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Research Article

Existence and Uniqueness of Periodic Solution for Nonlinear Second-Order Ordinary Differential Equations

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We study periodic solutions for nonlinear second-order ordinary differential problem $x'' + f(t,x,x') = 0$. By constructing upper and lower boundaries and using Leray-Schauder degree theory, we present a result about the existence and uniqueness of a periodic solution for second-order ordinary differential equations with some assumption.

1. Introduction

The study on periodic solutions for ordinary differential equations is a very important branch in the differential equation theory. Many results about the existence of periodic solutions for second-order differential equations have been obtained by combining the classical method of lower and upper solutions and the method of alternative problems (The Lyapunov-Schmidt method) as discussed by many authors [1–10]. In [11], the author gives a simple method to discuss the existence and uniqueness of nonlinear two-point boundary value problems. In this paper, we will extend this method to the periodic problem.

We consider the second-order ordinary differential equation

$$x'' + f(t,x,x') = 0. \tag{1.1}$$

Throughout this paper, we will study the existence of periodic solutions of (1.1) with the following assumptions:

(H_1) $f,\ f_x,\ and\ f_{x'}$ are continuous in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and

$$f(t,x\ x') = f(t+2\pi,x,x'), \tag{1.2}$$

(H_2)

$$\begin{gathered} N^2 < \alpha - \frac{\gamma^2}{4} \le \beta < (N+1)^2, \\ \sin\frac{\pi\sqrt{4\alpha-\gamma^2}}{4N} < \sqrt{1-\frac{\gamma^2}{4\alpha}} \quad \text{if } N > 0, \\ \gamma < \frac{4(N+1)}{\pi}\left[1-\frac{\beta}{(N+1)^2}\right], \end{gathered} \tag{1.3}$$

where N is some positive integer,

$$\alpha = \inf_{\mathbb{R}^3}(f_x), \qquad \beta = \sup_{\mathbb{R}^3}(f_x), \qquad \gamma = \sup_{\mathbb{R}^3}|f_{x'}|. \tag{1.4}$$

The following is our main result.

Theorem 1.1. *Assume that* (H_1) *and* (H_2) *hold, then* (1.1) *has a unique* 2π*-periodic solution.*

2. Basic Lemmas

The following results will be used later.

Lemma 2.1 (see [12]). *Let* $x \in C^1([0,h],\mathbb{R})$ $(h>0)$ *with*

$$x(0) = x(h) = 0, \quad x(t) > 0 \quad \textit{for } t \in (0,h), \tag{2.1}$$

then

$$\int_0^h |x(t)x'(t)|dt \le \frac{h}{4}\int_0^h x'^2(t)dt, \tag{2.2}$$

and the constant $h/4$ *is optimal.*

Lemma 2.2 (see [12]). *Let* $x \in C^1([a,b],\mathbb{R})$ $(a,b\in\mathbb{R},\ a<b)$ *with the boundary value conditions* $x(a)=x(b)=0$, *then*

$$\int_a^b x^2(t)dt \le \frac{(b-a)^2}{\pi^2}\int_a^b x'^2(t)dt. \tag{2.3}$$

Consider the periodic boundary value problem

$$\begin{gathered} x'' + p(t)x' + q(t)x = 0, \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi). \end{gathered} \tag{2.4}$$

Lemma 2.3. *Suppose that p, q are L^2-integrable 2π-periodic function, where p, q satisfy the condition (H_2), with*

$$\alpha = \inf_{[0,2\pi]} q(t), \qquad \beta = \sup_{[0,2\pi]} q(t), \qquad \gamma = \sup_{[0,2\pi]} |p(t)|, \tag{2.5}$$

then (2.4) *has only the trivial 2π-periodic solution $x(t) \equiv 0$.*

Proof. If on the contrary, (2.4) has a nonzero 2π-periodic solution $x(t)$, then using (2.4), we have

$$\left(e^{\int_{t_0}^{t} p(s)ds} x'\right)' + e^{\int_{t_0}^{t} p(s)ds} q(t)x = 0, \tag{2.6}$$

where $t_0 \in [0, 2\pi]$ is undetermined.

