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Research Article

Multiplicity Results via Topological Degree for Impulsive Boundary Value Problems under Non-Well-Ordered Upper and Lower Solution Conditions

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Some multiplicity results for solutions of an impulsive boundary value problem are obtained under the condition of non-well-ordered upper and lower solutions. The main ideas of this paper are to associate a Leray-Schauder degree with the lower or upper solution.

1. Introduction

In this paper, we study multiplicity of solutions of the impulsive boundary value problem

$$\begin{gathered}
y'' + f(t, y(t), y'(t)) = 0, \quad t \neq t_k, \\
\Delta y|_{t=t_k} = I_k(y(t_k)), \quad k = 1, 2, \dots, m, \\
\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k)), \quad k = 1, 2, \dots, m, \\
y(0) = 0 = y(1) - \alpha y(\eta),
\end{gathered} \tag{1.1}$$

where $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $J = [0,1]$, $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\alpha \in [0,1)$.

Impulsive differential equations arise naturally in a wide variety of applications, such as spacecraft control, inspection processes in operations research, drug administration, and threshold theory in biology. In the past twenty years, a significant development in the theory of impulsive differential equations was seen. Many authors have studied impulsive differential equations using a variety of methods (see [1–5] and the references therein).

The purpose of this paper is to study the multiplicity of solutions of the impulsive boundary value problems (1.1) by the method of upper and lower solutions. The method of lower and upper solutions has a very long history. Some of the ideas can be traced back to Picard [6]. This method deals mainly with existence results for various boundary value problems. For an overview of this method for ordinary differential equations, the reader is referred to [7]. Usually, when one uses the method of upper and lower solutions to study the existence and multiplicity of solutions of impulsive differential equations, one assumes that the upper solution is larger than the lower solution, that is, the condition that upper and lower solutions are well ordered. For example, Guo [1] studied the PBVP for second-order integrodifferential equations of mixed type in real Banach space E:

$$\begin{gathered}-u'' = f(t,u,Tu,Su) \quad \forall t \in [0,2\pi],\ t \neq t_i,\\ \Delta u|_{t=t_i} = L_i u'(t_i),\\ \Delta u'|_{t=t_i} = L_i^* u(t_i), \quad i = 1,2,\ldots,m,\\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi),\end{gathered} \tag{1.2}$$

where $f \in C([0,2\pi]\times E\times E\times E, E)$, T and $S : E \mapsto E$ are two linear operators, $0 < t_1 < t_2 < \cdots < t_m < 1$, L_i, L_i^* $(i = 1,2,\ldots,m)$ are constants. In [1] Guo first obtained a comparison result, and then, by establishing two increasing and decreasing sequences, he proved an existence result for maximal and minimal solutions of the PBVP (1.2) in the ordered interval defined by the lower and upper solutions.

However, to the best of our knowledge, only in the last few years, it was shown that existence and multiplicity for impulsive differential equation under the condition that the upper solution is not larger than the lower solution, that is, the condition of non-well-ordered upper and lower solutions. In [8], Rachůnková and Tvrdý studied the existence of solutions of the nonlinear impulsive periodic boundary value problem

$$\begin{gathered}u'' = f(t,u,u'), \quad t \neq t_i\\ u(t_i^+) = \mathcal{J}(u(t_i)), \quad i = 1,2,\ldots,m,\\ u'(t_i^+) = \mathcal{M}_i(u'(t_i)), \quad i = 1,2,\ldots,m,\\ u(0) = u(T), \qquad u'(0) = u'(T),\end{gathered} \tag{1.3}$$

where $f \in C([0,T]\times\mathbb{R}^2)$, $\mathcal{J}_i, \mathcal{M}_i \in C(\mathbb{R})$. Using Leray-Schauder degree, the authors of [8] showed some existence results for (1.3) under the non-well-ordered upper and lower solutions condition. For other results related to non-well-ordered upper and lower solutions, the reader is referred to [7, 9–14]. Also, here we mention the main results of a very recent paper [15]. In that paper, we studied the second-order three-point boundary value problem

$$\begin{gathered}y''(t) + f(t,y) = 0, \quad 0 \le t \le 1,\\ y(0) = 0, \qquad y(1) - \alpha y(\eta) = 0,\end{gathered} \tag{1.4}$$

where $0 < \eta < 1$, $0 < \alpha < 1$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$. In [15], we made the following assumption.

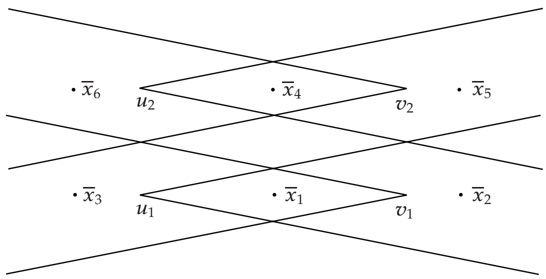

Figure 1: The positions of u_1, u_2, v_1, v_2 and six solutions $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_6$ in Theorem 1.1.

(A_0) There exists $M > 0$ such that

$$f(t, x_2) - f(t, x_1) \geq -M(x_2 - x_1), \quad t \in [0,1],\ x_2 \geq x_1. \tag{1.5}$$

Let the function e be $e = e(t) = t$ for $t \in [0,1]$. In [15], we proved the following theorem (see, [15, Theorem 3.4]).

Theorem 1.1. *Suppose that* (A_0) *holds,* u_1 *and* u_2 *are two strict lower solutions of* (1.4), v_1 *and* v_2 *are two strict upper solutions of* (1.4), *and* $u_1 < v_1$, $u_2 < v_2$, $u_2 \nleq v_1$, $u_1 \nleq v_2$. *Moreover, assume*

$$-\varsigma_0 e \leq u_2 - u_1 \leq \varsigma_0 e, \qquad -\varsigma_0 e \leq v_2 - v_1 \leq \varsigma_0 e \tag{1.6}$$

for some $\varsigma_0 > 0$. *Then the three-point boundary value problem* (1.4) *has at least six solutions* $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_6$.

Theorem 1.1 establishes the existence of at least six solutions of the three-point boundary value problem (1.4) only under the condition of two pairs of strict lower and upper solutions. The positions of u_1, u_2, v_1, v_2 and six solutions $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_6$ in Theorem 1.1 can be illustrated roughly by Figure 1.

In some sense, we can say that these two pairs of lower and upper solutions are parallel to each other. The position of these two pairs of lower and upper solutions is sharply different from that of the lower and upper solutions of the main results in [14, 16, 17]. The technique to prove our main results of [15] is to use the fixed-point index of some increasing operator with respect to some closed convex sets, which are translations of some special cones (see Q_c, $Q_{\overline{c}}$ of [15]).

This paper is a continuation of the paper [15]. The aim of this paper is to study the multiplicity of solutions of the impulsive boundary value problem (1.1) under the conditions of non-well-ordered upper and lower solutions. In this paper, we will permit the presence of impulses and the first derivative. The main ideas of this paper are to associate a Leray-Schauder degree with the lower or upper solution. We will give some multiplicity results for at least eight solutions. To obtain this multiplicity result, an additional pair of lower and upper solutions is needed, that is, we will employ a condition of three pairs of lower and upper solutions. The position of these three pairs of lower and upper solutions will be illustrated in Remark 2.16.

2. Results for at least eight solutions

Let $J' = J\setminus\{t_1,t_2,\dots,t_m\}$, $\mathrm{PC}[J,\mathbb{R}] = \{x \mid x$ is a map from J into $\mathbb{R}$ such that $x(t)$ is continuous at $t\neq t_k$, left continuous at $t=t_k$ and its right-hand limit $x(t_k^+)$ at $t=t_k$ exits$\}$, and $\mathrm{PC}^1[J,\mathbb{R}] = \{x \mid x$ is a map from J into $\mathbb{R}$ such that $x(t)$ and $x'(t)$ are continuous at $t\neq t_k$, left continuous at $t=t_k$ and their right-hand limits $x(t_k^+)$ and $x'(t_k^+)$ at $t=t_k$ exits$\}$. For each $x\in \mathrm{PC}^1[J,\mathbb{R}]$, let

$$\|x\|_{\mathrm{PC}^1} = \max\left\{\|x\|_{\mathrm{PC}}, \|x'\|_{\mathrm{PC}}\right\}, \tag{2.1}$$

where $\|x\|_{\mathrm{PC}} = \sup_{t\in J}|x(t)|$ and $\|x'\|_{\mathrm{PC}} = \sup_{t\in J}|x'(t)|$. Then, $\mathrm{PC}^1[J,\mathbb{R}]$ is a real Banach space with the norm $\|\cdot\|_{\mathrm{PC}^1}$. The function $x\in \mathrm{PC}^1[J,\mathbb{R}]\cap C^2[J',\mathbb{R}]$ is called a solution of the boundary value problem (1.1) if it satisfies all the equalities of (1.1).

