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Research Article

Interior Controllability of a 2×2 Reaction-Diffusion System with Cross-Diffusion Matrix

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We prove the interior approximate controllability for the following 2×2 reaction-diffusion system with cross-diffusion matrix $u_t = a\Delta u - \beta(-\Delta)^{1/2}u + b\Delta v + 1_\omega f_1(t,x)$ in $(0,\tau)\times\Omega$, $v_t = c\Delta u - d\Delta v - \beta(-\Delta)^{1/2}v + 1_\omega f_2(t,x)$ in $(0,\tau)\times\Omega$, $u = v = 0$, on $(0,T)\times\partial\Omega$, $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$, $x \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $u_0, v_0 \in L^2(\Omega)$, the 2×2 diffusion matrix $D = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has semisimple and positive eigenvalues $0 < \rho_1 \leq \rho_2$, β is an arbitrary constant, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, and the distributed controls $f_1, f_2 \in L^2([0,\tau]; L^2(\Omega))$. Specifically, we prove the following statement: if $\lambda_1^{1/2}\rho_1 + \beta > 0$ (where λ_1 is the first eigenvalue of $-\Delta$), then for all $\tau > 0$ and all open nonempty subset ω of Ω the system is approximately controllable on $[0,\tau]$.

1. Introduction

In this paper we prove the interior approximate controllability for the following 2×2 reaction-diffusion system with cross-diffusion matrix

$$\begin{gathered} u_t = a\Delta u - \beta(-\Delta)^{1/2}u + b\Delta v + 1_\omega f_1(t,x) \quad \text{in } (0,\tau)\times\Omega, \\ v_t = c\Delta u - d\Delta v - \beta(-\Delta)^{1/2}v + 1_\omega f_2(t,x) \quad \text{in } (0,\tau)\times\Omega, \\ u = v = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $u_0, v_0 \in L^2(\Omega)$, the 2×2 diffusion matrix

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.2}$$

has semisimple and positive eigenvalues, β is an arbitrary constant, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, and the distributed controls $f_1, f_2 \in L^2([0,\tau]; L^2(\Omega))$. Specifically, we prove the following statement: if $\lambda_1^{1/2}\rho_1 + \beta > 0$ (the first eigenvalue of $-\Delta$), then for all $\tau > 0$ and all open nonempty subset ω of Ω, the system is approximately controllable on $[0,\tau]$.

When $\Omega = (0,1)$ this system takes the following particular form:

$$\begin{gathered}
u_t = a\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + b\frac{\partial^2 v}{\partial x^2} + 1_\omega f_1(t,x) \quad \text{in } (0,\tau)\times(0,1), \\
v_t = c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial x} + 1_\omega f_2(t,x) \quad \text{in } (0,\tau)\times(0,1), \\
u(t,0) = v(t,0) = u(t,1) = v(t,1) = 0, \quad t \in (0,\tau), \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (0,1).
\end{gathered} \tag{1.3}$$

This paper has been motivated by the work done Badraoui in [1], where author studies the asymptotic behavior of the solutions for the system (1.3) on the unbounded domain $\Omega = \mathbb{R}$. That is to say, he studies the system:

$$\begin{gathered}
u_t = a\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + b\frac{\partial^2 v}{\partial x^2} + f(t,u,v), \quad x \in \mathbb{R},\ t > 0, \\
v_t = c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial x} + g(t,u,v), \quad x \in \mathbb{R},\ t > 0,
\end{gathered} \tag{1.4}$$

supplemented with the initial conditions:

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where the diffusion coefficients a and d are assumed positive constants, while the diffusion coefficients b, c and the coefficient β are arbitrary constants. He assume also the following three conditions:

(H1) $(a-d)^2 + 4bc > 0$, $cd \neq 0$ and $ad > bc$,

(H2) $u_0, v_0 \in X = C_{UB}(\mathbb{R})$, where C_{UB} is the space of bounded and uniformly continuous real-valued functions,

(H3) $f(t,u,v)$ and $g(t,u,v) \in X$, for all $t > 0$ and $u, v \in X$. Moreover, f and g are locally Lipshitz; namely, for all $t_1 \geq 0$ and all constant $k > 0$, there exist a constant $L = L(k,t_1) > 0$ such that

$$\left| f(t,w_1) - f(t,w_2) \right| \leq L|w_1 - w_2|, \tag{1.6}$$

is verified for all $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in \mathbb{R}\times\mathbb{R}$ with $|w_1| \leq k$, $|w_2| \leq k$ and $t \in [0,t_1]$.