Firstly, we prove that $x(t)$ has at least one zero in $(0, 2\pi)$. If $x(t) \neq 0$, we may assume $x(t) > 0$. Since $x(t)$ is a 2π-periodic solution, there exists a $t_0 \in [0, 2\pi]$ with $x'(t_0) = 0 = x'(t_0 + 2\pi)$. Then,

$$0 = \int_{t_0}^{t_0+2\pi} \left(e^{\int_{t_0}^{t} p(s)ds} x'\right)' dt = -\int_{t_0}^{t_0+2\pi} e^{\int_{t_0}^{t} p(s)ds} q(t)x\,dt < 0, \tag{2.7}$$

we could get a contradiction.

Without loss of generality, we may assume that $x(0) = x(2\pi) = 0$, $x'(0) = x'(2\pi) = A > 0$; then there exists a sufficiently small $\delta > 0$ such that $x(\delta/2) > 0$, $x(2\pi - \delta/2) < 0$. Since $x(t)$ is a continuous function, there must exist a $t' \in [\delta/2, 2\pi - \delta/2]$ with $x(t') = 0$.

Secondly, we prove that $x(t)$ has at least $2N+2$ zeros on $[0, 2\pi]$. Considering the initial value problem

$$\varphi'' - \gamma\varphi' + \alpha\varphi = 0, \qquad \varphi(0) = 0, \qquad \varphi'(0) = A. \tag{2.8}$$

Obviously,

$$\varphi(t) = \frac{2A}{\sqrt{4\alpha - \gamma^2}} e^{\gamma t/2} \sin \frac{\sqrt{4\alpha - \gamma^2}}{2} t \tag{2.9}$$

is the solution of (2.8) and

$$\varphi'(t) = 2A\sqrt{\frac{\alpha}{4\alpha - \gamma^2}} e^{\gamma t/2} \sin\left(\frac{\sqrt{4\alpha - \gamma^2}}{2} t + \theta\right), \tag{2.10}$$

where $\theta \in (0, \pi/2]$ with $\sin\theta = \sqrt{(4\alpha - \gamma^2)/4\alpha}$. Since

$$N < \frac{\sqrt{4\alpha - \gamma^2}}{2} < N + 1 \tag{2.11}$$

holds under the assumptions of (H_2), there is a $t_0 \in (0, \pi)$, such that

$$\frac{\sqrt{4\alpha - \gamma^2}}{2} t_0 + \theta = \pi, \qquad \text{i.e., } \frac{\pi}{2} \leq \frac{\sqrt{4\alpha - \gamma^2}}{2} t_0 < \pi. \tag{2.12}$$

Now, let $N > 0$. By the conditions (H_2), (2.11), and (2.12), we have

$$\sin \frac{\sqrt{4\alpha - \gamma^2}}{2} t_0 = \sin\theta = \sqrt{\frac{4\alpha - \gamma^2}{4\alpha}} > \sin \frac{\pi\sqrt{4\alpha - \gamma^2}}{4N}, \tag{2.13}$$

$$\frac{\pi}{2} < \frac{\pi\sqrt{4\alpha - \gamma^2}}{4N} < \pi. \tag{2.14}$$

Since $\sin t$ is decreasing in $[\pi/2, \pi)$, we have $0 < t_0 < \pi/2N$. Therefore,

$$\varphi'(t) > 0, \quad \varphi(t) > 0, \quad \text{for } t \in (0, t_0),\ \varphi'(t_0) = 0. \tag{2.15}$$

We also consider the initial value problem

$$\psi'' + \gamma\psi' + \alpha\psi = 0, \qquad \psi(t_0) = \varphi(t_0), \qquad \psi'(t_0) = 0. \tag{2.16}$$

Clearly,

$$\psi(t) = 2\sqrt{\frac{\alpha}{4\alpha - \gamma^2}} \varphi(t_0) e^{-\gamma(t - t_0)/2} \sin\left(\frac{\sqrt{4\alpha - \gamma^2}}{2}(t - t_0) + \theta\right) \tag{2.17}$$

is the solution of (2.16), where θ is the same as the previous one, and

$$\psi'(t) = -\frac{2\alpha}{\sqrt{4\alpha - \gamma^2}} \varphi(t_0) e^{-\gamma(t - t_0)/2} \sin \frac{\sqrt{4\alpha - \gamma^2}}{2}(t - t_0). \tag{2.18}$$

