

Research Article

A Note on the Relaxation-Time Limit of the Isothermal Euler Equations

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This work is concerned with the relaxation-time limit of the multidimensional isothermal Euler equations with relaxation. We show that Coulombel-Goudon's results (2007) can hold in the *weaker* and *more general* Sobolev space of fractional order. The method of proof used is the Littlewood-Paley decomposition.

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1. Introduction

The multidimensional isothermal Euler equation with relaxation describing the perfect gas flow is given by

$$\begin{aligned}n_t + \nabla \cdot (n\mathbf{u}) &= 0, \\(n\mathbf{u})_t + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla p(n) &= -\frac{1}{\tau} n\mathbf{u}\end{aligned}\tag{1.1}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^d$, $d \geq 3$, where n , $\mathbf{u} = (u^1, u^2, \dots, u^d)^\top$ (\top represents transpose) denote the density and velocity of the flow, respectively, and the constant τ is the momentum relaxation time for some physical flow. Here, we assume that $0 < \tau \leq 1$. The pressure $p(n)$ satisfies $p(n) = An$, and $A > 0$ is a physical constant. The symbols ∇ , \otimes are the gradient operator and the symbol for the tensor products of two vectors, respectively. The system is supplemented with the initial data

$$(n, \mathbf{u})(x, 0) = (n_0, \mathbf{u}_0)(x), \quad x \in \mathbb{R}^d.\tag{1.2}$$

2 Boundary Value Problems

To be concerned with the small relaxation-time analysis, we define the scaled variables

$$(n^\tau, \mathbf{u}^\tau)(x, s) = (n, \mathbf{u})\left(x, \frac{s}{\tau}\right). \quad (1.3)$$

Then the new variables satisfy the following equations:

$$\begin{aligned} n_s^\tau + \nabla \cdot \left(\frac{n^\tau \mathbf{u}^\tau}{\tau} \right) &= 0, \\ \tau^2 \left(\frac{n^\tau \mathbf{u}^\tau}{\tau} \right)_s + \tau^2 \left(\frac{n^\tau \mathbf{u}^\tau \otimes \mathbf{u}^\tau}{\tau^2} \right) + \frac{n^\tau \mathbf{u}^\tau}{\tau} &= -A \nabla n^\tau \end{aligned} \quad (1.4)$$

with initial data

$$(n^\tau, \mathbf{u}^\tau)(x, 0) = (n_0, \mathbf{u}_0). \quad (1.5)$$

Let $\tau \rightarrow 0$, formally, we obtain the heat equation

$$\begin{aligned} \mathcal{N}_s - A \Delta \mathcal{N} &= 0, \\ \mathcal{N}(x, 0) &= n_0. \end{aligned} \quad (1.6)$$

The above formal derivation of heat equation has been justified by many authors, see [1–3] and the references therein. In [2], Junca and Rascle studied the convergence of the solutions to (1.1) towards those of (1.6) for arbitrary large initial data in $BV(\mathbb{R}^d)$ space. Marcati and Milani [3] showed the derivation of the porous media equation as the limit of the isentropic Euler equations in one space dimension. Recently, Coulombel and Goudon [1] constructed the uniform smooth solutions to (1.1) in the multidimensional case and proved this relaxation-time limit in some Sobolev space $H^k(\mathbb{R}^d)$ ($k > 1 + d/2$, $k \in \mathbb{N}$). In this paper, we *weaken* the regularity assumptions on the initial data and establish a similar relaxation result in the more general Sobolev space of fractional order ($H^{\sigma+\varepsilon}(\mathbb{R}^d)$, $\sigma = 1 + d/2$, $\varepsilon > 0$) with the aid of Littlewood-Paley decomposition theory.

If fixed $\tau > 0$, there are some efforts on the global existence of smooth solutions to the system (1.1)-(1.2) for the isentropic gas or the general hyperbolic system, the interested readers can refer to [4–7]. Now, we state main results as follows.