Now, for convenience, we make the following assumptions.

(H_0) $0=t_0<t_1<\cdots<t_m<\eta<t_{m+1}=1$, $\alpha\in[0,1)$.

(H_1) I_k $(k=1,2,\dots,m)$ is increasing on $\mathbb{R}$.

Let $x,y\in \mathrm{PC}[J,\mathbb{R}]$. Now, we define the ordering $\prec$ by

$$x\prec y \quad \text{iff } x(t)<y(t)\ \forall t\in J, \qquad x(t_k^+)<y(t_k^+) \quad \text{for each } k=1,2,\dots,m. \tag{2.2}$$

Definition 2.1. The function $u\in \mathrm{PC}^1[J,\mathbb{R}]\cap C^2[J',\mathbb{R}]$ is called a strict lower solution of (1.1) if

$$\begin{gathered} u''(t)+f\left(t,u(t),u'(t)\right)>0, \quad t\neq t_k, \\ u(0)<0, \qquad u(1)-\alpha u(\eta)<0, \end{gathered} \tag{2.3}$$

whenever $I_{i_0}(x)\neq 0$ or $\overline{I}_{j_0}(x)\neq 0$ for some $i_0,j_0\in\{1,2,\dots,m\}$ and some $x\in\mathbb{R}$

$$\begin{aligned} \Delta u|_{t=t_k} &= I_k\left(u(t_k)\right), \quad k=1,2,\dots,m, \\ \Delta u'|_{t=t_k} &> \overline{I}_k\left(u(t_k)\right), \quad k=1,2,\dots,m, \end{aligned} \tag{2.4}$$

whenever $I_k(x)=\overline{I}_k(x)=0$ for each $x\in\mathbb{R}$ and $k\in\{1,2,\dots,m\}$, $\Delta u|_{t=t_k}=\Delta u'|_{t=t_k}=0$ for each $k\in\{1,2,\dots,m\}$.

The function $v\in \mathrm{PC}^1[J,\mathbb{R}]\cap C^2[J',R]$ is called a strict upper solution of (1.1) if

$$\begin{gathered} v''(t)+f\left(t,v(t),v'(t)\right)<0, \quad t\neq t_k, \\ v(0)>0, \qquad v(1)-\alpha v(\eta)>0, \end{gathered} \tag{2.5}$$

whenever $I_{i_0}(x)\neq 0$ or $\overline{I}_{j_0}(x)\neq 0$ for some $i_0,j_0\in\{1,2,\dots,m\}$ and some $x\in\mathbb{R}$

$$\begin{aligned} \Delta v|_{t=t_k} &= I_k\left(v(t_k)\right), \quad k=1,2,\dots,m, \\ \Delta v'|_{t=t_k} &< \overline{I}_k\left(v(t_k)\right), \quad k=1,2,\dots,m, \end{aligned} \tag{2.6}$$

and whenever $I_k(x)=\overline{I}_k(x)=0$ for each $x\in\mathbb{R}$ and $k\in\{1,2,\dots,m\}$, $\Delta v|_{t=t_k}=\Delta v'|_{t=t_k}=0$ for each $k\in\{1,2,\dots,m\}$.

Definition 2.2. Let $u(t), v(t) \in \mathrm{PC}^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$, $u(t) \le v(t)$ for all $t \in J$. We say that f satisfies Nagumo condition with respect to $[u, v]$ if there exists function $\phi \in C([0, \infty), (0, \infty))$ such that

$$\begin{gathered} |f(t, x, y)| \le \phi(|y|), \quad \forall (t, x, y) \in J \times [u(t), v(t)] \times \mathbb{R}, \\ \int_0^\infty \frac{s}{\phi(s)} ds = \infty. \end{gathered} \tag{2.7}$$

Definition 2.3. Let $r_1(t), r_2(t) \cdots r_n(t)$ be strict upper solutions of (1.1) and $r(t) = \min\{r_1(t), r_2(t), \dots r_n(t)\}$ for each $t \in J$. Then, we say the upper solutions $r_1(t), r_2(t), \dots r_n(t)$ are well ordered if for each $k \in \{1, 2, \dots, m\}$, there exist $i_0, j_0 \in \{1, 2, \dots, n\}$ and $\delta_0 > 0$ small enough such that

$$r(t) = \begin{cases} r_{i_0}(t), & t \in (t_k - \delta_0, t_k], \\ r_{j_0}(t), & t \in (t_k, t_k + \delta_0]. \end{cases} \tag{2.8}$$

Definition 2.4. Let $l_1(t), l_2(t), \dots l_n(t)$ be strict lower solutions of (1.1) and $l(t) = \max\{l_1(t), l_2(t), \dots l_n(t)\}$ for each $t \in J$. Then, we say the lower solutions $l_1(t), l_2(t), \dots l_n(t)$ are well ordered if for each $k \in \{1, 2, \dots, m\}$, there exist $i_0, j_0 \in \{1, 2, \dots, n\}$ and $\delta_0 > 0$ small enough such that

$$l(t) = \begin{cases} l_{i_0}(t), & t \in (t_k - \delta_0, t_k], \\ l_{j_0}(t), & t \in (t_k, t_k + \delta_0]. \end{cases} \tag{2.9}$$

From [18, Lemma 5.4.1], we have the following lemma.

Lemma 2.5. *$H \subset \mathrm{PC}^1[J, \mathbb{R}]$ is a relative compact set if and only if for all $x \in H$, $x(t)$ and $x'(t)$ are uniformly bounded on J and equicontinuous on each J_k ($k = 1, 2, \dots, m$), where $J_1 = [0, t_1]$, $J_i = (t_{i-1}, t_i]$, $i = 2, 3, \dots, m+1$.*

The following lemma can be easily proved.

Lemma 2.6. *Suppose that $x \in \mathrm{PC}^1[J, \mathbb{R}] \cap C^2[J', \mathbb{R}]$ satisfies*

$$-x''(t) = f(t, x(t), x'(t)), \quad t \ne t_k \ (k = 1, 2, \dots, m). \tag{2.10}$$

Then

$$\begin{aligned} x'(t) &= x'(0) - \int_0^t f(s, x(s), x'(s)) ds + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right] \quad \forall t \in J, \\ x(t) &= x(0) + x'(0)t - \int_0^t (t-s) f(s, x(s), x'(s)) ds + \sum_{0<t_k<t} \left[x(t_k^+) - x(t_k)\right] \\ &\quad + \sum_{0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right](t - t_k) \quad \forall t \in J. \end{aligned} \tag{2.11}$$

Lemma 2.7. *Let $g \in \mathrm{PC}[J,\mathbb{R}]$ and $a_k, b_k \in \mathbb{R}$ $(k = 0,1,2,\dots,m)$. Then, $x \in \mathrm{PC}^1[J,\mathbb{R}] \cap C^2[J',\mathbb{R}]$ is a solution of*

$$\begin{aligned} -x''(t) &= g(t), \quad t \neq t_k,\ k = 1,2,\dots,m, \\ \Delta x|_{t=t_k} &= a_k, \quad k = 1,2,\dots,m, \\ \Delta x'|_{t=t_k} &= b_k, \quad k = 1,2,\dots,m, \\ x(0) = a_0, &\qquad x(1) - \alpha x(\eta) = b_0 \end{aligned} \tag{2.12}$$

if and only if $x \in \mathrm{PC}[J,\mathbb{R}]$ satisfies

$$\begin{aligned} x(t) = {}& a_0\left[1 - \frac{1-\alpha}{1-\alpha\eta}t\right] + \frac{b_0 t}{1-\alpha\eta} + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)g(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)g(s)ds \\ & - \frac{t}{1-\alpha\eta}\sum_{k=1}^m \left\{(1-\alpha)a_k + \left[1 - t_k - \alpha\left(\eta - t_k\right)\right]b_k\right\} - \int_0^t (t-s)g(s)ds \\ & + \sum_{0<t_k<t}\left[a_k + b_k\left(t - t_k\right)\right], \quad t \in J. \end{aligned} \tag{2.13}$$

Proof. Let $x \in \mathrm{PC}^1[J,\mathbb{R}] \cap C^2[J',\mathbb{R}]$be a solution of (2.12). From Lemma 2.6, we have

$$\begin{aligned} x(t) &= x(0) + x'(0)t - \int_0^t (t-s)g(s)ds + \sum_{0<t_k<t} \Delta x|_{t=t_k} + \sum_{0<t_k<t} \Delta x'|_{t=t_k}\left(t - t_k\right) \\ &= a_0 + x'(0)t - \int_0^t (t-s)g(s)ds + \sum_{0<t_k<t} a_k + \sum_{0<t_k<t} b_k\left(t - t_k\right). \end{aligned} \tag{2.14}$$

Thus,

$$\begin{aligned} x(1) &= a_0 + x'(0) - \int_0^1 (1-s)g(s)ds + \sum_{k=1}^m a_k + \sum_{k=1}^m b_k\left(1 - t_k\right), \\ x(\eta) &= a_0 + x'(0)\eta - \int_0^\eta (\eta - s)g(s)ds + \sum_{k=1}^m a_k + \sum_{k=1}^m b_k\left(\eta - t_k\right). \end{aligned} \tag{2.15}$$

Using the boundary value condition $x(1) - \alpha x(\eta) = b_0$, we have

$$\begin{aligned} x'(0) = {}& \frac{1}{1-\alpha\eta}b_0 - \frac{1-\alpha}{1-\alpha\eta}a_0 + \frac{1}{1-\alpha\eta}\int_0^1 (1-s)g(s)ds - \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta - s)g(s)ds \\ & - \frac{1-\alpha}{1-\alpha\eta}\sum_{k=1}^m a_k - \frac{1}{1-\alpha\eta}\sum_{k=1}^m b_k\left(1 - t_k\right) + \frac{\alpha}{1-\alpha\eta}\sum_{k=1}^m b_k\left(\eta - t_k\right). \end{aligned} \tag{2.16}$$

The equality (2.13) now follows from (2.14) and (2.16).

