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Uniform blow-up rate for a porous medium equation with a weighted localized source

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Abstract

In this article, we investigate the Dirichlet problem for a porous medium equation with a more complicated source term. In some cases, we prove that the solutions have global blow-up and the rate of blow-up is uniform in all compact subsets of the domain. Moreover, in each case, the blow-up rate of $|u(t)|_\infty$ is precisely determined.

Keywords: porous medium equation, localized source, blow-up, uniform blow-up rate

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. We consider the following parabolic equation with a localized reaction term

$$v_\tau - \Delta v^m = a(x)v^{q_1}(x,\tau)v^{s_1}(x_0,\tau), \quad x \in \Omega, \tau > 0, \tag{1.1}$$

$$v(x,\tau) = 0, \quad x \in \partial\Omega, \tau > 0, \tag{1.2}$$

$$v(x,\ 0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

where $m \geq 1$, $q_1 \geq 0$, $s_1 > 0$ and $x_0 \in \Omega$ is a fixed point. Throughout this article, we assume the functions $a(x)$ and $v_0(x)$ satisfy the following conditions:

(A1) $a(x)$ and $v_0(x) \in C^2(\Omega)$; $a(x)$, $v_0(x) > 0$ in Ω and $a(x) = v_0(x) = 0$ on $\partial\Omega$.

When $\Omega = B = \{x \in \mathbb{R}^N;\ |x| < R\}$, we sometimes assume

(A2) $a(x)$ and $v_0(x)$ are radially symmetric; $a(r)$ and $v_0(r)$ are non-increasing for $r \in [0, R]$.

Problems (1.1)-(1.3) arise in the study of the flow of a fluid through a porous medium with an internal localized source and in the study of population dynamics (see [1-3]). Porous medium equations ($m > 1$) with or without local sources have been studied by many authors [4-6].

Concerning (1.1)-(1.3), to the best of authors knowledge, a number of articles have studied it from the point of the view of blow-up and global existence [7-10]. Many studies have been devoted to the case $m = 1$ [10-13]. The case $m = 1$, $a(x) = 1$, $q_1 = 0$, $s_1 \geq 1$ and $m = 1$, $a(x) = 1$, q_1, $s_1 > 1$ were studied by Souple [10,11]. Souple [10] demonstrated that the positive solution blows up in finite time if the initial value v_0 is large enough. In the case $a(x) = 1$, $q_1 = 0$, and $s_1 > 1$, Souple [11] showed that the solution $v(x, \tau)$ blows up globally and the blow-up rate is precisely determined. The case $q_1 = 0$ and $s_1 > 0$ was studied by Cannon and Yin [12] and Chandam et al. [13]. Cannon and Yin [12] studied its local solvability and Chandam et al. [13] investigated its blow-up properties.

The study of this article is motivated by some recent results of related problems (see [14][15][16]. In the case of $a(x)$(= constant), the global existence and blow-up behavior have been considered by Chen and Xie [15]. It turns out that if $q_1 + s_1 < m$ or $q_1 + s_1 = m$ and $a(x)$(= constant) is sufficiently small, there exists a global solution of problem (1.1)-(1.3); if $q_1 + s_1 > m$, the solution of problem (1.1)-(1.3) blows up for large initial datum while it admits a global solution for small initial datum. Furthermore, Du and Xiang [16] obtained the blow-up rate estimates under some appropriate hypotheses on initial datum. For some related localized models arising in physical phenomena, we refer the readers to [17-19] and the references therein.

For the localized semi-linear parabolic equation of the form

$$v_\tau - \Delta v = v^{q_1}(x,\tau)v^{s_1}(x_0,\tau), \quad x \in \Omega, \tau > 0, \tag{1.4}$$

with the Dirichlet boundary condition (1.2) and the initial condition (1.3). In [20], Li and Wang proved that the blow-up set to system (1.2)-(1.4): (a) the system possesses total blow-up when $q_1 \le 1$; (b) the system presents single point blow-up patterns when $q_1 > 1$.