We note that the hypothesis (H1) implies that the eigenvalues of the matrix D are simple and positive. But, this condition is not necessary for the eigenvalues of D to be positive, in fact we can find matrices D with a and d been negative and having positive eigenvalues. For example, one can consider the following matrix:

$$D = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}, \tag{1.7}$$

whose eigenvalues are $\rho_1 = 1$ and $\rho_2 = 2$.

The system (1.1) can be written in the following matrix form:

$$\begin{gathered} z_t = D\Delta z - \beta I_{2\times 2}(-\Delta)^{1/2} z + 1_\omega f(t,x), \quad \text{in } (0,\tau)\times\Omega, \\ z = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), \quad x\in\Omega, \end{gathered} \tag{1.8}$$

where $z = [u,v]^T \in \mathbb{R}^2$, the distributed controls $f = [f_1, f_2]^T \in L^2([0,\tau]; L^2(\Omega;\mathbb{R}^2))$, and $I_{2\times 2}$ is the identity matrix of dimension 2×2.

Our technique is simple and elegant from mathematical point of view, it rests on the shoulders of the following fundamental results.

Theorem 1.1. *The eigenfunctions of $-\Delta$ with Dirichlet boundary condition are real analytic functions.*

Theorem 1.2 (see [2, Theorem 1.23, page 20]). *Suppose $\Omega \subset \mathbb{R}^n$ is open, nonempty, and connected set, and f is real analytic function in Ω with $f = 0$ on a nonempty open subset ω of Ω. Then, $f = 0$ in Ω.*

Lemma 1.3 (see [3, Lemma 3.14, page 62]). *Let $\{\alpha_j\}_{j\geq 1}$ and $\{\beta_{i,j} : i = 1,2,\ldots,m\}_{j\geq 1}$ be two sequences of real numbers such that $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j} = 0, \quad \forall t \in [0,t_1],\ i = 1,2,\ldots,m \tag{1.9}$$

if and only if

$$\beta_{i,j} = 0, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,\infty. \tag{1.10}$$

Finally, with this technique those young mathematicians who live in remote and inhospitable places, far from major research centers in the world, can also understand and enjoy the interior controllability with a minor effort.

2. Abstract Formulation of the Problem

In this section we choose a Hilbert space where system (1.8) can be written as an abstract differential equation; to this end, we consider the following notations:

Let us consider the Hilbert space $H = L^2(\Omega, \mathbb{R})$ and $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then, we have the following well-known properties (see [3, pages 45-46]).

(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$, we have

$$-\Delta\xi = \sum_{j=1}^{\infty}\lambda_j \sum_{k=1}^{\gamma_j}\langle \xi, \phi_{j,k}\rangle \phi_{j,k} = \sum_{j=1}^{\infty}\lambda_j E_j \xi, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in H and

$$E_n\xi = \sum_{k=1}^{\gamma_j}\langle \xi, \phi_{j,k}\rangle \phi_{j,k}. \tag{2.2}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in H and $\xi = \sum_{j=1}^{\infty} E_j \xi$, $\xi \in H$.

(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi. \tag{2.3}$$

Now, we denote by Z the Hilbert space $H^2 = L^2(\Omega; \mathbb{R}^2)$ and define the following operator:

$$A : D(A) \subset Z \longrightarrow Z, \qquad A\psi = -D\Delta\psi + \beta I_{2\times 2}(-\Delta)^{1/2}\psi \tag{2.4}$$

with $D(A) = H^2(\Omega, \mathbb{R}^2) \cap H_0^1(\Omega, \mathbb{R}^2)$. Therefore, for all $z \in D(A)$, we obtain

$$Az = \sum_{j=1}^{\infty} \lambda_j^{1/2}\left(\lambda_j^{1/2} D + \beta I_{2\times 2}\right) P_j z, \tag{2.5}$$

$$z = \sum_{j=1}^{\infty} P_j z, \quad \|z\|^2 = \sum_{j=1}^{\infty} \left\|P_j z\right\|^2, \quad z \in Z, \tag{2.6}$$

where

$$P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix} \tag{2.7}$$

is a family of complete orthogonal projections in Z.

