

Research Article

Existence of Positive Solutions to a Boundary Value Problem for a Delayed Nonlinear Fractional Differential System

Zigen Ouyang,¹ Yuming Chen,² and Shuliang Zou³

¹ School of Mathematics and Physics, School of Nuclear Science and Technology, University of South China, Hengyang 421001, China

² Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada N2L 3C5

³ School of Nuclear Science and Technology, University of South China, Hengyang 421001, China

Correspondence should be addressed to Zigen Ouyang, zigenouyang@yahoo.com.cn

Received 14 November 2010; Accepted 24 February 2011

Academic Editor: Gary Lieberman

Copyright © 2011 Zigen Ouyang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Though boundary value problems for fractional differential equations have been extensively studied, most of the studies focus on scalar equations and the fractional order between 1 and 2. On the other hand, delay is natural in practical systems. However, not much has been done for fractional differential equations with delays. Therefore, in this paper, we consider a boundary value problem of a general delayed nonlinear fractional system. With the help of some fixed point theorems and the properties of the Green function, we establish several sets of sufficient conditions on the existence of positive solutions. The obtained results extend and include some existing ones and are illustrated with some examples for their feasibility.

1. Introduction

In the past decades, fractional differential equations have been intensively studied. This is due to the rapid development of the theory of fractional differential equations itself and the applications of such construction in various sciences such as physics, mechanics, chemistry, and engineering [1, 2]. For the basic theory of fractional differential equations, we refer the readers to [3–7].

Recently, many researchers have devoted their attention to studying the existence of (positive) solutions of boundary value problems for differential equations with fractional order [8–23]. We mention that the fractional order α involved is generally in $(1, 2]$ with the exception that $\alpha \in (2, 3]$ in [12, 23] and $\alpha \in (3, 4]$ in [8, 17]. Though there have been extensive

study on systems of fractional differential equations, not much has been done for boundary value problems for systems of fractional differential equations [18–20].

On the other hand, we know that delay arises naturally in practical systems due to the transmission of signal or the mechanical transmission. Though theory of ordinary differential equations with delays is mature, not much has been done for fractional differential equations with delays [24–31].

As a result, in this paper, we consider the following nonlinear system of fractional order differential equations with delays,

$$\begin{aligned} D^{\alpha_i} u_i(t) + f_i(t, u_1(\tau_{i1}(t)), \dots, u_N(\tau_{iN}(t))) &= 0, \quad t \in (0, 1), \\ u_i^{(j)}(0) &= 0, \quad j = 0, 1, \dots, n_i - 2, \quad i = 1, 2, \dots, N, \\ u_i^{(n_i-1)}(1) &= \eta_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (1.1)$$

where D^{α_i} is the standard Riemann-Liouville fractional derivative of order $\alpha_i \in (n_i - 1, n_i]$ for some integer $n_i > 1$, $\eta_i \geq 0$ for $i = 1, \dots, N$, $0 \leq \tau_{ij}(t) \leq t$ for $i, j = 1, 2, \dots, N$, and f_i is a nonlinear function from $[0, 1] \times \mathbb{R}_+^N$ to $\mathbb{R}_+ = [0, \infty)$. The purpose is to establish sufficient conditions on the existence of positive solutions to (1.1) by using some fixed point theorems and some properties of the Green function. By a positive solution to (1.1) we mean a mapping with positive components on $(0, 1)$ such that (1.1) is satisfied. Obviously, (1.1) includes the usual system of fractional differential equations when $\tau_{ij}(t) \equiv t$ for all i and j . Therefore, the obtained results generalize and include some existing ones.

The remaining part of this paper is organized as follows. In Section 2, we introduce some basics of fractional derivative and the fixed point theorems which will be used in Section 3 to establish the existence of positive solutions. To conclude the paper, the feasibility of some of the results is illustrated with concrete examples in Section 4.

2. Preliminaries

We first introduce some basic definitions of fractional derivative for the readers' convenience.

Definition 2.1 (see [3, 32]). The fractional integral of order $\alpha (> 0)$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \quad (2.1)$$

provided that the integral exists on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is the Gamma function.

Note that I^α has the semigroup property, that is,

$$I^\alpha I^\beta = I^{\alpha+\beta} = I^\beta I^\alpha \quad \text{for } \alpha > 0, \beta > 0. \quad (2.2)$$

Definition 2.2 (see [3, 32]). The Riemann-Liouville derivative of order $\alpha (> 0)$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{\alpha + 1 - n}} ds \quad (2.3)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$.

It is well known that if $n - 1 < \alpha \leq n$ then $D^\alpha t^{\alpha - k} = 0, k = 1, 2, \dots, n$. Furthermore, if $y(t) \in L^1[0, T]$ and $\alpha > 0$ then $D^\alpha I^\alpha y(t) = y(t)$ for $t \in (0, T]$.

The following results on fractional integral and fractional derivative will be needed in establishing our main results.

Lemma 2.3 (see [10]). *Let $\alpha > 0$. Then solutions to the fractional equation $D^\alpha h(t) = 0$ can be written as*

$$h(t) = c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \dots + c_n t^{\alpha - n}, \quad (2.4)$$

where $c_i \in \mathbb{R}, i = 1, 2, \dots, n = [\alpha] + 1$.