On the other hand, if $x \in \mathrm{PC}[J,\mathbb{R}]$ satisfies (2.13), by direct computation, we can easily show that x satisfies (2.12). The proof is complete. □

Let us define the operator $A : \mathrm{PC}^1[J,\mathbb{R}] \mapsto \mathrm{PC}^1[J,\mathbb{R}]$ by

$$\begin{aligned}(Ax)(t) = {} & \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s,x(s),x'(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds \\ & - \int_0^t (t-s)f(s,x(s),x'(s))ds + \sum_{0<t_k<t}\left[I_k(x(t_k)) + \overline{I}_k(x(t_k))(t-t_k)\right] \\ & - \frac{t}{1-\alpha\eta}\sum_{k=1}^m \left\{(1-\alpha)I_k(x(t_k)) + \left[1-t_k-\alpha(\eta-t_k)\right]\overline{I}_k(x(t_k))\right\}.\end{aligned} \tag{2.17}$$

From Lemma 2.5, $A : \mathrm{PC}^1[J,\mathbb{R}] \mapsto \mathrm{PC}^1[J,\mathbb{R}]$ is a completely continuous operator.

Theorem 2.8. *Suppose that (H_0) and (H_1) hold. Let α_i, β_i $(i = 1,2,\dots,n)$ be n pairs of strict lower and upper solution, and*

$$\begin{aligned}\overline{\alpha}(t) &= \max\left\{\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)\right\}, \quad t \in J, \\ \overline{\beta}(t) &= \min\left\{\beta_1(t), \beta_2(t), \dots, \beta_n(t)\right\}, \quad t \in J.\end{aligned} \tag{2.18}$$

Suppose that $\alpha_i \prec \beta_i$ $(i = 1,2,\dots,n)$, $\overline{\alpha} \prec \overline{\beta}$, f satisfies Nagumo condition with respect to $[\alpha_1, \beta_1]$. Moreover, the strict lower solutions $\alpha_1, \alpha_2, \dots, \alpha_n$ and the strict upper solutions $\beta_1, \beta_2, \dots, \beta_n$ are well ordered whenever $I_{i_0}(x) \neq 0$ or $\overline{I}_{j_0}(x) \neq 0$ for some $i_0, j_0 \in \{1,2,\dots,m\}$ and some $x \in \mathbb{R}$. Then, there exist $R_0 > 0$ and $L_0 > 0$ sufficiently large such that for each $R \geq R_0$ and $L > L_0$

$$\deg\left(I - A, \Omega, \theta\right) = 1, \tag{2.19}$$

where

$$\begin{aligned}\Omega &= \left\{x \in B(\theta, R) \mid \overline{\alpha} \prec x \prec \overline{\beta},\ -L \prec x' \prec L\right\}, \\ B(\theta, R) &= \left\{x \in \mathrm{PC}^1[J, R] \mid \|x\|_{\mathrm{PC}^1} < R\right\}.\end{aligned} \tag{2.20}$$

Proof. We only prove the case when $I_{i_0}(x) \neq 0$ or $\overline{I}_{j_0}(x) \neq 0$ for some $i_0, j_0 \in \{1,2,\dots,m\}$ and some $x \in \mathbb{R}$. The conclusion is achieved in four steps.

Step 1. Since f satisfies Nagumo condition with respect to $[\alpha_1, \beta_1]$, then there exists $\phi \in C([0,\infty),(0,\infty))$ such that

$$\begin{aligned}\left|f(t,x,y)\right| \leq \phi(|y|), &\quad (t,x,y) \in J \times \left[\alpha(t), \beta(t)\right] \times \mathbb{R}, \\ \int_0^\infty \frac{s}{\phi(s)}ds &= \infty.\end{aligned} \tag{2.21}$$

Let $\mu_0 = \min_{1\le k\le m+1}(t_k - t_{k-1})$. Take $\lambda > 0$ such that

$$\lambda > \frac{\max_{1\le i\le n}\sup_{t\in J}\beta_i(t) - \min_{1\le i\le n}\inf_{t\in J}\alpha_i(t)}{\mu_0}, \tag{2.22}$$

and $N > 0$ such that

$$\int_\lambda^N \frac{s}{\phi(s)}ds > 2\lambda. \tag{2.23}$$

Let $L_0 = \max\{N, 2\lambda, \max_{1\le i\le n}\|\alpha_i'\|_{PC}, \max_{1\le i\le n}\|\beta_i'\|_{PC}\}$. Define the functions $g, h : J \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\begin{gathered}
g(t,x,y) = \begin{cases} f(t,x,L), & y > L, \\ f(t,x,y), & -L \le y \le L, \\ f(t,x,-L), & y < -L, \end{cases} \\
h(t,x,y) = \begin{cases} g\left(t,\overline{\beta}(t),y\right), & x > \overline{\beta}(t), \\ g(t,x,y), & \overline{\alpha}(t) \le x \le \overline{\beta}(t), \\ g\left(t,\overline{\alpha}(t),y\right), & x < \overline{\alpha}(t). \end{cases}
\end{gathered} \tag{2.24}$$

For each $k \in \{1,2,\ldots,m\}$, let us define the functions $J_k, \overline{J}_k : \mathbb{R} \mapsto \mathbb{R}$ by

$$\begin{gathered}
J_k(x) = \begin{cases} I_k\left(\overline{\beta}(t_k)\right), & x > \overline{\beta}(t_k), \\ I_k(x), & \overline{\alpha}(t_k) \le x \le \overline{\beta}(t_k), \\ I_k\left(\overline{\alpha}(t_k)\right), & x < \overline{\alpha}(t_k), \end{cases} \\
\overline{J}_k(x) = \begin{cases} \overline{I}_k\left(\overline{\beta}(t_k)\right), & x > \overline{\beta}(t_k), \\ \overline{I}_k(x), & \overline{\alpha}(t_k) \le x \le \overline{\beta}(t_k), \\ \overline{I}_k\left(\overline{\alpha}(t_k)\right), & x < \overline{\alpha}(t_k). \end{cases}
\end{gathered} \tag{2.25}$$

It is easy to see that there exists $M_1 > 0$ such that

$$\begin{gathered}
\left|h(t,x,y)\right| \le M_1, \quad (t,x,y) \in J \times \mathbb{R}^2, \\
\left|J_k(x)\right| \le M_1, \quad \left|\overline{J}_k(x)\right| \le M_1, \quad x \in \mathbb{R},\ k = 1,2,\ldots,m.
\end{gathered} \tag{2.26}$$

Let us define the operator $A^*: \mathrm{PC}^1[J,\mathbb{R}] \mapsto \mathrm{PC}^1[J,\mathbb{R}]$ by

$$\begin{aligned}(A^*x)(t) = {} & \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h(s,x(s),x'(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s,x(s),x'(s))ds \\ & - \int_0^t (t-s)h(s,x(s),x'(s))ds + \sum_{0<t_k<t}\left[J_k(x(t_k)) + (t-t_k)\overline{J}_k(x(t_k))\right] \\ & - \frac{t}{1-\alpha\eta}\sum_{k=1}^m \left\{(1-\alpha)J_k(x(t_k)) + \left[(1-t_k)-\alpha(\eta-t_k)\right]\overline{J}_k(x(t_k))\right\}.\end{aligned} \tag{2.27}$$

