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Research Article

Solvability of Second-Order m-Point Boundary Value Problems with Impulses

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By Leray-Schauder continuation theorem and the nonlinear alternative of Leray-Schauder type, the existence of a solution for an m-point boundary value problem with impulses is proved.

1. Introduction

The main purpose of this paper is to get results on the solvability of the following boundary value problem (BVP):

$$\begin{gathered} x''(t) = f(t,x(t),x'(t)), \\ \Delta x'(t_k) = b_k x'(t_k), \qquad \Delta x(t_k) = c_k x(t_k), \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{gathered} \tag{1.1}$$

where $\xi_i \in (0,1)$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in R$, $i = 1,2,\dots,m-2$, $\sum_{i=1}^{m-2} a_i \neq 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_T < t_{T+1} = 1$.

Such problems without impulses effects have been solved before, for example, in [1–3]. But as far as we know the publication on the solvability of m-point problems with impulses is fewer [4]. Our main goal is to find condition for $f, b_k, c_k, 1 \leq k \leq T$, which guarantees the existence of at least one solution of problem (1.1). The proofs are based on the Leray-Schauder continuation theorem [5] and the nonlinear alternative of Leray-Schauder type [6].

In order to define the concept of solution for BVP (1.1), we introduce the following spaces of functions:

(i) $PC[0,1] = \{u : [0,1] \to R, u$ is continuous at $t \neq t_k$, $u(t_k^+)$, $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)\}$;

(ii) $PC^1[0,1] = \{u \in PC[0,1] : u$ is continuously differentiable at $t \neq t_k$, $u'(0^+)$, $u'(t_k^+)$, $u'(t_k^-)$ exist and $u'(t_k^-) = u'(t_k)\}$;

(iii) $PC^2[0,1] = \{u \in PC^1[0,1] : u$ is twice continuously differentiable at $t \neq t_k\}$.

Note that $PC[0,1]$ and $PC^1[0,1]$ are Banach spaces with the norms

$$\|u\|_\infty = \sup\{|u(t)| : t \in [0,1]\}, \qquad \|u\|_1 = \max\{\|u\|_\infty, \|u'\|_\infty\}, \tag{1.2}$$

respectively.

Definition 1.1. The set $\mathcal{F}$ is said to be quasiequicontinuous in $[0,c]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathcal{F}$, $k \in Z$, $t^*, t^{**} \in (t_{k-1}, t_k] \cap [0,c]$, and $|t^* - t^{**}| < \delta$, then $|x(t^*) - x(t^{**})| < \varepsilon$.

Lemma 1.2 (compactness criterion [7]). *The set $\mathcal{F} \subset PC([0,c], R^n)$ is relatively compact if and only if one has the following:*

(1) *$\mathcal{F}$ is bounded;*

(2) *$\mathcal{F}$ is quasiequicontinuous in $[0,c]$.*

Lemma 1.3 [7]. *Let $s \in [0,T)$, $c_k \geq 0$, α_k, $k = 1,\dots,p$, are constants and let $p, q \in PC(J,R)$, $x \in PC^1(J,R)$. If*

$$\begin{aligned} x'(t) &\leq p(t)x(t) + q(t), \quad t \in [s,T),\ t \neq t_k, \\ x(t_k^+) &\leq c_k x(t_k) + \alpha_k, \quad t_k \in [s,T), \end{aligned} \tag{1.3}$$

then for $t \in [s,T]$,

$$\begin{aligned} x(t) \leq{}& x(s^+)\left(\prod_{s<t_k<t} c_k\right)\exp\left(\int_s^t p(u)\,du\right) \\ &+ \int_s^t \left(\prod_{u<t_k<t} c_k\right)\exp\left(\int_u^t p(\tau)\,d\tau\right) q(u)\,du \\ &+ \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right)\exp\left(\int_{t_k}^t p(\tau)\,d\tau\right)\alpha_k. \end{aligned} \tag{1.4}$$

The result also holds if the above inequalities are reversed.