THEOREM 1.1. *Let \bar{n} be a constant reference density. Suppose that $n_0 - \bar{n}$ and $\mathbf{u}_0 \in H^{\sigma+\varepsilon}(\mathbb{R}^d)$, there exist two positive constants δ_0 and C_0 independent of τ such that if*

$$\| (n_0 - \bar{n}, \mathbf{u}_0) \|_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}^2 \leq \delta_0, \quad (1.7)$$

then the system (1.1)-(1.2) admits a unique global solution (n, \mathbf{u}) satisfying

$$(n - \bar{n}, \mathbf{u}) \in \mathcal{C}([0, \infty), H^{\sigma+\varepsilon}(\mathbb{R}^d)). \quad (1.8)$$

Moreover, the uniform energy inequality holds:

$$\begin{aligned} & \| (n - \bar{n}, \mathbf{u})(\cdot, t) \|_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}^2 + \frac{1}{\tau} \int_0^t \| \mathbf{u}(\cdot, \zeta) \|_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}^2 d\zeta + \tau \int_0^t \| (\nabla n, \nabla \mathbf{u})(\cdot, \zeta) \|_{H^{\sigma-1+\varepsilon}(\mathbb{R}^d)}^2 d\zeta \\ & \leq C_0 \| (n_0 - \bar{n}, \mathbf{u}_0) \|_{H^{\sigma+\varepsilon}(\mathbb{R}^d)}^2, \quad t \geq 0. \end{aligned} \quad (1.9)$$

Based on Theorem 1.1, using the standard weak convergence method and compactness theorem [8], we can obtain the following relaxation-time limit immediately.

COROLLARY 1.2. *Let (n, \mathbf{u}) be the global solution of Theorem 1.1, then*

$$\begin{aligned} & n^\tau - \bar{n} \text{ is uniformly bounded in } \mathcal{C}([0, \infty), H^{\sigma+\varepsilon}(\mathbb{R}^d)), \\ & \frac{n^\tau \mathbf{u}^\tau}{\tau} \text{ is uniformly bounded in } L^2([0, \infty), H^{\sigma+\varepsilon}(\mathbb{R}^d)). \end{aligned} \quad (1.10)$$

Furthermore, there exists some function $\mathcal{N} \in \mathcal{C}([0, \infty), \bar{n} + H^{\sigma+\varepsilon}(\mathbb{R}^d))$ which is a global weak solution of (1.6). For any time $T > 0$, we have $n^\tau(x, s)$ strongly converges to $\mathcal{N}(x, s)$ in $\mathcal{C}([0, T], (H^{\sigma'+\varepsilon}(\mathbb{R}^d))_{\text{loc}})$ ($\sigma' < \sigma$) as $\tau \rightarrow 0$.

2. Preliminary lemmas

On the Littlewood-Paley decomposition and the definitions of Besov space, for brevity, we omit the details, see [9] or [7]. Here, we only present some useful lemmas.

LEMMA 2.1 ([9, 7]). *Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $B_{p,r}^s \cap L^\infty$ is an algebra and one has*

$$\| fg \|_{B_{p,r}^s} \lesssim \| f \|_{L^\infty} \| g \|_{B_{p,r}^s} + \| g \|_{L^\infty} \| f \|_{B_{p,r}^s} \quad \text{if } f, g \in B_{p,r}^s \cap L^\infty. \quad (2.1)$$

LEMMA 2.2 [9, 7]. *Let $1 \leq p, r \leq \infty$, and I be open interval of \mathbb{R} . Let $s > 0$ and ℓ be the smallest integer such that $\ell \geq s$. Let $F : I \rightarrow \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{\ell, \infty}(I; \mathbb{R})$. Assume that $v \in B_{p,r}^s$ takes values in $J \subset I$. Then $F(v) \in B_{p,r}^s$ and there exists a constant C depending only on s, I, J , and d such that*

$$\| F(v) \|_{B_{p,r}^s} \leq C(1 + \| v \|_{L^\infty})^\ell \| F' \|_{W^{\ell, \infty}(I)} \| v \|_{B_{p,r}^s}. \quad (2.2)$$

