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Research Article

On the Solvability of Second-Order Impulsive Differential Equations with Antiperiodic Boundary Value Conditions

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We prove existence results for second-order impulsive differential equations with antiperiodic boundary value conditions in the presence of classical fixed point theorems. We also obtain the expression of Green's function of related linear operator in the space of piecewise continuous functions.

1. Introduction and preliminaries

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known that many biological phenomena involving threshold, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulated systems do exhibit impulse effects. The branch of modern, applied analysis known as "impulsive" differential equations provides a natural framework to mathematically describe the aforementioned jumping processes. The reader is referred to monographs [1–4] and references therein for some nice examples and applications to the above areas.

In this paper, we mainly study the following second-order impulsive differential equations with antiperiodic boundary value conditions:

$$\begin{gathered} u'' = f(t,u,u'), \quad t \in [0,T] \setminus \Omega, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \quad u'(t_k^+) = u'(t_k) + J_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T), \end{gathered} \tag{1.1}$$

where $\Omega := \bigcup_{i=1}^{m} t_i$ and $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $[0,T] \setminus \Omega \times \mathbb{R}^n \times \mathbb{R}^n$, $I, J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

In [4–12], the authors studied the existence of antiperiodic solutions for first-order, second-order, or high-order differential equations without impulses, and in [3, 13–16] the authors were concerned with the antiperiodic solutions of first-order impulsive differential equations. Also we should mention the work by Cabada et al. in [17] which is concerned with a certain nth order linear differential equation with constant impulses at fixed times and nonhomogeneous periodic boundary conditions. So far, to the best of our knowledge, this is the first work to deal with the antiperiodic solutions to second-order differential equations with nonconstant impulses. Our method to prove the existence of antiperiodic solutions is based on the works in [13, 18, 19]. We should point out that it is Christopher C. Tisdell who started with this method.

The article is organized as follows. In Section 2, we present the expression of Green's functions of related linear operator in the space of piecewise continuous functions. Section 3 contains the main results of the paper and is devoted to the existence of solutions to (1.1). There, differential inequalities are developed and applied to prove the existence of at least one solution to (1.1). In Section 4, a couple of examples are given to illustrate how the main results work.

To understand the notation used above and the ideas in the remainder of the paper, we now briefly introduce some appropriate concepts connected with impulsive differential equations. Most of the following notation can be found in [1, 2, 4, 5]. We assume that $f(t_k^+, x, y) := \lim_{t \to t_k^+} f(t, x, y)$, $f(t_k^-, x, y) := \lim_{t \to t_k^-} f(t, x, y)$ exist and $f(t_k^-, x, y) = f(t_k, x, y)$, $k = 1, 2, \ldots, m$. We introduce and denote the Banach space $PC([0,T], \mathbb{R}^n)$ by

$$\begin{aligned} PC([0,T];\mathbb{R}^n) := \{&u : [0,T] \longrightarrow \mathbb{R}^n,\ u \in C([0,T] \setminus \Omega, \mathbb{R}^n), \\ &u \text{ is left continuous at } t = t_k,\ \text{the right-hand limit } u\left(t_k^+\right) \text{ exists}\} \end{aligned} \tag{1.2}$$

with the norm $\|u\|_{PC} := \sup_{t \in [0,T]} \|u(t)\|$, where $\|\cdot\|$ is the usual Euclidean norm and $\langle \cdot, \cdot \rangle$ will be the Euclidean inner product.

In a similar fashion to the above, define and denote the Banach space $PC^1([0,T], \mathbb{R}^n)$ by

$$\begin{aligned} PC^1([0,T];\mathbb{R}^n) := \{&u \in PC([0,T];\mathbb{R}^n) : u \in C^1([0,T] \setminus \Omega, \mathbb{R}^n), \\ &\text{the limits } u'(t_k^-), u'(t_k^+) \text{ exist with } u'(t_k^-) = u'(t_k)\} \end{aligned} \tag{1.3}$$

with the norm $\|u\|_{PC^1} := \sup_{t \in [0,T]} \{\|u(t)\|_{PC}, \|u'(t)\|_{PC}\}$.

The following fixed point theorem is our main tool to prove the existence of at least one solution to (1.1).

Schaefer's fixed point theorem [19]

Let X be a Banach space and let $A : X \to X$ be a completely continuous operator. Then, either

(i) the operator equation $x = \lambda A x$ has a solution for $\lambda = 1$, or

(ii) the set $S := \{x \in X,\ x = \lambda A x,\ \lambda \in]0,1[\}$ is unbounded.

2. Expression of Green's function

In this part, we present the expression of Green's functions for second order impulsive equations with antiperiodic conditions.

Lemma 2.1. *Assume $p \geq 0$ and $q > 0$ are two constants. Let $\alpha = (p + \sqrt{p^2 + 4q})/2$, $\beta = (p - \sqrt{p^2 + 4q})/2$. Then for any $h(t) \in PC([0,T], \mathbb{R}^n)$, $u(t)$ solves*

$$\begin{gathered} u'' - pu' - qu = h(t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,2,\dots,m, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \quad u'(t_k^+) = u'(t_k) + J_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T) \end{gathered} \tag{2.1}$$

if and only if $u(t)$ is the solution of integral equation

$$u(t) = \int_0^T G(t,s)h(s)ds + \sum_{i=1}^m H(t,t_i)I_i(u(t_i)) + \sum_{i=1}^m G(t,t_i)J_i(u(t_i)), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\alpha - \beta}\begin{cases} -\dfrac{e^{\alpha(T+t-s)}}{1+e^{\alpha T}} + \dfrac{e^{\beta(T+t-s)}}{1+e^{\beta T}}, & 0 \leq t \leq s \leq T, \\ \dfrac{e^{\alpha(t-s)}}{1+e^{\alpha T}} - \dfrac{e^{\beta(t-s)}}{1+e^{\beta T}}, & 0 \leq s < t \leq T, \end{cases} \tag{G}$$

$$H(t,s) = \frac{1}{\alpha - \beta}\begin{cases} \dfrac{\beta e^{\alpha(T+t-s)}}{1+e^{\alpha T}} - \dfrac{\alpha e^{\beta(T+t-s)}}{1+e^{\beta T}}, & 0 \leq t \leq s \leq T, \\ -\dfrac{\beta e^{\alpha(t-s)}}{1+e^{\alpha T}} + \dfrac{\alpha e^{\beta(t-s)}}{1+e^{\beta T}}, & 0 \leq s < t \leq T. \end{cases} \tag{H}$$