By (2.26), we have

$$\begin{aligned}(A^*x)(t) \le {} & \frac{1}{1-\alpha\eta}\int_0^1 (1-s)\left|h(s,x(s),x'(s))\right|ds + \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)\left|h(s,x(s),x'(s))\right|ds \\ & + \int_0^1 (1-s)\left|h(s,x(s),x'(s))\right|ds + \sum_{k=1}^m\left[\left|J_k(x(t_k))\right| + (1-t_k)\left|\overline{J}_k(x(t_k))\right|\right] \\ & + \frac{2}{1-\alpha\eta}\sum_{k=1}^m\left[\left|J_k(x(t_k))\right| + \left|\overline{J}_k(x(t_k))\right|\right] \\ \le {} & \frac{M_1}{1-\alpha\eta}\left[\frac{1}{2}+\frac{1}{2}\alpha\eta^2+\frac{1}{2}+2m+4m\right] \\ \le {} & \frac{M_1}{1-\alpha\eta}[3+6m], \quad t \in J, \\ (A^*x)'(t) \le {} & \frac{1}{1-\alpha\eta}\int_0^1 (1-s)\left|h(s,x(s),x'(s))\right|ds + \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta-s)\left|h(s,x(s),x'(s))\right|ds \\ & + \int_0^1 \left|h(s,x(s),x'(s))\right|ds + \sum_{k=1}^m\left|\overline{J}_k(x(t_k))\right| + \frac{2}{1-\alpha\eta}\sum_{k=1}^m\left[\left|J_k(x(t_k))\right| + \left|\overline{J}_k(x(t_k))\right|\right] \\ \le {} & \frac{M_1}{1-\alpha\eta}\left[\frac{1}{2}+\frac{1}{2}\alpha\eta^2+1+5m\right] \\ \le {} & \frac{M_1}{1-\alpha\eta}[5m+3], \quad t \in J.\end{aligned} \tag{2.28}$$

From (2.28), we have $\|A^*x\|_{\mathrm{PC}^1} \le (M_1/(1-\alpha\eta))[11m+6]$ for each $x \in \mathrm{PC}^1[J,\mathbb{R}]$. Let $R_0 = (M_1/(1-\alpha\eta))(11m+6)+1$. Then, $A^*(\mathrm{PC}^1[J,\mathbb{R}]) \subset B(\theta,R_0)$. By the properties of the Leray-Schauder degree, we have

$$\deg\left(I-A^*, B(\theta,R),\theta\right) = 1. \tag{2.29}$$

Thus, A^* has at least one fixed point x_0. From Lemma 2.7, x_0 satisfies

$$\begin{gathered} x_0''(t) + h(t, x_0(t), x_0'(t)) = 0, \quad t \neq t_k, \\ \Delta x_0|_{t=t_k} = J_k(x_0(t_k)), \quad k = 1, 2, \ldots, m \\ \Delta x_0'\big|_{t=t_k} = \overline{J}_k(x_0(t_k)), \quad k = 1, 2, \ldots, m, \\ x_0(0) = 0 = x_0(1) - \alpha x_0(\eta), \end{gathered} \tag{2.30}$$

Step 2. Next, we will show that

$$\overline{\alpha} \prec x_0 \prec \overline{\beta}, \tag{2.31}$$

$$-L \prec x_0' \prec L. \tag{2.32}$$

We first show that

$$\overline{\alpha}(t) \leq x_0(t) \leq \overline{\beta}(t) \quad \forall t \in J. \tag{2.33}$$

To begin, we show that $x_0(t) \leq \overline{\beta}(t)$ for all $t \in J$. Suppose not, then there exists $t' \in J$ such that $x_0(t') > \overline{\beta}(t')$. Set $w(t) = x_0(t) - \overline{\beta}(t)$ for $t \in J$. There are a number of cases to consider.

(1) $w(0) = \sup_{t \in J} w(t) > 0$, then, we have

$$0 < w(0) = x_0(0) - \overline{\beta}(0) = -\overline{\beta}(0) < 0, \tag{2.34}$$

which is a contradiction.

(2) $w(1) = \sup_{t \in J} w(t) > 0$; assume without loss of generality that $\alpha > 0$ and $\overline{\beta}(1) = \beta_{i_0}(1)$ for some $i_0 \in \{1, 2, \ldots, n\}$, then, we have

$$0 < w(1) = x_0(1) - \beta_{i_0}(1) \leq \alpha x_0(\eta) - \alpha \beta_{i_0}(\eta) \leq \alpha x_0(\eta) - \alpha \overline{\beta}(\eta) = \alpha w(\eta) \leq \alpha w(1), \tag{2.35}$$

which is a contradiction.

(3) There exist $k_0 \in \{1, 2, \ldots, m, m+1\}$ and $\tau_0 \in (t_{k_0-1}, t_{k_0})$ such that $w(\tau_0) = \sup_{t \in J} w(t) > 0$. Assume without loss of generality that $\overline{\beta}(\tau_0) = \beta_{i_0}(\tau_0)$ for some $i_0 \in \{1, 2, \ldots, n\}$. We have the following two cases:

(3A) $\beta_j(\tau_0) > \beta_{i_0}(\tau_0)$ for each $j \in \{1, 2, \ldots, n\}$ and $j \neq i_0$;

(3B) there exists $j_0 \in \{1, 2, \ldots, n\}$, $j_0 \neq i_0$ such that $\beta_{j_0}(\tau_0) = \beta_{i_0}(\tau_0)$.

For case (3A), there exists $\delta_0 > 0$ small enough such that $[\tau_0 - \delta_0, \tau_0 + \delta_0] \subset (t_{k_0-1}, t_{k_0})$ and

$$w(t) = x_0(t) - \beta_{i_0}(t), \quad t \in [\tau_0 - \delta_0, \tau_0 + \delta_0]. \tag{2.36}$$

Then, $w \in C^2[\tau_0 - \delta_0, \tau_0 + \delta_0]$, $w(\tau_0)$ is the maximum of w on $[\tau_0 - \delta_0, \tau_0 + \delta_0]$. Thus, $w'(\tau_0) = 0$, $w''(\tau_0) \le 0$. By (2.30), we have

$$\begin{aligned} 0 \ge w''(\tau_0) &= x_0''(\tau_0) - \beta_{i_0}''(\tau_0) = -h(\tau_0, x_0(\tau_0), x_0'(\tau_0)) - \beta_{i_0}''(\tau_0) \\ &= -f(\tau_0, \beta_{i_0}(\tau_0), \beta_{i_0}'(\tau_0)) - \beta_{i_0}''(\tau_0), \end{aligned} \tag{2.37}$$

which is a contradiction.

For case (3B), set $w_1(t) = x_0(t) - \beta_{j_0}(t)$ for $t \in (t_{k_0-1}, t_{k_0})$. For any $t' \in (t_{k_0-1}, t_{k_0})$, we have

$$\begin{aligned} w_1(\tau_0) &= x_0(\tau_0) - \beta_{j_0}(\tau_0) = x_0(\tau_0) - \beta_{i_0}(\tau_0) = w(\tau_0) \\ &\ge w(t') = x_0(t') - \overline{\beta}(t') \ge x_0(t') - \beta_{j_0}(t') = w_1(t'). \end{aligned} \tag{2.38}$$

This implies that $w_1(\tau_0)$ is a local maximum. Since $w_1 \in C^2(t_{k_0-1}, t_{k_0})$, then $w_1'(\tau_0) = 0$, $w_1''(\tau_0) \le 0$. Therefore,

$$0 \ge w_1''(\tau_0) = x_0''(\tau_0) - \beta_{j_0}''(\tau_0) = -f(\tau_0, \beta_{j_0}(\tau_0), \beta_{j_0}'(\tau_0)) - \beta_{j_0}''(\tau_0) > 0, \tag{2.39}$$

which is a contradiction.

(4) There exists $k_0 \in \{1, 2, \ldots, m\}$ such that $w(t_{k_0}) = \sup_{t\in J} w(t) > 0$. Without loss of generality, we may assume $w(\tau) < \sup_{t\in J} w(t)$ for each $\tau \in (t_{k-1}, t_k)$ and $k \in \{1, 2, \ldots, m, m+1\}$. (Otherwise, if there exists $\tau_0 \in (t_{k_0-1}, t_{k_0})$ for some $k_0 \in \{1, 2, \ldots, m, m+1\}$ such that $w(\tau_0) = \sup_{t\in J} w(t)$, then we can get a contradiction as in case (3)). In this case, we have the following two subcases:

(4A) there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta_{i_0}(t_{k_0}) < \beta_j(t_{k_0})$ for $j = 1, 2, \ldots, n$ and $j \ne i_0$;

(4B) there exists a subset $\{n_1, n_2, \ldots, n_s\} \subset \{1, 2, \ldots, n\}$ such that

$$\overline{\beta}(t_{k_0}) = \beta_{n_1}(t_{k_0}) = \beta_{n_2}(t_{k_0}) = \cdots = \beta_{n_s}(t_{k_0}), \tag{2.40}$$

while $\beta_l(t_{k_0}) > \overline{\beta}(t_{k_0})$ for each $l \in \{1, 2, \ldots, n\} \setminus \{n_1, n_2, \ldots, n_s\}$, $s \ge 2$.

