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Uniqueness of the potential function for the vectorial Sturm-Liouville equation on a finite interval

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Abstract

In this paper, the vectorial Sturm-Liouville operator $L_Q = -\dfrac{d^2}{dx^2} + Q(x)$ is considered, where $Q(x)$ is an integrable $m \times m$ matrix-valued function defined on the interval $[0,\pi]$ The authors prove that m^2+1 characteristic functions can determine the potential function of a vectorial Sturm-Liouville operator uniquely. In particular, if $Q(x)$ is real symmetric, then $\dfrac{m(m+1)}{2} + 1$ characteristic functions can determine the potential function uniquely. Moreover, if only the spectral data of self-adjoint problems are considered, then $m^2 + 1$ spectral data can determine $Q(x)$ uniquely.

Keywords: Inverse spectral problems, Sturm-Liouville equation

1. Introduction

The study on inverse spectral problems for the vectorial Sturm-Liouville differential equation

$$\vec{y}'' + (\lambda I_m - Q(x))\vec{y} = 0, \quad 0 < x < \pi, \tag{1.1}$$

on a finite interval is devoted to determine the potential matrix $Q(x)$ from the spectral data of (1.1) with boundary conditions

$$U(\vec{y}) := \vec{y}'(0) - h\vec{y}(0) = 0, \quad V(\vec{y}) := \vec{y}'(\pi) + H\vec{y}(\pi) = 0, \tag{1.2}$$

where λ is the spectral parameter, $h = [h_{ij}]_{i,j=\overline{1,m}}$ and $H = [H_{ij}]_{i,j=\overline{1,m}}$ are in $M_n(\mathbb{C})$ and $Q(x) = [Q_{ij}(x)]_{i,j=\overline{1,m}}$ is an integrable matrix-valued function. We use $L_m = L(Q, h, H)$ to denote the boundary problem (1.1)-(1.2). For the case $m = 1$, (1.1)-(1.2) is a scalar Sturm-Liouville equation. The scalar Sturm-Liouville equation often arises from some physical problems, for example, vibration of a string, quantum mechanics and geophysics. Numerous research results for this case have been established by renowned mathematicians, notably Borg, Gelfand, Hochstadt, Krein, Levinson, Levitan, Marchenko, Gesztesy, Simon and their coauthors and followers (see [1-9] and references therein). For the case $m \geq 2$, some interesting results had been obtained (see [10-20]). In particular, for $m = 2$ and $Q(x)$ is a two-by-two real symmetric matrix-valued smooth functions defined in the interval $[0, \pi]$ Shen [18] showed that five spectral data can determine $Q(x)$ uniquely. More precisely speaking, he considered the inverse spectral problems of the vectorial Sturm-Liouville equation:

$$\vec{y}'' + (\lambda I_2 - Q_2(x))\vec{y}(x) = 0, \quad 0 < x < \pi, \tag{1.3}$$

where $Q_2(x)$ is a real symmetric matrix-valued function defined in the interval $[0, \pi]$. Let $\sigma_D(Q)$ denotes the Dirichlet spectrum of (1.3), $\sigma_{ND}(Q)$ the Neumann-Dirichlet spectrum of (1.3) and $\sigma_j(Q)$ the spectrum of (1.3) with boundary condition

$$\vec{y}'(0) - B_j\vec{y}(0) = \vec{y}(\pi) = \vec{0}, \tag{1.4}$$

for $j = 1, 2, 3$, where

$$B_j = \begin{bmatrix} \alpha_j & \beta_j \\ \beta_j & \gamma_j \end{bmatrix}$$

is a real symmetric matrix and $\{(\alpha_j, \beta_j, \gamma_j), j = 1, 2, 3\}$ is linearly independent over $\mathbb{R}$. Then

Theorem 1.1 ([18], Theorem 4.1). *Let* $Q_2(x)$ *and* $\widetilde{Q}_2(x)$ *be two continuous two-by-two real symmetric matrix-valued functions defined on* $[0, \pi]$. *Suppose that* $\sigma_{ND}(\widetilde{Q}) = \sigma_{ND}(\widetilde{Q})\sigma_{ND}(\widetilde{Q}) = \sigma_{ND}(\widetilde{Q})$ *and* $\sigma_j(Q) = \sigma_j(\widetilde{Q})$ *for* $j = 1, 2, 3$, *then* $Q(x) = \widetilde{Q}(x)$ *on* $[0, \pi]$.

