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New error bounds for Newton's formula associated with tempered fractional integrals

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Abstract

In this paper, we first construct an integral identity associated with tempered fractional operators. By using this identity, we have found the error bounds for Simpson's second formula, namely Newton–Cotes quadrature formula for differentiable convex functions in the framework of tempered fractional integrals and classical calculus. Furthermore, it is also shown that the newly established inequalities are the extension of comparable inequalities inside the literature.

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Keywords: Newton-type inequalities; Quadrature formulae; Fractional integrals; Tempered fractional integrals

1 Introduction

One of the most famous inequalities for convex functions is Hermite–Hadamard-type inequality on account of its rich geometrical importance and applications. Thus, remarkable number of mathematicians have considered the Hermite–Hadamard-type inequalities and related these inequalities such as trapezoid, midpoint, and Simpson's inequality. In last decades, the fractional calculus has application areas in various fields such as engineering, chemistry, and physics as well as mathematics. Because of its basic properties and applications in domains of science, fractional calculus has been the center of attraction in applied and pure mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very significant in order to obtain more realistic results in the solution of many problems. It can be established the bounds of new formulas by using not only Hermite–Hadamard and Simpson type inequalities but also Newton-type inequalities. Because of the importance of fractional calculus mentioned in this paragraph, one can examined distinct fractional integral inequalities extensively. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches.

Simpson's second rule has the rule of three-point Newton–Cotes quadrature, thus evaluations for the case of three steps quadratic kernel are generally called Newton-type results. These results are also known as Newton-type inequalities in the literature. Many researchers have been investigated to Newton-type inequalities extensively. For example, in paper [1], some Newton-type inequalities for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex. In addition to this, some new Newton-type inequalities for the case of differentiable convex functions involving Riemann–Liouville fractional integrals were proved in paper [2]. Moreover, the authors also presented some Newton-type inequalities including Riemann-Liouville fractional integrals for functions of bounded variation. Furthermore, new Newton-type inequalities based on convexity were given in paper [3]. It can be referred to [4–11], and the references therein to the case of more informations.

Tempered fractional calculus can be specified as the generalization of fractional calculus. The definitions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [12]. See the books [13–15] and references therein for more information about the different definitions of the tempered fractional integration. Mohammed et al. [16] established several Hermite–Hadamard-type inequalities connected with the tempered fractional integrals for the case of convex functions, which cover the previously published result such as, Riemann–Liouville fractional integrals.

This paper is organized with respect to the following plans: In Sect. 2, the fundamental definitions of fractional calculus and other relevant research in this discipline are presented. In Sect. 3, we prove an integral equality that is critical in establishing our primary results. With the help of this identity, we establish several Newton-type inequalities involving the tempered fractional integrals. In Sect. 4, we provide our results by using special cases of obtained theorems. In other words, we find the error bound of Newton's rule with the help of the obtained results. Finally, in Sect. 5, summary and concluding remarks are noted.

2 Preliminaries

Simpson-type inequalities are inequalities that are created from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_{\sigma_1}^{\sigma_2} \mathfrak{F}(x)\,dx \approx \frac{\sigma_2-\sigma_1}{6}\left[\mathfrak{F}(\sigma_1)+4\mathfrak{F}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\mathfrak{F}(\sigma_2)\right]. \tag{1}$$

ii. Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule (cf. [17])) is formulated as follows:

$$\int_{\sigma_1}^{\sigma_2} \mathfrak{F}(x)\,dx \approx \frac{\sigma_2-\sigma_1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]. \tag{2}$$

Formulae (1) and (2) are satisfied for any function $\mathfrak{F}$ with continuous $4^{\mu h}$ derivative on $[\sigma_1,\sigma_2]$.

The most popular and familiar Newton–Cotes quadrature involving three-point is Simpson-type inequality is as follows:

Theorem 1 *If* $\mathfrak{F}:[\sigma_1,\sigma_2]\to\mathbb{R}$ *is a four times continuously differentiable function on* (σ_1,σ_2), *and* $\|\mathfrak{F}^{(4)}\|_\infty=\sup_{x\in(\sigma_1,\sigma_2)}|\mathfrak{F}^{(4)}(x)|<\infty$, *then the following inequality holds*:

$$\left|\frac{1}{6}\left[\mathfrak{F}(\sigma_1)+4\mathfrak{F}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\mathfrak{F}(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{F}(x)\,dx\right|\le\frac{1}{2880}\left\|\mathfrak{F}^{(4)}\right\|_\infty(\sigma_2-\sigma_1)^4.$$

One of the classical closed type quadrature rules is the Simpson 3/8 rule based on the Simpson 3/8 inequality as follows:

Theorem 2 (See [17]) *Let us consider that $\mathfrak{F}:[\sigma_1,\sigma_2]\to\mathbb{R}$ is a four times continuously differentiable function on (σ_1,σ_2), and $\|\mathfrak{F}^{(4)}\|_\infty=\sup_{x\in(\sigma_1,\sigma_2)}|\mathfrak{F}^{(4)}(x)|<\infty$. Then, one has the inequality*

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{F}(x)\,dx\right|\\&\quad\le\frac{1}{6480}\left\|\mathfrak{F}^{(4)}\right\|_\infty(\sigma_2-\sigma_1)^4.\end{aligned}$$

Recall that the *gamma function, incomplete gamma function, λ-incomplete gamma function* are described by

$$\Gamma(\alpha):=\int_0^\infty\mu^{\alpha-1}e^{-\mu}\,d\mu,$$

$$\Upsilon(\alpha,x):=\int_0^x\mu^{\alpha-1}e^{-\mu}\,d\mu,$$

and

$$\Upsilon_\lambda(\alpha,x):=\int_0^x\mu^{\alpha-1}e^{-\lambda\mu}\,d\mu,$$

respectively. Here, $0<\alpha<\infty$ and $\lambda\ge 0$.

Remark 1 [16] For the real numbers $\alpha>0$; $x,\lambda\ge 0$ and $\sigma_1<\sigma_2$, we readily have

i. $\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)=\int_0^1\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu}\,d\mu=\frac{1}{(\sigma_2-\sigma_1)^\alpha}\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)$,
ii. $\int_0^1\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,x)\,dx=\frac{\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^\alpha}-\frac{\Upsilon_\lambda(\alpha+1,\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^{\alpha+1}}$.

Recall also that the *Riemann–Liouville integrals* of order $\alpha>0$ are given by

$$J_{\sigma_1+}^{\alpha}\mathfrak{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{\sigma_1}^{x}(x-\mu)^{\alpha-1}\mathfrak{F}(\mu)\,d\mu,\quad x>\sigma_1 \tag{3}$$

and

$$J_{\sigma_2-}^{\alpha}\mathfrak{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\sigma_2}(\mu-x)^{\alpha-1}\mathfrak{F}(\mu)\,d\mu,\quad x<\sigma_2. \tag{4}$$

for $\mathfrak{F}\in L_1[\sigma_1,\sigma_2]$. See [18, 19] for further information and unexplained subjects. Note that the Riemann–Liouville integrals reduce classical integrals for the condition $\alpha=1$.

We shall now give the basic definitions and new notations of tempered fractional operators.

Definition 1 [20, 21] The fractional tempered integral operators $\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}$ and $\mathcal{J}_{\sigma_2-}^{(\sigma_1,\lambda)}\mathfrak{F}$ of order $\alpha>0$ and $\lambda\ge 0$ are given by

$$\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{\sigma_1}^{x}(x-\mu)^{\alpha-1}e^{-\lambda(x-\mu)}\mathfrak{F}(\mu)\,d\mu,\quad x\in[\sigma_1,\sigma_2] \tag{5}$$

and

$$\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\sigma_2}(\mu-x)^{\alpha-1}e^{-\lambda(\mu-x)}\mathfrak{F}(\mu)\,d\mu,\quad x\in[\sigma_1,\sigma_2],\tag{6}$$

respectively for $\mathfrak{F}\in L_1[\sigma_1,\sigma_2]$.

If we choose $\lambda=0$, then the fractional integrals in (5) and (6) are equal to the Riemann–Liouville fractional integrals in (3) and (4), respectively.

3 Main results

Lemma 1 *Let* $\mathfrak{F}:[\sigma_1,\sigma_2]\to\mathbb{R}$ *be an absolutely continuous mapping on* (σ_1,σ_2) *so that* $\mathfrak{F}'\in L_1[\sigma_1,\sigma_2]$. *Then, the following equality holds*:

$$\begin{aligned}&\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\\&\quad-\frac{\Gamma(\alpha)}{2\curlyvee_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]=\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\curlyvee_\lambda(\alpha,\sigma_2-\sigma_1)}\sum_{i=1}^{3}I_i.\end{aligned}\tag{7}$$

Here,

$$\begin{cases}I_1=\int_0^{\frac{1}{3}}\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\}\\\qquad\times[\mathfrak{F}'(\mu\sigma_2+(1-\mu)\sigma_1)-\mathfrak{F}'(\mu\sigma_1+(1-\mu)\sigma_2)]\,d\mu,\\I_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\}\\\qquad\times[\mathfrak{F}'(\mu\sigma_2+(1-\mu)\sigma_1)-\mathfrak{F}'(\mu\sigma_1+(1-\mu)\sigma_2)]\,d\mu,\\I_3=\int_{\frac{2}{3}}^{1}\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\}\\\qquad\times[\mathfrak{F}'(\mu\sigma_2+(1-\mu)\sigma_1)-\mathfrak{F}'(\mu\sigma_1+(1-\mu)\sigma_2)]\,d\mu.\end{cases}$$