We now restrict ourselves to the problem of the form

$$v_\tau - \Delta v^m = a(x)v^{q_1}(x,\tau)v^{s_1}(0,\tau), \quad x \in B, \tau > 0, \tag{1.5}$$

$$v(x,\tau) = 0, \quad x \in \partial B, \tau > 0, \tag{1.6}$$

$$v(x,0) = v_0(x), \quad x \in B, \tag{1.7}$$

where $q_1 \ge 0$, $s_1 > 0$, and $q_1 + s_1 > m > 0$. When $m = 1$, it was proved in [14] that

(1) If $0 \le q_1 \le 1$ and $q_1 + s_1 > 1$, then the solution of (1.5)-(1.7) blows up in a finite time T.

(2) If $q_1 > 1$, then $x = 0$ is the only blow-up point for (1.5)-(1.7).

In the meantime, they obtained the blow-up rate estimate but less precise. Namely,

(i) If $0 \le q_1 < 1$, then for any $x \in B$

$$C_1(a(x))^{1/(1-q_1)} \le v(x,\tau)(T-\tau)^{1/(q_1+s_1-1)} \le C_2, \quad as\ \tau \to T,$$

where $C_1 = ((a(0))^{s_1/(1-q_1)}(q_1+s_1-1))^{1/(1-q_1-s_1)}$, $C_2 = (a(0)(p+q-1))^{1/(1-q_1-s_1)}$.

(ii) If $q_1 = 1$, then for any $x \in B$

$$\frac{a(x)}{a(0)} \ln(T-\tau)^{-1/s_1} \le \ln v(x,\tau) \le \ln(T-\tau)^{-1/s_1}, \quad as\ \tau \to T.$$

It seems that the results of [14] can be extended to $m \ge 1$ and the blow-up rate can be precisely determined. Motivated by this, in this article, we will extend and improve the results of [14].

The purpose of this article is to determine the blow-up rate of solutions for a nonlinear parabolic equation with a weighted localized source, that is, we investigate how the localized source and the local term affect the blow-up properties of the problem

(1.5)-(1.7). Indeed, we find that when $q_1 \le 1$, the solution of (1.5)-(1.7) blows up at the whole domain with a uniformly blow-up profile.

The rest of this article is organized as follows. The results are stated in Section 2. We then prove these results in Section 3.

2 Preliminaries and Main Results

The following two theorems are our main results.

Theorem 2.1 *Assume* $q_1 + s_1 > m$, (A1) *and* (A2) *hold. Let* $v(x, t)$ *be the solution of problem* (1.5)-(1.7), *then* $v(x,t)$ *blows up provided that the initial value* $v_0(x)$ *is sufficiently large.*

The method used in the proof Theorem 2.1 is originally due to [8,18], and bears much resemblance to that of Theorem 3.2 in [15] and Theorem 1.3 in [16]. Therefore, we omitted them here.

For the case $q_1 > 1$, we do not know how to deal with the uniform blow-up rate of problem (1.5)-(1.7). In the following, we focus only on the case of $0 \le q_1 \le 1$.

Theorem 2.2 *Assume* (A1) *and* (A2) *hold. Let* $v(x, t)$ *be the blow-up solution of* (1.5)-(1.7), *which blows up in finite time* T *and* $v(x, t)$ *is non-decreasing in time, then the following limits hold uniformly in all compact subsets of* B.

(i) *If* $0 \le q_1 < 1$, *then*

$$\lim_{\tau\to T}(T-\tau)^{1/(q_1+s_1-1)}v(x,\tau) = C(a(x))^{1/(1-q_1)}, \tag{2.1}$$

where $C = ((q_1+s_1-1)(a(0))^{s_1/(1-q_1)})^{1/(1-q_1-s_1)}$.

(ii) *If* $q_1 = 1$, *then*

$$\lim_{\tau\to T}\ln v(x,\tau) = \frac{a(x)}{a(0)}\ln(T-\tau)^{-1/s_1}. \tag{2.2}$$

Remark 2.1 *The domain we considered here is a ball, it seems that the results of Theorem* 2.2 *remain valid for the general domain. (It is an open problem in this case.)*

To get the blow-up profiles for problem (1.5)-(1.7), we need some transformations. Let $u(x, t) = v^m(x, \tau)$, $t = m\tau$, then (1.5)-(1.7) becomes

$$\begin{cases} u_t = u^p(\Delta u + a(x)u^q(x,t)u^s(0,t)), & x \in B, t > 0, \\ u(x,t) = 0, & x \in \partial B, t > 0, \\ u(x,0) = u_0(x) = v_0^m(x), & x \in B, \end{cases} \tag{2.3}$$

where $0 \le p = (m - 1)/m < 1$, $q = q_1/m$, and $s = s_1/m$.