LEMMA 2.3 [7]. *Let $s > 0, 1 < p < \infty$, the following inequalities hold.*

(I) $q \geq -1$:

$$2^{qs} \| [f, \Delta_q] \mathcal{A}g \|_{L^p} \leq \begin{cases} Cc_q \| f \|_{B_{p,2}^s} \| g \|_{B_{p,2}^s}, & f, g \in B_{p,2}^s, s = 1 + \frac{d}{p} + \varepsilon (\varepsilon > 0), \\ Cc_q \| f \|_{B_{p,2}^s} \| g \|_{B_{p,2}^{s+1}}, & f \in B_{p,2}^s, g \in B_{p,2}^{s+1}, s = \frac{d}{p} + \varepsilon (\varepsilon > 0), \\ Cc_q \| f \|_{B_{p,2}^{s+1}} \| g \|_{B_{p,2}^s}, & f \in B_{p,2}^{s+1}, g \in B_{p,2}^s, s = \frac{d}{p} + \varepsilon (\varepsilon > 0). \end{cases} \quad (2.3)$$

4 Boundary Value Problems

If $f = g$, then

$$2^{qs} \| [f, \Delta_q] \mathcal{A}g \|_{L^p} \leq Cc_q \| \nabla f \|_{L^\infty} \| g \|_{B_{p,2}^s}, \quad s > 0. \quad (2.4)$$

(II) $q = -1$:

$$2^{-s} \| [f, \Delta_q] \mathcal{A}g \|_{L^{2d/(d+2)}} \leq Cc_{-1} \| f \|_{B_{2,2}^s} \| g \|_{B_{2,2}^s}, \quad f, g \in B_{2,2}^s, \quad s = 1 + \frac{d}{2} + \varepsilon \quad (\varepsilon > 0), \quad (2.5)$$

where the operator $\mathcal{A} = \text{div}$ or ∇ , the commutator $[f, h] = fh - hf$, C is a harmless constant, and c_q denotes a sequence such that $\| (c_q) \|_{l^1} \leq 1$. (In particular, Besov space $B_{2,2}^s \equiv H^s$.)

3. Reformulation and local existence

Let us introduce the enthalpy $\mathcal{H}(\rho) = A \ln \rho$ ($\rho > 0$), and set

$$m(t, x) = A^{-1/2} (\mathcal{H}(n(t, x)) - \mathcal{H}(\bar{n})). \quad (3.1)$$

Then (1.1) can be transformed into the symmetric hyperbolic form

$$\partial_t U + \sum_{j=1}^d A_j(\mathbf{u}) \partial_{x_j} U = -\frac{1}{\tau} \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix}, \quad (3.2)$$

where

$$U = \begin{pmatrix} m \\ \mathbf{u} \end{pmatrix}, \quad A_j(\mathbf{u}) = \begin{pmatrix} u^j & \sqrt{A} e_j^\top \\ \sqrt{A} e_j & u^j \end{pmatrix}. \quad (3.3)$$

The initial data (1.2) become into

$$U_0 = (\sqrt{A} (\ln n_0 - \ln \bar{n}), \mathbf{u}_0)^\top. \quad (3.4)$$

Remark 1. The variable change is from the open set $\{(n, \mathbf{u}) \in (0, +\infty) \times \mathbb{R}^d\}$ to the whole space $\{(m, \mathbf{u}) \in \mathbb{R}^d \times \mathbb{R}^d\}$. It is easy to show that the system (1.1)-(1.2) is equivalent to (3.2)-(3.4) for classical solutions (n, \mathbf{u}) away from vacuum.

First, we recall a local existence and uniqueness result of classical solutions to (3.2)-(3.4) which has been obtained in [7].