The purpose of this paper is to generalize the above theorem for the case $m \geq 3$. The idea we use is the Weyl's matrix for matrix-valued Sturm-Liouville equation

$$Y'' + (\lambda I_m - Q(x))Y = 0, \quad 0 < x < \pi. \tag{1.5}$$

Some uniqueness theorems for vectorial Sturm-Liouville equation are obtained in the last section.

2. Main Results

Let $C(x, \lambda) = [C_{ij}(x, \lambda)]_{i,j=\overline{1,m}}$ and $S(x, \lambda) = [S_{ij}(x, \lambda)]_{i,j=\overline{1,m}}$ be two solutions of equation (1.5) which satisfy the initial conditions

$$C(0, \lambda) = S'(0, \lambda) = I_m,$$
$$C'(0, \lambda) = S(0, \lambda) = 0_m,$$

where 0_m is the $m \times m$ zero matrix, $I_m = [\delta_{ij}]_{i,j=\overline{1,m}}$ is the $m \times m$ identity matrix and δ_{ij} is the Kronecker symbol. For given complex-valued matrices h and H, we denote

$$\varphi(x, \lambda) = \left[\varphi_{ij}(x, \lambda)\right]_{i,j=\overline{1,m}} \text{ and } \Phi(x, \lambda) = \left[\Phi_{ij}(x, \lambda)\right]_{i,j=\overline{1,m}}$$

be two solutions of equation (1.5) so that $\phi(x, \lambda) = C(x, \lambda) + S(x, \lambda)h$ and $\Phi(x, \lambda) = S(x, \lambda) + \varphi(x, \lambda)\mathcal{M}(\lambda)$ which satisfy the boundary conditions

$$\begin{cases} U(\Phi) = \Phi'(0) - h\Phi(0) = I_m, \\ V(\Phi) = \Phi'(\pi) + H\Phi(\pi) = 0_m. \end{cases} \tag{2.1}$$

Then, $\mathcal{M}(\lambda) = \Phi(0, \lambda)$. The matrix $\mathcal{M}(\lambda) = [\mathcal{M}_{ij}(\lambda)]_{i,j=\overline{1,m}}$ is called the Weyl matrix for $L_m(Q, h, H)$. In 2006, Yurko proved that:

Theorem 2.1 ([20], Theorem 1). *Let* $\mathcal{M}(\lambda)$ *and* $\widetilde{\mathcal{M}}(\lambda)$ *denote Weyl matrices of the problems* $L_m(Q, h, H)$ *and* $L_m(\widetilde{Q}, \tilde{h}, \widetilde{H})$ *separately. Suppose* $\mathcal{M}(\lambda) = \widetilde{\mathcal{M}}(\lambda)$, *then* $h = \tilde{h}$, $h = \tilde{h}$ *and* $H = \widetilde{H}$.

Also note that from [20], we have

$$\Phi(x,\lambda) = S(x,\lambda) + \varphi(x,\lambda)\mathcal{M}(\lambda) = \psi(x,\lambda)(U(\psi))^{-1}, \tag{2.2}$$

$$\mathcal{M}(\lambda) = -(V(\varphi))^{-1}V(S) = \psi(0,\lambda)(U(\psi))^{-1} \tag{2.3}$$

where $\psi(x,\lambda) = [\psi_{ij}(x,\lambda)]_{i,j=\overline{1,m}}$ is a matrix solution of equation (1.5) associated with the conditions $\psi(\pi, \lambda) = I_m$ and $\psi'(\pi, \lambda) = -H$. It is not difficult to see that both $\Phi(x, \lambda)$ and $\mathcal{M}(\lambda)$ are meromorphic in λ and the poles of $\mathcal{M}(\lambda)$ are coincided with the eigenvalues of $L_m(Q, h, H)$. Moreover, we have

$$\mathcal{M}(\lambda) = -(V(\varphi))^{-1}V(S) = -\frac{\operatorname{Adj}(\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda))}{\det(\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda))}(S'(\pi,\lambda) + HS(\pi,\lambda)),$$