Lemma 2.4 (see [10]). *Let $\alpha > 0$. Then*

$$I^\alpha D^\alpha h(t) = h(t) + c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2} + \dots + c_n t^{\alpha - n} \quad (2.5)$$

for some $c_i \in \mathbb{R}, i = 1, 2, \dots, n = [\alpha] + 1$.

Now, we cite the fixed point theorems to be used in Section 3.

Lemma 2.5 (the Banach contraction mapping theorem [33]). *Let M be a complete metric space and let $T : M \rightarrow M$ be a contraction mapping. Then T has a unique fixed point.*

Lemma 2.6 (see [16, 34]). *Let C be a closed and convex subset of a Banach space X . Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \rightarrow C$ is completely continuous. Then at least one of the following two properties holds:*

- (i) T has a fixed point in \overline{U} ;
- (ii) there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda Tu$.

Lemma 2.7 (the Krasnosel'skii fixed point theorem [33, 35]). *Let P be a cone in a Banach space X . Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$. Suppose that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ is a completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$

or

- (ii) $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega_2} \setminus \Omega_1$.

3. Existence of Positive Solutions

Throughout this paper, we let $E = C([0, 1], \mathbb{R}^N)$. Then $(E, \|\cdot\|_E)$ is a Banach space, where

$$\|u\|_E = \max_{1 \leq i \leq N} \max_{0 \leq t \leq 1} |u_i(t)| \quad \text{for } u = (u_1, \dots, u_N)^T \in E. \quad (3.1)$$

In this section, we always assume that $f = (f_1, \dots, f_N)^T \in C([0, 1] \times \mathbb{R}_+^N, \mathbb{R}_+^N)$.

Lemma 3.1. *System (1.1) is equivalent to the following system of integral equations:*

$$u_i(t) = \int_0^1 G_i(t, s) f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + \frac{\eta_i t^{\alpha_i - 1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)}, \quad i = 1, 2, \dots, N, \quad (3.2)$$

where

$$G_i(t, s) = \begin{cases} \frac{t^{\alpha_i - 1} (1 - s)^{\alpha_i - n_i} - (t - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha_i - 1} (1 - s)^{\alpha_i - n_i}}{\Gamma(\alpha_i)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.3)$$

Proof. It is easy to see that if $(u_1, u_2, \dots, u_N)^T$ satisfies (3.2) then it also satisfies (1.1). So, assume that $(u_1, u_2, \dots, u_N)^T$ is a solution to (1.1). Integrating both sides of the first equation of (1.1) of order α_i with respect to t gives us

$$u_i(t) = -\frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + c_{1i} t^{\alpha_i - 1} + c_{2i} t^{\alpha_i - 2} + \cdots + c_{n_i i} t^{\alpha_i - n_i} \quad (3.4)$$

for $0 \leq t \leq 1, i = 1, 2, \dots, N$. It follows that

$$u_i'(t) = -\frac{\alpha_i - 1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 2} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + (\alpha_i - 1) c_{1i} t^{\alpha_i - 2} + (\alpha_i - 2) c_{2i} t^{\alpha_i - 3} + \cdots + (\alpha_i - n_i + 1) c_{n_i - 1, i} t^{\alpha_i - n_i} \quad (3.5)$$

for $0 \leq t \leq 1, i = 1, 2, \dots, N$. This, combined with the boundary conditions in (1.1), yields

$$c_{n_i - 1, i} = 0, \quad i = 1, 2, \dots, N. \quad (3.6)$$

Similarly, one can obtain

$$c_{n_i-2,i} = c_{n_i-3,i} = \cdots = c_{2,i} = 0, \quad (3.7)$$

$$u_i^{(n_i-1)}(t) = -\frac{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-n_i} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds \\ + (\alpha_i - 1) \cdots (\alpha_i - n_i + 1) c_{1i} t^{\alpha_i-n_i}, \quad (3.8)$$

$i = 1, 2, \dots, N$. Then it follows from (3.8) and the boundary condition $u_i^{(n_i-1)}(1) = \eta_i$ that

$$c_{1,i} = \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} + \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-n_i} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds. \quad (3.9)$$

Therefore, for $i = 1, 2, \dots, N$,

$$u_i(t) = -\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\ + \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha_i-n_i} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds \\ = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t^{\alpha_i-1} (1-s)^{\alpha_i-n_i} - (t-s)^{\alpha_i-1}) f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds \\ + \frac{1}{\Gamma(\alpha_i)} \int_t^1 t^{\alpha_i-1} (1-s)^{\alpha_i-n_i} f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\ = \int_0^1 G_i(t, s) f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds + \frac{\eta_i t^{\alpha_i-1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)}. \quad (3.10)$$

This completes the proof. \square

The following two results give some properties of the Green functions $G_i(t, s)$.

Lemma 3.2. For $i = 1, 2, \dots, N$, $G_i(t, s)$ is continuous on $[0, 1] \times [0, 1]$ and $G_i(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$.