Hence, there exists a $t_1 \in (0, 2\pi)$ with $t_1 - t_0 \in (0, \pi)$, such that

$$\frac{\sqrt{4\alpha - \gamma^2}}{2}(t_1 - t_0) + \theta = \pi. \tag{2.19}$$

Then,

$$\varphi(t_1) = 0. \tag{2.20}$$

From (2.12) and (2.19), it follows that

$$\frac{\sqrt{4\alpha-\gamma^2}}{4}t_1 = \pi-\theta, \qquad \text{i.e., } \frac{\pi}{2} \le \frac{\sqrt{4\alpha-\gamma^2}}{4}t_1 < \pi. \tag{2.21}$$

By (H_2) and (2.21), we have

$$\sin\frac{\sqrt{4\alpha-\gamma^2}}{4}t_1 = \sin\theta = \sqrt{\frac{4\alpha-\gamma^2}{4\alpha}} > \sin\frac{\pi\sqrt{4\alpha-\gamma^2}}{4N}. \tag{2.22}$$

Since $\sin t$ is decreasing on $[\pi/2,\pi)$, we have $0 < t_1 < \pi/N$, and

$$\varphi'(t) < 0, \quad \varphi(t) > 0, \quad \text{for } t \in (t_0, t_1). \tag{2.23}$$

We now prove that $x(t)$ has a zero point in $(0, t_1]$. If on the contrary $x(t) > 0$ for $t \in (0, t_1]$, then we would have the following inequalities:

$$x(t) \le \varphi(t), \quad \text{for } t \in [0, t_0], \tag{2.24}$$

$$x(t) \le \varphi(t), \quad \text{for } t \in [t_0, t_1]. \tag{2.25}$$

In fact, from(2.4), (2.8), and (2.15), we have

$$\begin{aligned}
&\left(\varphi'(t)x(t)-\varphi(t)x'(t)\right)' \\
&\quad = \varphi''(t)x(t)+\varphi'(t)x'(t)-\varphi'(t)x'(t)-\varphi(t)x''(t) \\
&\quad = \left(\gamma\varphi'(t)-\alpha\varphi(t)\right)x(t)-\varphi(t)\left(-p(t)x'(t)-q(t)x(t)\right) \\
&\quad = \left(\gamma+p(t)\right)\varphi'(t)x(t)+\left(-p(t)\right)\left(\varphi'(t)x(t)-\varphi(t)x'(t)\right)+\left(q(t)-\alpha\right)\varphi(t)x(t) \\
&\quad \ge \left(-p(t)\right)\left(\varphi'(t)x(t)-\varphi(t)x'(t)\right),
\end{aligned} \tag{2.26}$$

with $t \in [0, t_0]$. Setting $y = \varphi'(t)x(t) - \varphi(t)x'(t)$, and since

$$y' \ge -p(t)y, \tag{2.27}$$

we obtain

$$\left(ye^{\int_0^t p(s)ds}\right)' \ge 0, \quad t \in [0, t_0]. \tag{2.28}$$

Notice that $\varphi(0) = x(0) = 0$, which implies

$$y(0) = 0, \quad ye^{\int_0^t p(s)ds} \geq 0, \quad t \in [0, t_0]. \tag{2.29}$$

So, we have

$$\varphi'(t)x(t) - \varphi(t)x'(t) \geq 0, \quad t \in [0, t_0], \quad \text{i.e.,} \left(\frac{\varphi(t)}{x(t)}\right)' \geq 0, \quad t \in (0, t_0]. \tag{2.30}$$

Integrating from 0 to $t \in (0, t_0]$, we obtain

$$0 \leq \int_0^t \left(\frac{\varphi(s)}{x(s)}\right)' ds = \frac{\varphi(t)}{x(t)} - \lim_{t \to 0^+} \frac{\varphi(t)}{x(t)} = \frac{\varphi(t)}{x(t)} - \frac{\varphi'(0)}{x'(0)}. \tag{2.31}$$

Therefore,

$$\frac{\varphi(t)}{x(t)} \geq 1, \quad t \in (0, t_0], \tag{2.32}$$

which implies (2.24). By a similar argument, we have (2.25). Therefore, $0 < x(t_1) \leq \varphi(t_1) = 0$, a contradiction, which shows that $x(t)$ has at least one zero in $(0, t_1]$, with $t_1 < \pi/N$.