Under above transformation, assumptions (A1) and (A2) become

(B1) $a(x)$ and $u_0(x) \in C^2(B)$; $a(x)$, $u_0(x) > 0$ in B and $a(x) = u_0(x) = 0$ on ∂B.

(B2) $a(x)$ and $u_0(x)$ are radially symmetric; $a(r)$ and $u_0(r)$ are non-increasing for $r \in [0, R]$.

In our consideration, a crucial role is played by the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & \text{in } B, \\ \varphi(x) = 0, & \text{on } \partial B. \end{cases} \tag{2.4}$$

Denote λ be the first eigenvalue and by ϕ the corresponding eigenfunction with $\phi(x) > 0$ in B, normalized by $\int_B a(x)\varphi(x)dx = 1$.

3 Proof of Theorem 2.2

For convenience, we denote

$$g(t) = u^s(0,t) \quad and \quad G(t) = \int_0^t g(s)\mathrm{d}s.$$

Before proving our result, we would like to give a property of the following problem

$$\begin{cases} w_t = w^\alpha(\Delta w + a(x)g(t)), & x \in B, t > 0, \\ w(x,t) = 0, & x \in \partial B, t > 0, \\ w(x,0) = w_0(x) = u_0^{1-q}(x), x \in B, \end{cases} \tag{3.1}$$

where $0 \le \alpha \le 1$ and $w = u^{1-q}(x, t)$.

Lemma 3.1 *Assume* (B1) *and* (B2) *hold. Let $w(x, t)$ be the solution of Equation* (3.1)*, which blows up in a finite time T^* and non-decreasing in time t, then the following limits hold uniformly in all compact subsets of B.*

(i) *If* $0 \le \alpha < 1$*, then*

$$\lim_{t\to T^*} \frac{w^{1-\alpha}(x,t)}{G(t)} = (1-\alpha)a(x).$$

(ii) *If* $\alpha = 1$*, then*

$$\lim_{t\to T^*} \frac{\ln w(x,t)}{G(t)} = a(x).$$

Proof. (i) Assumption (B2) implies $w_r \le 0$ $(r = |x|)$, it then follows that $w(0,t) = \max_{x\in\bar{B}} w(x,t)$ and $\Delta w(0, t) \le 0$ for $t > 0$. From (3.1), we then get

$$\frac{\mathrm{d}w^{1-\alpha}(0,t)}{\mathrm{d}t} \le (1-\alpha)a(0)g(t), \quad 0 < t < T^*.$$

.

Consequently,

$$\lim_{t\to T^*} \sup \frac{w^{1-\alpha}(0,t)}{G(t)} \le (1-\alpha)a(0), \tag{3.2}$$

which implies

$$\lim_{t\to T^*} G(t) = \infty \quad and \quad \lim_{t\to T^*} g(t) = \infty.$$

Moreover, it is apparent that $\lim_{t\text{-}T^*} w(0, t)/g(t) = 0$, since $s > 1 - q$.

Set $R_1 \in (0, R)$, $B_1 = \{x \in \mathbb{R}^N, \mid x \mid < R_1\}$ and $b(x) = 1/a(x)$, $x \in B_1$. Since $a'(r) \le 0$, we obtain that $b'(r) \ge 0$, for $0 \le r \le R_1$.

We now introduce the function

$$w_1(x,t) = b^{1/(1-\alpha)}(x)w(x,t), \quad x \in B_1, 0 < t < T^*.$$

By a simple calculation, and note that $\nabla w(x, t)\nabla b(x) = u_r(r, t)b'(r) \le 0$, then there exist m_1, $m_2 > 0$ such that

$$b(x)\Delta w(x,\ t) \ge m_1\Delta w_1(x,t) - m_2 w(x,t) \quad x \in B_1, 0 < t < T^*.$$

Setting $\varepsilon(t) = m_2 w(0, t)/g(t)$. From $\lim_{t\to T^*} w(0, t)/g(t) = 0$, we infer that there exists $t_1 \in (0, T^*)$ such that $0 < \varepsilon(t) \le 1/2$ for $t_1 \le t < T^*$.