Proof. Obviously, $G_i(t, s)$ is continuous on $[0, 1] \times [0, 1]$. It remains to show that $G_i(t, s) > 0$ for $(t, s) \in (0, 1) \times (0, 1)$. It is easy to see that $G_i(t, s) > 0$ for $0 < t \leq s < 1$. We only need to show that $G_i(t, s) > 0$ for $0 < s \leq t < 1$. For $0 < s \leq t \leq 1$, let

$$g_i(t, s) = t^{\alpha_i-1} (1-s)^{\alpha_i-n_i} - (t-s)^{\alpha_i-1}, \quad (3.11)$$

$$h_i(t, s) = (1-s)^{\alpha_i-n_i} - \left(1 - \frac{s}{t}\right)^{\alpha_i-1}. \quad (3.12)$$

Then

$$g_i(t, s) = t^{\alpha_i-1}h_i(t, s), \quad 0 < s \leq t < 1. \quad (3.13)$$

Note that $h_i(s, s) > 0$ and $(\partial h_i / \partial t)(t, s) = -(\alpha_i - 1)(1 - s/t)^{\alpha_i-2}st^{-2} < 0$ for $0 < s \leq t < 1$. It follows that $h_i(t, s) > 0$ and hence $g_i(t, s) > 0$ for $0 < s \leq t < 1$.

Therefore, $G_i(t, s) > 0$ for $0 < s \leq t < 1$ and the proof is complete. \square

Lemma 3.3. (i) If $n_i = 2$, then $G_i(t, s) \leq G_i(s, s)$ for $(t, s) \in (0, 1) \times (0, 1)$.

(ii) If $n_i > 2$, then $G_i(t, s) < G_i(1, s)$ for $(t, s) \in (0, 1) \times (0, 1)$.

Proof.

(i) Obviously, $G_i(t, s) \leq G_i(s, s)$ for $0 < t \leq s < 1$. Now, for $0 < s \leq t < 1$, we have

$$\frac{\partial g_i(t, s)}{\partial t} = (\alpha_i - 1)t^{\alpha_i-2} \left[(1 - s)^{\alpha_i-2} - \left(1 - \frac{s}{t}\right)^{\alpha_i-2} \right] \leq 0, \quad (3.14)$$

where g_i is the function defined by (3.11). It follows that $G_i(t, s) \leq G_i(s, s)$ for $0 < s \leq t < 1$. In summary, we have proved (i).

(ii) Again, one can easily see that $G_i(t, s) < G_i(1, s)$ for $0 < t \leq s < 1$. When $0 < s \leq t \leq 1$, we have in this case that

$$\begin{aligned} \frac{\partial g_i(t, s)}{\partial t} &= (\alpha_i - 1)t^{\alpha_i-2} \left[(1 - s)^{\alpha_i-n_i} - \left(1 - \frac{s}{t}\right)^{\alpha_i-2} \right] \\ &\geq (\alpha_i - 1)t^{\alpha_i-2} \left[(1 - s)^{\alpha_i-n_i} - (1 - s)^{\alpha_i-2} \right] \\ &> 0, \end{aligned} \quad (3.15)$$

which implies that $G_i(t, s) \leq G_i(1, s)$ for $0 < s \leq t < 1$. To summarize, we have proved (ii) and this completes the proof. \square

Now, we are ready to present the main results.

Theorem 3.4. Suppose that there exist functions $\lambda_{ij}(t) \in C([0, 1], \mathbb{R}_+)$, $i, j = 1, 2, \dots, N$, such that

$$|f_i(t, u_1, \dots, u_N) - f_i(t, v_1, \dots, v_N)| \leq \sum_{j=1}^N \lambda_{ij}(t) |u_j - v_j| \quad (3.16)$$

for $t \in [0, 1]$, $i = 1, 2, \dots, N$. If

$$\max_{1 \leq i \leq N, n_i > 2} \int_0^1 G_i(1, s) \left(\sum_{j=1}^N \lambda_{ij}(s) \right) ds < 1, \quad (3.17)$$

$$\max_{1 \leq i \leq N, n_i=2} \int_0^1 G_i(s, s) \left(\sum_{j=1}^N \lambda_{ij}(s) \right) ds < 1, \quad (3.18)$$

then (1.1) has a unique positive solution.

Proof. Let

$$\Omega = \{u \in E \mid u_i(t) \geq 0 \text{ for } t \in [0, 1], i = 1, 2, \dots, N\}. \quad (3.19)$$

It is easy to see that Ω is a complete metric space. Define an operator T on Ω by

$$Tu(t) = \int_0^1 G(t, s)g(s)ds + \text{diag} \left(\dots, \frac{\eta_i t^{\alpha_i - 1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)}, \dots \right), \quad (3.20)$$

where $G(t, s) = \text{diag}(G_1(t, s), G_2(t, s), \dots, G_N(t, s))$ and

$$g(t) = \begin{pmatrix} f_1(t, u_1(\tau_{11}(t)), u_2(\tau_{12}(t)), \dots, u_N(\tau_{1N}(t))) \\ f_2(t, u_1(\tau_{21}(t)), u_2(\tau_{22}(t)), \dots, u_N(\tau_{2N}(t))) \\ \vdots \\ f_N(t, u_1(\tau_{N1}(t)), u_2(\tau_{N2}(t)), \dots, u_N(\tau_{NN}(t))) \end{pmatrix}. \quad (3.21)$$

Because of the continuity of G and f , it follows easily from Lemma 3.2 that T maps Ω into itself. To finish the proof, we only need to show that T is a contraction. Indeed, for $u, v \in \Omega$, by (3.16) we have

$$\begin{aligned} & |(Tu(t))_i - (Tv(t))_i| \\ &= \left| \int_0^1 G_i(t, s) (f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) - f_i(s, v_1(\tau_{i1}(s)), \dots, v_N(\tau_{iN}(s)))) ds \right| \\ &\leq \int_0^1 G_i(t, s) |f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) - f_i(s, v_1(\tau_{i1}(s)), \dots, v_N(\tau_{iN}(s)))| ds \\ &\leq \int_0^1 G_i(t, s) \left(\sum_{j=1}^N \lambda_{ij}(s) |u_j(\tau_{ij}(s)) - v_j(\tau_{ij}(s))| \right) ds. \end{aligned} \quad (3.22)$$

This, combined with Lemma 3.3 and (3.17) and (3.18), immediately implies that $T : \Omega \rightarrow \Omega$ is a contraction. Therefore, the proof is complete with the help of Lemmas 3.1 and 2.5. \square

The following result can be proved in the same spirit as that for Theorem 3.4.