Proof. Assume $u(t)$ is a solution of (2.1) and let $v(t) = u'(t) - \beta u(t)$ for $t \neq t_k$, $k = 1,2,\dots,m$. We have

$$v'(t) - \alpha v(t) = u''(t) - (\alpha+\beta)u'(t) + \alpha\beta u(t) = u''(t) - pu'(t) - qu(t) = h(t). \tag{2.3}$$

Then for $t \in [0, t_1]$,

$$v(t) = e^{\alpha t}\left[e^0 v(0) + \int_0^t e^{-\alpha s}h(s)ds\right] = e^{\alpha t}\left[v(0) + \int_0^t e^{-\alpha s}h(s)ds\right]. \tag{2.4}$$

This implies $v(t_1) = e^{\alpha t_1}[v(0) + \int_0^{t_1} e^{-\alpha s} h(s)ds]$. Consequently, from the impulsive condition in (2.1) we get that

$$v(t_1^+) = v(t_1) + J_1(u(t_1)) - \beta I_1(u(t_1)) = e^{\alpha t_1}\left(v(0) + \int_0^{t_1} e^{-\alpha s} h(s)ds\right) + \Delta_1, \tag{2.5}$$

where $\Delta_i := J_i(u(t_i)) - \beta I_i(u(t_i))$, $i = 1, 2, \ldots, m$. Now we integrate (2.3) from t_1 to $t \in (t_1, t_2]$ and use (2.5) to obtain

$$v(t) = e^{\alpha t}\left[e^{-\alpha t_1} v(t_1^+) + \int_{t_1}^{t} e^{-\alpha s} h(s)ds\right] = e^{\alpha t} v(0) + e^{\alpha(t-t_1)}\Delta_1 + \int_0^t e^{\alpha(t-s)} h(s)ds. \tag{2.6}$$

It follows that

$$v(t_2^+) = v(t_2) + \Delta_2 = e^{\alpha t_2} v(0) + e^{\alpha(t_2-t_1)}\Delta_1 + \int_0^{t_2} e^{\alpha(t_2-s)} h(s)ds + \Delta_2. \tag{2.7}$$

Similarly, we have for $t \in (t_2, t_3]$ that

$$v(t) = e^{\alpha t} v(0) + e^{\alpha(t-t_1)}\Delta_1 + e^{\alpha(t-t_2)}\Delta_2 + \int_0^t e^{\alpha(t-s)} h(s)ds. \tag{2.8}$$

To sum up, we have for $t \in [0, T]$ that

$$v(t) = e^{\alpha t} v(0) + \sum_{t_i \in [0,t)} e^{\alpha(t-t_i)}\Delta_i + \int_0^t e^{\alpha(t-s)} h(s)ds. \tag{2.9}$$

Since $v(t) = u'(t) - \beta u(t)$, we can deduce in a similar way as to deal with $h(t) = v'(t) - \alpha v(t)$ to obtain

$$u(t) = e^{\beta t} u(0) + \sum_{t_i \in [0,t)} e^{\beta(t-t_i)} I_i(u(t_i)) + \int_0^t e^{\beta(t-s)} v(s)ds. \tag{2.10}$$

Now we are in position to show the expression of $u(t)$ for $t \in [0, T]$. To do that, we need to compute $\int_0^t e^{\beta(t-s)} v(s)ds$ in (2.10). In what follows we present the expression of $u(t)$ for $t \in (t_1, t_2], (t_2, t_3]$ step by step and then obtain the general form of $u(t)$ for $t \in [0, T]$.

First of all, for $t \in (t_1, t_2]$, we have

$$\int_0^t e^{\beta(t-s)} v(s)ds = \int_0^{t_1} e^{\beta(t-s)} v(s)ds + \int_{t_1}^t e^{\beta(t-s)} v(s)ds. \tag{2.11}$$

See that

$$\begin{aligned}
\int_0^{t_1} e^{\beta(t-s)} v(s)ds &= \int_0^{t_1} e^{\beta(t-s)} \left[e^{\alpha s} v(0) + \int_0^s e^{\alpha(s-\tau)} h(\tau)d\tau \right] ds, \\
\int_{t_1}^t e^{\beta(t-s)} v(s)ds &= \int_{t_1}^t e^{\beta(t-s)} \left[e^{\alpha s} v(0) + \Delta_1 e^{\alpha(s-t_1)} + \int_0^s e^{\alpha(s-\tau)} h(\tau)d\tau \right] ds.
\end{aligned} \tag{2.12}$$

Consequently,

$$\int_0^t e^{\beta(t-s)} v(s)ds = \int_0^t e^{\beta(t-s)} \left[e^{\alpha s} v(0) + \int_0^s e^{\alpha(s-\tau)} h(\tau)d\tau \right] ds + \Delta_1 \int_{t_1}^t e^{\beta(t-s)} e^{\alpha(s-t_1)} ds. \tag{2.13}$$

Integrate $\int_0^t e^{\beta(t-s)} [\int_0^s e^{\alpha(s-\tau)} h(\tau)d\tau] ds$ by parts to get

$$\frac{e^{\beta t}}{\alpha - \beta} \int_0^t h(s) e^{-\alpha s} \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)s} \right] ds. \tag{2.14}$$