First, we consider case (4A). Since I_{k_0} is increasing on $\mathbb{R}$, then

$$\beta_{i_0}(t_{k_0}^+) = \beta_{i_0}(t_{k_0}) + I_{k_0}(\beta_{i_0}(t_{k_0})) < \beta_j(t_{k_0}) + I_{k_0}(\beta_j(t_{k_0})) = \beta_j(t_{k_0}^+), \quad j \ne i_0. \tag{2.41}$$

Then, there exists $\delta_0 > 0$ small enough such that $\overline{\beta}(t) = \beta_{i_0}(t)$ for $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$ and so $w(t) = x_0(t) - \beta_{i_0}(t)$ for $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$. Since $\beta_{i_0}(t)$ is a strict upper solution, we have

$$\begin{aligned} w\left(t_{k_0}^+\right) &= x_0\left(t_{k_0}^+\right) - \beta_{i_0}\left(t_{k_0}^+\right) \\ &= \left[x_0\left(t_{k_0}\right) + J_{k_0}\left(x_0\left(t_{k_0}\right)\right)\right] - \left[\beta_{i_0}\left(t_{k_0}\right) + I_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right)\right] \\ &= \left[x_0\left(t_{k_0}\right) - \beta_{i_0}\left(t_{k_0}\right)\right] + \left[J_{k_0}\left(x_0\left(t_{k_0}\right)\right) - I_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right)\right] \\ &= w\left(t_{k_0}\right) + \left[I_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right) - I_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right)\right] \\ &= w\left(t_{k_0}\right). \end{aligned} \tag{2.42}$$

Since $w(\tau) < w(t_{k_0})$ for each $\tau \in (t_{k_0-1}, t_{k_0})$, then we have $w'(t_{k_0}) \geq 0$. Similarly, we have $w'(t_{k_0}^+) \leq 0$. Therefore,

$$\begin{aligned} 0 \geq w'\left(t_{k_0}^+\right) &= x'\left(t_{k_0}^+\right) - \beta_{i_0}'\left(t_{k_0}^+\right) \\ &> \left[x_0'\left(t_{k_0}\right) + \overline{J}_{k_0}\left(x_0\left(t_{k_0}\right)\right)\right] - \left[\beta_{i_0}'\left(t_{k_0}\right) + \overline{I}_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right)\right] \\ &= w'\left(t_{k_0}\right) + \left[\overline{I}_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right) - \overline{I}_{k_0}\left(\beta_{i_0}\left(t_{k_0}\right)\right)\right] = w'\left(t_{k_0}\right) \geq 0, \end{aligned} \tag{2.43}$$

which is contradiction.

Now we consider case (4B). Since I_{k_0} is increasing, then we have

$$\overline{\beta}\left(t_{k_0}^+\right) = \beta_{n_1}\left(t_{k_0}^+\right) = \beta_{n_2}\left(t_{k_0}^+\right) = \cdots = \beta_{n_s}\left(t_{k_0}^+\right), \tag{2.44}$$

while $\beta_l(t_{k_0}^+) > \beta(t_{k_0}^+)$ for each $l \in \{1, 2, \ldots, n\} \setminus \{n_1, n_2, \ldots, n_s\}$. For case (4B), we have two subcases:

(4Ba) there exists $\delta_0 > 0$ small enough and $i_0 \in \{n_1, n_2, \ldots, n_s\}$ such that $\beta_{i_0}(t) = \beta(t)$ for $t \in [t_{k_0} - \delta_0, t_{k_0} + \delta_0]$;

(4Bb) there exists $\delta_0 > 0$ small enough and $i_0 \neq j_0$, $i_0, j_0 \in \{n_1, n_2, \ldots, n_s\}$ such that

$$\overline{\beta}(t) = \begin{cases} \beta_{i_0}(t), & t \in \left[t_{k_0} - \delta_0, t_{k_0}\right], \\ \beta_{j_0}(t), & t \in \left(t_{k_0}, t_{k_0} + \delta_0\right]. \end{cases} \tag{2.45}$$

For case (4Ba) as in case (4A), we can easily obtain a contradiction. For case (4Bb), we have

$$w(t) = \begin{cases} x_0(t) - \beta_{i_0}(t), & t \in \left[t_{k_0} - \delta_0, t_{k_0}\right], \\ x_0(t) - \beta_{j_0}(t), & t \in \left(t_{k_0}, t_{k_0} + \delta_0\right]. \end{cases} \tag{2.46}$$

In the same way as in the proof of case (4A), we see that $w(t_{k_0}^+) = w(t_{k_0})$, $w'(t_{k_0}) \geq 0$ and we have $w'(t_{k_0}^+) \leq 0$. Note that $\beta'_{j_0}(t_{k_0}) \leq \beta'_{i_0}(t_{k_0})$, and we have

$$\begin{aligned}
0 \geq w'(t_{k_0}^+) &= x_0'(t_{k_0}^+) - \beta'_{j_0}(t_{k_0}^+) \\
&= \left[x_0'(t_{k_0}) + \Delta x_0'|_{t=t_{k_0}}\right] - \left[\beta'_{j_0}(t_{k_0}) + \Delta \beta'_{j_0}|_{t=t_{k_0}}\right] \\
&> \left[x_0'(t_{k_0}) - \beta'_{j_0}(t_{k_0})\right] + \left[\overline{J}_{k_0}(x_0(t_{k_0})) - \overline{I}_{k_0}(\beta_{j_0}(t_{k_0}))\right] \\
&\geq x_0'(t_{k_0}) - \beta'_{i_0}(t_{k_0}) + \left[\overline{I}_{k_0}(\beta_{j_0}(t_{k_0})) - \overline{I}_{k_0}(\beta_{j_0}(t_{k_0}))\right] \\
&= w'(t_{k_0}) \geq 0,
\end{aligned} \tag{2.47}$$

which is a contradiction.

(5) There exists a $k_0 \in \{1, 2, \ldots, m\}$ such that $w(t_{k_0}^+) = \sup_{t \in J} w(t) > 0$. Without loss of generality, we may assume that $w(\tau) < w(t_{k_0}^+)$ for each $k \in \{1, 2, \ldots, m, m+1\}$ and $\tau \in (t_{k-1}, t_k)$. We have two subcases:

(5A) there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta_{i_0}(t_{k_0}^+) < \beta_j(t_{k_0}^+)$ for each $j \neq i_0$;

(5B) there exists a subset $\{n_1, n_2, \ldots, n_s\} \subset \{1, 2, \ldots, n\}$ such that

$$\overline{\beta}(t_{k_0}^+) = \beta_{n_1}(t_{k_0}^+) = \beta_{n_2}(t_{k_0}^+) = \cdots = \beta_{n_s}(t_{k_0}^+), \tag{2.48}$$

while $\beta_l(t_{k_0}^+) > \overline{\beta}(t_{k_0}^+)$ for each $l \in \{1, 2, \ldots, n\} \setminus \{n_1, n_2, \ldots, n_s\}$, $s \geq 2$.

Since I_{k_0} is increasing, then for case (5A), we have

$$\beta_{i_0}(t_{k_0}) < \beta_j(t_{k_0}) \quad (j \neq i_0), \qquad x_0(t_{k_0}) > \beta_{i_0}(t_{k_0}), \tag{2.49}$$

and for case (5B), we have $x_0(t_{k_0}) > \beta_{i_0}(t_{k_0})$ and

$$\overline{\beta}(t_{k_0}) = \beta_{n_1}(t_{k_0}) = \beta_{n_2}(t_{k_0}) = \cdots = \beta_{n_s}(t_{k_0}), \tag{2.50}$$

while $\beta_l(t_{k_0}) > \overline{\beta}(t_{k_0})$ for each $l \in \{1, 2, \ldots, n\} \setminus \{n_1, n_2, \ldots, n_s\}$. Therefore, we can use the same method as in case (4) to obtain a contradiction.

From the discussions of (1)–(5), we see that $x_0(t) \leq \overline{\beta}(t)$ for $t \in J$. Similarly, we can prove that $\overline{\alpha}(t) \leq x_0(t)$ for $t \in J$. Thus, (2.33) holds.

Next, we prove that $\overline{\alpha} \prec x_0 \prec \overline{\beta}$. If the inequality $x_0 \prec \overline{\beta}$ does not hold, then either there exists $\tau_0 \in J$ such that $x_0(\tau_0) = \overline{\beta}(\tau_0)$ or there exists $k_0 \in \{1, 2, \ldots, m\}$ such that $x_0(t_{k_0}^+) = \overline{\beta}(t_{k_0}^+)$. Set $w(t) = x_0(t) - \overline{\beta}(t)$ for $t \in J$. Then, we have either $w(\tau_0) = \sup_{t \in J} w(t)$ or $w(t_{k_0}^+) = \sup_{t \in J} w(t)$ for some $k_0 \in \{1, 2, \ldots, m\}$. Essentially the same reasoning as in (1)–(5) above yields a contradiction. Thus, $x_0 \prec \overline{\beta}$. Similarly, $\overline{\alpha} \prec x_0$. Consequently, (2.31) holds.