where Adj(A) denotes the adjoint matrix of A and det(A) denotes the determinant of A. In the remaining of this section, we shall prove some uniqueness theorems for vectorial Sturm-Liouville equations. Let $B(i,j) = \left[b_{rs}\right]_{r,s=\overline{1,m}}$,

$$b_{rs} = \begin{cases} 0, & (r,s) \neq (i,j), \\ 1, & (r,s) = (i,j), \end{cases} \quad 1 \leq i,j \leq m,$$

and $B(0, 0) = 0_m$ The characteristic function for this boundary value problem $L_m(Q, h + B(i, j), H)$ is

$$\Delta_{ij}(\lambda) = \det(V(\varphi + SB(i,j))), \quad 1 \leq i,j \leq m \text{ or } (i,j) = (0,0). \tag{2.4}$$

The first problem we want to study is as following:

Problem 1. *How many $\Delta_{ij}(\lambda)$ can uniquely determine Q, h and H? where $(i, j) = (0, 0)$ or $1 \leq i, j \leq m$*

To find the solution of Problem 1, we start with the following lemma

Lemma 2.2. *Let $B(i, j) = [b_{rs}]_{m\times m}$ and Δ_{ij} be defined as above. Then*

$$\begin{aligned}\Delta_{ij}(\lambda) = &\Delta_{00}(\lambda) + \det(Augment[\varphi_1'(\pi,\lambda) + H\varphi_1(\pi,\lambda), \ldots, \\ &\overset{(j\text{th column})}{S_i'(\pi,\lambda) + HS_i(\pi,\lambda)}, \ldots, \varphi_m'(\pi,\lambda) + H\varphi_m(\pi,\lambda)]),\end{aligned}$$

where $\phi_k(\pi, \lambda)$ is the kth column of $\phi(\pi, \lambda)$ and $S_k(\pi, \lambda)$ the kth column of $S(\pi, \lambda)$ for $k = 1, 2, 3, \ldots, m$.

Proof. Let

$$\begin{aligned}Y(x,\lambda) &= [C(x,\lambda) + S(x,\lambda)(h + B(i,j))] \\ &= [(C(x,\lambda) + S(x,\lambda)h) + S(x,\lambda)B(i,j)] \\ &= [\varphi(x,\lambda) + S(x,\lambda)B(i,j)]\end{aligned}$$

Then

$$\begin{aligned}\Delta_{ij}(\lambda) &= \det(Y'(\pi,\lambda) + HY(\pi,\lambda)) \\ &= \det((\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)) + (S'(\pi,\lambda) + HS(\pi,\lambda)B(i,j)) \\ &= \det((\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)) + [0, \overset{(j\text{th column})}{S_i'(\pi,\lambda) + HS_i(\pi,\lambda)}0]) \\ &= \det(\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)) + \det(\varphi_1'(\pi,\lambda) + H\varphi_1(\pi,\lambda), \ldots, \\ &\quad \overset{(j\text{th column})}{S_i'(\pi,\lambda) + HS_i(\pi,\lambda)}, \ldots, \varphi_m'(\pi,\lambda) + H\varphi_m(\pi,\lambda)) \\ &= \Delta_{00}(\lambda) + \det(\varphi_m'(\pi,\lambda) + H\varphi_1(\pi,\lambda), \ldots, \\ &\quad \overset{(j\text{th column})}{S_i'(\pi,\lambda) + HS_i(\pi,\lambda)}, \ldots, \varphi_m'(\pi,\lambda) + H\varphi_m(\pi,\lambda)).\end{aligned}$$

□

Next, we shall prove the first main theorem. For simplicity, if a symbol α denotes an object related to $L_m(Q, h, H)$, then the symbol $\tilde{\alpha}$ denotes the analogous object related to $L_m(\widetilde{Q}, \tilde{h}, \widetilde{H})$.

Theorem 2.3. *Suppose that* $\Delta_{ij}(\lambda) = \widetilde{\Delta}_{ij}(\lambda)$ *for* $(i, j) = (0, 0)$ *or* $1 \le i, j \le m$ *then* $h = \tilde{h}$, $h = \tilde{h}$ *and* $H = \widetilde{H}$.