PROPOSITION 3.1. *For any fixed relaxation time $\tau > 0$, assume that $U_0 \in B_{2,1}^\sigma$, then there exist a time $T_0 > 0$ (only depending on the initial data U_0) and a unique solution $U(t, x)$ to (3.2)-(3.4) such that $U \in \mathcal{C}^1([0, T_0] \times \mathbb{R}^d)$ and $U \in \mathcal{C}([0, T_0], B_{2,1}^\sigma) \cap \mathcal{C}^1([0, T_0], B_{2,1}^{\sigma-1})$.*

4. A priori estimate and global existence

In this section, we will establish a uniform a priori estimate, which is used to derive the global existence of classical solutions to (3.2)–(3.4). Defining the energy function

$$E_\tau(T)^2 := \sup_{0 \leq t \leq T} \|U(t)\|_{H^{\sigma+\varepsilon}}^2 + \frac{1}{\tau} \int_0^T \|\mathbf{u}(t)\|_{H^{\sigma+\varepsilon}}^2 dt + \tau \int_0^T \|\nabla_x U(t)\|_{H^{\sigma-1+\varepsilon}}^2 dt, \quad (4.1)$$

then we have the following a priori estimate.

PROPOSITION 4.1. *For any given time $T > 0$, if $U \in \mathcal{C}([0, T], H^{\sigma+\varepsilon})$ is a solution to the system (3.2)–(3.4), then the following inequality holds:*

$$E_\tau(T)^2 \leq C(S(T)) (E_\tau(0)^2 + E_\tau(T)^2 + E_\tau(T)^4), \quad (4.2)$$

where $S(T) = \sup_{0 \leq t \leq T} \|U(\cdot, t)\|_{H^{\sigma+\varepsilon}}$, $C(S(T))$ denotes an increasing function from \mathbb{R}^+ to \mathbb{R}^+ , which is independent of τ, T, U .

Proof. The proof of Proposition 4.1 is divided into two steps. First, we estimate the $L^\infty([0, T], H^{\sigma+\varepsilon})$ norm of U , and the $L^2([0, T], H^{\sigma+\varepsilon})$ one of \mathbf{u} . Then, we estimate the $L^2([0, T], H^{\sigma-1+\varepsilon})$ norm of ∇U .

Step 1. Applying the operator Δ_q to (3.2), multiplying the resulting equations by $\Delta_q m$ and $\Delta_q \mathbf{u}$, respectively, and then integrating them over \mathbb{R}^d , we get

$$\begin{aligned} & \frac{1}{2} \left(\|\Delta_q m\|_{L^2}^2 + \|\Delta_q \mathbf{u}\|_{L^2}^2 \right) \Big|_0^t + \frac{1}{\tau} \int_0^t \|\Delta_q \mathbf{u}(\varsigma)\|_{L^2}^2 d\varsigma \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \operatorname{div} \mathbf{u} \left(|\Delta_q m|^2 + |\Delta_q \mathbf{u}|^2 \right) dx d\varsigma \\ & \quad + \int_0^t \int_{\mathbb{R}^d} \{ [\mathbf{u}, \Delta_q] \cdot \nabla m \Delta_q m + [\mathbf{u}, \Delta_q] \cdot \nabla \mathbf{u} \Delta_q \mathbf{u} \} dx d\varsigma. \end{aligned} \quad (4.3)$$

In what follows, we first deal with the low-frequency case. By performing integration by parts, then using Hölder- and Gagliardo-Nirenberg-Sobolev inequality, we have ($d \geq 3$)

$$\begin{aligned} & \left(\|\Delta_{-1} m\|_{L^2}^2 + \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) \Big|_0^t + \frac{2}{\tau} \int_0^t \|\Delta_{-1} \mathbf{u}(\varsigma)\|_{L^2}^2 d\varsigma \\ & \leq \int_0^t \left(2 \|\mathbf{u}\|_{L^d} \|\Delta_{-1} m\|_{L^{2d/(d-2)}} \|\Delta_{-1} \nabla m\|_{L^2} + \|\nabla \mathbf{u}\|_{L^\infty} \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) d\varsigma \\ & \quad + 2 \int_0^t \left(\|[\mathbf{u}, \Delta_{-1}] \cdot \nabla m\|_{L^{2d/(d+2)}} \|\Delta_{-1} m\|_{L^{2d/(d-2)}} + \|[\mathbf{u}, \Delta_{-1}] \cdot \nabla \mathbf{u}\|_{L^2} \|\Delta_{-1} \mathbf{u}\|_{L^2} \right) d\varsigma \\ & \leq \int_0^t \left(2 \|\mathbf{u}\|_{L^d} \|\Delta_{-1} \nabla m\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty} \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) d\varsigma \\ & \quad + 2 \int_0^t \left(\|[\mathbf{u}, \Delta_{-1}] \cdot \nabla m\|_{L^{2d/(d+2)}} \|\Delta_{-1} \nabla m\|_{L^2} + \|[\mathbf{u}, \Delta_{-1}] \cdot \nabla \mathbf{u}\|_{L^2} \|\Delta_{-1} \mathbf{u}\|_{L^2} \right) d\varsigma. \end{aligned} \quad (4.4)$$