Thus,

$$\begin{aligned}
\int_0^t e^{\beta(t-s)} v(s)ds = \frac{e^{\beta t}}{\alpha - \beta} \Bigg\{ & v(0) \left[e^{(\alpha-\beta)t} - 1 \right] + e^{-\alpha t_1} \Delta_1 \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)t_1} \right] \\
& + \int_0^t h(s) e^{-\alpha s} \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)s} \right] ds \Bigg\}.
\end{aligned} \tag{2.15}$$

Similarly, we have for $t \in (t_2, t_3]$ that

$$\begin{aligned}
\int_0^t e^{\beta(t-s)} v(s)ds = \frac{e^{\beta t}}{\alpha - \beta} \Bigg\{ & v(0) \left[e^{(\alpha-\beta)t} - 1 \right] + e^{-\alpha t_1} \Delta_1 \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)t_1} \right] \\
& + e^{-\alpha t_2} \Delta_2 \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)t_2} \right] + \int_0^t h(s) e^{-\alpha s} \left[e^{(\alpha-\beta)t} - e^{(\alpha-\beta)s} \right] ds \Bigg\}.
\end{aligned} \tag{2.16}$$

Now we consider $u(t)$ for $t \in (t_1, t_2]$. Clearly,

$$u(t) = e^{\beta t}u(0) + e^{\beta(t-t_1)} I_1(u(t_1)) + \frac{e^{\beta t}}{\alpha-\beta}\left\{ v(0)\left[e^{(\alpha-\beta)t}-1\right] + e^{-\alpha t_1}\Delta_1\left[e^{(\alpha-\beta)t}-e^{(\alpha-\beta)t_1}\right] + \int_0^t h(s)e^{-\alpha s}\left[e^{(\alpha-\beta)t}-e^{(\alpha-\beta)s}\right]ds \right\}. \tag{2.17}$$

Noting that $v(0) = u'(0) - \beta u(0)$, we have

$$u(t) = \frac{1}{\alpha-\beta}\left\{ (u'(0)-\beta u(0))e^{\alpha t} + (\alpha u(0) - u'(0))e^{\beta t} + \int_0^t h(s)\left[e^{\alpha(t-s)} - e^{\beta(t-s)}\right] + e^{\alpha(t-t_1)}\Delta_1 - e^{\beta(t-t_1)}\widetilde{\Delta}_1 \right\}, \tag{2.18}$$

where $\widetilde{\Delta}_i$ is denoted by $\widetilde{\Delta}_i = J_1(u(t_1)) - \alpha I_1(u(t_1)),\ i = 1, 2, \ldots, m$. Similarly, for $t \in (t_2, t_3]$ there holds

$$u(t) = \frac{1}{\alpha-\beta}\left\{ (u'(0)-\beta u(0))e^{\alpha t} + (\alpha u(0) - u'(0))e^{\beta t} + \int_0^t h(s)\left[e^{\alpha(t-s)} - e^{\beta(t-s)}\right] + e^{\alpha(t-t_1)}\Delta_1 + e^{\alpha(t-t_2)}\Delta_2 - e^{\beta(t-t_1)}\widetilde{\Delta}_1 - e^{\beta(t-t_2)}\widetilde{\Delta}_2 \right\}. \tag{2.19}$$

Thus, for $t \in [0, T]$,

$$u(t) = \frac{1}{\alpha-\beta}\left\{ (u'(0)-\beta u(0))e^{\alpha t} + (\alpha u(0) - u'(0))e^{\beta t} + \int_0^t h(s)\left[e^{\alpha(t-s)} - e^{\beta(t-s)}\right] + \sum_{t_i \in [0,t)} e^{\alpha(t-t_i)}\Delta_i - \sum_{t_i \in [0,t)} e^{\beta(t-t_i)}\widetilde{\Delta}_i \right\}. \tag{2.20}$$

By the boundary condition of (2.1), we have

$$\begin{aligned} u'(0) - \beta u(0) &= -\frac{1}{1+e^{\alpha T}}\left[\int_0^T e^{\alpha(T-s)}h(s)ds + \sum_{i=1}^m \Delta_i e^{\alpha(T-t_i)}\right], \\ \alpha u(0) - u'(0) &= \frac{1}{1+e^{\beta T}}\left[\int_0^T e^{\beta(T-s)}h(s)ds + \sum_{i=1}^m \widetilde{\Delta}_i e^{\beta(T-t_i)}\right]. \end{aligned} \tag{2.21}$$

Substituting (2.21) into (2.20), and also noting that for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} -\sum_{i=1}^{m} \frac{\Delta_i}{1+e^{\alpha T}} e^{\alpha(T+t-t_i)} + \sum_{t_i \in [0,t]} e^{\alpha(t-t_i)} \Delta_i &= -\sum_{i=k+1}^{m} \frac{e^{\alpha(T+t-t_i)} \Delta_i}{1+e^{\alpha T}} + \sum_{i=1}^{k} \frac{e^{\alpha(t-t_i)} \Delta_i}{1+e^{\alpha T}}, \\ \sum_{i=1}^{m} \frac{\widetilde{\Delta}_i}{1+e^{\beta T}} e^{\beta(T+t-t_i)} - \sum_{t_i \in [0,t]} e^{\beta(t-t_i)} \widetilde{\Delta}_i &= \sum_{i=k+1}^{m} \frac{e^{\alpha(T+t-t_i)} \widetilde{\Delta}_i}{1+e^{\alpha T}} - \sum_{i=1}^{k} \frac{e^{\beta(t-t_i)} \widetilde{\Delta}_i}{1+e^{\beta T}}, \end{aligned} \tag{2.22}$$

we see that $u(t)$ is the solution of (2.2).

