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Remarks on uniform attractors for the 3D non-autonomous Navier-Stokes-Voight equations

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Abstract

In this paper, we show the existence of pullback attractors for the non-autonomous Navier-Stokes-Voight equations by using contractive functions, which is more simple than the weak continuous method to establish the uniformly asymptotical compactness in H^1 and H^2.

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1 Introduction

Let $\Omega \subset R^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider the non-autonomous 3D Navier-Stokes-Voight (NSV) equations that govern the motion of a Klein-Voight linear viscoelastic incompressible fluid:

$$u_t - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + \nabla p = f(t,x), \quad (x,t)\in\Omega\times R_\tau, \tag{1.1}$$

$$\nabla\cdot u = 0, \quad x\in\Omega, \quad t\in R_\tau, \tag{1.2}$$

$$u(t,x)|_{\partial\Omega} = 0, \quad t\in R_\tau, \tag{1.3}$$

$$u(\tau,x) = u_\tau(x), \quad x\in\Omega, \quad \tau\in R_\tau. \tag{1.4}$$

Here $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity, and the length scale α is a characterizing parameter of the elasticity of the fluid.

When $\alpha = 0$, the above system reduce to the well-known 3D incompressible Navier-Stokes system:

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t,x),\ x\in\Omega,\ t\in R_\tau, \tag{1.5}$$

$$\nabla\cdot u = 0,\ x\in\Omega,\ t\in R_\tau. \tag{1.6}$$

For the well-posedness of 3D incompressible Navier-Stokes equations, in 1934, Leray [1-3] derived the existence of weak solution by weak convergence method; Hopf [4] improved Leray's result and obtained the familiar Leray-Hopf weak solution in 1951. Since the 3D Navier-Stokes equations lack appropriate priori estimate and the strong nonlinear property, the existence of strong solution remains open. For the infinite-dimensional dynamical systems, Sell [5] constructed the semiflow generated by the weak solution which lacks the global regularity and obtained the existence of global attractor of the 3D incompressible Navier-Stokes equations on any bounded smooth domain. Chepyzhov and Vishik [6] investigated the trajectory attractors for 3D non-autonomous incompressible Navier-Stokes system which is based on the works of Leray and Hopf. Using the weak convergence topology of the space H (see below for the definition), Kapustyan and Valero [7] proved the existence of a weak attractor in both autonomous and non-autonomous cases and gave a existence result of strong attractors. Kapustyan, Kasyanov and Valero [8] considered a revised 3D incompressible Navier-Stokes equations generated by an optimal control problem and proved the existence of pullback attractors by constructing a dynamical multivalued process.

However, the infinite-dimensional systems for 3D incompressible Navier-Stokes equations have not yet completely resolved, so many mathematicians pay attention to this challenging problem. In this regard, Kalantarov and Titi [9] investigated the Navier-Stokes-Voight equations as an inviscid regularization of the 3D incompressible Navier-Stokes equations, and further obtained the existence of global attractors for Navier-Stokes-Voight equations. Recently, Qin, Yang and Liu [10] showed the existence of uniform attractors by uniform condition-(C) and weak continuous method to obtain uniformly asymptotical compactness in H^1 and H^2, Yue and Zhong [11] investigated the attractors for autonomous and nonautonomous 3D Navier-Stokes-Voight equations in different methods. More details about the infinite-dimensional dynamics systems, we can refer to [12-27].

Using the contractive functions, we have in this paper established the uniformly asymptotical compactness of the processes $\{U(t, \tau)\}(t \geq \tau, \tau \in R)$ to obtain the existence of the uniform attractor of the 3D non-autonomous NSV equations.

Main difficulties we encountered are as follows:

(1) how to obtain a contractive function,

(2) how to deduce the uniformly asymptotical compactness from a contractive function,

(3) how to obtain the convergence of contractive function.

2 Main results

Notations: Throughout this paper, we set $R_\tau = [\tau, +\infty)$, $\tau \in R$. C stands for a generic positive constant, depending on Ω, but independent of t. $L^p(\Omega)(1 \leq p \leq +\infty)$ is the generic Lebesgue space, $H^s(\Omega)$ is the general Sobolev space. We set $E := \{u | u \in (C_0^\infty(\Omega))^3, divu = 0\}$, H, V, W is the closure of the set E in the topology of $(L^2(\Omega))^3$, $(H^1(\Omega))^3$, $(H^2(\Omega))^3$ respectively. "$\rightharpoonup$" stands for weak convergence of sequence.