6 Boundary Value Problems

Multiplying the factor $2^{-2(\sigma+\varepsilon)}$ on both sides of (4.4), from Lemma 2.3 and Young inequality, we obtain

$$\begin{aligned}
& 2^{-2(\sigma+\varepsilon)} \left(\|\Delta_{-1} m\|_{L^2}^2 + \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) \Big|_0^t + \frac{2}{\tau} \int_0^t 2^{-2(\sigma+\varepsilon)} \|\Delta_{-1} \mathbf{u}(\varsigma)\|_{L^2}^2 d\varsigma \\
& \leq \int_0^t \left(\frac{1}{2} \|\mathbf{u}\|_{L^d} 2^{-2(\sigma-1+\varepsilon)} \|\Delta_{-1} \nabla m\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty} 2^{-2(\sigma+\varepsilon)} \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) d\varsigma \\
& \quad + C \int_0^t \left(c_{-1} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}} \|m\|_{H^{\sigma+\varepsilon}} 2^{-(\sigma-1+\varepsilon)} \|\Delta_{-1} \nabla m\|_{L^2} + c_{-1} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 2^{-(\sigma+\varepsilon)} \|\Delta_{-1} \mathbf{u}\|_{L^2} \right) d\varsigma \\
& \leq \int_0^t \left(\frac{1}{2} \|\mathbf{u}\|_{L^d} 2^{-2(\sigma-1+\varepsilon)} \|\Delta_{-1} \nabla m\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^\infty} 2^{-2(\sigma+\varepsilon)} \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) d\varsigma \\
& \quad + C \int_0^t \|m\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} c_{-1}^2 \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \tau 2^{-2(\sigma-1+\varepsilon)} \|\Delta_{-1} \nabla m\|_{L^2}^2 \right) d\varsigma \\
& \quad + C \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} c_{-1}^2 \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \frac{1}{\tau} 2^{-2(\sigma+\varepsilon)} \|\Delta_{-1} \mathbf{u}\|_{L^2}^2 \right) d\varsigma \quad \left(\tau \leq \frac{1}{\tau} \right),
\end{aligned} \tag{4.5}$$

where C is some positive constant independent of τ . For the high-frequency case, we can also achieve the similar inequality:

$$\begin{aligned}
& 2^{2q(\sigma+\varepsilon)} \left(\|\Delta_q m\|_{L^2}^2 + \|\Delta_q \mathbf{u}\|_{L^2}^2 \right) \Big|_0^t + \frac{2}{\tau} \int_0^t 2^{2q(\sigma+\varepsilon)} \|\Delta_q \mathbf{u}(\varsigma)\|_{L^2}^2 d\varsigma \\
& \leq C \int_0^t \|\nabla \mathbf{u}\|_{L^\infty} \left(2^{2q(\sigma-1+\varepsilon)} \|\Delta_q \nabla m\|_{L^2}^2 + 2^{2q(\sigma+\varepsilon)} \|\Delta_q \mathbf{u}\|_{L^2}^2 \right) d\varsigma \\
& \quad + C \int_0^t \|m\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} c_q^2 \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \tau 2^{2q(\sigma-1+\varepsilon)} \|\Delta_q \nabla m\|_{L^2}^2 \right) d\varsigma \\
& \quad + C \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} c_q^2 \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \frac{1}{\tau} 2^{2q(\sigma+\varepsilon)} \|\Delta_q \mathbf{u}\|_{L^2}^2 \right) d\varsigma \quad \left(\tau \leq \frac{1}{\tau} \right),
\end{aligned} \tag{4.6}$$

where we have taken the advantage of the fact $\|\Delta_q \nabla m\|_{L^2} \approx 2^q \|\Delta_q m\|_{L^2}$ ($q \geq 0$).