Now assume $u(t)$ is a solution of (2.2). Then for $t \neq t_k,\ k = 1, 2, \ldots, m$.

$$u'(t) = \int_0^T G_t(t,s) h(s) ds + \sum_{i=1}^{m} H_t(t,t_i) I_i(u(t_i)) + \sum_{i=1}^{m} G_t(t,t_i) J_i(u(t_i)), \tag{2.23}$$

$$u''(t) = \int_0^T G_{tt}(t,s) h(s) ds + h(t) + \sum_{i=1}^{m} H_{tt}(t,t_i) I_i(u(t_i)) + \sum_{i=1}^{m} G_{tt}(t,t_i) J_i(u(t_i)). \tag{2.24}$$

It is easy to verify

$$u''(t) - pu'(t) - q(t) = h(t). \tag{2.25}$$

For $t = t_k,\ k = 1, 2, \ldots, m$, we compute straightforwardly to get

$$\begin{aligned} H(t_k^+, t_k) - H(t_k, t_k) &= 1, \qquad G(t_k^+, t_k) - G(t_k, t_k) = 0, \\ G_t(t_k^+, t_k) - G_t(t_k, t_k) &= 1, \qquad H_t(t_k^+, t_k) - H_t(t_k, t_k) = 0, \end{aligned} \tag{2.26}$$

which implies

$$\Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)). \tag{2.27}$$

Now, we prove $u(t)$ is a solution of (2.1). Then the proof is completed. □

For later use, we present the following estimations:

$$\max_{(t,s) \in [0,T] \times [0,T]} |G(t,s)| \leq \frac{1}{\alpha - \beta} \left(\frac{e^{\alpha T}}{1+e^{\alpha T}} + \frac{e^{\beta T}}{1+e^{\beta T}} \right) := G_0,$$

$$\max_{(t,s) \in [0,T] \times [0,T]} |H(t,s)| \leq \frac{\alpha}{\alpha - \beta} \left(\frac{e^{\alpha T}}{1+e^{\alpha T}} + \frac{e^{\beta T}}{1+e^{\beta T}} \right) = \alpha G_0,$$

$$\max_{(t,s)\in[0,T]\times[0,T]}|G(t,s)| \leq \frac{\alpha}{\alpha-\beta}\left(\frac{e^{\alpha T}}{1+e^{\alpha T}}+\frac{e^{\beta T}}{1+e^{\beta T}}\right)=\alpha G_0,$$
$$\max_{(t,s)\in[0,T]\times[0,T]}|H_t(t,s)| \leq \frac{|\alpha|\cdot|\beta|}{\alpha-\beta}\left(\frac{e^{\alpha T}}{1+e^{\alpha T}}+\frac{e^{\beta T}}{1+e^{\beta T}}\right)\leq \alpha^2 G_0. \tag{2.28}$$

Corollary 2.2. *Assume in* (2.1) *that* $p=0$ *and* $q=M^2>0$. *Then for any* $h(t)\in PC[0,T]$, $u(t)$ *is the solution of*

$$\begin{gathered} u''-M^2u=h(t), \quad t\in[0,T],\ t\neq t_k,\ k=1,2,\ldots,m, \\ u(t_k^+)=u(t_k)+I_k(u(t_k)), \quad u'(t_k^+)=u'(t_k)+J_k(u(t_k)), \quad k=1,2,\ldots,m, \\ u(0)=-u(T), \qquad u'(0)=-u'(T) \end{gathered} \tag{2.29}$$

if and only if $u(t)$ *is the solution of integral equation*

$$u(t)=\int_0^T G(t,s)h(s)ds+\sum_{i=1}^m H(t,t_i)I_i(u(t_i))+\sum_{i=1}^m G(t,t_i)J_i(u(t_i)), \tag{2.30}$$

where

$$G(t,s)=\frac{1}{2M}\begin{cases} -\dfrac{e^{M(T+t-s)}}{1+e^{MT}}+\dfrac{e^{-M(T+t-s)}}{1+e^{-MT}}, & 0\leq t\leq s\leq T, \\ \dfrac{e^{M(t-s)}}{1+e^{MT}}-\dfrac{e^{-M(t-s)}}{1+e^{-MT}}, & 0\leq s<t\leq T, \end{cases}$$
$$H(t,s)=G_t(t,s)=\frac{1}{2M}\begin{cases} \dfrac{-Me^{M(T+t-s)}}{1+e^{MT}}-\dfrac{Me^{-M(T+t-s)}}{1+e^{-MT}}, & 0\leq t\leq s\leq T, \\ \dfrac{Me^{M(t-s)}}{1+e^{MT}}+\dfrac{Me^{-M(t-s)}}{1+e^{-MT}}, & 0\leq s<t\leq T. \end{cases} \tag{2.31}$$

Obviously, there hold

$$\begin{gathered} \max_{(t,s)\in[0,T]\times[0,T]}|G(t,s)|\leq\frac{e^{MT}}{M(1+e^{MT})}, \qquad \max_{(t,s)\in[0,T]\times[0,T]}|H(t,s)|\leq\frac{e^{MT}}{1+e^{MT}}, \\ \max_{(t,s)\in[0,T]\times[0,T]}|G_t(t,s)|\leq\frac{e^{MT}}{1+e^{MT}}, \qquad \max_{(t,s)\in[0,T]\times[0,T]}|H_t(t,s)|\leq\frac{Me^{MT}}{1+e^{MT}}. \end{gathered} \tag{2.32}$$

We now give Green's function of (2.1) for $p=q=0$.