Theorem 3.5. For $i = 1, 2, \dots, N$, suppose that there exist nonnegative function $\lambda_i(t)$ and nonnegative constants $q_{i1}, q_{i2}, \dots, q_{iN}$ such that $\sum_{j=1}^N q_{ij} = 1$ and

$$|f_i(t, u_1, \dots, u_N) - f_i(t, v_1, \dots, v_N)| \leq \lambda_i(t) \prod_{j=1}^N |u_j - v_j|^{q_{ij}} \quad (3.23)$$

for $t \in [0, 1]$, $(u_1, u_2, \dots, u_N)^T, (v_1, v_2, \dots, v_N)^T \in \mathbb{R}_+^N$. If

$$\max_{1 \leq i \leq N, n_i > 2} \int_0^1 G_i(1, s) \lambda_i(s) ds < 1, \quad \max_{1 \leq i \leq N, n_i = 2} \int_0^1 G_i(s, s) \lambda_i(s) ds < 1, \quad (3.24)$$

then (1.1) has a unique positive solution.

Theorem 3.6. For $i = 1, 2, \dots, N$, suppose that there exist nonnegative real-valued functions $m_i, n_{i1}, \dots, n_{iN} \in L[0, 1]$ such that

$$f_i(t, u_1, \dots, u_N) \leq m_i(t) + \sum_{j=1}^N n_{ij}(t) u_j \quad (3.25)$$

for almost every $t \in [0, 1]$ and all $(u_1, u_2, \dots, u_N)^T \in \mathbb{R}_+^N$. If

$$\begin{aligned} \max_{1 \leq i \leq N, n_i > 2} \left\{ \int_0^1 G_i(1, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} < 1, \\ \max_{1 \leq i \leq N, n_i = 2} \left\{ \int_0^1 G_i(s, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} < 1, \end{aligned} \quad (3.26)$$

then (1.1) has at least one positive solution.

Proof. Let Ω and $T : \Omega \rightarrow \Omega$ be defined by (3.19) and (3.20), respectively. We first show that T is completely continuous through the following three steps.

Step 1. Show that $T : \Omega \rightarrow \Omega$ is continuous. Let $\{u^k(t)\}$ be a sequence in Ω such that $u^k(t) \rightarrow u(t) \in \Omega$. Then $\Omega_0 = [0, 1] \times \{u(t) \mid u^k(t), u(t) \in \Omega, t \in [0, 1], k \geq 1\}$ is bounded in $[0, 1] \times \mathbb{R}_+^N$. Since f is continuous, it is uniformly continuous on any compact set. In particular, for any $\varepsilon > 0$, there exists a positive integer K_0 such that

$$\begin{aligned} & \left| f_i \left(t, u_1^k(\tau_{i1}(t)), \dots, u_N^k(\tau_{iN}(t)) \right) - f_i \left(t, u_1(\tau_{i1}(t)), \dots, u_N(\tau_{iN}(t)) \right) \right| \\ & < \frac{\varepsilon}{\max_{1 \leq i \leq N} \max_{t \in [0, 1]} \int_0^1 G_i(t, s) ds} \end{aligned} \quad (3.27)$$

for $t \in [0, 1]$ and $k \geq K_0, i = 1, 2, \dots, N$. Then, for $k \geq K_0$, we have

$$\begin{aligned}
 & \left| (Tu^k(t))_i - (Tu(t))_i \right| \\
 &= \left| \int_0^1 G_i(t, s) f_i \left(s, u_1^k(\tau_{i1}(s)), \dots, u_N^k(\tau_{iN}(s)) \right) \right. \\
 &\quad \left. - f_i \left(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s)) \right) \right| ds \\
 &\leq \int_0^1 G_i(t, s) \left| f_i \left(s, u_1^k(\tau_{i1}(s)), \dots, u_N^k(\tau_{iN}(s)) \right) \right. \\
 &\quad \left. - f_i \left(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s)) \right) \right| ds \\
 &< \frac{\varepsilon}{\max_{1 \leq i \leq N} \max_{t \in [0, 1]} \int_0^1 G_i(t, s) ds} \int_0^1 G_i(t, s) ds \leq \varepsilon
 \end{aligned} \tag{3.28}$$

for $k \geq K_0$ and $t \in [0, 1], i = 1, 2, \dots, N$. Therefore,

$$\left\| Tu^k(t) - Tu(t) \right\| < \varepsilon \quad \text{for } k \geq K_0, \tag{3.29}$$

which implies that T is continuous.