By summing (4.6) on $q \in \mathbb{N} \cup \{0\}$ and adding (4.5) together, then according to the imbedding property in Sobolev space, we have

$$\begin{aligned}
& (\|m\|_{H^{\sigma+\varepsilon}}^2 + \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2) \Big|_0^t + \frac{2}{\tau} \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 d\varsigma \\
& \leq C \int_0^t \|m\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \tau \|\nabla m\|_{H^{\sigma-1+\varepsilon}}^2 \right) d\varsigma + C \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}} \frac{1}{\tau} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 d\varsigma \\
& \quad + C \int_0^t \|m\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \tau \|\nabla m\|_{H^{\sigma-1+\varepsilon}}^2 \right) d\varsigma \\
& \quad + C \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}} \left(\frac{1}{\tau} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 + \frac{1}{\tau} \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 \right) d\varsigma.
\end{aligned} \tag{4.7}$$

Therefore, for any $t \in [0, T]$, the following inequality holds:

$$\|U(t)\|_{H^{\sigma+\varepsilon}}^2 + \frac{2}{\tau} \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 d\zeta \leq C(S(t)) (E_\tau(0)^2 + E_\tau(t)^2). \quad (4.8)$$

Step 2. Thanks to the important skew-symmetric lemma developed in [1, 6, 10], we are going to estimate the $L^2([0, T], H^{\sigma-1+\varepsilon})$ norm of ∇U .

LEMMA 4.2 (Shizuta-Kawashima). *For all $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the system (3.2) admits a real skew-symmetric smooth matrix $K(\xi)$ which is defined in the unit sphere \mathcal{S}^{d-1} :*

$$K(\xi) = \begin{pmatrix} 0 & \frac{\xi^\top}{|\xi|} \\ \frac{\xi}{|\xi|} & 0 \end{pmatrix}, \quad (4.9)$$

then

$$K(\xi) \sum_{j=1}^d \xi_j A_j(0) = \begin{pmatrix} \sqrt{A} |\xi| & 0 \\ 0 & -\sqrt{A} \frac{\xi \otimes \xi}{|\xi|} \end{pmatrix}. \quad (4.10)$$

The system (3.2) can be written as the linearized form

$$\partial_t U + \sum_{j=1}^d A_j(0) \partial_{x_j} U = \sum_{j=1}^d \{A_j(0) - A_j(\mathbf{u})\} \partial_{x_j} U - \frac{1}{\tau} \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix}. \quad (4.11)$$

Let

$$\mathcal{G} = \sum_{j=1}^d \{A_j(0) - A_j(\mathbf{u})\} \partial_{x_j} U. \quad (4.12)$$

From Lemma 2.1, we have

$$\|\mathcal{G}\|_{H^{\sigma-1+\varepsilon}} \leq C \|\mathbf{u}\|_{H^{\sigma-1+\varepsilon}} \|\nabla U\|_{H^{\sigma-1+\varepsilon}}. \quad (4.13)$$

Apply the operator Δ_q to the system (4.11) to get

$$\partial_t \Delta_q U + \sum_{j=1}^d A_j(0) \partial_{x_j} \Delta_q U = \Delta_q \mathcal{G} - \frac{1}{\tau} \begin{pmatrix} 0 \\ \Delta_q \mathbf{u} \end{pmatrix}. \quad (4.14)$$