Consequently, system (1.8) can be written as an abstract differential equation in Z:

$$z' = -Az + B_\omega f, \quad z \in Z, \ t \geq 0, \tag{2.8}$$

where $f \in L^2([0,\tau];U)$, $U = Z$, and $B_\omega : U \to Z$, $B_\omega f = 1_\omega f$ is a bounded linear operator.

Now, we will use the following Lemma from [4] to prove the following theorem.

Lemma 2.1. *Let Z be a Hilbert separable space and $\{A_j\}_{j\geq 1}$, $\{P_j\}_{j\geq 1}$ two families of bounded linear operator in Z, with $\{P_j\}_{j\geq 1}$ a family of complete orthogonal projection such that*

$$A_j P_j = P_j A_j, \quad j \geq 1. \tag{2.9}$$

Define the following family of linear operators:

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \ t \geq 0. \tag{2.10}$$

Then the following hold.

(a) *$T(t)$ is a linear and bounded operator if $\|e^{A_j t}\| \leq g(t)$, $j = 1, 2, \ldots$, with $g(t) \geq 0$, continuous for $t \geq 0$.*

(b) *Under the above (a), $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup in the Hilbert space Z, whose infinitesimal generator A is given by*

$$Az = \sum_{j=1}^{\infty} A_j P_j z, \quad z \in D(A) \tag{2.11}$$

with

$$D(A) = \left\{ z \in Z : \sum_{j=1}^{\infty} \left\| A_j P_j z \right\|^2 < \infty \right\}. \tag{2.12}$$

(c) *The spectrum $\sigma(A)$ of A is given by*

$$\sigma(A) = \overline{\bigcup_{j=1}^{\infty} \sigma\left(\overline{A_j}\right)}, \tag{2.13}$$

where $\overline{A_j} = A_j P_j : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$.

Theorem 2.2. *The operator $-A$ define by (2.5) is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ given by:*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \ t \geq 0, \tag{2.14}$$

where $P_j = \operatorname{diag}[E_j, E_j]$ *and* $A_j = R_j P_j$ *with*

$$R_j = \begin{bmatrix} -a\lambda_j & -b\lambda_j \\ -c\lambda_j & -d\lambda_j \end{bmatrix} - \beta \begin{bmatrix} \lambda_j^{1/2} & 0 \\ 0 & \lambda_j^{1/2} \end{bmatrix}. \tag{2.15}$$

Moreover, if $\lambda_1^{1/2}\rho_1 + \beta > 0$*, then there exists* $M > 0$ *such that*

$$\|T(t)\| \leq M \exp\left\{-\lambda_1^{1/2}\left(\lambda_1^{1/2}\rho_1 + \beta\right)t\right\}, \quad t \geq 0. \tag{2.16}$$

Proof. In order to apply the foregoing Lemma, we observe that $-A$ can be written as follows:

$$-Az = \sum_{j=1}^{\infty} A_j P_j z, \quad z \in D(A) \tag{2.17}$$

with

$$A_j = -\lambda_j^{1/2}\left(\lambda_j^{1/2} D + \beta I_{2\times 2}\right)P_j, \qquad P_j = \operatorname{diag}[E_j, E_j]. \tag{2.18}$$

Therefore, $A_j = R_j P_j$ with

$$R_j = \begin{bmatrix} -a\lambda_j & -b\lambda_j \\ -c\lambda_j & -d\lambda_j \end{bmatrix} - \beta \begin{bmatrix} \lambda_j^{1/2} & 0 \\ 0 & \lambda_j^{1/2} \end{bmatrix}, \qquad A_j P_j = P_j A_j. \tag{2.19}$$

Clearly that A_j is a bounded linear operator (linear and continuous). That is, there exists $M_j > 0$ such that

$$\left\|A_j z\right\| \leq M_j \|z\|, \quad \forall z \in Z. \tag{2.20}$$

In fact, $\|A_j z\| = \|R_j P_j z\| \leq \|R_j\| \|P_j z\| \leq \|R_j\| \|z\|$.