Step 2. Show that T maps bounded sets of Ω into bounded sets. Let A be a bounded subset of Ω . Then $[0, 1] \times \{u(t) \mid t \in [0, 1], u \in A\} \subseteq [0, 1] \times \mathbb{R}_+^N$ is bounded. Since f is continuous, there exists an $M > 0$ such that

$$f_i(t, u_1(\tau_{i1}(t)), \dots, u_N(\tau_{iN}(t))) \leq M \quad \text{for } u \in A, t \in [0, 1], 1 \leq i \leq N. \tag{3.30}$$

It follows that, for $u \in A, t \in [0, 1]$ and $1 \leq i \leq N$,

$$\begin{aligned}
 (Tu(t))_i &= \int_0^1 G_i(t, s) f_i \left(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s)) \right) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\
 &\leq M \int_0^1 G_i(t, s) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\
 &\leq \max_{1 \leq i \leq N} \left[M \max_{t \in [0, 1]} \int_0^1 G_i(t, s) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \right].
 \end{aligned} \tag{3.31}$$

Immediately, we can easily see that TA is a bounded subset of Ω .

Step 3. Show that T maps bounded sets of Ω into equicontinuous sets. Let B be a bounded subset of Ω . Similarly as in Step 2, there exists $L > 0$ such that

$$f_i(t, u_1(\tau_{i1}(t)), \dots, u_N(\tau_{iN}(t))) \leq L \quad \text{for } u \in B, t \in [0, 1], 1 \leq i \leq N. \quad (3.32)$$

Then, for any $u \in B$ and $t_1, t_2 \in [0, 1]$ and $1 \leq i \leq N$,

$$\begin{aligned} |(Tu(t_2))_i - (Tu(t_1))_i| &= \left| \frac{\eta_i (t_2^{\alpha_i-1} - t_1^{\alpha_i-1})}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \right. \\ &\quad \left. + \int_0^1 (G_i(t_2, s) - G_i(t_1, s)) f_i(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))) ds \right| \\ &\leq \frac{\eta_i |t_2^{\alpha_i-1} - t_1^{\alpha_i-1}|}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} + \int_0^1 |G_i(t_2, s) - G_i(t_1, s)| L ds \\ &\leq \frac{\eta_i |t_2^{\alpha_i-1} - t_1^{\alpha_i-1}|}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} + \max_{0 \leq s \leq 1} |G_i(t_2, s) - G_i(t_1, s)| L. \end{aligned} \quad (3.33)$$

Now the equicontinuity of T on B follows easily from the fact that G_i is continuous and hence uniformly continuous on $[0, 1] \times [0, 1]$.

Now we have shown that T is completely continuous. To apply Lemma 2.6, let

$$\begin{aligned} \mu &= \frac{\max_{1 \leq i \leq N, n_i > 2} \left\{ \int_0^1 G_i(1, s) m_i(s) ds + \eta_i / (\alpha_i - 1) \cdots (\alpha_i - n_i + 1) \right\}}{1 - \max_{1 \leq i \leq N, n_i > 2} \left\{ \int_0^1 G_i(1, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\}}, \\ \nu &= \frac{\max_{1 \leq i \leq N, n_i = 2} \left\{ \int_0^1 G_i(s, s) m_i(s) ds + \eta_i / (\alpha_i - 1) \cdots (\alpha_i - n_i + 1) \right\}}{1 - \max_{1 \leq i \leq N, n_i = 2} \left\{ \int_0^1 G_i(s, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\}}. \end{aligned} \quad (3.34)$$

Fix $r > \max\{\mu, \nu\}$ and define

$$U = \{u \in \Omega \mid \|u\|_E < r\}. \quad (3.35)$$

We claim that there is no $u \in \partial U$ such that $u = \lambda Tu$ for some $\lambda \in (0, 1)$. Otherwise, assume that there exist $\lambda \in (0, 1)$ and $u \in \partial U$ such that $u = \lambda Tu$. Then

$$\begin{aligned}
 |u_i(t)| &= |\lambda(Tu(t))_i| \leq |(Tu(t))_i| \\
 &\leq \int_0^1 G_i(t, s) f_i(s, u_1(\tau_{i1}(t)), u_2(\tau_{i2}(t)), \dots, u_N(\tau_{iN}(t))) ds + \frac{\eta_i t^{\alpha_i - 1}}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\
 &\leq \int_0^1 G_i(t, s) \left(m_i(s) + \sum_{j=1}^N n_{ij}(s) u_j(\tau_{ij}(s)) \right) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\
 &\leq \int_0^1 G_i(t, s) m_i(s) ds + r \int_0^1 G_i(t, s) \sum_{j=1}^N n_{ij}(s) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)}.
 \end{aligned} \tag{3.36}$$

If $n_i = 2$, then

$$\begin{aligned}
 |u_i(t)| &< \int_0^1 G_i(s, s) m_i(s) ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} + r \int_0^1 G_i(s, s) \sum_{j=1}^N n_{ij}(s) ds \\
 &\leq r \left(1 - \max_{1 \leq i \leq N, n_i=2} \left\{ \int_0^1 G_i(s, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} \right) + r \int_0^1 G_i(s, s) \sum_{j=1}^N n_{ij}(s) ds \\
 &< r \left(1 - \max_{1 \leq i \leq N, n_i=2} \left\{ \int_0^1 G_i(s, s) \left(\sum_{j=1}^N n_{ij}(s) \right) ds \right\} \right) + r \int_0^1 G_i(s, s) \sum_{j=1}^N n_{ij}(s) ds \leq r.
 \end{aligned} \tag{3.37}$$

Similarly, we can have $|u_i(t)| < r$ if $n_i > 2$. To summarize, $\|u\| < r$, a contradiction to $u \in \partial U$. This proves the claim. Applying Lemma 2.6, we know that T has a fixed point in \bar{U} , which is a positive solution to (1.1) by Lemma 3.1. Therefore, the proof is complete. \square

As a consequence of Theorem 3.6, we have the following.