Hence, in view of (3.1), we observe

$$\begin{aligned}\frac{1}{1-\alpha}({w_1}^{1-\alpha})_t &= b(x)\Delta w + g(t)\\ &\ge m_1\Delta w_1 + (1-\varepsilon(t))g(t) + \varepsilon(t)g(t) - m_2 w(0,t)\\ &= m_1\Delta w_1 + (1-\varepsilon(t))g(t), \quad x\in B_1, \quad t_1 < t < T^*.\end{aligned}$$

Set $g_1(t) = (1 - \varepsilon(t))g(t)$, $G_1(t) = \int_{t_1}^t g(s)ds$, we then obtain

$$\lim_{t\to T^*} G_1(t) = \infty \quad \text{and} \quad \lim_{t\to T^*}\frac{G_1(t)}{G(t)} = 1.$$

Obviously, $w_1(x, t)$ is a sup-solution of the following equation

$$\begin{cases}(w^*)_t = (w^*)^\alpha((m_1 \triangle w^* + g_1(t)), & x\in B_1, \quad t_1 < t < T^*,\\ w^*(x,t) = 0, & x\in \partial B_1,\ t > t_1,\\ w^*(x,t_1) = b^{1/1-\alpha}(x)w(x,t_1), & x\in B_1.\end{cases}$$

By the maximum principle, $w_1(x, t) \ge w^*(x, t)$ and $w_r^* \le 0$. Similar to the proof of (4.15) in [15] that

$$\lim_{t\to T^*}\frac{(w^*)^{1-\alpha}(x,t)}{G(t)} = (1-\alpha),$$

uniformly in all compact subsets of B_1,

Therefore, by the arbitrariness of B_1, we obtain that the following inequatlity holds uniformly in all compact subsets of B

$$\lim_{t\to T^*}\inf\frac{w^{(1-\alpha)}(x,t)}{G(t)} \ge (1-\alpha)a(x). \tag{3.3}$$

In particular,

$$\lim_{t\to T^*}\inf\frac{w^{(1-\alpha)}(0,t)}{G(t)} \ge (1-\alpha)a(0). \tag{3.4}$$

From (3.2) and (3.4), we deduce

$$\lim_{t\to T^*}\frac{w^{(1-\alpha)}(0,t)}{G(t)} = (1-\alpha)a(0). \tag{3.5}$$

Multiplying both sides of (3.1) by ϕ and integrating over $B \times (0, t)$, $0 < t < T^*$

$$\frac{1}{1-\alpha}\left(\int_B w^{1-\alpha}\varphi \mathrm{d}x - \int_B w_0^{1-\alpha}\varphi \mathrm{d}x\right) = -\lambda\int_0^t\int_B w\varphi \mathrm{d}x\mathrm{d}s + G(t).$$

Since $\int_0^t\int_B w\varphi \mathrm{d}x\mathrm{d}s \le \int_B \varphi \mathrm{d}x \int_0^t w(0,s)\mathrm{d}s$, so we have

$$\lim_{t\to T^*}\frac{\int_0^t\int_B w\varphi \mathrm{d}x\mathrm{d}s}{G(t)} = 0.$$

It then follows that

$$\lim_{t\to T^*}\frac{\int_B w^{1-\alpha}\varphi \mathrm{d}x}{G(t)}=(1-\alpha). \tag{3.6}$$

Note that $w_r \le 0$, (3.3) and (3.6), it is sufficient to prove

$$\lim_{t\to T^*}\sup\frac{w^{1-\alpha}(x,t)}{G(t)}\le(1-\alpha)a(x),\quad \forall x\in B. \tag{3.7}$$