Now, we have to verify condition (a) of Lemma 2.1. To this end, without loss of generality, we will suppose that $0 < \rho_1 < \rho_2$. Then, there exists a set $\{Q_1, Q_2\}$ of complementary projections on $\mathbb{R}^2$ such that

$$e^{Dt} = e^{\rho_1 t} Q_1 + e^{\rho_2 t} Q_2. \tag{2.21}$$

Hence,

$$e^{R_j t} = e^{\Gamma_{1j} t} Q_1 + e^{\Gamma_{2j} t} Q_2, \quad \text{with } \Gamma_{js} = -\lambda_j^{1/2}\left[\lambda_j^{1/2}\rho_s + \beta\right], \ s = 1, 2. \tag{2.22}$$

This implies the existence of positive numbers α, M such that

$$\left\| e^{A_j t} \right\| \leq M e^{\alpha t}, \quad j = 1, 2, \ldots. \tag{2.23}$$

Therefore, $-A$ generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ given by (2.14).

Finally, if $\lambda_1^{1/2} \rho_1 + \beta > 0$, then

$$-\lambda_1^{1/2} \left(\lambda_1^{1/2} \rho_1 + \beta \right) \geq -\lambda_j^{1/2} \left(\lambda_j^{1/2} \rho_i + \beta \right), \quad j = 1, 2, 3, \ldots; \; i = 1, 2, \tag{2.24}$$

and using (2.14) we obtain (2.16). □

3. Proof of the Main Theorem

In this section we will prove the main result of this paper on the controllability of the linear system (2.8). But, before we will give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z$ and $f \in L^2(0, \tau; U)$, the initial value problem

$$\begin{aligned} z' &= -Az + B_\omega f(t), \quad z \in Z, \\ z(0) &= z_0, \end{aligned} \tag{3.1}$$

where the control function f belonging to $L^2(0, \tau; U)$ admits only one mild solution given by

$$z(t) = T(t) z_0 + \int_0^t T(t-s) B_\omega f(s) ds, \quad t \in [0, \tau]. \tag{3.2}$$

Definition 3.1 (approximate controllability). The system (2.8) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$, $\varepsilon > 0$, there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (3.2) corresponding to u verifies:

$$\| z(\tau) - z_1 \| < \varepsilon. \tag{3.3}$$

The following result can be found in [5] for the general evolution equation:

$$z' = \mathcal{A} z + B f(t), \quad z \in Z, \; u \in U, \tag{3.4}$$

where Z, U are Hilbert spaces, $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ is the infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in $Z, B \in L(U, Z)$, the control function f belongs to $L^2(0, \tau; U)$.

Theorem 3.2. *System* (3.4) *is approximately controllable on* $[0, \tau]$ *if and only if*

$$B^* T^*(t) z = 0, \quad \forall t \in [0, \tau] \Longrightarrow z = 0. \tag{3.5}$$

Now, one is ready to formulate and prove the main theorem of this work.

Theorem 3.3 (main theorem). *If $\lambda_1^{1/2}\rho_1 + \beta > 0$, then for all $\tau > 0$ and all open nonempty subset ω of Ω the system,* (2.8) *is approximately controllable on* $[0,\tau]$.

Proof. We will apply Theorem 3.2 to prove the approximate controllability of system (2.8). With this purpose, we observe that

$$B_\omega^* = B_\omega, \qquad T^*(t)z = \sum_{j=1}^{\infty} e^{R_j^* t} P_j^* z, \quad z \in Z, \ t \geq 0. \tag{3.6}$$

On the other hand,

$$R_j = -\lambda_j^{1/2}\left\{\lambda_j^{1/2}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = -\lambda_j^{1/2}\left\{\lambda_j^{1/2} D + \beta I_{2\times 2}\right\}. \tag{3.7}$$