Let $\Sigma \subseteq L^2_{loc}(R, L^2(\Omega))$ be the hull of f_0 as a symbol space:

$$\Sigma = H_+(f_0) = \left[f_0(t+h) | h \in R\right]_{L^2_{loc}(R, L^2(\Omega))} \tag{2.1}$$

for all $f_0 \in L^2_{loc}(R, L^2(\Omega))$, where $[\cdot]_{L^2_{loc}(R, L^2(\Omega))}$ denotes the closure in the topology of $L^2_{loc}(R, L^2(\Omega))$.

Under the assumptions of the initial data, the problem (1.1)-(1.4) has a global solution $u \in C([\tau, +\infty), V)$. $U_f(t, \tau, u_\tau)$: $V \to V$ denotes the processes generated by the global solutions and satisfies

$$u(t,\tau;u_\tau) = U_f(t,\tau,u_\tau)u_\tau, \tag{2.2}$$

$$U_f(t,s,u_\tau)\cdot U_f(s,\tau,u_\tau) = U_f(t,\tau,u_\tau),\ \forall\, u_\tau \in \Sigma,\ t \geq \tau,\ \tau \geq 0, \tag{2.3}$$

$$U_f(\tau,\tau) = Id,\ \forall\, \tau \in R. \tag{2.4}$$

Let $\{T(s)\}$ be the translation semigroup on Σ, we see that the family of processes $\{U_f(t, \tau)\}$ $(f \in \Sigma)$ satisfies the translation identity if

$$U_f(t+s,\tau+s) = U_{T(s)f}(t,\tau),\ \forall\, f \in \Sigma,\ t \geq \tau,\ \tau \in R, \tag{2.5}$$

$$T(s)\Sigma = \Sigma,\ \forall s \geq 0. \tag{2.6}$$

Next, we recall a simple method to derive uniformly asymptotical compactness which can be found in [28].

Definition 2.1 *Let X be a Banach space and B be a bounded subset of X, Σ be a symbol space. We call a function $\varphi(\cdot,\cdot;\cdot,\cdot)$ defined on $(X \times X) \times (\Sigma \times \Sigma)$ to be a contractive function on $B \times B$ if for any sequence $\{x_n\}_{n=1}^{\infty} \subset B$ and any $\{g_n\} \subset \Sigma$, there are subsequences $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and $\{g_{n_k}\}_{k=1}^{\infty} \subset \{g_n\}_{n=1}^{\infty}$ such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi(x_{n_k},x_{n_l};g_{n_k},g_{n_l}) = 0. \tag{2.7}$$

We denote the set of all contractive functions on $B \times B$ by $Contr(B, \Sigma)$.

Lemma 2.2 *Let $\{U_f(t, \tau)\}(f \in \Sigma)$ be a family of processes satisfying the translation identity (2.5) and (2.6) on Banach space X and has a bounded uniform (w.r.t $f \in \Sigma$) absorbing set $B_0 \subset X$. Moreover, assume that for any $\varepsilon > 0$, there exist $T = T(B_0, \varepsilon)$ and $\varphi_T \in Contr(B_0, \Sigma)$ such that*

$$\begin{aligned}\|U_{f_1}(T,0)x - U_{f_2}(T,0)y\| &\leq \varepsilon + \phi_T(x,y;f_1,f_2),\\ &\qquad \forall\, x,y \in B_0,\ \forall f_1,f_2 \in \Sigma.\end{aligned} \tag{2.8}$$

Then $\{U_f(t, \tau)\}$ $(f \in \Sigma)$ is uniformly (w.r.t. $f \in \Sigma$) asymptotically compact in X.

Theorem 2.3 *Assume that $f \in \Sigma \subseteq L^2(R, H)$, $u_\tau \in V$, then the problem (1.1)-(1.4) possesses uniform attractors $\mathcal{A}_f^1(t)$ in V.*

Theorem 2.4 *Assume that $f \in \Sigma \subseteq L^2(R, H)$, $u_\tau \in W$, then the problem (1.1)-(1.4) possesses uniform attractors $\mathcal{A}_f^2(t)$ in W.*

3 Proof of Theorem 2.3

In this section, we shall prove Theorem 2.3 by two steps as follows, the first one is to get the existence of an absorbing ball, the second is to prove the asymptotical compactness by means of a contractive function.