Assume on the contrary that there exists a point $x_1 \in B$, $x_1 \neq 0$ such that

$$\lim_{t\to T^*}\sup w^{1-\alpha}(x_1,t)/G(t)=c>(1-\alpha)a(x_1).$$

Then there exists a sequence $\{t_n\}$ such that $t_n \to T^*$

$$\lim_{t_n\to T^*}\sup w^{1-\alpha}(x_1,t_n)/G(t_n)=c>(1-\alpha)a(x_1).$$

By the continuity of a(x), we deduce that there exists $x_2 \in B$ such that $(1 - \alpha)a(x) < c$ for $B_1 = \{x \in \mathbb{R}^n : |x_2| \le |x| \le |x_1|\}$. Using $w_r \le 0$, (3.3) and (3.6), it is easy to check that

$$\begin{aligned}\lim_{t_n\to T^*}\frac{\int_B w^{1-\alpha}(x,t_n)\varphi(x)\mathrm{d}x}{G(t_n)}&=\lim_{t_n\to T^*}\frac{\int_{B\setminus B_1} w^{1-\alpha}(x,t_n)\varphi(x)\mathrm{d}x+\int_{B_1} w^{1-\alpha}(x,t_n)\varphi(x)\mathrm{d}x}{G(t_n)}\\&\ge\int_{B\setminus B_1}(1-\alpha)a(x)\varphi(x)\mathrm{d}x+\lim_{t_n\to T^*}c\int_B\varphi(x)\mathrm{d}x\\&>(1-\alpha),\end{aligned}$$

which is a contradiction to (3.6). Combining (3.3) and (3.7), Lemma 3.1 (i) is proved. Case (ii) can be treated similarly.

The key step in establishing the result of Theorem 2.2 is the following lemma.

Lemma 3.2 *Under the assumption of Lemma* 3.1, *let $u(x, t)$ be the blow-up solution of* (2.3), *which blows up in a finite time T^* and non-decreasing in time t, then the following statements hold uniformly in all compact subsets of B:*

(i) *If $p + q < 1$, then*

$$\lim_{t\to T^*}\frac{u^{1-q-p}(x,t)}{G(t)}=(1-q-p)a(x).$$

(ii) *If $p + q = 1$, then*

$$\lim_{t\to T^*}\frac{\ln u(x,t)}{G(t)}=a(x).$$

Proof. (i) Since $u_r \le 0$ and $u_t \ge 0$, it then follows that $u(0,t)=\max_{x\in\bar{B}}u(x,t)$ and $\Delta u(0, t) \le 0$ for $t > 0$, which imply $\lim_{t\to T^*} u(0, t) = \infty$. Obviously,

$$u_t(0,t)\le a(0)u^{p+q}(0,t)g(t),\quad 0<t<T^*,$$

which implies

$$\lim_{t\to T^*}\sup\frac{u^{1-p-q}(0,t)}{G(t)}\le(1-p-q)a(0). \tag{3.8}$$

Notice that $p + q < 1$ and (3.8), hence $\lim_{t\to T^*} G(t) = \infty$ and $\lim_{t\to T^*} g(t) = \infty$.
A simple calculation yields

$$\frac{1}{1-r}\Delta u^{1-r} = -ru^{-(1+r)}|\nabla u|^2 + u^{-r}\Delta u \quad (\text{if} \quad 0 < r < 1).$$

In view of (2.3), we have, for $x \in \Omega$, $0 < t < T^*$

$$\frac{1-q}{1-p-q}\frac{\mathrm{d}u^{1-p-q}}{\mathrm{d}t} = \Delta u^{1-q} + q(1-q)u^{-q-1}|\nabla u|^2 + (1-q)a(x)g(t). \tag{3.9}$$