Proof From fundamental rules of integration by parts, we have

$$\begin{aligned}I_1&=\int_0^{\frac{1}{3}}\left\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\}\\&\quad\times\left[\mathfrak{F}'\big(\mu\sigma_2+(1-\mu)\sigma_1\big)-\mathfrak{F}'\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right]d\mu\\&=\frac{1}{\sigma_2-\sigma_1}\left\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\}\\&\quad\times\left[\mathfrak{F}\big(\mu\sigma_2+(1-\mu)\sigma_1\big)+\mathfrak{F}\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right]\Big|_0^{\frac{1}{3}}\\&\quad-\frac{\alpha}{\sigma_2-\sigma_1}\int_0^{\frac{1}{3}}\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu}\left[\mathfrak{F}\big(\mu\sigma_2+(1-\mu)\sigma_1\big)+\mathfrak{F}\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right]d\mu\\&=\frac{1}{\sigma_2-\sigma_1}\left\{\curlyvee_{\lambda(\sigma_2-\sigma_1)}\left(\alpha,\frac{1}{3}\right)-\frac{1}{8}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\}\left[\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)\right]\\&\quad+\frac{1}{8(\sigma_2-\sigma_1)}\curlyvee_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\left[\mathfrak{F}(\sigma_1)+\mathfrak{F}(\sigma_2)\right]\\&\quad-\frac{1}{\sigma_2-\sigma_1}\int_0^{\frac{1}{3}}\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu}\left[\mathfrak{F}\big(\mu\sigma_2+(1-\mu)\sigma_1\big)+\mathfrak{F}\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right]d\mu.\end{aligned}\tag{8}$$

In a similar manner, applying the fundamental rules of integration by parts, we obtain

$$\begin{aligned}
I_2 &= \frac{1}{\sigma_2-\sigma_1}\left\{\Upsilon_{\lambda(\sigma_2-\sigma_1)}\left(\alpha,\frac{2}{3}\right)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\} \\
&\quad\times\left[\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)\right] \\
&\quad-\frac{1}{(\sigma_2-\sigma_1)}\left\{\Upsilon_{\lambda(\sigma_2-\sigma_1)}\left(\alpha,\frac{1}{3}\right)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\} \\
&\quad\times\left[\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)\right] \\
&\quad-\frac{1}{\sigma_2-\sigma_1}\int_{\frac{1}{3}}^{\frac{2}{3}}\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu}\left[\mathfrak{F}\left(\mu\sigma_2+(1-\mu)\sigma_1\right)+\mathfrak{F}\left(\mu\sigma_1+(1-\mu)\sigma_2\right)\right]d\mu \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \frac{1}{8(\sigma_2-\sigma_1)}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\left[\mathfrak{F}(\sigma_1)+\mathfrak{F}(\sigma_2)\right] \\
&\quad-\frac{1}{\sigma_2-\sigma_1}\left\{\Upsilon_{\lambda(\sigma_2-\sigma_1)}\left(\alpha,\frac{2}{3}\right)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right\}\left[\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)\right] \\
&\quad-\frac{1}{\sigma_2-\sigma_1}\int_{\frac{2}{3}}^{\frac{1}{3}}\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu} \\
&\quad\times\left[\mathfrak{F}\left(\mu\sigma_2+(1-\mu)\sigma_1\right)+\mathfrak{F}\left(\mu\sigma_1+(1-\mu)\sigma_2\right)\right]d\mu. \qquad (10)
\end{aligned}$$

Let us collect from the equality (8) to (10). Then, it yields

$$\begin{aligned}
\sum_{i=1}^{3}I_i &= \frac{\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\sigma_2-\sigma_1)}{4(\sigma_2-\sigma_1)^{\alpha+1}}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right] \\
&\quad-\frac{1}{\sigma_2-\sigma_1}\int_0^1\mu^{\alpha-1}e^{-\lambda(\sigma_2-\sigma_1)\mu} \\
&\quad\times\left[\mathfrak{F}\left(\mu\sigma_2+(1-\mu)\sigma_1\right)+\mathfrak{F}\left(\mu\sigma_1+(1-\mu)\sigma_2\right)\right]d\mu. \qquad (11)
\end{aligned}$$

By using the equality (11) and with the help of the change of the variable $x=\mu\sigma_2+(1-\mu)\sigma_1$ and $x=\mu\sigma_1+(1-\mu)\sigma_2$ for $\mu\in[0,1]$ respectively, it can be rewritten as follows

$$\begin{aligned}
\sum_{i=1}^{3}I_i &= \frac{\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\sigma_2-\sigma_1)}{4(\sigma_2-\sigma_1)^{\alpha+1}}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right] \\
&\quad-\frac{\Gamma(\alpha)}{(\sigma_2-\sigma_1)^{\alpha+1}}\left[\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)+\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)\right]. \qquad (12)
\end{aligned}$$