Without lose of generality, we will suppose that $0 < \rho_1 < \rho_2$. Then, there exists a set $\{Q_1, Q_2\}$ of complementary projections on $\mathbb{R}^2$ such that

$$e^{Dt} = e^{\rho_1 t} Q_1 + e^{\rho_2 t} Q_2. \tag{3.8}$$

Hence,

$$e^{R_j t} = e^{\Gamma_{j1} t} Q_1 + e^{\Gamma_{j2} t} Q_2, \quad \text{with } \Gamma_{js} = -\lambda_j^{1/2}\left[\lambda_j^{1/2}\rho_s + \beta\right], \ s = 1, 2. \tag{3.9}$$

Therefore,

$$B_\omega^* T^*(t)z = \sum_{j=1}^{\infty} B_\omega^* e^{R_j * t} P_j^* z = \sum_{j=1}^{\infty}\sum_{s=1}^{2} e^{\Gamma_{js} t} B_\omega^* P_{s,j}^* z, \tag{3.10}$$

where $P_{s,j} = Q_s P_j = P_j Q_s$.

Now, suppose for $z \in Z$ that $B_\omega^* T^*(t) z = 0$, for all $t \in [0,\tau]$. Then,

$$\begin{aligned} B_\omega^* T^*(t)z &= \sum_{j=1}^{\infty} B_\omega^* e^{R_j^* t} P_j^* z = \sum_{j=1}^{\infty}\sum_{s=1}^{2} e^{\Gamma_{js} t} B_\omega^* P_{s,j}^* z = 0 \\ &\Longleftrightarrow \sum_{j=1}^{\infty}\sum_{s=1}^{2} e^{\Gamma_{js} t}\left(B_\omega^* P_{s,j}^*\right) z(x) = 0, \qquad \forall x \in \Omega. \end{aligned} \tag{3.11}$$

Clearly that, $\{\Gamma_{js}\}$ is a decreasing sequence. Then, from Lemma 1.3, we obtain for all x $\in \Omega$ that

$$\left(B_\omega^* P_{s,j}^* z\right)(x) = Q_s^* \begin{bmatrix} \sum_{k=1}^{\gamma_j} \langle z_1, \phi_{j,k}\rangle 1_\omega \phi_{j,k}(x) \\ \sum_{k=1}^{\gamma_j} \langle z_2, \phi_{j,k}\rangle 1_\omega \phi_{j,k}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1, 2, 3, 4, \ldots; \ s = 1, 2. \tag{3.12}$$

Since $Q_1 + Q_2 = I_{\mathbb{R}^2}$, we get that

$$\begin{bmatrix} \sum_{k=1}^{\gamma_j} \langle z_1, \ \phi_{j,k} \rangle \phi_{j,k}(x) \\ \sum_{k=1}^{\gamma_j} \langle z_2, \ \phi_{j,k} \rangle \phi_{j,k}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1,2,3,4,\ldots; \ s = 1,2, \ \forall x \in \omega. \tag{3.13}$$

On the other hand, from Theorem 1.1 we know that $\phi_{n,k}$ are analytic functions, which implies the analyticity of $E_j z_i = \sum_{k=1}^{\gamma_j} < z_i, \phi_{j,k} > \phi_{j,k}$. Then, from Theorem 1.2 we get that

$$\begin{bmatrix} \sum_{k=1}^{\gamma_j} \langle z_1, \ \phi_{j,k} \rangle \phi_{j,k}(x) \\ \sum_{k=1}^{\gamma_j} \langle z_2, \ \phi_{j,k} \rangle \phi_{j,k}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1,2,3,4,\ldots, \ \forall x \in \Omega, \ s = 1,2. \tag{3.14}$$

Hence $P_j z = 0$, $j = 1,2,3,4,\ldots$, which implies that $z = 0$. This completes the proof of the main theorem. □

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References

[1] S. Badraoui, "Asymptotic behavior of solutions to a 2 × 2 reaction-diffusion system with a cross diffusion matrix on unbounded domains," *Electronic Journal of Differential Equations*, vol. 2006, no. 61, pp. 1–13, 2006.

[2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, vol. 137 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1992.

[3] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, vol. 8 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 1978.

[4] H. Leiva, "A lemma on C_0-semigroups and applications PDEs systems," *Quaestiones Mathematicae*, vol. 26, no. 3, pp. 247–265, 2003.

[5] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, vol. 21 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 1995.