists ζ_0 large enough such that $\mathcal{J}(\zeta_0 u_0,\zeta_0 v_0) < 0$. Let $u^* = \zeta_0 u_0$ and $v^* = \zeta_0 v_0$, then $J(u^*,v^*) \leq 0$. □

Proof of Theorem 1.1. Obviously, $\mathcal{J}(0,0) = 0$. By Lemma 2.2 and Lemmas 3.1–3.3, $\mathcal{J}$ has a critical value c such that $\mathcal{J}(u,v) = c$, $\mathcal{J}'(u,v) = 0$. Hence, (u,v) is a desired nontrivial periodic solution of (1.1). □

4 Example

Example 4.1 Let

$$\begin{aligned}
K(t,u,v) &= b_1|u|^{\lambda_1} + b_2|v|^{\lambda_2} + \theta_1(t)|u|^{\kappa_1} + \theta_2(t)|v|^{\kappa_2}, \\
W(t,u,v) &= \theta_3(t)|u|^{\max\{p,q\}}\left(1 - \frac{1}{\ln(e+|u|^p)}\right) + \theta_4(t)|v|^{\max\{p,q\}}\left(1 - \frac{1}{\ln(e+|v|^q)}\right),
\end{aligned}$$

where $b_1, b_2 > 0$, $\theta_i \in l^1(\mathbb{Z},[0,+\infty))(i = 1,2,3,4)$ and are T-periodic, $1 < \lambda_1 < \kappa_1 \leq p$, $1 < \lambda_2 < \kappa_2 \leq q$, then K and W satisfy $(K1)$, $(K2)$, $(W1)$, and $(W3)$.

$$\begin{aligned}
\nabla_u W(t,u,v) &= \theta_3(t)\max\{p,q\} u|u|^{\max\{p,q\}-2}\left(1 - \frac{1}{\ln(e+|u|^p)}\right) \\
&\quad + u|u|^{\max\{p,q\}+p-2}\frac{p\theta_3(t)}{(e+|u|^p)(\ln(e+|u|^p))^2},
\end{aligned}$$

$$\begin{aligned}
\nabla_v W(t,u,v) &= \theta_4(t)\max\{p,q\} v|v|^{\max\{p,q\}-2}\left(1 - \frac{1}{\ln(e+|v|^q)}\right) \\
&\quad + v|v|^{\max\{p,q\}+q-2}\frac{q\theta_4(t)}{(e+|v|^q)(\ln(e+|v|^q))^2},
\end{aligned}$$

so

$$\begin{aligned}&\big(\nabla_u W(t,u,v),u\big)+\big(\nabla_v W(t,u,v),v\big)-\max\{p,q\}W(t,u,v)\\&\quad=|u|^{\max\{p,q\}+p}\frac{p\theta_3(t)}{(e+|u|^p)(\ln(e+|u|^p))^2}+|v|^{\max\{p,q\}+q}\frac{q\theta_4(t)}{(e+|v|^q)(\ln(e+|v|^q))^2},\end{aligned}$$

then it is easy to test that $(W2)$ holds.

$$\begin{aligned}&\sum_{t=1}^{T}\left[K(t,u_0,v_0)-W(t,u_0,v_0)-\frac{g(t)}{\max\{p,q\}}\right]\\&\quad=\sum_{t=1}^{T}\left[b_1|u|^{\lambda_1}+b_2|v|^{\lambda_2}+\theta_1(t)|u|^{\kappa_1}+\theta_2(t)|v|^{\kappa_2}-\theta_3(t)|u|^{\max\{p,q\}}\left(1-\frac{1}{\ln(e+|u|^p)}\right)\right.\\&\qquad\left.-\theta_4(t)|v|^{\max\{p,q\}}\left(1-\frac{1}{\ln(e+|v|^q)}\right)-\frac{g(t)}{\max\{p,q\}}\right]\\&\quad=b_1T|u|^{\lambda_1}+b_2T|v|^{\lambda_2}+|u|^{\kappa_1}\sum_{t=1}^{T}\theta_3(t)+|v|^{\kappa_2}\sum_{t=1}^{T}\theta_4(t)-\frac{\|g\|_{l^1}}{\max\{p,q\}}\\&\qquad-|u|^{\max\{p,q\}}\left(1-\frac{1}{\ln(e+|u|^p)}\right)\sum_{t=1}^{T}\theta_1(t)-|v|^{\max\{p,q\}}\left(1-\frac{1}{\ln(e+|v|^q)}\right)\sum_{t=1}^{T}\theta_2(t)\end{aligned}$$

if

$$\sum_{t=1}^{T}\theta_1(t)>\sum_{t=1}^{T}\theta_3(t),\qquad\sum_{t=1}^{T}\theta_2(t)>\sum_{t=1}^{T}\theta_4(t),$$

there exists $(u_0,v_0)\in\mathbb{R}^N\times\mathbb{R}^N$ such that $(W4)$ holds. Hence, system (1.1) has one nontrival T-periodic solution.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

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The authors declare no competing interests.

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