2. Main results

Theorem 2.1. *Let $f : [0,1] \times R^2 \to R$ be a continuous function. Assume that there exist $p(t)$, $q(t)$, and $r(t) : [0,1] \to [0,\infty)$ such that*

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t) \tag{2.1}$$

for $t \in [0,1]$ *and all* $(u,v) \in R^2$. *Then the BVP (1.1) has at least one solution in* $PC^1[0,1]$ *provided*

$$Q + B < 1, \tag{2.2}$$

$$\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right)\left(\frac{P}{1-Q-B} + C\right) < 1, \tag{2.3}$$

where $P = \int_0^1 p(t)dt$, $Q = \int_0^1 q(t)dt$, $B = \sum_{k=1}^{T} |b_k|$, $C = \sum_{k=1}^{T} |c_k|$.

Proof. Let $Y = X = PC^1[0,1]$. Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \left\{x \in PC^2[0,1],\ x'(0) = 0,\ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)\right\}, \tag{2.4}$$

and for $x \in D(L) : Lx = (x'', \Delta x'(t_k), \Delta x(t_k))$. We also define a nonlinear mapping $F : X \to Y$ by setting

$$(Fx)(t) = \big(f(t,x(t),x'(t)), b_k x'(t_k), c_k x(t_k)\big). \tag{2.5}$$

From the assumption on f, we see that F is a bounded mapping from X to Y. Next, it is easy to see that $L : D(L) \to Y$ is one-to-one mapping. Moreover, it follows easily using Lemma 1.2 that $L^{-1}F : X \to X$ is a compact mapping.

We note that $x \in PC^1[0,1]$ is a solution of (1.1) if and only if x is a fixed point of the equation

$$x = L^{-1}Fx. \tag{2.6}$$

We apply the Leray-Schauder continuation theorem to obtain the existence of a solution for $x = L^{-1}Fx$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations:

$$\begin{gathered} x''(t) = \lambda f(t,x(t),x'(t)), \\ \Delta x'(t_k^+) = \lambda b_k x'(t_k), \qquad \Delta x(t_k) = \lambda c_k x(t_k), \\ x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \end{gathered} \tag{2.7}$$

Integrate (2.7) from 0 to t to obtain

$$x'(t) = \lambda \int_0^t f(s,x(s),x'(s))ds + \lambda \sum_{0<t_k<t} b_k x'(t_k). \tag{2.8}$$

By condition (2.1), we have

$$\begin{aligned} |x'(t)| &\leq \int_0^t [p(s)\|x\| + q(s)\|x'\| + r(s)]ds + \sum_{k=1}^{T} |b_k|\|x'\| \\ &\leq (Q+B)\|x'\| + P\|x\| + R_1, \end{aligned} \tag{2.9}$$

where $R_1 = \int_0^1 r(t)dt$. Thus,

$$\|x'\| \leq \frac{1}{1-Q-B}(P\|x\| + R_1). \tag{2.10}$$

Integrate (2.8) from t to 1 to obtain

$$\begin{aligned} &-x(t) \\ &= \lambda\Bigg\{\int_0^1 H(t,s)f(s,x(s),x'(s))ds + \int_t^1 \sum_{0<t_k<s} b_k x'(t_k)ds + \sum_{t<t_k<1} c_k x(t_k) \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i\Bigg[\int_0^1 H(\xi_i,s)f(s,x(s),x'(s))ds \int_{\xi_i}^1 \sum_{0<t_k<s} b_k x'(t_k)ds + \sum_{\xi_i<t_k<1} c_k x(t_k)\Bigg]\Bigg\}, \end{aligned} \tag{2.11}$$

where

$$H(t,s) = \begin{cases} 1-t, & 0 \leq s \leq t \leq 1, \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.12}$$