Lemma 2.3. *For any $h(t) \in PC[0,T]$, $u(t)$ is the solution of*

$$\begin{gathered} u'' = h(t), \quad t \in [0,T],\ t \neq t_k,\ k = 1,2,\dots,m, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \qquad u'(t_k^+) = u'(t_k) + J_k(u(t_k)), \\ u(0) = -u(T), \qquad u'(0) = -u'(T) \end{gathered} \tag{2.33}$$

if and only if $u(t)$ satisfies the integral equation

$$u(t) = \int_0^T G(t,s)^* h(s)ds + \sum_{i=1}^m H^*(t,t_i) I_i(u(t_i)) + \sum_{i=1}^m G^*(t,t_i) J_i(u(t_i)), \tag{2.34}$$

where

$$\begin{gathered} G^*(t,s) = -\frac{1}{2}\begin{cases} \dfrac{T}{2} + t - s, & 0 \le t \le s \le T, \\ \dfrac{T}{2} - t + s, & 0 \le s < t \le T, \end{cases} \\ H^*(t,s) = G^*(t,s)_t = \begin{cases} -\dfrac{1}{2}, & 0 \le t \le s \le T, \\ \dfrac{1}{2}, & 0 \le s < t \le T. \end{cases} \end{gathered} \tag{2.35}$$

Since the proof is very similar to that of Lemma 2.1, we omit it here. We can check easily that $u(t)$ satisfies (2.34) and hence $u(t)$ is a solution of (2.33). Also we get by straightforward computation that

$$\max_{(t,s)\in[0,T]\times[0,T]} |G^*(t,s)| \le \frac{T}{4}, \qquad \max_{(t,s)\in[0,T]\times[0,T]} |G_t^*(t,s)| = \max_{(t,s)\in[0,T]\times[0,T]} |H^*(t,s)| \le \frac{1}{2}. \tag{2.36}$$

Recall that a mapping between Banach spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

Lemma 2.4. *Suppose that $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $I, J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Define an operator $A : PC^1([0,T],\mathbb{R}^n) \to PC^1([0,T],\mathbb{R}^n)$ as*

$$\begin{aligned} Au(t) := &\int_0^T G(t,s)\big(f(s,u(s),u'(s)) - pu'(s) - qu(s)\big)ds \\ &+ \sum_{i=1}^m H(t,t_i) I_i(u(t_i)) + \sum_{i=1}^m G(t,t_i) J_i(u(t_i)), \end{aligned} \tag{2.37}$$

where $G(t,s)$ and $H(t,s)$ are as given in Lemma 2.1. Then A is a compact map.

Proof. Noting the continuity of f and I_k, J_k, this follows in a standard step-by-step process and so it is omitted. □

3. Main results

In this section, we prove the existence results for (1.1) in presence of Schaefer's fixed-point theorem.

Theorem 3.1. *Suppose that $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $I, J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. If for some $p \geq 0$ and $q > 0$, there exist nonnegative constants γ, δ_k, ζ_k, L_k, N_k, and M such that*

$$\begin{gathered}\|f(t,x,y) - py - qx\| \leq \gamma\left[\langle x+y, f(t,x,y)\rangle + \|y\|^2\right] + M, \\ \forall (t,x,y) \in \left([0,T] \setminus \bigcup_{i=0}^{m} t_i\right) \times \mathbb{R}^n \times \mathbb{R}^n,\end{gathered} \tag{3.1}$$

$$\left\|I_k(x)\right\| \leq \delta_k\|x\| + L_k, \quad \left\|J_k(x)\right\| \leq \zeta_k\|x\| + N_k, \quad \forall x \in \mathbb{R}^n, \tag{3.2}$$

$$\sum_{k=1}^{m}\delta_k + \sum_{k=1}^{m}\zeta_k < \frac{1}{H}, \tag{3.3}$$

where $\langle\cdot\rangle$ is the Euclidean inner product, $H = \max\{G_0, \alpha G_0, \alpha^2 G_0\}$. Then (1.1) *has at least one solution.*

Proof. Define an integral operator A as

$$Au = \int_0^T G(t,s)\left(f(s,u(s),u'(s)) - pu'(s) - qu(s)\right)ds + \sum_{i=1}^{m} H(t,t_i)I_i(u(t_i)) + \sum_{i=1}^{m} G(t,t_i)J_i(u(t_i)), \tag{3.4}$$

where $G(t,s)$ and $H(t,s)$ follow the forms of (G) and (H) in Lemma 2.1. By Lemma 2.4, A is a compact mapping. Also, it follows from Lemma 2.1 that $u(t)$ is a fixed point of A if and only if $u(t)$ satisfies

$$\begin{gathered}u''(t) - pu'(t) - qu(t) = f(t,u(t),u'(t)) - pu'(t) - qu(t), \quad t \neq t_k, \ k = 1,2,\ldots,m, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \quad u'(t_k^+) = u'(t_k) + J_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T),\end{gathered} \tag{3.5}$$

which is equivalent to (1.1). Consequently, all that we need to do is to verify that A has at least one fixed point. With this in mind, we assume $u(t)$ is a solution of

$$u = \lambda A u, \quad \lambda \in (0,1). \tag{3.6}$$

That is,

$$\begin{aligned} u(t) = {} & \int_0^T G(t,s)\lambda\big[f(s,u(s),u'(s)) - pu'(s) - qu(s)\big]ds \\ & + \lambda\sum_{i=1}^{m} H(t,t_i)I_i(u(t_i)) + \lambda\sum_{i=1}^{m} G(t,t_i)J_i(u(t_i)). \end{aligned} \tag{3.7}$$

It is equivalent to say that $u(t)$ satisfies

$$\begin{gathered} u''(t) - pu'(t) - qu(t) = \lambda\big[f(t,u(t),u'(t)) - pu'(t) - qu(t)\big], \quad t \neq t_k, \ k = 1,2,\dots,m, \\ u(t_k^+) = u(t_k) + \lambda I_k(u(t_k)), \quad u'(t_k^+) = u'(t_k) + \lambda J_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T). \end{gathered} \tag{3.8}$$