If we multiply both sides of (12) by $\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\sigma_2-\sigma_1)}$, then the equality (7) is obtained readily. This finishes the proof of Lemma 1. □

Theorem 3 *Let us consider that the assumptions of Lemma 1 are valid and the function* $|\mathfrak{F}'|$ *is convex on* $[\sigma_1,\sigma_2]$. *Then, the following Newton's rule inequality holds*:

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\big(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda)+\Omega_3(\alpha,\lambda)\big)\big[\left|\mathfrak{F}'(\sigma_1)\right|+\left|\mathfrak{F}'(\sigma_2)\right|\big],
\end{aligned}\tag{13}$$

where

$$\begin{cases}
\Omega_1(\alpha,\lambda)=\int_0^{\frac{1}{3}}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\,d\mu,\\
\Omega_2(\alpha,\lambda)=\int_{\frac{1}{3}}^{\frac{2}{3}}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\,d\mu,\\
\Omega_3(\alpha,\lambda)=\int_{\frac{2}{3}}^{1}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\,d\mu.
\end{cases}\tag{14}$$

Proof We shall first take modulus in Lemma 1. Then, we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\\
&\quad\times\left\{\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\\
&\quad\times\left|\mathfrak{F}'\big(\mu\sigma_2+(1-\mu)\sigma_1\big)-\mathfrak{F}'\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right|d\mu\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\\
&\quad\times\left|\mathfrak{F}'\big(\mu\sigma_2+(1-\mu)\sigma_1\big)-\mathfrak{F}'\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right|d\mu\\
&\quad+\int_{\frac{2}{3}}^{1}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\\
&\quad\left.\times\left|\mathfrak{F}'\big(\mu\sigma_2+(1-\mu)\sigma_1\big)-\mathfrak{F}'\big(\mu\sigma_1+(1-\mu)\sigma_2\big)\right|d\mu\right\}.
\end{aligned}\tag{15}$$

With the aid of the convexity of $|\mathfrak{F}'|$, it follows

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\sigma_2-\sigma_1)^\alpha}\left[J_{\sigma_1+}^{\alpha}\mathfrak{F}(\sigma_2)+J_{\sigma_2-}^{\alpha}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left\{\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|+\mu\left|\mathfrak{F}'(\sigma_1)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|\right]d\mu\\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\\
&\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|+\mu\left|\mathfrak{F}'(\sigma_1)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|\right]d\mu\\
&+\int_{\frac{2}{3}}^{1}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\\
&\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|+\mu\left|\mathfrak{F}'(\sigma_1)\right|+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|\right]d\mu\Bigg\}\\
&=\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda)+\Omega_3(\alpha,\lambda)\right)\left[\left|\mathfrak{F}'(\sigma_1)\right|+\left|\mathfrak{F}'(\sigma_2)\right|\right].
\end{aligned}
$$

This completes the proof of Theorem 3. □

Theorem 4 *Suppose that the assumptions of Lemma 1 hold. Suppose also that the function $|\mathfrak{F}'|^q$, $q>1$ is convex on $[\sigma_1,\sigma_2]$. Then, we have the following Newton's rule inequality*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\left.\quad-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\Bigg\{\left(\varphi_1(\alpha,\lambda,p)+\varphi_3(\alpha,\lambda,p)\right)\\
&\quad\times\left[\left(\frac{5|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+5|\mathfrak{F}'(\sigma_2)|^q}{18}\right)^{\frac{1}{q}}\right]\\
&\quad+2\varphi_2(\alpha,\lambda,p)\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{6}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$
\begin{cases}
\varphi_1(\alpha,\lambda,p)=\left(\int_0^{\frac{1}{3}}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|^p\,d\mu\right)^{\frac{1}{p}},\\
\varphi_2(\alpha,\lambda,p)=\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|^p\,d\mu\right)^{\frac{1}{p}},\\
\varphi_3(\alpha,\lambda,p)=\left(\int_{\frac{2}{3}}^{1}|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|^p\,d\mu\right)^{\frac{1}{p}}.
\end{cases}
$$

Proof Now, applying Hölder inequality in inequality (15), it follows

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\left.\quad-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\\
&\quad\times\left\{\left(\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|^p d\mu\right)^{\frac{1}{p}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_0^{\frac{1}{3}} \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\int_0^{\frac{1}{3}} \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\int_{\frac{2}{3}}^{1} \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right)\right|^q d\mu \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\times \left(\int_{\frac{2}{3}}^{1} \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right)\right|^q d\mu \right)^{\frac{1}{q}} \Bigg\}.
\end{aligned}
$$