Proof. Since

$$0_m = \Phi'(\pi, \lambda) + H\Phi(\pi, \lambda)$$

and

$$\Phi(x, \lambda) = S(x, \lambda) + \varphi(x, \lambda)\mathcal{M}(\lambda),$$

we have that

$$-(S'(\pi, \lambda) + HS(\pi, \lambda))e_i = (\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda))\mathcal{M}(\lambda)e_i$$

for each $i = 1, ..., m$, that is,

$$-(S_i'(\pi, \lambda) + HS_i(\pi, \lambda)) = (\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda))\mathcal{M}_i(\lambda).$$

By Crammer's rule,

$$\begin{aligned}
&\mathcal{M}_{ji}(\lambda) \\
&= \frac{-\det(\varphi_1'(\pi, \lambda) + H\varphi_1(\pi, \lambda), \ldots, \overset{(j\text{th column})}{S_i'(\pi, \lambda)} + HS_i(\pi, \lambda), \ldots, \varphi_m'(\pi, \lambda) + H\varphi_m(\pi, \lambda))}{\det(\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda))} \\
&= \frac{\Delta_{00}(\lambda) - \Delta_{ij}(\lambda)}{\Delta_{00}(\lambda)} \\
&= \frac{\widetilde{\Delta}_{00}(\lambda) - \widetilde{\Delta}_{ij}(\lambda)}{\widetilde{\Delta}_{00}(\lambda)} \\
&= \widetilde{\mathcal{M}}_{ji}(\lambda) \text{ for } 1 \le i, j \le m.
\end{aligned}$$

Applying Theorem 2.1, we conclude that $Q = \widetilde{Q}$, $h = \tilde{h}$ and $H = \widetilde{H}$. □

Lemma 2.4. *Suppose that h, H are real symmetric matrices and Q(x) is a real symmetric matrix-valued function. Then,* $\mathcal{M}(\lambda) = -V(\varphi)^{-1}V(S)$ *is real symmetric for all* $\lambda \in \mathbb{R}$.

Proof. Let

$$U(x, \lambda) = \begin{bmatrix} \varphi'(x, \lambda) & S'(x, \lambda) \\ \varphi(x, \lambda) & S(x, \lambda) \end{bmatrix}. \tag{2.5}$$

For $\lambda \in \mathbb{R}$,

$$\begin{cases} (S'^*\varphi - S^*\varphi')(x, \lambda) = (S'^*\varphi - S^*\varphi')(0, \lambda) = I_m, \\ (S'^*S - S^*S')(x, \lambda) = (S'^*S - S^*S')(0, \lambda) = 0_m, \\ (\varphi^*\varphi' - \varphi'^*\varphi)(x, \lambda) = (\varphi^*\varphi' - \varphi'^*\varphi)(0, \lambda) = 0_m, \\ (\varphi^*S' - \varphi'^*S)(x, \lambda) = (\varphi^*S' - \varphi'^*S)(0, \lambda) = I_m, \end{cases}$$

This leads to

$$U^{-1}(x, \lambda) = \begin{bmatrix} -(S)^*(x, \lambda) & (S^*)'(x, \lambda) \\ \varphi^*(x, \lambda) & -(\varphi^*)'(x, \lambda) \end{bmatrix}. \tag{2.6}$$

Now let

$$U_2(x,\lambda) = \begin{bmatrix} I_m & H \\ 0 & I_m \end{bmatrix} U(x,\lambda).$$

Then

$$U_2(1,\lambda) = \begin{bmatrix} I_m & H \\ 0 & I_m \end{bmatrix} U(1,\lambda) = \begin{bmatrix} V(\varphi) & V(S) \\ \varphi(1,\lambda) & S(1,\lambda) \end{bmatrix}$$

and

$$U_2^{-1}(1,\lambda) = (\begin{bmatrix} I_m & H \\ 0 & I_m \end{bmatrix} U(1,\lambda))^{-1} = \begin{bmatrix} -S^*(1;\lambda) & [V(S)]^* \\ (\varphi)^*(1,\lambda) & -[V(\varphi)]^* \end{bmatrix}.$$

Since

$$U(x,\lambda)U^{-1}(x,\lambda) = I_{2m},$$

we have

$$V(\varphi)[V(S)]^* = V(S)[V(\varphi)]^*,$$

i.e., $\mathcal{M}(\lambda) = V(\varphi)^{-1}V(S)$is real symmetric for all $\lambda \in \mathbb{R}$. □