From the property of solutions, we can easily derive that the set class $\{U_f(t, \tau, u_\tau)\}$ $(\tau, \leq t)$ is a process in V for all $\tau \leq t$. Moreover, the mapping $U_f(t, \tau, u_\tau)$: $V \to V$ is continuous.

Lemma 3.1 *We assume that $\{u_\tau^n\} \subset V$, $u_\tau \in V$ and $u_\tau^n \rightharpoonup u_\tau$, $f^n \to f$ in $L^2(R, H)$, then*

$$U_{f^n}(t,\tau,u_\tau^n)u_\tau^n \rightharpoonup U_f(t,\tau,u_\tau)u_\tau \quad \textit{weakly star in } L^\infty((\tau,T);H),\ \forall\, t\geq\tau, \tag{3.1}$$

$$U_{f^n}(\cdot,\tau,u_\tau^n)u_\tau^n \rightharpoonup U_f(\cdot,\tau,u_\tau)u_\tau \quad \textit{weakly in } L^2(\tau,T;V),\ \forall\, t\geq\tau. \tag{3.2}$$

Proof. From the boundedness of the solutions in corresponding topological spaces, we easily conclude the results. □

Lemma 3.2 *Assume* $f\in L^2(R,H)$, $u_\tau\in V$, *then there exists a uniform (w.r.t.* $f\in\Sigma$*) absorbing set* B_0 *of processes* $\{U_f(t,\tau,u_\tau)\}$.

Proof. For all $u\in V$, multiplying both sides of (1.1) with u and noting that $((u\cdot\nabla)u, u) = 0$, we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2+\alpha^2\|\nabla u(t)\|^2)+2\nu\|\nabla u(t)\|^2 &\leq 2(f(t),u(t))\\ &\leq 2\nu\|\nabla u(t)\|^2+\frac{2}{\nu\lambda}\|f(t)\|^2.\end{aligned}\tag{3.3}$$

Consequently, for all $\tau\in R$, there holds

$$\|u(t)\|^2+\alpha^2\|\nabla u(t)\|^2 \leq (\|u_\tau\|^2+\alpha^2\|\nabla u(\tau)\|^2)+\frac{2}{\nu\lambda}\int_\tau^t\|f(\xi)\|^2\mathrm{d}\xi. \tag{3.4}$$

Consider the property of the functional $\langle\cdot,\cdot\rangle+\alpha^1\langle\nabla\cdot,\nabla\cdot\rangle$, we get

$$C_1\|\cdot\|_V^2\leq\langle\cdot,\cdot\rangle+\alpha^2\langle\nabla\cdot,\nabla\cdot\rangle\leq C_2\|\cdot\|_V^2,\ 0<C_1\leq C_2,$$

and there exists a constant C_0 satisfying $C_1\leq C_0\leq C_2$, such that

$$C_0\|\cdot\|_V^2=\langle\cdot,\cdot\rangle+\alpha^2\langle\nabla\cdot,\nabla\cdot\rangle.$$

Setting the radius $r^2=\|u_\tau\|^2+\alpha^2\|\nabla u_\tau\|^2$, we easily get that there exists a constant $C>0$ such that

$$\|U_f(t,\tau,u_\tau)\|_V^2\leq Cr^2+\frac{2C}{\nu\lambda}\int_{-\infty}^t\|f(\xi)\|^2\mathrm{d}\xi, \tag{3.5}$$

for all $u_\tau\in V$, $t\geq\tau$.

Setting

$$r^2\leq\frac{2}{\nu\lambda}\int_{-\infty}^t\|f(\xi)\|^2\mathrm{d}\xi,$$

then we denote R the nonnegative number given by

$$R^2=\frac{2C}{\nu\lambda}\int_{-\infty}^t\|f(\xi)\|^2\mathrm{d}\xi, \tag{3.6}$$

and consider the family of closed balls B_0 in V defined by

$$B_0=\{v\in V\mid\|v\|_V\leq 2R\}. \tag{3.7}$$

It is straightforward to check that B_0 is a uniform absorbing ball for the processes $\{U_f(t,\tau,u_\tau)\}$. □

Lemma 3.3 *Under the condition of* $f \in L^2(R, H)$, *the process* $\{U_f(t, \tau, u_\tau)\}$ *generated by the global solutions for problem (1.1)-(1.4) is uniformly asymptotically compact in V.*