It is known that $|\mathfrak{F}'|^q$ is convex. Then, we have

$$
\begin{aligned}
&\Bigg| \frac{1}{8}\left[\mathfrak{F}(\sigma_1) + 3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right) + 3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right) + \mathfrak{F}(\sigma_2) \right] \\
&\quad - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda\,(\alpha,\sigma_2-\sigma_1)} \left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2) + \mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1) \right] \Bigg| \\
&\le \frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\,\Upsilon_\lambda\,(\alpha,\sigma_2-\sigma_1)} \\
&\quad \times \Bigg\{ \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \Bigg[\left(\int_0^{\frac{1}{3}} \mu\left|\mathfrak{F}'(\sigma_2)\right|^q + (1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|^q d\mu \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^{\frac{1}{3}} \mu\left|\mathfrak{F}'(\sigma_1)\right|^q + (1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|^q d\mu \right)^{\frac{1}{q}} \Bigg] \\
&\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}\,(\alpha,1) \right|^p d\mu \right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[\bigg(\int_{\frac{1}{3}}^{\frac{2}{3}}\mu\big|\mathfrak{F}'(\sigma_2)\big|^q+(1-\mu)\big|\mathfrak{F}'(\sigma_1)\big|^q\,d\mu\bigg)^{\frac{1}{q}}\\
&+\bigg(\int_{\frac{1}{3}}^{\frac{2}{3}}\mu\big|\mathfrak{F}'(\sigma_1)\big|^q+(1-\mu)\big|\mathfrak{F}'(\sigma_2)\big|^q\,d\mu\bigg)^{\frac{1}{q}}\Bigg]\\
&+\bigg(\int_{\frac{2}{3}}^{1}\Big|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\Big|^p\,d\mu\bigg)^{\frac{1}{p}}\\
&\times\Bigg[\bigg(\int_{\frac{2}{3}}^{1}\mu\big|\mathfrak{F}'(\sigma_2)\big|^q+(1-\mu)\big|\mathfrak{F}'(\sigma_1)\big|^q\,d\mu\bigg)^{\frac{1}{q}}\\
&+\bigg(\int_{\frac{2}{3}}^{1}\mu\big|\mathfrak{F}'(\sigma_1)\big|^q+(1-\mu)\big|\mathfrak{F}'(\sigma_2)\big|^q\,d\mu\bigg)^{\frac{1}{q}}\Bigg]\Bigg\}\\
&=\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\Bigg\{\Bigg[\bigg(\int_0^{\frac{1}{3}}\Big|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\Big|^p\,d\mu\bigg)^{\frac{1}{p}}\\
&+\bigg(\int_{\frac{2}{3}}^{1}\Big|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\Big|^p\,d\mu\bigg)^{\frac{1}{p}}\Bigg]\\
&\times\Bigg[\bigg(\frac{5|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{18}\bigg)^{\frac{1}{q}}+\bigg(\frac{|\mathfrak{F}'(\sigma_1)|^q+5|\mathfrak{F}'(\sigma_2)|^q}{18}\bigg)^{\frac{1}{q}}\Bigg]\\
&+\bigg(\int_{\frac{1}{3}}^{\frac{2}{3}}\Big|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\Big|^p\,d\mu\bigg)^{\frac{1}{p}}\\
&\times\Bigg[\bigg(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{6}\bigg)^{\frac{1}{q}}+\bigg(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{6}\bigg)^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}
$$

This ends the proof of Theorem 4. □

Theorem 5 *Assume that the assumptions of Lemma 1 are valid. Assume also that the function $|\mathfrak{F}'|^q$, $q\geq 1$ is convex on $[\sigma_1,\sigma_2]$. Then, the following Newton's rule inequality holds*:

$$
\begin{aligned}
&\Bigg|\frac{1}{8}\bigg[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\Big(\frac{2\sigma_1+\sigma_2}{3}\Big)+3\mathfrak{F}\Big(\frac{\sigma_1+2\sigma_2}{3}\Big)+\mathfrak{F}(\sigma_2)\bigg]\\
&\quad-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\Big[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\Big]\Bigg|\\
&\leq\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\Big\{\big(\Omega_1(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\big(\Omega_4(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_2)\big|^q\\
&\quad+\big(\Omega_1(\alpha,\lambda)-\Omega_4(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(\Omega_4(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_1)\big|^q+\big(\Omega_1(\alpha,\lambda)-\Omega_4(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_2)\big|^q\big)^{\frac{1}{q}}\Big]\\
&\quad+\big(\Omega_2(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\big(\Omega_5(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_2)\big|^q+\big(\Omega_2(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(\Omega_5(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_1)\big|^q+\big(\Omega_2(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_2)\big|^q\big)^{\frac{1}{q}}\Big]\\
&\quad+\big(\Omega_3(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\big(\Omega_6(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_2)\big|^q+\big(\Omega_3(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(\Omega_6(\alpha,\lambda)\big|\mathfrak{F}'(\sigma_1)\big|^q+\big(\Omega_3(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\big)\big|\mathfrak{F}'(\sigma_2)\big|^q\big)^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}
$$