Corollary 3.7. *If all f_i , $i = 1, 2, \dots, N$, are bounded, then (1.1) has at least one positive solution. To state the last result of this section, we introduce*

$$\begin{aligned}
 M_1 &= \frac{1}{\max_{1 \leq i \leq N} \int_0^1 G_i(1, s) ds}, \\
 N &= \max \left\{ \max_{1 \leq i \leq N, n_i > 2} \int_0^1 G_i(1, s) ds, \max_{1 \leq i \leq N, n_i = 2} \int_0^1 G_i(s, s) ds \right\}.
 \end{aligned} \tag{3.38}$$

Theorem 3.8. *Suppose that there exist $M_2 \in (0, 1/N)$ and positive constants $0 < r_1 < r_2$ with $r_2 \geq \max_{1 \leq i \leq N} \{\eta_i / (\alpha_i - 1) \cdots (\alpha_i - n_i + 1)\} / (1 - M_2 N)$ such that*

(i) $f_i(t, u_1, \dots, u_N) \leq M_2 r_2$ for $(t, u_1, \dots, u_N) \in [0, 1] \times B_{r_2}, i = 1, 2, \dots, N$

and

(ii) $f_i(t, u_1, \dots, u_N) \geq M_1 r_1$, for $(t, u_1, \dots, u_N) \in [0, 1] \times B_{r_1}, i = 1, 2, \dots, N$,

where $B_{r_i} = \{u = (u_1, \dots, u_N)^T \in \mathbb{R}_+^N \mid \max_{1 \leq i \leq N} u_i \leq r_i\}, i = 1, 2$. Then (1.1) has at least a positive solution.

Proof. Let Ω be defined by (3.19) and $\Omega_i = \{u \in E \mid \|u\| < r_i\}, i = 1, 2$. Obviously, Ω is a cone in E . From the proof of Theorem 3.6, we know that the operator T defined by (3.20) is completely continuous on Ω . For any $u \in \Omega \cap \partial\Omega_1$, it follows from Lemma 3.3 and condition (ii) that

$$\begin{aligned} \|Tu\|_E &= \max_{1 \leq i \leq N} \max_{0 \leq t \leq 1} (Tu(t))_i \geq \max_{1 \leq i \leq N} (Tu(1))_i \\ &= \max_{1 \leq i \leq N} \left\{ \int_0^1 G_i(1, s) f_i(s, u_1(\tau_{i1}(s)), u_2(\tau_{i2}(s)), \dots, u_N(\tau_{iN}(s))) ds \right. \\ &\quad \left. + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \right\} \\ &\geq \max_{1 \leq i \leq N} \left\{ \int_0^1 G_i(1, s) M_1 r_1 + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \right\} \geq r_1 = \|u\|_E, \end{aligned} \quad (3.39)$$

that is,

$$\|Tu\|_E \geq \|u\|_E \quad \text{for } u \in \Omega \cap \partial\Omega_1. \quad (3.40)$$

On the other hand, for any $u \in \Omega \cap \partial\Omega_2$, it follows from Lemma 3.3 and condition (i) that, for $t \in [0, 1]$,

$$\begin{aligned} (Tu(t))_i &\leq \int_0^1 G_i(1, s) M_2 r_2 ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\ &\leq M_2 r_2 \max_{1 \leq i \leq N, n_i > 2} \int_0^1 G_i(1, s) ds + \max_{1 \leq i \leq N} \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\ &\leq M_2 r_2 \max_{1 \leq i \leq N, n_i > 2} \int_0^1 G_i(1, s) ds + (1 - M_2 N) r_2 \leq r_2 = \|u\|_E \end{aligned} \quad (3.41)$$

if $n_i > 2$, whereas

$$\begin{aligned} (Tu(t))_i &\leq \int_0^1 G_i(s, s) M_2 r_2 ds + \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \\ &\leq M_2 r_2 \max_{1 \leq i \leq N, n_i = 2} \int_0^1 G_i(s, s) ds + \max_{1 \leq i \leq N} \frac{\eta_i}{(\alpha_i - 1) \cdots (\alpha_i - n_i + 1)} \leq r_2 = \|u\|_E \end{aligned} \quad (3.42)$$

if $n_i = 2$. In summary,

$$\|Tu\| \leq \|u\|_E \quad \text{for } u \in \Omega \cap \partial\Omega_2. \quad (3.43)$$

Therefore, we have verified condition (ii) of Lemma 2.7. It follows that T has a fixed point in $\Omega \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is a positive solution to (1.1). This completes the proof. \square

4. Examples

In this section, we demonstrate the feasibility of some of the results obtained in Section 3.