By performing the Fourier transform with respect to the space variable x for (4.14) and multiplying the resulting equation by $-i\tau(\widehat{\Delta_q U})^* K(\xi)$, “*” represents transpose and conjugator, then taking the real part of each term in the equality, we can obtain

$$\begin{aligned} & \tau \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q U} \right) + \tau (\widehat{\Delta_q U})^* K(\xi) \left(\sum_{j=1}^d \xi_j A_j(0) \right) \widehat{\Delta_q U} \\ & = -\operatorname{Im} \left((\widehat{\Delta_q m})^* \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \mathbf{u}} \right) + \tau \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) (\widehat{\Delta_q \mathcal{G}}) \right). \end{aligned} \quad (4.15)$$

8 Boundary Value Problems

Using the skew-symmetry of $K(\xi)$, we have

$$\operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q U} \right) = \frac{1}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \widehat{\Delta_q U} \right). \quad (4.16)$$

Substituting (4.10) into the second term on the left-hand side of (4.15), it is not difficult to get

$$\begin{aligned} & \tau \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \frac{d}{dt} \widehat{\Delta_q U} \right) + \tau (\widehat{\Delta_q U})^* K(\xi) \left(\sum_{j=1}^d \xi_j A_j(0) \right) \widehat{\Delta_q U} \\ & \geq \frac{\tau}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \widehat{\Delta_q U} \right) + \tau \sqrt{A} |\xi| |\widehat{\Delta_q U}|^2 - 2\sqrt{A} |\xi| |\widehat{\Delta_q \mathbf{u}}|^2. \end{aligned} \quad (4.17)$$

With the help of Young inequality, the right-hand side of (4.15) can be estimated as

$$\begin{aligned} & - \operatorname{Im} \left((\widehat{\Delta_q m})^* \frac{\xi^\top}{|\xi|} \widehat{\Delta_q \mathbf{u}} \right) + \tau \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) (\widehat{\Delta_q \mathcal{G}}) \right) \\ & \leq \tau \frac{\sqrt{A}}{2} |\xi| |\widehat{\Delta_q U}|^2 + \frac{C}{\tau |\xi|} |\widehat{\Delta_q \mathbf{u}}|^2 + \frac{C\tau}{|\xi|} |(\widehat{\Delta_q \mathcal{G}})|^2, \end{aligned} \quad (4.18)$$

where the positive constant C is independent of τ . Combining with the equality (4.15) and the inequalities (4.17)-(4.18), we deduce

$$\tau \frac{\sqrt{A}}{2} |\xi| |\widehat{\Delta_q U}|^2 \leq \frac{C}{\tau} \left(|\xi| + \frac{1}{|\xi|} \right) |\widehat{\Delta_q \mathbf{u}}|^2 + \frac{C\tau}{|\xi|} |(\widehat{\Delta_q \mathcal{G}})|^2 - \frac{\tau}{2} \frac{d}{dt} \operatorname{Im} \left((\widehat{\Delta_q U})^* K(\xi) \widehat{\Delta_q U} \right). \quad (4.19)$$

Multiplying (4.19) by $|\xi|$ and integrating it over $[0, t] \times \mathbb{R}^d$, from Plancherel's theorem, we reach

$$\begin{aligned} & \tau \int_0^t \|\Delta_q \nabla U\|_{L^2}^2 d\zeta \leq \frac{C}{\tau} \int_0^t \left(\|\Delta_q \mathbf{u}\|_{L^2}^2 + \|\Delta_q \nabla \mathbf{u}\|_{L^2}^2 \right) d\zeta + C\tau \int_0^t \|\Delta_q \mathcal{G}\|_{L^2}^2 d\zeta \\ & \quad - \frac{\tau}{2} \operatorname{Im} \int_{\mathbb{R}^d} |\xi| \left((\widehat{\Delta_q U})^* K(\xi) \widehat{\Delta_q U} \right) d\xi \Big|_0^t \\ & \leq \frac{C}{\tau} \int_0^t 2^{2q} \|\Delta_q \mathbf{u}\|_{L^2}^2 d\zeta + C\tau \int_0^t \|\Delta_q \mathcal{G}\|_{L^2}^2 d\zeta \\ & \quad + C\tau 2^{2q} \left(\|\Delta_q U(t)\|_{L^2}^2 + \|\Delta_q U(0)\|_{L^2}^2 \right), \end{aligned} \quad (4.20)$$

where we have used the uniform boundedness of the matrix $K(\xi)$ ($\xi \neq 0$).