Here, $\Omega_1(\alpha,\lambda)$, $\Omega_2(\alpha,\lambda)$ *and* $\Omega_3(\alpha,\lambda)$ *are described in* (14) *and*

$$\begin{cases} \Omega_4(\alpha,\lambda) = \int_0^{\frac{1}{3}} \mu |\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\, d\mu, \\ \Omega_5(\alpha,\lambda) = \int_{\frac{1}{3}}^{\frac{2}{3}} \mu |\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\, d\mu, \\ \Omega_6(\alpha,\lambda) = \int_{\frac{2}{3}}^{1} \mu |\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\, \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)|\, d\mu. \end{cases}$$

Proof Let us start with applying power-mean inequality in inequality (15). Then, we have

$$\begin{aligned} &\left| \frac{1}{8}\left[\mathfrak{F}(\sigma_1) + 3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right) + 3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right) + \mathfrak{F}(\sigma_2) \right] \right. \\ &\quad \left. - \frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)} \left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2) + \mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1) \right] \right| \\ &\le \frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\,\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)} \left\{ \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \right. \\ &\quad \times \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right) \right|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^{\frac{1}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right) \right|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right) \right|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{1}{2}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right) \right|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_2 + (1-\mu)\sigma_1\right) \right|^q d\mu \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| d\mu \right)^{1-\frac{1}{q}} \\ &\quad \left. \times \left(\int_{\frac{2}{3}}^{1} \left| \Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu) - \frac{7}{8}\,\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1) \right| \left| \mathfrak{F}'\left(\mu\sigma_1 + (1-\mu)\sigma_2\right) \right|^q d\mu \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

From the fact of convexity of $|\mathfrak{F}'|^q$, we can readily obtain

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left[\mathcal{J}_{\sigma_1+}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_2)+\mathcal{J}_{\sigma_2-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma_1)\right]\right|\\
&\le\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left\{\left(\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|d\mu\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left[\left(\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\right.\\
&\quad\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|^q\right]d\mu\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{1}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\\
&\quad\left.\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_1)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|d\mu\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\right.\\
&\quad\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|^q\right]d\mu\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{1}{2}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\\
&\quad\left.\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_1)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|d\mu\right)^{1-\frac{1}{q}}\\
&\quad\times\left[\left(\int_{\frac{2}{3}}^{1}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\right.\\
&\quad\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_2)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_1)\right|^q\right]d\mu\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left|\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,\mu)-\frac{7}{8}\Upsilon_{\lambda(\sigma_2-\sigma_1)}(\alpha,1)\right|\right.\\
&\quad\left.\left.\left.\times\left[\mu\left|\mathfrak{F}'(\sigma_1)\right|^q+(1-\mu)\left|\mathfrak{F}'(\sigma_2)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right]\right\}.\\
&=\frac{(\sigma_2-\sigma_1)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,\sigma_2-\sigma_1)}\left\{\left(\Omega_1(\alpha,\lambda)\right)^{1-\frac{1}{q}}\left[\left(\Omega_4(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_2)\right|^q\right.\right.\right.\\
&\quad\left.+\left(\Omega_1(\alpha,\lambda)-\Omega_4(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\Omega_4(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_1(\alpha,\lambda)-\Omega_4(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\Big]$$
$$+\left(\Omega_2(\alpha,\lambda)\right)^{1-\frac{1}{q}}\Big[\left(\Omega_5(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_2)\right|^q+\left(\Omega_2(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\Omega_5(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_2(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\Big]$$
$$+\left(\Omega_3(\alpha,\lambda)\right)^{1-\frac{1}{q}}\Big[\left(\Omega_6(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_2)\right|^q+\left(\Omega_3(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\Omega_6(\alpha,\lambda)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_3(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}. \qquad \square$$

4 Special cases

Remark 2 Let us consider $\lambda = 0$ in Theorem 3. Then, the following Newton-type inequality holds:

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\sigma_2-\sigma_1)^\alpha}\left[J^\alpha_{\sigma_1+}\mathfrak{F}(\sigma_2)+J^\alpha_{\sigma_2-}\mathfrak{F}(\sigma_1)\right]\right|\\&\leq\frac{\alpha(\sigma_2-\sigma_1)}{2}\left\{\Omega_1(\alpha,0)+\Omega_2(\alpha,0)+\Omega_3(\alpha,0)\right\}\left[\left|\mathfrak{F}'(\sigma_1)\right|+\left|\mathfrak{F}'(\sigma_2)\right|\right].\end{aligned}$$