Proof. For any initial data $u_\tau^i \in B_0 (i = 1, 2)$, let $u^i(t, x)$ be the corresponding solutions to the symbols f^i with u_τ^i, that is, $u^i(t)$ is the solution of the problem:

$$u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f^i(t, x), \quad x \in \Omega, \quad t \in R_\tau, \tag{3.8}$$

$$divu = 0, \quad t \in R_\tau, \tag{3.9}$$

$$u(t, x)|_{\partial\Omega} = 0, \quad t \in R_\tau, \tag{3.10}$$

$$u(\tau, x) = u_\tau^i(x), \quad \tau \in R. \tag{3.11}$$

Denote

$$w(t) = u^1(t) - u^2(t), \tag{3.12}$$

then $w(t)$ satisfies the equivalent abstract equations

$$w_t + \alpha A w_t - \nu A w + B(u^1) - B(u^2) = f^1(t, x) \ - f^2(t, x), \tag{3.13}$$

$$divw = 0, \tag{3.14}$$

$$w(t, x)|_{\partial\Omega} = 0, \tag{3.15}$$

$$w(\tau, x) = u_\tau^1(x) - u_\tau^2(x), \quad \tau \in R, \tag{3.16}$$

where $B(u) = (u \cdot \nabla) u$, p has disappeared by the projection operator P.
Setting

$$E_w(t) = \frac{1}{2} \int_\Omega |\ w(t)\ |^2 \mathrm{d}x + \frac{\alpha^2}{2} \int_\Omega |\ \nabla w(t)\ |^2 \mathrm{d}x. \tag{3.17}$$

Multiplying (3.13) by w and integrating over $[s, T] \times \Omega$, we deduce

$$\begin{aligned} &E_w(T) - E_w(s) + \nu \int_s^T \int_\Omega |\nabla w(h)|^2 \mathrm{d}x\mathrm{d}h \\ &+ \int_s^T \int_\Omega \left(B(u^1(h)) - B(u^2(h))\right) w(h) \mathrm{d}x\mathrm{d}h \\ &= \int_s^T \int_\Omega \left(f^1(h) - f^2(h)\right) w(h) \mathrm{d}x\mathrm{d}h, \end{aligned} \tag{3.18}$$

where $\tau \le s \le T$. Then we have

$$\begin{aligned} \nu \int_\tau^T \int_\Omega |\nabla w(h)\ |\ \mathrm{d}x\mathrm{d}h \le E_w(\tau) - \int_\tau^T \int_\Omega \left(B(u^1(h)) - B(u^2(h))\right) w(h) \mathrm{d}x\mathrm{d}h \\ + \int_\tau^T \int_\Omega \left(f^1(h) - f^2(h)\right) w(h) \mathrm{d}x\mathrm{d}h. \end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned}\int_\tau^T E_w(s)\mathrm{d}s \quad &= \quad \int_\tau^T \left(\frac{1}{2}\int_\Omega \text{—}w(s)\text{—}^2\mathrm{d}x + \frac{\alpha^2}{2}\int_\Omega \text{—}\nabla w(s)\text{—}^2\mathrm{d}x\right)\mathrm{d}s \\ &\le \quad C\int_\tau^T\int_\Omega \text{—}\nabla w(s)\text{—}^2\mathrm{d}x\mathrm{d}s \\ &\le \quad C\Bigg[E_w(\tau) - \int_\tau^T\int_\Omega \left(B(u^1(s)) - B(u^2(s))\right)w(s)\mathrm{d}x\mathrm{d}s \\ &\qquad + \int_\tau^T\int_\Omega \left(f^1(s) - f^2(s)\right)w(s)\mathrm{d}x\mathrm{d}s\Bigg].\end{aligned} \tag{3.20}$$