Definition 2.1. *We call $L_m(h, H, Q)$ a real symmetric problem if h, H are real symmetric matrices and $Q(x)$ is a real symmetric matrix-valued function.*

Corollary 2.5. *Let $L_m(h, H, Q)$ and $L(\tilde{h}, \tilde{H}, \tilde{Q})$ be two real symmetric problems. Suppose that $\Delta_{ij}(\lambda) = \tilde{\Delta}_{ij}(\lambda)$ for $(i, j) = (0, 0)$ or $1 \le i \le j \le m$, then $h = \tilde{h}$, $h = \tilde{H}$ and $Q = \tilde{Q}$.*

Proof. For $\lambda \in \mathbb{R}$. both $\mathcal{M}(\lambda)$ and $\tilde{\mathcal{M}}(\lambda)$ are real symmetric. Moreover,

$$\begin{aligned} \mathcal{M}_{ji}(\lambda) &= \frac{\Delta_{00}(\lambda) - \Delta ij(\lambda)}{\Delta_{00}(\lambda)} \\ &= \frac{\tilde{\Delta}_{00}(\lambda) - \tilde{\Delta} ij(\lambda)}{\tilde{\Delta}_{00}(\lambda)} \\ &= \tilde{\mathcal{M}}_{ji}(\lambda), \text{ for } 1 \le i \le j \le m. \end{aligned}$$

Hence, $\mathcal{M}_{ij}(\lambda) = \tilde{\mathcal{M}}_{ij}(\lambda)$ for $\lambda \in \mathbb{R}$ and $1 \le i, j \le m$. This leads to $\Delta_{ij}(\lambda) = \tilde{\Delta}_{ij}(\lambda)$ for $\lambda \in \mathbb{R}$. We conclude that $\Delta_{ij}(\lambda) = \tilde{\Delta}_{ij}(\lambda)$ and $\mathcal{M}_{ij}(\lambda) = \tilde{\mathcal{M}}_{ij}(\lambda)$ for $\lambda \in \mathbb{C}$. This completes the proof. □

From now on, we let $L_m(Q, h, H)$ be a real symmetric problem. We would like to know that how many spectral data can determine the problem $L_m(Q, h, H)$ if we require all spectral data come from real symmetric problems. Denote

$$\Gamma_{ij} = \left[e_1, \ldots, \overset{(i\text{th-column})}{0}, \ldots, \overset{(j\text{th-column})}{0}, \ldots, e_m \right],$$

$$\Gamma^{ij} = \left[0, \ldots, \overset{(i\text{th-column})}{e_i}, \ldots, \overset{(i\text{th-column})}{e_j}, \ldots, 0 \right],$$

where $e_i = (0, 0, \ldots, 0, \overset{(i\text{th-coordiante})}{1}, 0, \ldots, 0)^t$. Hence, $\Gamma_{ij} + \Gamma^{ij} = I_m$. Let $\Theta_{ij}(\lambda)$ be the characteristic function of the self-adjoint problem

$$y'' + (\lambda I_m - Q(x))y = 0, \ \ 0 < x < \pi \tag{2.7}$$

associated with some boundary conditions

$$\begin{cases} \Gamma_{ij}y'(0,\lambda) - (\Gamma_{ij}h + \Gamma^{ij})y(0,\lambda) = 0, \\ y'(\pi,\lambda) + Hy(\pi,\lambda) = 0, \end{cases} \tag{2.8}$$

then

$$\Theta_{ij}(\lambda) = \det[V(\varphi_1), \dots, \overset{(i\text{th-column})}{V(S_j)}, \dots, \overset{(j\text{th-column})}{V(S_i)}, \dots, V(\varphi_m)],$$

where $V\,(L_j)$ denotes the jth column of $(V(L))$ for a $m \times m$ matrix L. Similarly, we denote $\Omega_{ij}(\lambda)$ the characteristic function of the real symmetric problem $L_m(Q, h + \frac{1}{2}(B(i,j) + B(j,i)), H)$ for $1 \le i, j \le m$, then