Here,

$$\Omega_1(\alpha,0)=\frac{1}{\alpha}\int_0^{\frac{1}{3}}\left|\mu^\alpha-\frac{1}{8}\right|d\mu=\begin{cases}\frac{2}{1+\alpha}\left(\frac{1}{8}\right)^{1+\frac{1}{\alpha}}+\frac{1}{3^{\alpha+1}\alpha(\alpha+1)}-\frac{1}{24\alpha} & 0<\alpha\leq\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})}\\ \frac{1}{24\alpha}-\frac{1}{3^{\alpha+1}\alpha(\alpha+1)} & \alpha>\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\end{cases}$$

$$\Omega_2(\alpha,0)=\frac{1}{\alpha}\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mu^\alpha-\frac{1}{2}\right|d\mu=\begin{cases}\frac{2^{1+\alpha}-1}{3^{\alpha+1}\alpha(\alpha+1)}-\frac{1}{6\alpha} & 0<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}\\ \frac{1}{1+\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}+\frac{1+2^{1+\alpha}}{3^{\alpha+1}\alpha(\alpha+1)}-\frac{1}{2\alpha} & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}\\ \frac{1}{6\alpha}-\frac{2^{1+\alpha}-1}{3^{\alpha+1}\alpha(\alpha+1)} & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\end{cases}$$

and

$$\Omega_3(\alpha,0)=\frac{1}{\alpha}\int_{\frac{2}{3}}^{1}\left|\mu^\alpha-\frac{7}{8}\right|d\mu=\begin{cases}\frac{3^{1+\alpha}-2^{1+\alpha}}{3^{\alpha+1}\alpha(\alpha+1)}-\frac{7}{24\alpha} & 0<\alpha\leq\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}\\ \frac{2}{1+\alpha}\left(\frac{7}{8}\right)^{1+\frac{1}{\alpha}}+\frac{2^{1+\alpha}+3^{1+\alpha}}{3^{\alpha+1}\alpha(\alpha+1)}-\frac{35}{24\alpha} & \alpha>\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}.\end{cases}$$

This result is established by Hezenci et al. in [2, Theorem 7].

Remark 3 If we choose $\alpha = 1$ and $\lambda = 0$ in Theorem 3, then the following Newton-type inequality holds:

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{F}(\mu)\,d\mu\right|\\&\leq\frac{25(\sigma_2-\sigma_1)^2}{576}\left[\left|\mathfrak{F}'(\sigma_1)\right|+\left|\mathfrak{F}'(\sigma_2)\right|\right],\end{aligned}$$

which is given by [9, Remark 3]. This inequality helps us to find the error bound of Newton's rule.

Remark 4 Consider $\lambda = 0$ in Theorem 4. Then, the following Newton-type inequality holds:

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\sigma_2-\sigma_1)^{\alpha}}\left[J_{\sigma_1+}^{\alpha}\mathfrak{F}(\sigma_2)+J_{\sigma_2-}^{\alpha}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{\alpha(\sigma_2-\sigma_1)}{2}\left\{\left(\varphi_1(\alpha,0,p)+\varphi_3(\alpha,0,p)\right)\right.\\
&\quad\times\left[\left(\frac{5|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+5|\mathfrak{F}'(\sigma_2)|^q}{18}\right)^{\frac{1}{q}}\right]\\
&\quad\left.+2\varphi_2(\alpha,0,p)\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{6}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

which is presented by [2, Theorem 10].

Remark 5 If we assign $\alpha = 1$ and $\lambda = 0$ in Theorem 4, then we have the following Newton-type inequality

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{F}(\mu)\,d\mu\right|\\
&\quad\le\frac{\sigma_2-\sigma_1}{9}\left\{\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left[\left(\frac{5|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+5|\mathfrak{F}'(\sigma_2)|^q}{6}\right)^{\frac{1}{q}}\right]\\
&\quad\left.+\left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{2}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

which is given by Sitthiwirattham et al. in [9, Remark 5].

Remark 6 If we choose $\lambda = 0$ in Theorem 5, then we have the following Newton-type inequality

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\sigma_2-\sigma_1)^{\alpha}}\left[J_{\sigma_1+}^{\alpha}\mathfrak{F}(\sigma_2)+J_{\sigma_2-}^{\alpha}\mathfrak{F}(\sigma_1)\right]\right|\\
&\quad\le\frac{\alpha(\sigma_2-\sigma_1)}{2}\left\{\left(\Omega_1(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_4(\alpha,0)\left|\mathfrak{F}'(\sigma_2)\right|^q+\left(\Omega_1(\alpha,0)-\Omega_4(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\left(\Omega_4(\alpha,0)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_1(\alpha,0)-\Omega_4(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\Omega_2(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha,0)\left|\mathfrak{F}'(\sigma_2)\right|^q+\left(\Omega_2(\alpha,0)-\Omega_5(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\Omega_5(\alpha,0)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_2(\alpha,0)-\Omega_5(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\right]\\
&\quad+\left(\Omega_3(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha,0)\left|\mathfrak{F}'(\sigma_2)\right|^q+\left(\Omega_3(\alpha,0)-\Omega_6(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_1)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\Omega_6(\alpha,0)\left|\mathfrak{F}'(\sigma_1)\right|^q+\left(\Omega_3(\alpha,0)-\Omega_6(\alpha,0)\right)\left|\mathfrak{F}'(\sigma_2)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}.$$