We let $x(t^1) = 0$, $t^1 \in (0, t_1]$. If $t^1 + t_1 < 2\pi$, then from a similar argument, there is a $t^2 \in (t^1, t^1 + t_1)$, such that $x(t^2) = 0$ and so on. So, we obtain that $x(t)$ has at least $2N + 2$ zeros on $[0, 2\pi]$.

Thirdly, we prove that $x(t)$ has at least $2N + 3$ zeros on $[0, 2\pi]$. If, on the contrary, we assume that $x(t)$ only has $2N + 2$ zeros on $[0, 2\pi]$, we write them as

$$0 = t^0 < t^1 < \cdots < t^{2N+1} = 2\pi. \tag{2.33}$$

Obviously,

$$x'\left(t^i\right) \neq 0, \quad i = 0, 1, \ldots, 2N + 1. \tag{2.34}$$

Without loss of generality, we may assume that $x'(t^0) > 0$. Since

$$x'\left(t^i\right)x'\left(t^{i+1}\right) < 0, \quad i = 0, 1, \ldots, 2N, \tag{2.35}$$

we obtain $x'(t^{2N+1}) < 0$, which contradicts $x'(t^{2N+1}) = x'(t^0) > 0$. Therefore, $x(t)$ has at least $2N + 3$ zeros on $[0, 2\pi]$.

Finally, we prove Lemma 2.3. Since $x(t)$ has at least $2N+3$ zeros on $[0,2\pi]$, there are two zeros ξ_1 and ξ_2 with $0<\xi_2-\xi_1\le \pi/(N+1)$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\int_{\xi_1}^{\xi_2} x'^2(t)dt &= -\int_{\xi_1}^{\xi_2} x(t)x''(t)dt = \int_{\xi_1}^{\xi_2} p(t)x(t)x'(t)dt + \int_{\xi_1}^{\xi_2} q(t)x^2(t)dt \\ &\le \left[\frac{\gamma}{4}(\xi_2-\xi_1)+\frac{\beta}{\pi^2}(\xi_2-\xi_1)^2\right]\int_{\xi_1}^{\xi_2} x'^2(t)dt.\end{aligned} \tag{2.36}$$

From (H_2), it follows that

$$\frac{\gamma}{4}(\xi_2-\xi_1)+\frac{\beta}{\pi^2}(\xi_2-\xi_1)^2 \le \frac{\pi\gamma}{4(N+1)}+\frac{\beta}{(N+1)^2}<1. \tag{2.37}$$

Hence,

$$\int_{\xi_1}^{\xi_2} x'^2(t)dt = 0, \tag{2.38}$$

which implies $x'(t)=0$ for $t\in[\xi_1,\xi_2]$. Also $x(\xi_1)=0$. Therefore, $x(t)\equiv 0$ for $t\in[0,2\pi]$, a contradiction. The proof is complete. □

3. Proof of Theorem 1.1

Firstly, we prove the existence of the solution. Consider the homotopy equation

$$x''+\alpha x = \lambda(-f(t,x,x')+\alpha x)\equiv \lambda F(t,x,x'), \tag{3.1}$$

where $\lambda\in[0,1]$ and $\alpha=\inf_{\mathbb{R}^3}(f_x)$. When $\lambda=1$, it holds (1.1). We assume that $\Phi(t)$ is the fundamental solution matrix of $x''+\alpha x=0$ with $\Phi(0)=I$. Equation (3.1) can be transformed into the integral equation

$$\begin{pmatrix} x \\ x'\end{pmatrix}(t) = \Phi(t)\left(\begin{pmatrix} x(0)\\ x'(0)\end{pmatrix}+\int_0^t \Phi^{-1}(s)\begin{pmatrix}0\\ \lambda F(s,x(s),x'(s))\end{pmatrix}ds\right). \tag{3.2}$$

From (H_1), $x(t)$ is a 2π-periodic solution of (3.2), then

$$(I-\Phi(2\pi))\begin{pmatrix} x(0)\\ x'(0)\end{pmatrix} = \Phi(2\pi)\int_0^{2\pi}\Phi^{-1}(s)\begin{pmatrix}0\\ \lambda F(s,x(s),x'(s))\end{pmatrix}ds. \tag{3.3}$$