Integrating (3.18) over $[\tau, T]$ with respect to s, we get

$$\begin{aligned}TE_w(T) \quad &+ \nu\int_\tau^T\int_s^T\int_\Omega |\nabla w(h)|\,\mathrm{d}x\mathrm{d}h\mathrm{d}s \\ &\le \int_\tau^T E_w(s)\mathrm{d}s - \int_\tau^T\int_s^T\int_\Omega \left(B(u^1(h)) - B(u^2(h))\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\ &\quad + \int_\tau^T\int_s^T\int_\Omega \left(f^1(h) - f^2(h)\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\ &\le C\Bigg[E_w(\tau) - \int_\tau^T\int_\Omega \left(B(u^1(s)) - B(u^2(s))\right)w(s)\mathrm{d}x\mathrm{d}s \\ &\quad + \int_\tau^T\int_\Omega \left(f^1(s) - f^2(s)\right)w(s)\mathrm{d}x\mathrm{d}s\Bigg] \\ &\quad - \int_\tau^T\int_s^T\int_\Omega \left(B(u^1(h)) - B(u^2(h))\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\ &\quad + \int_\tau^T\int_s^T\int_\Omega \left(f^1(h) - f^2(h)\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s.\end{aligned} \tag{3.21}$$

If we set

$$\begin{aligned}&C_0 = CE_w(\tau), \\ &\phi(u_0^1, u_0^2; g^1(t), g^2(t)) = C\Bigg[-\int_\tau^T\int_\Omega \left(B(u^1(s)) - B(u^2(s))\right)w(s)\mathrm{d}x\mathrm{d}s \\ &\qquad + \int_\tau^T\int_\Omega \left(f^1(s) - f^2(s)\right)w(s)\mathrm{d}x\mathrm{d}s\Bigg] \\ &\qquad - \int_\tau^T\int_s^T\int_\Omega \left(B(u^1(h)) - B(u^2(h))\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\ &\qquad + \int_\tau^T\int_s^T\int_\Omega \left(f^1(h) - f^2(h)\right)w(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s,\end{aligned} \tag{3.22}$$

then we have

$$E_w(T) \le \frac{C_0}{T} + \frac{1}{T}\phi(u_0^1, u_0^2; f^1(t), f^2(t)). \tag{3.23}$$

Since the family of processes has a uniformly bounded absorbing set, we choose T large enough such that

$$\frac{C_0}{T} \le \varepsilon, \tag{3.24}$$

i.e., $T \ge \frac{C_0}{\varepsilon}$.

Let u^n, u^m be the solutions with respect to the initial data u_0^n, u_0^m and symbols $f^n(t)$, $f^m(t) \in \Sigma$, m, $n = 1, 2, \ldots$ respectively. Then from Lemma 3.1, we can derive

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \left(f^n(h)-f^m(h)\right)\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}h\mathrm{d}s = 0,$$
$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_\Omega \left(f^n(s)-f^n(s)\right)\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}s = 0,$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_\Omega \left(B(u^n(s))-B(u^m(s))\right)\\
&\times\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega \left((u^n(s)\cdot\nabla)u^n(s)-(u^m(s)\cdot\nabla)u^m(s)\right)\\
&\times\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega \left(((u^n(s)-u^m(s))\cdot\nabla)u^n(s)-(u^m(s)\cdot\nabla)\right.\\
&\left.\times(u^m(s)-u^n(s))\right)\times(u^n(s)-u^m(s))\mathrm{d}x\mathrm{d}s\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \left(B(u^n(h))-B(u^m(h))\right)\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \left((u^n(h)\cdot\nabla)u^n(h)-(u^m(h)\cdot\nabla)u^m(h)\right)\\
&\times\left(u^n(s)-u^m(s)\right)\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \left(((u^n(h)-u^m(h))\cdot\nabla)u^n(h)-(u^m(h)\cdot\nabla)\right.\\
&\left.\times(u^m(h)-u^n(h))\right)\times(u^n(s)-u^m(s))\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=0.
\end{aligned}$$

Hence $\phi(u_0^1, u_0^2; f^1(t), f^2(t)) \in Contr(B_0, \Sigma)$ for the above T. By Lemma 2.2 and the property of the functional $\langle\cdot,\cdot\rangle + \alpha^2\langle\nabla\cdot,\nabla\cdot\rangle$, the conclusion holds. □

Proof of Theorem 2.3 From Lemmas 3.1-3.3, we can deduce the result easily. □

4 Proof of Theorem 2.4

Similarly to the proof of Theorem 2.3, we easily obtain that the set class $\{U_f(t, \tau, u_\tau)\}$ ($\tau \le t$) is a process in W for all $\tau \le t$. Moreover, the mapping $U_f(t, \tau, u_\tau)$: $W \to W$ is continuous. If we assume that $\{u_\tau^n\}$ is a sequence in W and weakly converges to $u_\tau \in W$, $f^n \to f$ in $L^2(R, H)$, then

$$U_{f^n}(t,\tau,u_\tau^n)u_\tau^n \rightharpoonup U_f(t,\tau,u_\tau)u_\tau \textit{ weakly in } W,\ \forall \textit{ fixed } t\ge\tau, \tag{4.1}$$

$$U_{f^n}(\cdot,\tau,u_\tau^n)u_\tau^n \rightharpoonup U_f(\cdot,\tau,u_\tau)u_\tau \textit{ weakly in } L^2(\tau,T;W), \forall\ t\ge\tau. \tag{4.2}$$