Step 3. Now, we show (2.32). Suppose not, then we have the following two subcases:

(I) there exists $s_1 \in J$ such that $|x_0'(s_1)| \geq L$;

(II) there exists $k_0 \in \{1, 2, \ldots, m\}$ such that $|x_0'(t_{k_0}^+)| \geq L$.

We only consider case (II). A similar argument works for case (I). We may assume without loss of generality that $x_0'(t_{k_0}^+) \geq L$. By the mean-value theorem, there exists $s_2 \in (t_{k_0}, t_{k_0+1})$ such that

$$x_0'(s_2) = \frac{x_0(t_{k_0+1}) - x_0(t_{k_0})}{t_{k_0+1} - t_{k_0}} \leq \frac{\max_{1\leq i\leq n}\sup_{t\in J}\beta_i(t) - \min_{1\leq i\leq n}\inf_{t\in J}\alpha_i(t)}{\mu_0} < \lambda < L. \tag{2.51}$$

Let L_1 be such that $L_0 < L_1 < L$, then, there exist $s_3, s_4 \in (t_{k_0}, s_2]$ such that $s_3 < s_4$, $x_0'(s_3) = L_1$, $x_0'(s_4) = \lambda$, and $\lambda \leq x_0'(s) \leq L_1$ for $s \in [s_3, s_4]$. Therefore,

$$\left|x_0''(s)\right| = \left|h(s, x_0(s), x_0'(s))\right| = \left|f(s, x_0(s), x_0'(s))\right| \leq \phi(x_0'(s)), \quad s \in [s_3, s_4]. \tag{2.52}$$

Consequently,

$$\left|\int_{s_3}^{s_4} \frac{x_0'(s)x_0''(s)}{\phi(x_0'(s))}ds\right| \leq \int_{s_3}^{s_4}\left|x_0'(s)\right|ds = \int_{s_3}^{s_4} x_0'(s)ds = x_0(s_4) - x_0(s_3) < \lambda. \tag{2.53}$$

On the other hand,

$$\left|\int_{s_3}^{s_4} \frac{x_0'(s)x_0''(s)}{\phi(x_0'(s))}ds\right| = \left|\int_{\lambda}^{L_1} \frac{s}{\phi(s)}ds\right| \geq \int_{\lambda}^{N} \frac{s}{\phi(s)}ds > \lambda, \tag{2.54}$$

which is a contradiction. Thus, (2.32) holds.

Step 4. From the excision property of Leray-Schauder degree and (2.29), we have

$$\deg\left(I - A^*, \Omega, \theta\right) = \deg\left(I - A^*, B(\theta, R), \theta\right) = 1. \tag{2.55}$$

From (2.31) and (2.32), we see that $Ax = A^*x$ for each $x \in \overline{\Omega}$, and so

$$\deg\left(I - A, \Omega, \theta\right) = 1. \tag{2.56}$$

The proof is complete. □

Remark 2.9. From the proof of Theorem 2.8, we see that A has no fixed point on $\partial\Omega$.

Theorem 2.10. *Suppose that (H_0), (H_1) hold, $u_1(t)$, $u_2(t)$ are strict lower solutions, $v_1(t), v_0(t)$ are strict upper solutions, $u_1 \prec v_1 \prec v_0$, $u_2 \prec v_0$, $u_2(t') > v_1(t')$ for some $t' \in J$, and f satisfies Nagumo condition with respect to $[u_1, v_0]$. Moreover, the strict lower solutions $u_1(t)$, $u_2(t)$ are well ordered whenever $I_{i_0}(x) \neq 0$ or $\overline{I}_{j_0}(x) \neq 0$ for some $i_0, j_0 \in \{1, 2, \ldots, m\}$ and some $x \in \mathbb{R}$. Then,* (1.1) *has at least three solutions x_1, x_2, and x_3, such that*

$$u_1 \prec x_1 \prec v_1, \qquad u_1 \prec x_2 \prec v_0, \qquad u_2 \prec x_2 \prec v_0, \qquad u_1 \prec x_3 \prec v_0, \tag{2.57}$$

and $v_1(s_1) < x_3(s_1)$, $x_3(s_2) < u_2(s_2)$ for some $s_1, s_2 \in J$.

Proof. Set $\overline{\alpha}(t) = \max\{u_1(t), u_2(t)\}$ for $t \in J$, and $\overline{\alpha}(t_k^+) = \max\{u_1(t_k^+), u_2(t_k^+)\}$ for each $k \in \{1, 2, \ldots, m\}$. From Theorem 2.8, we see that there exist $R > 0$ and $L > 0$ large enough such that

$$\begin{gathered}\deg(I - A, G_0, \theta) = 1, \\ \deg(I - A, G_1, \theta) = 1, \\ \deg(I - A, G_2, \theta) = 1,\end{gathered} \tag{2.58}$$

where $G_0 = \{x \in B(\theta, R) \mid u_1 \prec x \prec v_0,\ -L \prec x' \prec L\}$, $G_1 = \{x \in B(\theta, R) \mid u_1 \prec x \prec v_1,\ -L \prec x' \prec L\}$, and $G_2 = \{x \in B(\theta, R) \mid \overline{\alpha} \prec x \prec v_0,\ -L \prec x' \prec L\}$. Then, A has fixed points $x_1 \in G_1$ and $x_2 \in G_2$, respectively. From the conditions of Theorem 2.10, we see that $\overline{G}_1 \cap \overline{G}_2 = \emptyset$. Let $\omega_0(t)$ be a continuous function on J such that its graph passes the points $(0, (v_0(0) + v_1(0))/2)$ and $(t', (v_1(t') + u_1(t'))/2)$, and satisfies $u_1 \prec \omega_0 \prec v_0$. By the well-known Weierstrass approximation theorem, there exists $\omega_1 \in C^1[0, 1]$ such that

$$|\omega_1(t) - \omega_0(t)| < \min\left\{\frac{v_1(t') - u_1(t')}{4}, \frac{v_0(0) - v_1(0)}{4}, \frac{1}{4}\|\omega_0 - v_0\|_{PC}, \frac{1}{4}\|\omega_0 - u_1\|_{PC}\right\}, \quad t \in J. \tag{2.59}$$

It is easy to see that $\omega_1 \in G_0 \setminus (\overline{G}_1 \cup \overline{G}_2)$, and so $G_0 \setminus (\overline{G}_1 \cup \overline{G}_2)$ is a nonempty open set. Note A has no fixed point on $\partial G_0, \partial G_1$, and ∂G_2. From (2.58), we have

$$\deg(I - A, G_0 \setminus (\overline{G}_1 \cup \overline{G}_2), \theta) = \deg(I - A, G_0, \theta) - \deg(I - A, G_1, \theta) - \deg(I - A, G_2, \theta) = -1. \tag{2.60}$$

Thus, A has at least one fixed point $x_3 \in G_0 \setminus (\overline{G}_1 \cup \overline{G}_2)$. Since $x_3 \notin \overline{G}_1 \cup \overline{G}_2$, then there exist $s_1, s_2 \in J$ such that $v_1(s_1) < x_3(s_1)$ and $x_3(s_2) < u_2(s_2)$. The proof is complete. □

Remark 2.11. Theorem 2.10 is a partial generalization of the main results of [16, Theorem 2.2]. Here, we do not need to assume that u_2 satisfies $u_1 \leq u_2 \leq v_0$.

Remark 2.12. The position of u_1, u_2, v_1, v_0 in Theorem 2.10 can be illustrated roughly by Figure 2.

Remark 2.13. The relationship of u_1, u_2, v_1, $v_0$is different from that of [12, Theorems 9 and 10].

Similarly, we have the following result.

Theorem 2.14. *Suppose that* (H_0), (H_1) *hold,* u_0, u_1 *are strict lower solutions of* (1.1), v_1 *and* v_2 *are strict upper solutions of* (1.1), $u_0 \prec u_1 \prec v_1$, $u_0 \prec v_2$, $v_2(t') < u_1(t')$ *for some* $t' \in J$, *and* f *satisfies Nagumo condition with respect to* $[u_0, v_1]$. *Moreover, the strict upper solutions* $v_1(t)$, $v_2(t)$ *are well ordered whenever* $I_{i_0}(x) \neq 0$ *or* $\overline{I}_{j_0}(x) \neq 0$ *for some* $i_0, j_0 \in \{1, 2, \ldots, m\}$ *and some* $x \in \mathbb{R}$.