So

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)[(P+C)\|x\| + (Q+B)\|x'\| + R_1]. \tag{2.13}$$

Equations (2.10) and (2.13) imply

$$\|x\| \leq \left(1 + \frac{\sum_{i=1}^{m-2}|a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)\left[\left(\frac{P}{1-Q-B} + C\right)\|x\| + R_1\right]. \tag{2.14}$$

It follows from the assumption (2.3) that there is a constant M_1 in dependent of $\lambda \in [0,1]$ such that $\|x\| \leq M_1$. Furthermore, by (2.10), there is a constant M_2 such that $\|x'\| \leq M_2$. It is now immediate that the set of solutions of the family of equations (2.7) is, a priori, bounded in $PC^1[0,1]$ by a constant independent of $\lambda \in [0,1]$. This completes the proof of the theorem.

Theorem 2.2. *Let $f : [0,1] \times R^2 \to R$. Assume that the following conditions hold:*

(H_1) $|f(t,u,v)| \leq q(t)w(\max\{|u|,|v|\})$ *on* $[0,1] \times R^2$ *with* $w > 0$ *continuous and nondecreasing on* $[0,\infty)$, $q(t) : [0,1] \to [0,\infty)$ *is continuous;*

(H_2) $b_k \geq 0$, *and*

$$C\left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\right|}\right)<1,$$
$$\sup_{r\geq 0}\frac{r}{w(r)}>M_3=\left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\right|}\right)\left[1-C\left(1+\frac{\sum_{i=1}^{m-2}|a_i|}{\left|1-\sum_{i=1}^{m-2}a_i\right|}\right)\right]^{-1}Q, \tag{2.15}$$

where $Q=\int_0^1\prod_{0<t_k<1}(1+b_k)q(s)ds$.

Then (1.1) has at least one solution.

Choose $\widetilde{M}>0$ such that

$$\frac{\widetilde{M}}{w(\widetilde{M})}>M_3. \tag{2.16}$$

To show that (1.1)) has at least one solution, we consider the operator

$$x=\lambda L^{-1}Fx,\quad \lambda\in[0,1], \tag{2.17}$$

which is equivalent to (2.7). Let $x\in PC^1[0,1]$ be any solution of (2.7), from (H_1), we have

$$-q(t)w(\|x\|_1)\leq x''(t)\leq q(t)w(\|x\|_1). \tag{2.18}$$

Consider the inequalities

$$\begin{aligned}
x''(t)&\leq q(t)w(\|x\|_1),\\
x'(t_k)&=(1+b_k)x(t_k),\\
x'(0)&=0,\\
x''(t)&\geq -q(t)w(\|x\|_1),\\
x'(t_k)&=(1+b_k)x(t_k),\\
x'(0)&=0.
\end{aligned} \tag{2.19}$$

By Lemma 1.3, we have

$$\begin{aligned}
x'(t)&\leq w(\|x\|_1)\int_0^t\prod_{0<t_k<t}(1+b_k)q(s)ds\\
&\leq Qw(\|x\|_1),\\
x'(t)&\geq -w(\|x\|_1)\int_0^t\prod_{0<t_k<t}(1+b_k)q(s)ds\\
&\geq -Qw(\|x\|_1).
\end{aligned} \tag{2.20}$$

From (2.20), we can deduce

$$|x'(t)|\leq Qw(\|x\|_1), \tag{2.21}$$

and so

$$\|x'\| \le Qw(\|x\|_1). \tag{2.22}$$

Using $x(t) = x(1) - \int_t^1 x'(s)ds - \sum_{t<t_k<1} c_k x(t_k)$ and $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, we have

$$x(t) = -\frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \left[\int_{\xi_i}^1 x'(s)ds + \sum_{\xi_i<t_k<1} c_k x(t_k) \right] - \int_t^1 x'(s)ds - \sum_{t<t_k<1} c_k x(t_k), \tag{2.23}$$

which implies

$$|x(t)| \le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)(\|x'\| + C\|x\|), \tag{2.24}$$

and so

$$\begin{aligned} \|x\| &\le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right) \left[1 - C\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)\right]^{-1} \|x'\| \\ &\le \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right) \left[1 - C\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1-\sum_{i=1}^{m-2} a_i|}\right)\right]^{-1} Qw(\|x\|_1). \end{aligned} \tag{2.25}$$