Firstly, we see that for $\lambda \in (0,1)$,

$$\begin{aligned} & \lambda\|f(t,u(t),u'(t)) - pu'(t) - qu(t)\| \\ & \quad \le \lambda\{\gamma\big[\langle u(t)+u'(t), f(t,u(t),u'(t))\rangle + \|u'(t)\|^2\big] + M\} \\ & \quad = \gamma\big[\langle u(t)+u'(t), \lambda f(t,u(t),u'(t))\rangle + \lambda\|u'(t)\|^2\big] + \lambda M \\ & \quad = \gamma\big[\langle u(t)+u'(t), u''(t) - (1-\lambda)(pu'(t)+qu(t))\rangle + \lambda\|u'(t)\|^2\big] + \lambda M \\ & \quad = \gamma\big[\langle u(t)+u'(t), u''(t)\rangle - (1-\lambda)(p+q)\langle u(t), u'(t)\rangle \\ & \qquad\quad - q(1-\lambda)\|u(t)\|^2 - p(1-\lambda)\|u'(t)\|^2 + \lambda\|u'(t)\|^2\big] + \lambda M \\ & \quad \le \gamma\big[\langle u(t)+u'(t), u''(t)\rangle - (1-\lambda)(p+q)\langle u(t), u'(t)\rangle + \|u'(t)\|^2\big] + M \\ & \quad = \gamma\big[\langle u(t)+u'(t), u'(t)+u''(t)\rangle - (1-\lambda)(p+q)\langle u(t), u'(t)\rangle - \langle u(t), u'(t)\rangle\big] + M. \end{aligned} \tag{3.9}$$

Further more, by the antiperiodic boundary condition we have

$$\begin{gathered} \int_0^T \langle u(t), u'(t)\rangle dt = \frac{1}{2}\int_0^T \frac{d}{dt}(\|u(t)\|^2) = \frac{1}{2}(\|u(T)\|^2 - \|u(0)\|^2) = 0, \\ \int_0^T \langle u(t)+u'(t), u'(t)+u''(t)\rangle dt = \frac{1}{2}(\|u(T)+u'(T)\|^2 - \|u(0)+u'(0)\|^2) = 0. \end{gathered} \tag{3.10}$$

As a result,

$$\int_0^T \lambda \left\| f(t,u(t),u'(t)) - pu'(t) - qu(t)) \right\| dt \le MT. \tag{3.11}$$

Now we show that any potential solution of (3.6) is bounded a priori. By (3.2) and (3.11), we obtain

$$\begin{aligned}
\|u(t)\| &= \lambda \|Au(t)\| \\
&= \Bigg\| \int_0^T G(t,s)\lambda\left[f(t,u(t),u'(t)) - pu'(t) - qu(t)\right] dt \\
&\quad + \lambda \sum_{i=1}^m H(t,t_i) I_i(u(t_i)) + \lambda \sum_{i=1}^m G(t,t_i) J_i(u(t_i)) \Bigg\| \\
&\le G_0 \int_0^T \lambda \left\| f(t,u(t),u'(t)) - pu'(t) - qu(t) \right\| dt + \lambda\alpha G_0 \sum_{i=1}^m \|I_i(u(t_i))\| + \lambda G_0 \sum_{i=1}^m \|J_i(u(t_i))\| \\
&\le G_0\left(MT + \alpha \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right) + G_0\left(\alpha \sum_{i=1}^m \zeta_i \|(u(t_i))\| + \sum_{i=1}^m \delta_i \|(u(t_i))\| \right) \\
&\le G_1\left(MT + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right) + G_1\left(\sum_{i=1}^m \zeta_i \|(u(t_i))\| + \sum_{i=1}^m \delta_i \|(u(t_i))\| \right).
\end{aligned} \tag{3.12}$$

Taking the supremum and rearranging, we get by (3.3) that

$$\sup_{t\in[0,T]} \|u(t)\| \le \frac{G_1\left(TM + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k\right)}{1 - G_1\left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k\right)}. \tag{3.13}$$

Differentiating both sides of (3.7) and noting (2.23), we obtain

$$\sup_{t\in[0,T]} \|u'(t)\| \le \frac{G_2\left(TM + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k\right)}{1 - G_2\left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k\right)}, \tag{3.14}$$

where

$$G_2 = \max\left\{\alpha G_0, \alpha^2 G_0\right\}. \tag{3.15}$$

Thus,

$$\|u(t)\|_{PC^1} = \max\left\{ \frac{G_1\left(TM + \sum_{k=1}^{m} L_k + \sum_{k=1}^{m} N_k\right)}{1 - G_1\left(\sum_{k=1}^{m}\delta_k + \sum_{k=1}^{m}\zeta_k\right)}, \frac{G_2\left(TM + \sum_{k=1}^{m} L_k + \sum_{k=1}^{m} N_k\right)}{1 - G_2\left(\sum_{k=1}^{m}\delta_k + \sum_{k=1}^{m}\zeta_k\right)} \right\} := R. \tag{3.16}$$

Now we have shown that any possible solution of (3.6) is bounded by R which is independent of λ. By Scheafer's fixed theorem we know that A has at least one fixed point. Therefore, the proof is completed. □

Suppose both $p = 0$ and $q = M^2$ in Theorem 3.1. We obtain the following theorem.

Theorem 3.2. *Assume that $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $I, J : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. If for some $M > 0$ there exist nonnegative constants γ, δ_k, ζ_k, L_k, N_k, and M^* such that*

$$\begin{gathered} \left\| f(t,x,y) - M^2 x \right\| \le \gamma\left[\langle x, f(t,x,y)\rangle + \|y\|^2\right] + M^*, \\ \forall (t,x,y) \in \left([0,T] \setminus \bigcup_{i=0}^{m} t_i \right) \times \mathbb{R}^n \times \mathbb{R}^n, \\ \left\| I_k(x) \right\| \le \delta_k \|x\| + L_k, \quad \left\| J_k(x) \right\| \le \zeta_k \|x\| + N_k, \quad \forall x \in \mathbb{R}^n, \\ \sum_{k=1}^{m} \delta_k + \sum_{k=1}^{m} \zeta_k < \frac{1}{\check{H}}, \end{gathered} \tag{3.17}$$

where $\langle\cdot\rangle$ is the Euclidean inner product, $\check{H} = \max\{e^{MT}/M(1+e^{MT}),\ e^{MT}/(1+e^{MT}),\ Me^{MT}/M(1+e^{MT})\}$, then (1.1) has at least one solution.