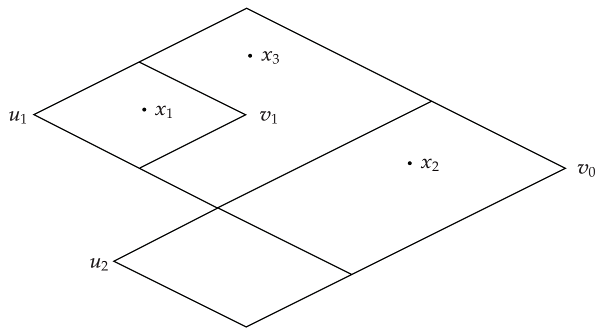

Figure 2: The position of u_1, u_2, v_1, v_0 in Theorem 2.10.

Then, (1.1) *has at least three solutions* x_1, x_2, x_3 *such that*

$$u_1 \prec x_1 \prec v_1, \qquad u_0 \prec x_2 \prec v_1, \qquad u_0 \prec x_2 \prec v_2, \qquad u_0 \prec x_3 \prec v_1, \tag{2.61}$$

and $v_2(s_1) < x_3(s_1)$, $x_3(s_2) < u_1(s_2)$ *for some* $s_1, s_2 \in J$.

From Theorems 2.10 and 2.14, we have the following Theorem 2.15.

Theorem 2.15. *Suppose that* (H_0), (H_1) *hold,* u_0, u_1, u_2 *are three strict lower solutions of* (1.1), v_0, v_1, v_2 *are three strict upper solutions of* (1.1), $u_0 \prec u_1 \prec v_1 \prec v_0$, $u_0 \prec u_2 \prec v_2 \prec v_0$, $u_2(t') > v_1(t')$, $v_2(t'') < u_1(t'')$ *for some* $t', t'' \in J$, *and* f *satisfies Nagumo conditions with respect to* $[u_0, v_0]$. *Moreover, the strict lower solutions* u_0, u_1, u_2 *and the strict upper solutions* v_0, v_1, v_2 *are well ordered whenever* $I_{i_0}(x) \neq 0$ *or* $\overline{I}_{j_0}(x) \neq 0$ *for some* $i_0, j_0 \in \{1, 2, \ldots, m\}$ *and some* $x \in \mathbb{R}$. *Then,* (1.1) *has at least eight solutions.*

Proof. Now Theorem 2.10 guarantees that (1.1) has at least three solutions x_1, x_2, x_3 such that

$$u_1 \prec x_1 \prec v_1, \qquad u_1 \prec x_2 \prec v_0, \qquad u_2 \prec x_2 \prec v_0, \qquad u_1 \prec x_3 \prec v_0, \tag{2.62}$$

and $v_1(s_1) < x_3(s_1)$, $x_3(s_2) < u_2(s_2)$ for some $s_1, s_2 \in J$.

Also (1.1) has at least two solutions x_4 and x_5 such that

$$u_2 \prec x_4 \prec v_2, \qquad u_2 \prec x_5 \prec v_0, \tag{2.63}$$

and $v_2(s_3) < x_5(s_3), x_5(s_4) < u_1(s_4)$.

Now Theorem 2.14 guarantees that (1.1) has at least two solutions x_6, x_7 such that

$$u_0 \prec x_6 \prec v_2, \qquad u_0 \prec x_6 \prec v_1, \qquad u_0 \prec x_7 \prec v_2, \tag{2.64}$$

and $v_1(s_5) < x_7(s_5)$, $x_7(s_6) < u_2(s_6)$.

Also (1.1) has at least one solution x_8 such that $u_0 \prec x_8 \prec v_1$ and $v_2(s_7) < x_8(s_7)$, $x_8(s_8) < u_1(s_8)$ for some $s_7, s_8 \in J$. It is easy to see that $x_1, x_2, \ldots, x_8$ are distinct eight solutions of (1.1). The proof is complete. □

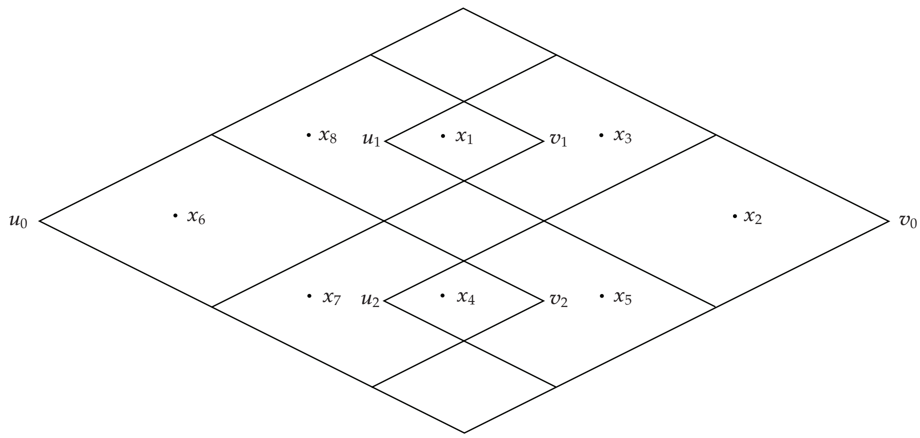

Figure 3: The position of u_0, u_1, u_2, v_0, v_1, v_2 in Theorem 2.15.

Remark 2.16. The position of u_0, u_1, u_2, v_0, v_1, v_2 in Theorem 2.15 can be illustrated roughly by Figure 3.

3. Further discussions

For simplicity, in this section, we will always assume that

$$I_k(x) = \overline{I}_k(x) = 0, \quad x \in \mathbb{R},\ k \in \{1, 2, \ldots, m\}. \tag{3.1}$$

In this case, (1.1) can be reduced to the following three-point boundary value problem

$$\begin{gathered} y'' + f\left(t, y(t), y'(t)\right) = 0, \quad t \in (0,1), \\ y(0) = 0 = y(1) - \alpha y(\eta), \end{gathered} \tag{3.2}$$

where $0 < \eta < 1$ and $0 \le \alpha < 1$.

In this section, we will use the following assumptions.

(A_1) Suppose that u_1, u_2 are two strict lower solutions, v_1, v_2 are two strict upper solutions of (1.1), $u_1 \prec v_1$, $u_2 \prec v_2$, and $u_2(s_1) > v_1(s_1)$, $u_1(s_2) > v_2(s_2)$ for some $s_1, s_2 \in J$.

Recently, this multipoint boundary value problem has been studied by many authors, see [16, 17, 19–21] and the references therein. The goal of this section is to prove some multiplicity results for (3.2) using the condition of two pairs of strict upper and lower solutions. As we can see from [13], some bounding condition on the nonlinear term is needed. Instead of the space $PC^1[J, \mathbb{R}]$, in this section we will use the space $C^1(J)$. First, we have the following theorem.

Theorem 3.1. *Suppose that* (A_1) *holds, and*

$$\left|f(t, x, y)\right| < D_0, \quad (t, x, y) \in J \times \mathbb{R}^2 \tag{3.3}$$

for some $D_0 > 0$*. Then,* (3.2) *has at least eight solutions.*

Proof. First, we show that there exist strict lower and upper solutions u_0, v_0 such that

$$u_0 \prec u_i \prec v_i \prec v_0, \quad i = 1, 2. \tag{3.4}$$

Let $a_0 > \|u_1\|_C + \|u_2\|_C + \|v_1\|_C + \|v_2\|_C + 1$. Now, we consider the following boundary value problem:

$$\begin{gathered} v_0''(t) + D_0 = 0, \quad t \in (0,1), \\ v_0(0) = a_0 = v_0(1) - \alpha v_0(\eta). \end{gathered} \tag{3.5}$$

Let

$$\begin{gathered} g_1(t) = \frac{1 - \alpha\eta + \alpha t}{1 - \alpha\eta}, \quad t \in J, \\ g_2(t) = \frac{t}{2(1-\alpha\eta)} - \frac{\alpha\eta^2 t}{2(1-\alpha\eta)} - \frac{1}{2}t^2, \quad t \in J. \end{gathered} \tag{3.6}$$

By Lemma 2.7, we have

$$\begin{aligned} v_0(t) &= g_1(t)a_0 + \frac{D_0 t}{1-\alpha\eta}\int_0^1 (1-s)\,ds - \frac{\alpha D_0 t}{1-\alpha\eta}\int_0^\eta (\eta - s)\,ds - D_0\int_0^t (t-s)\,ds \\ &= g_1(t)a_0 + g_2(t)D_0, \quad t \in J. \end{aligned} \tag{3.7}$$

It is easy to see that $g_1(t) \geq 1$ and $g_2(t) \geq 0$ for each $t \in J$. Thus, $v_0(t) \geq a_0$ for each $t \in J$, and therefore, $u_i \prec v_0$, $v_i \prec v_0$ for $i = 1, 2$. On the other hand, from (3.5), it is easy to see that v_0 is a strict upper solution of (1.1). Similarly, we can show the existence of u_0. Then, by Theorem 2.15, the conclusion holds. □

Remark 3.2. Obviously, the condition (3.3) is restrictive. In the following, we will make use of a weaker condition. We study the multiplicity of solutions of (3.2) under a Nagumo-Knobloch-Schmitt condition. For this kind of bounding condition, the reader is referred to [13].