Multiplying both sides of Equation (3.9) by ϕ and integrating over $B \times (0, t)$, it follows that

$$\begin{aligned}&\frac{1}{1-p-q}\left(\int_B u^{1-p-q}\varphi \mathrm{d}x - \int_B u_0^{1-p-q}\varphi \mathrm{d}x\right)\\ &\quad = -\frac{\lambda}{1-q}\int_0^t\int_B u^{1-q}\varphi \mathrm{d}x\mathrm{d}s + G(t) + \int_0^t\int_B qu^{-q-1}|\nabla u|^2\varphi \mathrm{d}x\mathrm{d}s,\end{aligned} \tag{3.10}$$

for $0 < t < T^*$. Clearly,

$$\textstyle\int_0^t\int_B u^{1-q}\varphi \mathrm{d}x\mathrm{d}s \le \int_0^t u^{1-q}(0,t)\mathrm{d}s\int_B \varphi(x)\mathrm{d}x, \tag{3.11}$$

which yields

$$\lim_{t\to T^*}\frac{\int_0^t\int_B u^{1-q}\varphi \mathrm{d}x\mathrm{d}s}{G(t)} = 0. \tag{3.12}$$

Setting $u_1(r, t) = u^{(1-q)/2}(r, t)(r = |x|)$. We may claim that

$$\lim_{t\to T^*}\frac{(u_1(r,t))_r}{(g(t))^{1/2}} = 0, \qquad a.e.\ r \in (0, R).$$

Indeed, due to $\lim_{t\to T^*} g(t) = \lim_{t\to T^*} u^s(0, t) = \infty$, $u_r \le 0$, and $s > 1 - q$, we then have

$$\lim_{t\to T^*}\frac{\int_0^R (u_1(r,t))_r\mathrm{d}r}{(g(t))^{1/2}} = \lim_{t\to T^*}\frac{u_1(R,t)-u_1(0,t)}{(g(t))^{1/2}} = 0.$$

Therefore, by Lebesgue's dominated convergence theorem, we infer that

$$\begin{aligned}\lim_{t\to T^*}\frac{\int_B qu^{-q-1}|\nabla u|^2\varphi(x)\mathrm{d}x}{g(t)} &= q\omega_n\lim_{t\to T^*}\frac{\int_0^R u^{-q-1}(r,t)u_r^2\varphi(r)r^{n-1}\mathrm{d}r}{g(t)}\\ &\le q\omega_n R^{n-1}\lim_{t\to T^*}\frac{\int_0^R u^{-q-1}(r,t)u_r^2\varphi(r)\mathrm{d}r}{g(t)}\\ &= \frac{4q}{(1-q)^2}R^{n-1}\omega_n\lim_{t\to T^*}\frac{\int_0^R ((u^{(1-q)/2})_r)^2\varphi(r)\mathrm{d}r}{g(t)}\\ &\le C\int_0^R \lim_{t\to T^*}(\frac{(u^{(1-q)/2})_r}{(g(t))^{1/2}})^2\mathrm{d}r = 0,\end{aligned} \tag{3.13}$$

where w_n is the surface area of unit ball in $\mathbb{R}^N$.

Now according to (3.10)-(3.12), we obtain

$$\lim_{t\to T^*}\frac{\int_B u^{1-p-q}\varphi \mathrm{d}x}{G(t)} = (1-p-q). \tag{3.14}$$

On the other hand, By (3.9), we find

$$\frac{\mathrm{d}u^{1-q}}{\mathrm{d}t} \geq u^p(\Delta u^{1-q} + (1-q)a(x)g(t)), \quad x\in B,\ 0<t<T^*,$$

where $\gamma = p/(1 - q)$. Consequently, u^{1-q} is a sup-solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = v^\gamma(\Delta v + (1-q)a(x)g(t)), & x\in B,\ 0<t<T^*,\\ v(x,t) = 0, & x\in\partial B,\ t>0,\\ v(x,0) = u_0^{1-q}(x), & x\in B.\end{cases}$$

By the maximum principle, $u^{1-q} \geq v$ in $B \times (0, T^*)$. Note that $0 \leq \gamma <1$, we know from Lemma 3.1 (i) that

$$\lim_{t\to T^*}\frac{v^{(1-p-q)/(1-q)}(x,t)}{G(t)} = (1-p-q)a(x),$$

uniformly in all compact subsets of B.

Thus,

$$\lim_{t\to T^*}\inf\frac{u^{(1-p-q)}(x,t)}{G(t)} \geq (1-p-q)a(x), \tag{3.15}$$

uniformly in all compact subsets of B.

Next, we prove that

$$\lim_{t\to T^*}\sup\frac{u^{(1-p-q)}(x,t)}{G(t)} \leq (1-p-q)a(x), \tag{3.16}$$

uniformly in all compact subsets of B.