Lemma 4.1 *Assume $f \in L^2(R, H)$, then there exists a global uniform (w.r.t. $f \in \Sigma$) absorbing set B_0 of the process $\{U_f(t, \tau, u_\tau)\}$.*

Proof. By the Faedo-Galerkin method, the standard elliptic operator theory and the Poincaré inequality, we get that u belongs to $L^2((\tau, T); D(A)) \cap L^\infty((\tau, T); W)$, then using the Gronwall inequality and similar energy method to the proof of Theorem 3.1 in Qin, Yang and Liu [10], we can deduce the boundedness of u and the existence of absorbing set. □

Lemma 4.2 *Under the condition of* $f \in L^2(R, H)$, $u_\tau \in W$, *the process* $\{U_f(t, \tau, u_\tau)\}$ *generated by the global solutions for problem (1.1)-(1.4) is asymptotically compact in* W.

Proof. For any initial data $u_\tau^i \in B_0(i = 1, 2)$, let $u^i(t, x)$ be the corresponding solutions to the symbols f^i with u_τ^i, that is, $u^i(t)$ is the solution of the problem (3.8)-(3.11). Denote $A = -\Delta$ and $w(t) = u^1(t) - u^2(t)$, then $w(t)$ satisfies the equivalent abstract equations (3.13)-(3.14).

Setting

$$E_w(t) = \frac{1}{2}\int_\Omega |\nabla w(t)|^2 \mathrm{d}x + \frac{\alpha^2}{2}\int_\Omega |Aw(t)|^2 \mathrm{d}x. \tag{4.3}$$

Multiplying (3.13) by Aw and integrating over $[s, T] \times \Omega$, we deduce

$$\begin{aligned} & E_w(T) - E_w(s) + \nu \int_s^T \int_\Omega |Aw(h)|^2 \mathrm{d}x\mathrm{d}h \\ & + \int_s^T \int_\Omega (B(u^1(h)) - B(u^2(h)))Aw(h)\mathrm{d}x\mathrm{d}h \\ & = \int_s^T \int_\Omega (f^1(h) - f^2(h))Aw(h)\mathrm{d}x\mathrm{d}h, \end{aligned} \tag{4.4}$$

where $\tau \le s \le T$. Then we have

$$\begin{aligned} \nu \int_\tau^T \int_\Omega |\, Aw(h)|\, \mathrm{d}x\mathrm{d}h \le & E_w(\tau) - \int_\tau^T \int_\Omega (B(u^1(h)) - B(u^2(h)))Aw(h)\mathrm{d}x\mathrm{d}h \\ & + \int_\tau^T \int_\Omega (f^1(h) - f^2(h))Aw(h)\mathrm{d}x\mathrm{d}h. \end{aligned} \tag{4.5}$$

Hence,

$$\begin{aligned} \int_\tau^T E_w(s)\mathrm{d}s &= \int_\tau^T \left(\frac{1}{2}\int_\Omega |Aw(s)|^2 \mathrm{d}x + \frac{\alpha^2}{2}\int_\Omega |Aw(s)|^2 \mathrm{d}x\right)\mathrm{d}s \\ &\le C\int_\tau^T \int_\Omega |Aw(s)|^2 \mathrm{d}x\mathrm{d}s \\ &\le C\left[E_w(\tau) - \int_\tau^T \int_\Omega (B(u^1(s)) - B(u^2(s)))Aw(s)\mathrm{d}x\mathrm{d}s \right.\\ &\qquad \left. + \int_\tau^T \int_\Omega (f^1(s) - f^2(s))Aw(s)\mathrm{d}x\mathrm{d}s\right]. \end{aligned} \tag{4.6}$$