Theorem 3.3. *Suppose (A_1) holds, and there exists function $\beta_1, \beta_2 \in C^1(J)$, $\beta_1 \leq \beta_2$ such that*

$$\beta_1(t) \leq u_i'(t) \leq \beta_2(t), \quad \beta_1(t) \leq v_i'(t) \leq \beta_2(t), \quad t \in J, \tag{3.8}$$

$$-f(t, x, \beta_1(t)) < \beta_1'(t), \quad -f(t, x, \beta_2(t)) > \beta_2'(t), \quad (t, x) \in J \times [s_2, s_1], \tag{3.9}$$

$$\beta_1(1) \leq -\frac{2}{1-\alpha\eta}D_0 \leq \frac{2}{1-\alpha\eta}D_0 \leq \beta_2(1), \tag{3.10}$$

where $D_0 = \max_{(t,x,y)\in J\times[s_2,s_1]\times\mathbb{R}} |f(t,y,\phi(t,y))|$, $\gamma = \|u_1\|_C + \|v_1\|_c + \|u_2\|_c + \|v_2\|_C$,

$$s_2 = -\int_0^1 |\beta_1(t)|dt - \gamma, \qquad s_1 = \int_0^1 |\beta_2(t)|dt + \gamma,$$
$$\phi(t,y) = \begin{cases} \beta_2(t), & y > \beta_2(t), \\ y, & \beta_1(t) \le y \le \beta_2(t), \\ \beta_1(t), & y < \beta_1(t). \end{cases} \tag{3.11}$$

Then, (3.2) *has at least eight solutions.*

Proof. Let $\rho(x) = \max\{s_2, \min\{x, s_1\}\}$ for each $x \in \mathbb{R}$, and

$$f^*(t,x,y) = f(t,\rho(x),\phi(t,y)), \quad (t,x,y) \in J \times \mathbb{R}^2. \tag{3.12}$$

Now, we consider the following boundary value problem:

$$\begin{gathered} y''(t) + f^*(t,y(t),y'(t)) = 0, \quad t \in (0,1), \\ y(0) = 0 = y(1) - \alpha y(\eta). \end{gathered} \tag{3.13}$$

From (A_1) and (3.8), we see that u_1, u_2 are strict lower solutions of (3.13), and v_1 and v_2 are two strict upper solutions of (3.13). By Theorem 3.1, (3.13) has at least eight solutions $x_1, x_2, \ldots, x_8$. We need only to show that $x_1, x_2, \ldots, x_8$ are solutions of (3.2). We claim that

$$\beta_1(t) \le x_i'(t) \le \beta_2(t), \quad t \in J,\ i = 1,2,\ldots,8. \tag{3.14}$$

We only show that $x_1'(t) \le \beta_2(t)$ for $t \in J$. If $x_1'(t') > \beta_2(t')$ for some $t' \in J$, then $\max_{t\in J} z(t) = z(t_0) > 0$ for some $t_0 \in J$, where $z(t) = x_1'(t) - \beta_2(t)$ for $t \in J$. If $t_0 \in [0,1)$, then $z'(t_0) \le 0$, and so

$$\begin{aligned} 0 \ge z'(t_0) &= x_1''(t_0) - \beta_2'(t_0) = -f^*(t_0, x_1(t_0), x_1'(t_0)) - \beta_2'(t_0) \\ &= -f(t_0, \rho(x_1(t_0)), \phi(t_0, x_1'(t_0))) - \beta_2'(t_0) \\ &= -f(t_0, \rho(x_1(t_0)), \beta_2(t_0)) - \beta_2'(t_0), \end{aligned} \tag{3.15}$$

which contradicts (3.9).

From Lemma 2.6, we have

$$\begin{aligned} x_1(t) = \frac{t}{1-\alpha\eta}\int_0^1 (1-s) f^*(s,x(s),x'(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s) f^*(s,x(s),x_1'(s))ds \\ - \int_0^t (t-s) f^*(s,x(s),x_1'(s))ds, \end{aligned} \tag{3.16}$$

and so

$$x_1'(t) \le \left|x_1'(t)\right| \le \left[\frac{1}{1-\alpha\eta}\int_0^1 (1-s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta - s)ds + 1\right] \le \frac{2}{1-\alpha\eta} D_0 \le \beta_2(1). \tag{3.17}$$

This implies that $t_0 \neq 1$. Therefore, (3.14) holds. Integrating (3.14), we have

$$s_2 \le -\int_0^1 \left|\beta_1(t)\right| dt \le x_1(t) \le \int_0^1 \left|\beta_2(t)\right| dt \le s_1, \quad t \in J. \tag{3.18}$$

From (3.13)–(3.18), we see that $x_1, x_2, \ldots, x_8$ are eight solutions of (3.2). The proof is complete. □

Remark 3.4. We also can replace (3.3) by other bounding conditions, see [13].

Remark 3.5. To end this paper, we point out that the results of this paper can be applied to study the multiplicity of radial solutions of elliptic differential equation in an annulus with impulses at some radii.

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References

[1] D. Guo, "Periodic boundary value problems for second order impulsive integro-differential equations in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 28, no. 6, pp. 983–997, 1997.

[2] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific, Teaneck, NJ, USA, 1989.

[3] D. Guo, "Multiple positive solutions of a boundary value problem for *n*th-order impulsive integro-differential equations in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 56, no. 7, pp. 985–1006, 2004.

[4] L. H. Erbe and X. Liu, "Quasi-solutions of nonlinear impulsive equations in abstract cones," *Applicable Analysis*, vol. 34, no. 3-4, pp. 231–250, 1989.

[5] X. Liu, "Nonlinear boundary value problems for first order impulsive integro-differential equations," *Applicable Analysis*, vol. 36, no. 1-2, pp. 119–130, 1990.

[6] E. Picard, "Sur l'application des méthodes d'approximations successives à l' étude de certaines équations différentielles ordinaries," *Journal de Mathématiques Pures et Appliquées*, vol. 9, pp. 217–271, 1893.

[7] C. De Coster and P. Habets, "An overview of the method of lower and upper solutions for ODEs," in *Nonlinear Analysis and Its Applications to Differential Equations*, vol. 43, pp. 3–22, Birkhǎuser, Boston, Mass, USA, 2001.

[8] I. Rachůnková and M. Tvrdý, "Non-ordered lower and upper functions in second order impulsive periodic problems," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A*, vol. 12, no. 3-4, pp. 397–415, 2005.

[9] H. Amann, A. Ambrosetti, and G. Mancini, "Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities," *Mathematische Zeitschrift*, vol. 158, no. 2, pp. 179–194, 1978.

[10] C. De Coster and M. Henrard, "Existence and localization of solution for second order elliptic BVP in presence of lower and upper solutions without any order," *Journal of Differential Equations*, vol. 145, no. 2, pp. 420–452, 1998.

[11] I. Rachůnková and M. Tvrdý, "Periodic problems with φ-Laplacian involving non-ordered lower and upper functions," *Fixed Point Theory*, vol. 6, no. 1, pp. 99–112, 2005.

[12] I. Rachůnková, “Upper and lower solutions and topological degree,” *Journal of Mathematical Analysis and Applications*, vol. 234, no. 1, pp. 311–327, 1999.
[13] I. Rachůnková, “Upper and lower solutions and multiplicity results,” *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 446–464, 2000.
[14] P. Habets and P. Omari, “Existence and localization of solutions of second order elliptic problems using lower and upper solutions in the reversed order,” *Topological Methods in Nonlinear Analysis*, vol. 8, no. 1, pp. 25–56, 1996.
[15] X. Xu, D. O’Regan, and J. Sun, “Multiplicity results for three-point boundary value problems with a non-well-ordered upper and lower solution condition,” *Mathematical and Computer Modelling*, vol. 45, no. 1-2, pp. 189–200, 2007.
[16] R. A. Khan and J. R. L. Webb, “Existence of at least three solutions of a second-order three-point boundary value problem,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 6, pp. 1356–1366, 2006.
[17] X. Xian, “Three solutions for three-point boundary value problems,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 1053–1066, 2005.
[18] D. Guo, J. Sun, and Z. Liu, *The Funtional Method for Nonlinear Ordinary Differential Equations*, Shandong Science and Technology Press, Jinan, China, 1995.
[19] B. Liu, “Positive solutions of second-order three-point boundary value problems with change of sign,” *Computers & Mathematics with Applications*, vol. 47, no. 8-9, pp. 1351–1361, 2004.
[20] C. P. Gupta and S. I. Trofimchuk, “Existence of a solution of a three-point boundary value problem and the spectral radius of a related linear operator,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 34, no. 4, pp. 489–507, 1998.
[21] R. Ma and N. Castaneda, “Existence of solutions of nonlinear m-point boundary-value problems,” *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 556–567, 2001.