For $(I-\Phi(2\pi))$ is invertible,

$$\begin{pmatrix} x(0)\\ x'(0)\end{pmatrix} = (I-\Phi(2\pi))^{-1}\Phi(2\pi)\int_0^{2\pi}\Phi^{-1}(s)\begin{pmatrix}0\\ \lambda F(s,x(s),x'(s))\end{pmatrix}ds. \tag{3.4}$$

We substitute (3.4) into (3.2),

$$\begin{aligned}\begin{pmatrix} x \\ x' \end{pmatrix}(t) &= \Phi(t)(I-\Phi(2\pi))^{-1}\Phi(2\pi)\int_0^{2\pi}\Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,x(s),x'(s)) \end{pmatrix}ds \\ &\quad + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,x(s),x'(s)) \end{pmatrix}ds.\end{aligned} \tag{3.5}$$

Define an operator

$$P_\lambda : C^1[0,2\pi] \longrightarrow C^1[0,2\pi], \tag{3.6}$$

such that

$$\begin{aligned}P_\lambda\left[\begin{pmatrix} x \\ x' \end{pmatrix}\right](t) &\equiv \Phi(t)(I-\Phi(2\pi))^{-1}\Phi(2\pi)\int_0^{2\pi}\Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,x(s),x'(s)) \end{pmatrix}ds \\ &\quad + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,x(s),x'(s)) \end{pmatrix}ds.\end{aligned} \tag{3.7}$$

Clearly, P_λ is a completely continuous operator in $C^1[0,2\pi]$.

There exists $B > 0$, such that every possible periodic solution $x(t)$ satisfies $\|x\| \le B$ ($\|\cdot\|$ denote the usual normal in $C^1[0,2\pi]$). If not, there exists $\lambda_k \to \lambda_0$ and the solution $x_k(t)$ with $\|x_k\| \to \infty$ $(k \to \infty)$.

We can rewrite (3.1) in the following form:

$$x_k'' + \alpha x_k = -\lambda_k \int_0^1 f_{x'}(t, x_k, \theta x_k')d\theta x_k' - \lambda_k \int_0^1 f_x(t,\theta x_k, 0)d\theta x_k - \lambda_k f(t,0,0) + \lambda_k \alpha x_k. \tag{3.8}$$

Let $y_k = x_k/\|x_k\|$ $(t \in \mathbb{R})$, obviously $\|y_k\| = 1$ $(k = 1,2,\ldots)$. It satisfies the following problem:

$$y_k'' + \alpha y_k = -\lambda_k \textstyle\int_0^1 f_{x'}(t, x_k, \theta x_k')d\theta y_k' - \lambda_k \int_0^1 f_x(t,\theta x_k, 0)d\theta y_k - \lambda_k f(t,0,0)/\|x_k\| + \lambda_k \alpha y_k, \tag{3.9}$$

in which we have

$$\frac{f(t,0,0)}{\|x_k\|} \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{3.10}$$

Since $\{y_k\}$, $\{y_k'\}$ are uniformly bounded and equicontinuous, there exists continuous function $u(t)$, $v(t)$ and a subsequence of $\{k\}_1^\infty$ (denote it again by $\{k\}_1^\infty$), such that $\lim_{k\to\infty} y_k(t) = u(t)$, $\lim_{k\to\infty} y_k'(t) = v(t)$ uniformly in $\mathbb{R}$. Using (H_1) and (H_2), $\{\int_0^1 f_x(t,\theta x_k,0)d\theta\}_1^\infty$ and

$\{\int_0^1 f_{x'}(t, x_k, \theta x_k')d\theta\}_1^\infty$ are uniformly bounded. By the Hahn-Banach theorem, there exists L^2-integrable function $p(t)$, $q(t)$, and a subsequence of $\{k\}_1^\infty$ (denote it again by $\{k\}_1^\infty$), such that

$$\int_0^1 f_x(t, \theta x_k, 0)d\theta \xrightarrow{\omega} q(t), \qquad \int_0^1 f_{x'}(t, x_k, \theta x'_k)d\theta \xrightarrow{\omega} p(t), \tag{3.11}$$

where $\xrightarrow{\omega}$ denotes "weakly converges to" in $L^2[0, 2\pi]$. As a consequence, we have

$$u''(t) + \alpha u(t) = -\lambda_0 p(t)u'(t) - \lambda_0 q(t)u(t) + \lambda_0 \alpha u(t), \tag{3.12}$$

that is,

$$u''(t) + \lambda_0 p(t)u'(t) + \left(\lambda_0 q(t) + (1 - \lambda_0)\alpha\right)u(t) = 0. \tag{3.13}$$