Multiplying the factor $2^{2q(\sigma-1+\varepsilon)}$ ($q \geq -1$) on both sides of (4.20) and summing it on q , we have

$$\begin{aligned} & \tau \int_0^t \|\nabla U\|_{H^{\sigma-1+\varepsilon}}^2 d\zeta \leq \frac{C}{\tau} \int_0^t \|\mathbf{u}\|_{H^{\sigma+\varepsilon}}^2 d\zeta + C\tau \int_0^t \|\mathcal{G}\|_{H^{\sigma-1+\varepsilon}}^2 d\zeta + C\tau \left(\|U(t)\|_{H^{\sigma+\varepsilon}}^2 + \|U(0)\|_{H^{\sigma+\varepsilon}}^2 \right) \\ & \leq C(S(t)) (E_\tau(0)^2 + E_\tau(t)^2 + E_\tau(t)^4). \end{aligned} \quad (4.21)$$

Together with the inequalities (4.8) and (4.21), (4.2) follows immediately, which completes the proof of Proposition 4.1. \square

Proof of Theorem 1.1. In fact, Proposition 3.1 also holds on the framework of the functional space $H^{\sigma+\varepsilon}(\equiv B_{2,2}^{\sigma+\varepsilon})$. There exists a sufficiently small number ϵ_0 independent of τ such that $E_\tau(T) \leq \epsilon_0 \leq 1$ from (4.1), we have

$$E_\tau(T)^2 \leq \tilde{C}(E_\tau(0)^2 + E_\tau(T)^3), \tag{4.22}$$

where the constant \tilde{C} is independent of τ . Without loss of generality, we may assume $\tilde{C} \geq 1$. Similar to that in [1], we achieve that

$$E_\tau(t) \leq \min \left\{ \epsilon_0, \frac{1}{2\tilde{C}}, \sqrt{2\tilde{C}E_\tau(0)} \right\} \tag{4.23}$$

for any $t \geq 0$ if

$$\|U_0\|_{H^{\sigma+\varepsilon}} \leq \frac{1}{2(2\tilde{C})^{3/2}}. \tag{4.24}$$

Note that the density

$$n - \bar{n} = \bar{n} \{ \exp(A^{-1/2}m) - 1 \}; \tag{4.25}$$

from Lemma 2.2, the definition of $E_\tau(t)$, and the standard continuity argument, we can obtain the following result: there exist two positive constants δ_0, C_0 independent of τ if the initial data satisfy

$$\|n_0 - \bar{n}\|_{H^{\sigma+\varepsilon}}^2 + \|\mathbf{u}_0\|_{H^{\sigma+\varepsilon}}^2 \leq \delta_0, \tag{4.26}$$

then the system (1.1)-(1.2) exists as a unique global solution (n, \mathbf{u}) . Moreover, the uniform energy estimate holds:

$$\begin{aligned} & \| (n - \bar{n}, \mathbf{u})(\cdot, t) \|_{H^{\sigma+\varepsilon}}^2 + \frac{1}{\tau} \int_0^t \| \mathbf{u}(\cdot, \zeta) \|_{H^{\sigma+\varepsilon}}^2 d\zeta + \tau \int_0^t \| (\nabla n, \nabla \mathbf{u})(\cdot, \zeta) \|_{H^{\sigma-1+\varepsilon}}^2 d\zeta \\ & \leq C_0 \| (n_0 - \bar{n}, \mathbf{u}_0) \|_{H^{\sigma+\varepsilon}}^2, \quad t \geq 0, \end{aligned} \tag{4.27}$$

which completes the proof of Theorem 1.1. \square

The proof of Corollary 1.2 is similar to that in [1]; here, we omit the details, the interested readers can refer to [1].

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