Now, (2.22) together with (2.25) imply $\|x\|_1 \ne \widetilde{M}$. Set

$$U = \{u \in PC^1[0,1] : \|u\|_1 < \widetilde{M}\}, \qquad K = E = PC^1[0,1], \tag{2.26}$$

then the nonlinear alternative of Leray-Schauder type [6] guarantees that $L^{-1}F$ has a fixed point, that is, (1.1) has a solution $x \in PC^1[0,1]$, which completes the proof. □

3. Examples

Example 3.1. Consider the boundary value problem

$$\begin{gathered} x'' = f(t,x,x'), \quad t \in [0,1],\ t \ne \frac{1}{2}, \\ \Delta x'(t_k) = \frac{1}{6}x'(t_k), \quad \Delta x(t_k) = \frac{1}{4}x(t_k), \quad t_k = \frac{1}{2}, \\ x'(0) = 0,\ x(1) = \frac{1}{2}x\left(\frac{1}{3}\right) - \frac{1}{3}x\left(\frac{2}{3}\right), \end{gathered} \tag{3.1}$$

where

$$f(t,u,v) = t^5 u + \frac{1}{2}t^3 v + t^2\left[1 + \cos\left(u^{200} + v^{30}\right)\right]. \tag{3.2}$$

It is easy to see that

$$|f(t,u,v)| \le p(t)|u| + q(t)|v| + r(t) \tag{3.3}$$

with $p(t) = t^5$, $q(t) = (1/2)t^3$, $r(t) = 2t^2$. Clearly, $P = 1/6$, $Q = 1/8$, $B = 1/6$, $C = 1/4$, and

$$Q + B = \frac{7}{24} < 1, \qquad \left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right)\left(\frac{P}{1 - Q - B} + C\right) = \frac{33}{34} < 1. \tag{3.4}$$

By Theorem 2.1, (3.1) has at least one solution.

Example 3.2. Consider the boundary value problem

$$\begin{gathered} x'' = f(t, x, x'), \quad t \in [0,1],\ t \neq \frac{1}{2}, \\ \Delta x'(t_k) = x'(t_k), \quad \Delta x(t_k) = \frac{1}{3} x(t_k), \quad t_k = \frac{1}{2}, \\ x'(0) = 0, \quad x(1) = \frac{1}{2} x\left(\frac{1}{3}\right) - \frac{1}{2} x\left(\frac{2}{3}\right), \end{gathered} \tag{3.5}$$

where

$$f(t, u, v) = e^{-t}(u^\alpha + v^\beta) + \mu e^{-t} \tag{3.6}$$

with $\alpha \in [0,1]$, $\beta \in [0,1]$, $\mu > 0$. It is easy to see that

$$|f(t, u, v)| \leq q(t) w(\max\{|u|, |v|\}) \tag{3.7}$$

with $q(t) = e^{-t}$, $w(s) = s^\alpha + s^\beta + \mu$. Clearly

$$\begin{gathered} C\left(1 + \frac{\sum_{i=1}^{m-2} |a_i|}{|1 - \sum_{i=1}^{m-2} a_i|}\right) = \frac{2}{3} < 1, \\ \sup_{r \geq 0} \frac{r}{w(r)} = \sup_{r \geq 0} \frac{r}{r^\alpha + r^\beta + \mu} = \infty, \end{gathered} \tag{3.8}$$

so (H_2) is true. Theorem 2.2 shows that (3.5) has at least one solution.

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