Proof. Consider the mapping

$$A : PC^1\left([0,T], \mathbb{R}^n\right) \longrightarrow PC^1\left([0,T], \mathbb{R}^n\right), \tag{3.18}$$

$$Au(t) = \int_0^T G(t,s)h(s)ds + \sum_{i=1}^{m} H\left(t,t_i\right) I_i\left(u\left(t_i\right)\right) + \sum_{i=1}^{m} G\left(t,t_i\right) J_i\left(u\left(t_i\right)\right), \tag{3.19}$$

where

$$G(t,s) = \frac{1}{2M} \begin{cases} -\dfrac{e^{M(T+t-s)}}{1+e^{MT}} + \dfrac{e^{-M(T+t-s)}}{1+e^{-MT}}, & 0 \le t \le s \le T, \\[2ex] \dfrac{e^{M(t-s)}}{1+e^{MT}} - \dfrac{e^{-M(t-s)}}{1+e^{-MT}}, & 0 \le s < t \le T, \end{cases} \tag{3.20}$$

$$H(t,s) = G_t(t,s) = \frac{1}{2M} \begin{cases} \dfrac{-Me^{M(T+t-s)}}{1+e^{MT}} - \dfrac{Me^{-M(T+t-s)}}{1+e^{-MT}}, & 0 \le t \le s \le T, \\[2ex] \dfrac{Me^{M(t-s)}}{1+e^{MT}} + \dfrac{Me^{-M(t-s)}}{1+e^{-MT}}, & 0 \le s < t \le T. \end{cases} \tag{3.21}$$

By Lemma 2.4, A is a compact mapping. Consider the equation

$$u = Au. \tag{3.22}$$

To show that A has at least one fixed point, we apply Schaefer's theorem by showing that all potential solutions to

$$u = \lambda Au, \quad \lambda \in (0,1), \tag{3.23}$$

are bounded a priori, with the bound being independent of λ. With this in mind, let $u(t)$ be a solution of (3.23). Note that $u(t)$ is also a solution to

$$\begin{gathered} u''(t) - M^2u(t) = \lambda\left[f(t,u(t),u'(t)) - M^2u(t)\right], \quad t \neq t_k,\ k = 1,2,\dots,m, \\ u(t_k^+) = u(t_k) + \lambda I_k(u(t_k)), \quad u'(t_k^+) = u'(t_k) + \lambda J_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T). \end{gathered} \tag{3.24}$$

On one hand, we see that for $\lambda \in (0,1)$,

$$\begin{aligned} \lambda\|f(t,u(t),u'(t))\| &\leq \lambda\left\{\gamma\left[\langle u(t), f(t,u(t),u'(t))\rangle + \|u'(t)\|^2\right] + M^*\right\} \\ &= \gamma\left[\langle u(t), \lambda f(t,u(t),u'(t))\rangle + \lambda\|u'(t)\|^2\right] + \lambda M \\ &= \gamma\left[\langle u(t), u''(t) - (1-\lambda)M^2u(t))\rangle + \lambda\langle u'(t), u'(t)\rangle\right] + \lambda M^* \\ &= \gamma\left[\langle u(t), u''(t)\rangle - (1-\lambda)M^2\|u(t)\|^2 + \lambda\langle u'(t), u'(t)\rangle\right] + \lambda M^* \\ &\leq \gamma\left[\langle u(t), u''\rangle + \langle u'(t), u'(t)\rangle\right] + M^* \\ &= \gamma\frac{d}{dt}\langle u(t), u'(t)\rangle + M^*. \end{aligned} \tag{3.25}$$

On the other hand, by the antiperiodic boundary condition we have

$$\int_0^T \left[\langle u(t), u''\rangle + \langle u'(t), u'(t)\rangle\right]dt = \langle u(T), u'(T)\rangle - \langle u(0), u'(0)\rangle = \langle -u(0), -u'(0)\rangle - \langle u(0), u'(0)\rangle = 0. \tag{3.26}$$

It therefore follows that

$$\int_0^T \lambda \left\| f(t, u(t), u'(t)) \right\| dt \le M^* T. \tag{3.27}$$

Consequently,

$$\begin{aligned}
\|u(t)\| &= \lambda \|Au(t)\| \\
&= \left\| \int_0^T G(t,s)\lambda\left[f(t,u(t),u'(t))\right]dt + \lambda \sum_{i=1}^m H(t,t_i) I_i(u(t_i)) + \lambda \sum_{i=1}^m (t,t_i) J_i(u(t_i)) \right\| \\
&\le \check{G}_0 \int_0^T \lambda \left\| f(t,u(t),u'(t)) - pu'(t) - qu(t) \right\| dt + \lambda M \check{G}_0 \sum_{i=1}^m \left\| I_i(u(t_i)) \right\| + \lambda \check{G}_0 \sum_{i=1}^m \left\| (u(t_i)) \right\| \\
&\le \check{G}_1 \left(M^* T + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right) + \check{G}_1 \left(\sum_{i=1}^m \zeta_i \left\| (u(t_i)) \right\| + \sum_{i=1}^m \delta_i \left\| (u(t_i)) \right\| \right),
\end{aligned} \tag{3.28}$$

where $\check{G}_0 = e^{MT}/M(1+e^{MT})$, $\check{G}_1 = \max\{e^{MT}/M(1+e^{MT}),\ e^{MT}/(1+e^{MT})\}$.