Denote that $\tilde{p}(t) = \lambda_0 p(t)$, $\tilde{q}(t) = \lambda_0 q(t) + (1 - \lambda_0)\alpha$, then we get

$$\left|\tilde{p}(t)\right| = \lambda_0 \left|p(t)\right| \le \gamma, \qquad \lambda_0 \alpha + (1 - \lambda_0)\alpha \le \tilde{q}(t) \le \lambda_0 \beta + (1 - \lambda_0)\alpha, \tag{3.14}$$

which also satisfy the condition (H_2). Notice that $\tilde{p}(t)$ and $\tilde{q}(t)$ are L^2-integrable on $[0, 2\pi]$, so $u(t)$ satisfies Lemma 2.3. Hence, we have $u(t) \equiv 0$ for $t \in [0, 2\pi)$, which contradicts $\|u\| = 1$. Therefore, $PC^1[0, 2\pi]$ is bounded.

Denote

$$\begin{gathered} \Omega = \left\{x \in C^1[0, 2\pi], \|x\| < B + 1\right\}, \\ h_\lambda(x) = x - P_\lambda x. \end{gathered} \tag{3.15}$$

Because $0 \notin h_\lambda(\partial\Omega)$ for $\lambda \in [0, 1]$, by Leray-Schauder degree theory, we have

$$\deg(x - Px, \Omega, 0) = \deg(h_1(x), \Omega, 0) = \deg(h_0(x), \Omega, 0) \neq 0. \tag{3.16}$$

So, we conclude that P has at least one fixed point in Ω, that is, (1.1) has at least one solution.

Finally, we prove the uniqueness of the equation when the condition (H_1) and (H_2) holds. Let $x_1(t)$ and $x_2(t)$ be two 2π-periodic solutions of the problem. Denote $x_0(t) = x_1(t) - x_2(t)$, $t \in [0, 2\pi]$, then $x_0(t)$ is a solution of the following problem:

$$\begin{gathered} x'' + \int_0^1 f_{x'}(t, x_2 + x_0, x_2' + \theta x'_0)d\theta x' + \int_0^1 f_x(t, x_2 + \theta x_0, x_2')d\theta x = 0, \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi). \end{gathered} \tag{3.17}$$

By Lemma 2.3, we have $x_0(t) \equiv 0$ for $t \in [0, 2\pi]$.

Let $\tilde{x}(t+2k\pi) = x(t),\ t \in [0,2\pi],\ k \in \mathbb{Z}$. We have

$$\tilde{x}''(t+2k\pi) = x''(t) = -f(t,x,x') = -f(t,\tilde{x},\tilde{x}') = -f(t+2k\pi,\tilde{x},\tilde{x}'), \tag{3.18}$$

with $t \in [0,2\pi],\ k \in \mathbb{Z}$. Denote $\tilde{x}(t+2k\pi)\ (t \in [0,2\pi])$ by $x(t)\ (t \in \mathbb{R})$. So, $x(t)$ is the solution of the problem (1.1). The proof is complete.

4. An Example

Consider the system

$$x'' + \frac{2}{3}\sin t x' + 6x + \cos x = p(t), \tag{4.1}$$

where $p(t) = p(t+2\pi)$ is a continuous function. Obviously,

$$\begin{aligned} \alpha &= \inf_{\mathbb{R}^3}(f_x) = \inf_{\mathbb{R}^3}(6-\sin x) = 5, \\ \beta &= \sup_{\mathbb{R}^3}(f_x) = \sup_{\mathbb{R}^3}(6-\sin x) = 7, \\ \gamma &= \sup_{\mathbb{R}^3}|f_{x'}| = \sup_{\mathbb{R}^3}\left|\frac{2}{3}\sin t\right| = \frac{2}{3} \end{aligned} \tag{4.2}$$

satisfy Theorem 1.1, then there is a unique 2π-periodic solution in this system.

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