Here, $\Omega_1(\alpha,0)$, $\Omega_2(\alpha,0)$, and $\Omega_3(\alpha,0)$ are defined in Remark 2 and

$$\Omega_4(\alpha,0)=\frac{1}{\alpha}\int_0^{\frac{1}{3}}\mu\left|\mu^\alpha-\frac{1}{8}\right|d\mu=\begin{cases}\frac{1}{2+\alpha}(\frac{1}{8})^{1+\frac{2}{\alpha}}+\frac{1}{3^{\alpha+2}\alpha(\alpha+2)}-\frac{1}{144\alpha} & 0<\alpha\leq\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})}\\ \frac{1}{144\alpha}-\frac{1}{3^{\alpha+2}\alpha(\alpha+2)} & \alpha>\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\end{cases}$$

$$\Omega_5(\alpha,0)=\frac{1}{\alpha}\int_{\frac{1}{3}}^{\frac{2}{3}}\mu\left|\mu^\alpha-\frac{1}{2}\right|d\mu=\begin{cases}\frac{2^{2+\alpha}-1}{3^{\alpha+2}\alpha(\alpha+2)}-\frac{1}{12\alpha} & 0<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}\\ \frac{1}{2+\alpha}(\frac{1}{2})^{1+\frac{2}{\alpha}}+\frac{1+2^{2+\alpha}}{3^{\alpha+2}\alpha(\alpha+2)}-\frac{5}{36\alpha} & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}\\ \frac{1}{12\alpha}-\frac{2^{2+\alpha}-1}{3^{\alpha+2}\alpha(\alpha+2)} & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\end{cases}$$

and

$$\Omega_6(\alpha,0)=\frac{1}{\alpha}\int_{\frac{2}{3}}^{1}\mu\left|\mu^\alpha-\frac{7}{8}\right|d\mu=\begin{cases}\frac{3^{2+\alpha}-2^{2+\alpha}}{3^{\alpha+2}\alpha(\alpha+2)}-\frac{35}{144\alpha} & 0<\alpha\leq\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}\\ \frac{1}{2+\alpha}(\frac{7}{8})^{1+\frac{2}{\alpha}}+\frac{2^{2+\alpha}+3^{2+\alpha}}{3^{\alpha+2}\alpha(\alpha+2)}-\frac{91}{144\alpha} & \alpha>\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}.\end{cases}$$

This result is proved by Hezenci et al. in [2, Theorem 12].

Remark 7 For $\alpha=1$ and $\lambda=0$ in Theorem 5, we have the following Newton-type inequality

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{F}(\sigma_1)+3\mathfrak{F}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathfrak{F}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathfrak{F}(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathfrak{F}(\mu)\,d\mu\right|\\&\quad\leq\frac{\sigma_2-\sigma_1}{2}\left\{\frac{17}{32\cdot 9}\right.\\&\qquad\times\left[\left(\frac{251|\mathfrak{F}'(\sigma_2)|^q+973|\mathfrak{F}'(\sigma_1)|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{251|\mathfrak{F}'(\sigma_1)|^q+973|\mathfrak{F}'(\sigma_2)|^q}{72}\right)^{\frac{1}{q}}\right]\\&\qquad\left.+\frac{1}{18}\left(\frac{|\mathfrak{F}'(\sigma_1)|^q+|\mathfrak{F}'(\sigma_2)|^q}{2}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

which is proved by [9, Remark 4]. This inequality helps us to find the error bound of Newton's rule.

5 Summary & concluding remarks

In this paper, we first establish an integral equality connected with tempered fractional operators. With the help of this equality, we have found the error bounds for Simpson's second formula, namely Newton–Cotes quadrature formula for differentiable convex functions in the framework of tempered fractional integrals and classical calculus. More precisely, with the help of the Hölder and power-mean inequality, we prove several Newton-type inequalities involving tempered fractional operators. Furthermore, some results are presented by using special choices of obtained inequalities.

These type of inequalities will inspire new studies in various fields of mathematics. In the future works, mathematicians can try to generalize our results by utilizing a different version of convex function classes or another type fractional integral operators. Moreover, the researchers may derive new inequalities of different fractional types related to these Newton's rule type inequality. Furthermore, one can obtain these type of inequalities by tempered fractional integrals for convex functions by using quantum calculus.

Author contributions

Conceptualization, F.H. and H.B.; investigation, F.H. and H.B.; methodology, F.H.; validation, H.B. and F.H.; visualization, H.B. and F.H.; writing-original draft, F.H.; writing-review and editing, H.B. All authors read and approved the final manuscript.

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No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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