Integrating (4.4) over $[\tau, T]$ with respect to s, we get

$$
\begin{aligned}
TE_w(T) \quad &+ \nu \int_\tau^T \int_s^T \int_\Omega |Aw(h)|^2 \mathrm{d}x\mathrm{d}h\mathrm{d}s \\
&\le \int_\tau^T E_w(s)\mathrm{d}s - \int_\tau^T \int_s^T \int_\Omega (B(u^1(h)) - B(u^2(h)))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\
&\quad + \int_\tau^T \int_s^T \int_\Omega (f^1(h) - f^2(h))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\
&\le C\Big[E_w(\tau) - \int_\tau^T \int_\Omega (B(u^1(s)) - B(u^2(s)))Aw(s)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_\tau^T \int_\Omega (f^1(s) - f^2(s))Aw(s)\mathrm{d}x\mathrm{d}s\Big] \\
&\quad - \int_\tau^T \int_s^T \int_\Omega (B(u^1(h)) - B(u^2(h)))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\
&\quad + \int_\tau^T \int_s^T \int_\Omega (f^1(h) - f^2(h))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s.
\end{aligned}
\tag{4.7}
$$

If we set

$$
\begin{aligned}
C_0 &= CE_w(\tau), \\
\phi(u_0^1, u_0^2; g^1(t), g^2(t)) &= C\Big[-\int_\tau^T \int_\Omega (B(u^1(s)) - B(u^2(s)))Aw(s)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_\tau^T \int_\Omega (f^1(s) - f^2(s))Aw(s)\mathrm{d}x\mathrm{d}s\Big] \\
&\quad - \int_\tau^T \int_s^T \int_\Omega (B(u^1(h)) - B(u^2(h)))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s \\
&\quad + \int_\tau^T \int_s^T \int_\Omega (f^1(h) - f^2(h))Aw(h)\mathrm{d}x\mathrm{d}h\mathrm{d}s,
\end{aligned}
\tag{4.8}
$$

then we have

$$
E_w(T) \le \frac{C_0}{T} + \frac{1}{T}\phi(u_0^1, u_0^2; f^1(t), f^2(t)). \tag{4.9}
$$

Since the family of the processes has a uniformly bounded absorbing set, we choose T large enough such that

$$
\frac{C_0}{T} \le \varepsilon, \tag{4.10}
$$

i.e., $T \ge \frac{C_0}{\varepsilon}$.

Let u^n, u^m be the solutions with respect to the initial data u_0^n, u_0^m and symbols $f^n(t)$, $f^m(t) \in \Sigma$, m, $n = 1, 2, \ldots$ respectively. Then we can obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \lim_{m\to\infty} \int_\tau^T \int_s^T \int_\Omega (f^n(h) - f^m(h))(Au^n(s) - Au^m(s))\mathrm{d}x\mathrm{d}h\mathrm{d}s = 0, \\
&\lim_{n\to\infty} \lim_{m\to\infty} \int_\tau^T \int_\Omega (f^n(s) - f^n(s))(Au^n(s) - Au^m(s))\mathrm{d}x\mathrm{d}s = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_\Omega (B(u^n(s))-B(u^m(s)))(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega \big((u^n(s)\cdot\nabla)u^n(s)-(u^m(s)\cdot\nabla)u^m(s)\big)\\
&\times(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega \big(((u^n(s)-u^m(s))\cdot\nabla)u^n(s)-(u^m(s)\cdot\nabla)\\
&\times(u^m(s)-u^n(s))\big)\times(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}s\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega (B(u^n(h))-B(u^m(h)))(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \big((u^n(h)\cdot\nabla)u^n(h)-(u^m(h)\cdot\nabla)u^m(h)\big)\\
&\times(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\int_\tau^T\int_s^T\int_\Omega \big(((u^n(h)-u^m(h))\cdot\nabla)u^n(h)-(u^m(h)\cdot\nabla)\\
&\times(u^m(h)-u^n(h))\big)\times(Au^n(s)-Au^m(s))\mathrm{d}x\mathrm{d}h\mathrm{d}s\\
&=0.
\end{aligned}$$

Hence $\phi(u_0^1, u_0^2; f^1(t), f^2(t)) \in Contr(B_0, \Sigma)$ for the above T. By Lemma 2.2 and the property of the functional $\langle\cdot, A\cdot\rangle + \alpha^2\langle A\cdot, A\cdot\rangle$, the conclusion holds. □

Proof of Theorem 2.4 From Lemmas 4.1-4.2, we can deduce the result easily. □

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Authors' contributions
The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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