We compute directly to get

$$\sup_{t\in[0,T]} \|u(t)\| \le \frac{\check{G}_1 \left(TM^* + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right)}{1 - \check{G}_1 \left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k \right)}. \tag{3.29}$$

Differentiating both sides of (3.19), we obtain

$$\sup_{t\in[0,T]} \|u'(t)\| \le \frac{\check{G}_2 \left(T^* M + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right)}{1 - \check{G}_2 \left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k \right)}, \tag{3.30}$$

where

$$\check{G}_2 = \max\left\{ \frac{e^{MT}}{1+e^{MT}}, \frac{Me^{MT}}{1+e^{MT}} \right\}. \tag{3.31}$$

Thus,

$$\|u(t)\|_{PC^1} = \max\left\{ \frac{\check{G}_1 \left(TM^* + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right)}{1 - \check{G}_1 \left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k \right)}, \frac{\check{G}_2 \left(TM^* + \sum_{k=1}^m L_k + \sum_{k=1}^m N_k \right)}{1 - \check{G}_2 \left(\sum_{k=1}^m \delta_k + \sum_{k=1}^m \zeta_k \right)} \right\} := \check{R}. \tag{3.32}$$

Then the proof is completed. □

Similarly, we can prove the following existence result for $M = 0$ in Theorem 3.2.

Theorem 3.3. *Suppose that* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ *and* $I, J : \mathbb{R}^n \to \mathbb{R}^n$ *are continuous. If there exist nonnegative constants* $\gamma,\ \delta_k,\ \zeta_k,\ L_k,\ N_k$, *and* M *such that*

$$\begin{gathered}
\|f(t,x,y)\| \le \gamma\left[\langle x, f(t,x,y)\rangle + \|y\|^2\right] + M, \quad (t,x,y) \in \left([0,T] \setminus \bigcup_{i=0}^{m} t_i\right) \times \mathbb{R}^n \times \mathbb{R}^n, \\
\left\|I_k(x)\right\| \le \delta_k \|x\| + L_k, \quad \left\|J_k(x)\right\| \le \zeta_k \|x\| + N_k, \quad \forall x \in \mathbb{R}^n, \\
\sum_{k=1}^{m} \delta_k + \sum_{k=1}^{m} \zeta_k < \frac{1}{G_0^*},
\end{gathered} \tag{3.33}$$

where $\langle \cdot \rangle$ *is the Euclidean inner product,* $G_0^* = \max\{T/4, 1/2\}$, *then* (1.1) *has at least one solution.*

4. Examples

In this part, we show how our main theorems work by a couple of examples.

Example 4.1. The scalar second-order impulsive equations with antiperiodic boundary value condition

$$\begin{gathered}
u'' = (u(t) + u'(t))^3 + 2u(t) + u'(t) + t, \quad t \in [0,1],\ t \neq t_1, \\
u(t_1^+) = u(t_1) + \frac{1}{10} u(t_1) + 1, \qquad u'(t_1^+) = u'(t_1) - \frac{1}{10} u(t_1) + 2, \\
u(0) = -u(1), \qquad u'(0) = -u'(1),
\end{gathered} \tag{4.1}$$

where $t_1 \in (0,1)$, have at least one solution.

Proof. Let $T = 1$ and $f(t,x,y) = (x+y)^3 + 2x + y + t$ in Theorem 3.1. For $p = 1,\ q = 2$, we have $\alpha = 2,\ \beta = -1$, and

$$|f(t,x,y) - y - 2x| = |x+y|^3 + 1, \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}^2. \tag{4.2}$$

On the other hand, for $(t,x,y) \in [0,1] \times \mathbb{R}^2$,

$$\begin{aligned}
\langle (x+y), f(t,x,y) \rangle + y^2 &= (x+y)^4 + 2x^2 + 3xy + 2y^2 + (x+y)t \\
&\ge (x+y)^4 - |x+y| + 4|x| \cdot |y| + 3xy \\
&\ge (x+y)^4 - |x+y|.
\end{aligned} \tag{4.3}$$

Noting $\min_{v\geq 0}\{v^4 - v^3 - v\} > -2$, we have for $\gamma = 1$ and $M = 3$ that

$$\gamma\left[\langle (x+y), f(t,x,y)\rangle + y^2\right] + M \geq |f(t,x,y) - y - 2x|, \quad \forall (t,x,y) \in [0,1]\times\mathbb{R}^2. \tag{4.4}$$

Moreover, $H = \alpha^2 G_0 = (4/3)(e^2/(1+e^2) + e^{-1}/(1+e^{-1})) \approx 1.53298$, $\delta_1 + \zeta_1 = 0.2 < 1/H$. Then the conclusion follows from Theorem 3.1. □

Example 4.2. Consider antiperiodic value problem

$$\begin{gathered} u''(t) = u(t) + u(t)u'(t)^2 + \cos t, \quad t \in [0,1],\ t \neq t_1, \\ u(t_1^+) = u(t_1) + \frac{1}{4}u(t_1) + 4, \qquad u'(t_1^+) = u'(t_1) - \frac{1}{2}u(t_1), \\ u(0) = -u(1), \qquad u'(0) = -u'(1). \end{gathered} \tag{4.5}$$

We claim that (4.5) has at least one solution.

Proof. Let $T = 1$ and $f(t,x,y) = x + xy^2 + \cos t$ in Theorem 3.2. Choosing $M = 1$, we have for $(t,x,y) \in [0,1]\times\mathbb{R}^2$ that

$$\begin{gathered} |f(t,x,y) - x| = |x|y^2 + \cos t, \\ \langle x, f(t,x,y)\rangle + y^2 = x^2 + x^2y^2 + y^2 + x\cos t \geq x^2 + x^2y^2 + y^2 - |x|. \end{gathered} \tag{4.6}$$

Since $\min_{v\geq 0}\{v^2 - v\} > -1$, we have $x^2y^2 + y^2 - |x|y^2 = y^2(x^2 - |x| + 1) > 0$. Thus, for $\gamma = 1$ and $M^* = 2$,

$$\gamma\left[\langle x, f(t,x,y)\rangle + y^2\right] + M^* \geq |f(t,x,y) - x|, \quad \forall (t,x,y) \in [0,1]\times\mathbb{R}^2. \tag{4.7}$$

Moreover, $\check{H} = e/(1+e)$, $\delta_1 + \zeta_1 = 3/4 < 1/\check{H}$. Then the conclusion follows from Theorem 3.2. □

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