We can verify (3.15) by similar means of (3.7). Therefore, we conclude the proof of case (i).

(ii) Proceeding as (3.8), we have

$$\lim_{t\to T^*}\sup\frac{\ln u(0,t)}{G(t)} \leq a(0).$$

For any compact subset $B_1 \in B$, there exists $t_1 \in (0, T^*)$ such that $u(x, t_1) \geq 1$ for all $x\in\bar{B}_1$, and thus $\ln u(x, t) \geq 0$ in $\bar{B}_1\times(t_1,T^*)$.

Direct calculation shows

$$\frac{\mathrm{d}\ln u}{\mathrm{d}t} = \frac{1}{1-q}\Delta u^{1-q} + qu^{-q-1}|\nabla u|^2 + a(x)g(t). \tag{3.17}$$

Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(B_1)$ and by ϕ_1 >0 the corresponding eigenfunction, normalized by $\int_{B_1} a(x)\varphi_1(x)dx = 1$. Set $G_1(t) = \int_{t_1}^t g(s)ds$. Clearly, $\lim_{t\to T^*} G(t)/G_1(t) = 1$.

Multiplying both sides of Equation (3.16) by ϕ_1 and integrating over $B_1 \times (t_1, t)$, we get

$$\begin{aligned}&\int_{B_1} (\ln u)\varphi_1 dx - \int_{B_1} (\ln u(x,t_1))\varphi_1 dx \\ &= -\lambda \int_{t_1}^{t}\int_{B_1} u^{1-q}\varphi \mathrm{d}x\mathrm{d}s + \mathrm{G}(t) + \int_{t_1}^{t}\int_{B_1} qu^{-q-1}|\nabla u|^2\varphi \mathrm{d}x\mathrm{d}s, \qquad t_1 < t < T^*.\end{aligned} \tag{3.18}$$

The result of case (ii) follows by analogy with the argument used in the proof of case (i).

Proof of Theorem 2.2

(i) By Lemma 3.2 (i), we infer that

$$u(0,t) \sim ((1-q-p)a(0))^{1/(1-q-p)}G(t), \quad \text{as } t \to T^*,$$

hence

$$G'(t) = g(t) = u^q(0,t) \sim ((1-q-p)a(0))^{q/(1-q-p)}G(t)^{q/(1-q-p)}. \tag{3.19}$$

Integrating equivalence (3.18) between t and T^*, we obtain

$$G(t) \sim (a(0)(1-q-p))^{-1}(a(0)(p+q+s-1)(T^*-t))^{(1-q-p)/(1-p-q-s)}. \tag{3.20}$$

Using Lemma 3.2 (i) and substituting $p = (m - 1)/m$, $q = q_1/m$, $s = s_1/m$, $t^* = m\tau$, and $u(x, t) = v^m(x, \tau)$ into (3.19), we complete the proof of Theorem 2.2 (i).

(ii) To obtain the blow-up rate of the exponent type, we need to be more careful in this case, since exponentiation of equivalents is not permitted. Similar to the proof of Theorem 3 in [14] and Lemma 2.3 in [16], we get

$$\lim_{t\to T^*} G(t) = a^{-1}(0)\ln(T^*-t)^{(-1/s)}. \tag{3.21}$$

Thanks to Lemma 3.2(ii) and (3.20), we then get the desired result.

4 Discussion

This article deals with the porous medium equation with local and localized source terms, represented by two factors $v_{q_1}(x,\tau)$ and $v_{s_1}(0,\tau)$, respectively. As we all know that, in the absence of weight function, the solutions of model (1.5)-(1.7) have a global blow-up and the rate of blow-up is uniform in all compact subsets of the domain. A natural question is what happens in the model (1.5)-(1.7), where the source term is the product of localized source, local source, and weight function. It is shown by Theorem 2.2 that if $0 \le q_1 \le 1$, this equation possesses uniform blow-up profiles. In other words, the localized term plays a leading role in the blow-up profile for this case. Moreover, the blow-up rate estimates in time and space is obtained.

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Authors' contributions

All the authors typed, read, and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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