Example 4.1. Consider

$$\begin{aligned} D^{5/2}x_1(t) + \frac{e^{-t}(x_1(t/2) + x_2(\sin t))}{(9 + e^t)(1 + x_1(t/2) + x_2(\sin t))} &= 0, \quad t \in (0, 1), \\ D^{5/2}x_2(t) + \frac{t^2(x_1(t^2) + x_2(\sin t))}{10(1 + (x_1(t^2) + x_2(\sin t)))} &= 0, \quad t \in (0, 1), \\ x_1(0) = x_2(0) = x'_1(0) = x'_2(0) = 0, x''_1(1) = x''_2(1) &= \frac{1}{2}. \end{aligned} \quad (4.1)$$

Here

$$\begin{aligned} n_1 = n_2 = 3, \quad \alpha_1 = \alpha_2 = \frac{5}{2}, \quad \eta_1 = \eta_2 = \frac{1}{2}, \\ \tau_{11}(t) = \frac{t}{2}, \quad \tau_{12}(t) = \tau_{22}(t) = \sin t, \quad \tau_{21}(t) = t^2, \\ f_1(t, x_1, x_2) = \frac{e^{-t}(x_1 + x_2)}{(9 + e^t)(1 + x_1 + x_2)}, \quad f_2(t, x_1, x_2) = \frac{t^2(x_1 + x_2)}{10(1 + x_1 + x_2)}. \end{aligned} \quad (4.2)$$

One can easily see that (3.16) is satisfied with

$$\lambda_{11}(t) = \lambda_{12}(t) = \frac{e^{-t}}{9 + e^t}, \quad \lambda_{21}(t) = \lambda_{22}(t) = \frac{t^2}{10}. \quad (4.3)$$

Moreover,

$$G_1(1, s) = G_2(1, s) \frac{(1-s)^{-1/2} - (1-s)^{3/2}}{\Gamma(5/2)}, \quad 0 \leq s \leq 1 \quad (4.4)$$

and hence

$$\begin{aligned} \max_{1 \leq i \leq 2} \int_0^1 G_i(1, s) \left(\sum_{j=1}^2 \lambda_{ij}(s) \right) ds &\leq \int_0^1 \frac{(1-s)^{-1/2} - (1-s)^{3/2}}{\Gamma(5/2)} \max_{0 \leq s \leq 1} \left\{ \frac{2s^2}{10}, \frac{2}{e^s(9+e^s)} \right\} ds \\ &\leq \frac{1}{5} \int_0^1 \frac{(1-s)^{-1/2} - (1-s)^{3/2}}{\Gamma(5/2)} ds \\ &= \frac{2 - (2/5)}{5 \cdot (3/4)\sqrt{\pi}} = \frac{32}{75\sqrt{\pi}} < 1. \end{aligned} \quad (4.5)$$

It follows from Theorem 3.4 that (4.1) has a unique positive solution on $[0, 1]$.

Example 4.2. Consider

$$\begin{aligned} D^{5/2}x_1(t) + \frac{tx_1(t)}{20} + \frac{x_2(t)}{20} + \frac{t}{10} + \frac{1}{10} &= 0, \quad t \in (0, 1), \\ D^{3/2}x_2(t) + \frac{x_1(1)}{20} + \frac{tx_2(t)}{20} + \frac{t^2}{10} + \frac{1}{10} &= 0, \quad t \in (0, 1), \\ x_1(0) = x_2(0) = x_1'(0) = 0, \quad x_1''(1) = x_2''(1) &= \frac{1}{2}. \end{aligned} \quad (4.6)$$

Here

$$\begin{aligned} n_1 = 3, \quad n_2 = 2, \quad \alpha_1 = \frac{5}{2}, \quad \alpha_2 = \frac{3}{2}, \quad \eta_1 = \eta_2 = \frac{1}{2}, \\ f_i(t, x_1, x_2) = m_i(t) + \sum_{j=1}^2 n_{ij}(t)x_j, \quad i = 1, 2, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} m_1(t) = \frac{t}{10} + \frac{1}{10}, \quad m_2(t) = \frac{t^2}{10} + \frac{1}{10}, \\ n_{11}(t) = n_{22}(t) = \frac{t}{20}, \quad n_{12}(t) = n_{21}(t) = \frac{1}{20}. \end{aligned} \quad (4.8)$$

Hence, f_1 and f_2 satisfy (3.25). Moreover, simple calculations give us

$$\begin{aligned} \int_0^1 G_1(1, s) ds = \frac{32}{15\sqrt{\pi}}, \quad \int_0^1 G_2(1, s) ds = \frac{8}{3\sqrt{\pi}}, \\ \int_0^1 G_1(s, s) ds = \frac{\sqrt{\pi}}{2}, \quad \int_0^1 G_2(s, s) ds = \sqrt{\pi}. \end{aligned} \quad (4.9)$$

Then $M_1 = 3\sqrt{\pi}/8$ and

$$N = \max \left\{ \int_0^1 G_1(1, s) ds, \int_0^1 G_2(s, s) ds \right\} = \sqrt{\pi}. \quad (4.10)$$

Choose $M_2 = \sqrt{\pi}/10 \in (0, 1/N) = (0, 1/\sqrt{\pi})$, $r_1 = \sqrt{\pi}/12$ and

$$r_2 = \max \left\{ \frac{1/2}{(5/2-1)(5/2-3+1)}, \frac{1/2}{(3/2-2+1)} \right\} / \left(1 - \frac{\sqrt{\pi}}{10} \sqrt{\pi} \right) = \frac{10}{10-\pi}. \quad (4.11)$$