$$\begin{aligned} \Omega_{ij}(\lambda) &= \det\left[V(\varphi_1), \dots, \overset{(i\text{th-column})}{V(\varphi_i) + \frac{1}{2}V(S_j)}, \dots, \overset{(j\text{th-column})}{V(\varphi_j) + \frac{1}{2}V(S_i)}, \dots, V(\varphi_m) \right] \\ &= \det[V(\varphi_1), \dots, V(\varphi_i), \dots, V(\varphi_m)] \\ &\quad + \frac{1}{2}\det[V(\varphi_1), \dots, \overset{(i\text{th-column})}{V(S_j)}, \dots, \overset{(j\text{th-column})}{V(\varphi_j)}, \dots, V(\varphi_m)] \\ &\quad + \frac{1}{2}\det[V(\varphi_1), \dots, \overset{(i\text{th-column})}{V(\varphi_i)}, \dots, \overset{(j\text{th-column})}{V(S_i)}, \dots, V(\varphi_m)] \\ &\quad + \det[V(\varphi_1), \dots, \overset{(i\text{th-column})}{V(S_j)}, \dots, \overset{(j\text{th-column})}{V(S_i)}, \dots, V(\varphi_m)] \end{aligned} \tag{2.9}$$

for $1 \le i, j \le m$. For simplicity, we write

$$\Omega_{00}(\lambda) = \det[V(\varphi_1), \dots, V(\varphi_j), \dots, V(\varphi_m)].$$

Now, we are going to focus on self-adjoint problems. For a self-adjoint problem $L_m(Q, h, H)$ all its eigenvalues are real and the geometric multiplicity of an eigenvalue is equal to its algebraic multiplicity. Moreover, if we denote $\{(\lambda_i, m_i)\}_{i = 1,\infty}$ the spectral data of $L_m(Q, h, H)$ where m_i is the multiplicity of the eigenvalue λ_i of $L_m(Q, h, H)$ then the characteristic function of $L_m(Q, h, H)$ is

$$\Delta(\lambda) = C\Pi_{i=1}^{\infty}(1 - \frac{\lambda}{\lambda_i})^{m_i}$$

where C is determined by $\{(\lambda_i, m_i)\}_{i = 1,\infty}$. This means that the spectral data determined the corresponding characteristic function.

Theorem 2.6. *Assuming that* $L_m(Q, h, H)$ *and* $L_m(\tilde{Q}, \tilde{h}, \tilde{H})$ *are two real symmetric problems. If the conditions*

(1) $\Omega_{ij}(\lambda) = \widetilde{\Omega}_{ij}(\lambda)$ *for* $(i, j) = (0, 0)$ *or* $1 \le i \le j \le m$,
(2) $\Theta_{ij}(\lambda) = \widetilde{\Theta}_{ij}(\lambda)$ *for* $1 \le i < j \le m$.,

are satisfied, then $h = \tilde{h}$, $H = \widetilde{H}$ *and* $Q(x) = \widetilde{Q}(x)$ *a.e on* $[0, 1]$.

Proof. Note that for any problem $L_m(Q, h, H)$ we have

$$\begin{aligned}\Delta_{ij}(\lambda) &= \det[V(\varphi_1),\ldots,\overset{(i\text{th-column})}{V(\varphi_i)},\ldots,\overset{(j\text{th-column})}{V(\varphi_j)+V(S_i)},\ldots,V(\varphi_m)]\\ &= \det[V(\varphi_1),\ldots,V(\varphi_j),\ldots,V(\varphi_m)]\\ &+\det[V(\varphi_1),\ldots,\overset{(i\text{th-column})}{V(\varphi_i)},\ldots,\overset{(j\text{th-column})}{V(S_i)},\ldots,V(\varphi_m)]\\ &= \Delta_{00}(\lambda)+\det[V(\varphi_1),\ldots,\overset{(i\text{th-column})}{V(\varphi_i)},\ldots,\overset{(j\text{th-column})}{V(S_i)},\ldots,V(\varphi_m)]\\ &= \Delta_{00}(\lambda)-\Delta_{00}(\lambda)M_{ji}(\lambda).\end{aligned}$$

Similarly,

$$\tilde{\Delta}_{ij}(\lambda)=\tilde{\Delta}_{00}(\lambda)-\tilde{\Delta}_{00}(\lambda)\tilde{M}_{ji}(\lambda).$$

Moreover, by the assumptions and Lemma 2.4, we have $M_{ij}(\lambda) = M_{ji}(\lambda)$ Hence,

(1) $\Delta_{ij}(\lambda) = \Delta_{ji}(\lambda)$ and $\tilde{\Delta}_{ij}(\lambda)=\tilde{\Delta}_{ji}(\lambda)$ for $1 \le i \le j \le m$,
(2) $\Delta_{ii}(\lambda)=\Omega_{ii}(\lambda)=\widetilde{\Omega}_{ii}(\lambda)=\widetilde{\Delta}_{ii}(\lambda)$ for $i = 0, 1, \ldots, m$,
(3) $\Delta_{ij}(\lambda)=\Omega_{ij}(\lambda)-\Theta_{ij}(\lambda)=\widetilde{\Omega}_{ij}(\lambda)-\widetilde{\Theta}_{ij}(\lambda)=\widetilde{\Delta}_{ij}(\lambda)$ for $1 \le i < j \le m$.

This implies $L_m(Q,h,H)=L_m(\widetilde{Q},\tilde{h},\widetilde{H})$.

The authors want to emphasis that for $n = 1$, the result is classical; for $n = 2$, Theorem 2.6 leads to Theorem 1.1. Shen also shows by providing an example that 5 minimal number of spectral sets can determine the potential matrix uniquely (see [18]).

The readers may think that if all Q, h and H are diagonals then $L_m(Q, h, H)$ is an uncoupled system. Hence, everything for the operator $L_m(Q, h, H)$ can be obtained by applying inverse spectral theory for scalar Sturm-Liouville equation. Unfortunately, it is not true. We say $L_m(Q, h, H)$ diagonal if all Q, h and H are diagonals.

Corollary 2.7. *Suppose $L_m(Q, h, H)$ and $L_m(\widetilde{Q},\tilde{h},\widetilde{H})$ are both diagonals. If $\Delta_{kk}(\lambda)=\widetilde{\Delta}_{kk}(\lambda)$ for $k = 0, 1, \ldots, m$, then $Q=\widetilde{Q}$, $h=\tilde{h}$ and $H=\widetilde{H}$.*

Proof. Since $L_m(Q, h, H)$ and $L_m(\widetilde{Q},\tilde{h},\widetilde{H})$ are both diagonals, we know

$$\mathcal{M}(\lambda)=\frac{-\text{Adj}(\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda))}{\det(\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda))}(S'(\pi,\lambda)+HS(\pi,\lambda))$$

is diagonal and so is $\widetilde{\mathcal{M}}(\lambda)$. Hence,

$$\mathcal{M}_{ij}(\lambda)=0 \text{ for } i\neq j,\ 1\le i,j\le m.$$

Moreover,

$$\begin{aligned}\mathcal{M}_{kk}(\lambda) &= \frac{-1}{\Delta_{00}(\lambda)}(\varphi_1'(\pi,\lambda)+H_1\varphi_1(\pi,\lambda)\cdots\overset{(k)}{(S_k'(\pi,\lambda)+H_kS_k(\pi,\lambda))}\cdots\\ &= \frac{-1}{\Delta_{00}(\lambda)}(\Delta_{kk}(\lambda)-\Delta_{00}(\lambda))\\ &= \frac{\Delta_{00}(\lambda)-\Delta_{kk}(\lambda)}{\Delta_{00}(\lambda)}\\ &= \frac{\widetilde{\Delta}_{00}(\lambda)-\widetilde{\Delta}_{kk}(\lambda)}{\widetilde{\Delta}_{00}(\lambda)}\\ &= \widetilde{\mathcal{M}}_{kk}(\lambda).\end{aligned}$$

for $k = 1, 2, \ldots, m$. This implies. $\mathcal{M}(\lambda) = \widetilde{\mathcal{M}}(\lambda)$. Applying Theorem 2.1 again, we have $Q = \widetilde{Q}$, $h = \tilde{h}$ and $H = \widetilde{H}$. □

Footnote

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Authors' contributions

Both authors contributed to each part of this work equally and read and approved the final version of the manuscript.

Competing interests

The authors declare that they have no competing interests.

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