Then, for $\|(x_1, x_2)^T\| \leq r_2$ and $t \in [0, 1]$, we have

$$\begin{aligned} f_1(t, x_1, x_2) &= \frac{tx_1}{20} + \frac{x_2}{20} + \frac{t}{10} + \frac{1}{10} \leq \frac{r_2}{10} + \frac{1}{10} = \frac{1}{10-\pi} + \frac{1}{10} \\ &\leq 0.24581 < 0.2584 < M_2 r_2 \end{aligned} \quad (4.12)$$

$$f_2(t, x_1, x_2) = \frac{x_1}{20} + \frac{tx_2}{20} + \frac{t^2}{10} + \frac{1}{10} < M_2 r_2;$$

for $\|(x_1, x_2)^T\| \leq r_1$ and $t \in [0, 1]$, we have

$$f_1(t, x_1, x_2), f_2(t, x_1, x_2) \geq \frac{1}{10} > \frac{\pi}{32} = M_1 r_1. \quad (4.13)$$

By now we have verified all the assumptions of Theorem 3.8. Therefore, (4.6) has at least one positive solution $x = (x_1, x_2)^T$ satisfying $\sqrt{\pi}/12 \leq \|x\| \leq 10/(10-\pi)$.

Acknowledgment

Supported partially by the Doctor Foundation of University of South China under Grant no. 5-XQD-2006-9, the Foundation of Science and Technology Department of Hunan Province under Grant no. 2009RS3019 and the Subject Lead Foundation of University of South China no. 2007XQD13. Research was partially supported by the Natural Science and Engineering Re-search Council of Canada (NSERC) and the Early Researcher Award (ERA) Pro-gram of Ontario.

References

- [1] L. Debnath, "Recent applications of fractional calculus to science and engineering," *International Journal of Mathematics and Mathematical Sciences*, no. 54, pp. 3413–3442, 2003.
- [2] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.
- [3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.
- [4] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3337–3343, 2008.

- [5] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific, Cambridge, UK, 2009.
- [6] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.
- [7] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.
- [8] C. Bai, "Triple positive solutions for a boundary value problem of nonlinear fractional differential equation," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 24, 10 pages, 2008.
- [9] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.
- [10] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.
- [11] M. Benchohra, S. Hamani, and S. K. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2391–2396, 2009.
- [12] M. El-Shahed, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Abstract and Applied Analysis*, vol. 2007, Article ID 10368, 8 pages, 2007.
- [13] H. Jafari and V. Daftardar-Gejji, "Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 700–706, 2006.
- [14] E. R. Kaufmann and E. Mboumi, "Positive solutions of a boundary value problem for a nonlinear fractional differential equation," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 3, 11 pages, 2008.
- [15] N. Kosmatov, "A singular boundary value problem for nonlinear differential equations of fractional order," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 125–135, 2009.
- [16] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.
- [17] S. Liang and J. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5545–5550, 2009.
- [18] X. Su, "Boundary value problem for a coupled system of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 64–69, 2009.
- [19] J. Wang, H. Xiang, and Z. Liu, "Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations," *International Journal of Differential Equations*, vol. 2010, Article ID 186928, 12 pages, 2010.
- [20] A. Yang and W. Ge, "Positive solutions for boundary value problems of N -dimension nonlinear fractional differential system," *Boundary Value Problems*, vol. 2008, Article ID 437453, 15 pages, 2008.
- [21] S. Zhang, "Existence of solution for a boundary value problem of fractional order," *Acta Mathematica Scientia B*, vol. 26, no. 2, pp. 220–228, 2006.
- [22] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations," *Electronic Journal of Differential Equations*, no. 36, 12 pages, 2006.
- [23] Y. Zhao, S. Sun, Z. Han, and Q. Li, "The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2086–2097, 2011.
- [24] A. Babakhani, "Positive solutions for system of nonlinear fractional differential equations in two dimensions with delay," *Abstract and Applied Analysis*, vol. 2010, Article ID 536317, 16 pages, 2010.
- [25] A. Babakhani and E. Enteghami, "Existence of positive solutions for multiterm fractional differential equations of finite delay with polynomial coefficients," *Abstract and Applied Analysis*, vol. 2009, Article ID 768920, 12 pages, 2009.
- [26] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, "Existence results for fractional order functional differential equations with infinite delay," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1340–1350, 2008.
- [27] W. Deng, C. Li, and J. Lü, "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, vol. 48, no. 4, pp. 409–416, 2007.
- [28] L. Hu, Y. Ren, and R. Sakthivel, "Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays," *Semigroup Forum*, vol. 79, no. 3, pp. 507–514, 2009.

- [29] T. A. Maraaba, F. Jarad, and D. Baleanu, "On the existence and the uniqueness theorem for fractional differential equations with bounded delay within Caputo derivatives," *Science in China. Series A*, vol. 51, no. 10, pp. 1775–1786, 2008.
- [30] G. M. Mophou and G. M. N'Guérékata, "A note on a semilinear fractional differential equation of neutral type with infinite delay," *Advances in Difference Equations*, vol. 2010, Article ID 674630, 8 pages, 2010.
- [31] X. Zhang, "Some results of linear fractional order time-delay system," *Applied Mathematics and Computation*, vol. 197, no. 1, pp. 407–411, 2008.
- [32] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1993.
- [33] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, vol. 141 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2001.
- [34] A. Granas, R. B. Guenther, and J. W. Lee, "Some general existence principles in the Carathéodory theory of nonlinear differential systems," *Journal de Mathématiques Pures et Appliquées*, vol. 70, no. 2, pp. 153–196, 1991.
- [35] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, The Macmillan, New York, NY, USA, 1964.