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Existence result for semilinear elliptic systems involving critical exponents

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Abstract

In this paper we deal with the existence of a positive solution for a class of semilinear systems of multi-singular elliptic equations which involve Sobolev critical exponents. In fact, by the analytic techniques and variational methods, we prove that there exists at least one positive solution for the system.

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1 Introduction

We consider the following elliptic system:

$$\begin{cases} Lu = \frac{\sigma\alpha}{2^*} u|u|^{\alpha-2}|v|^{\beta} + \eta u|u|^{2^*-2} + a_1 u + a_2 v, & \Omega, \\ Lv = \frac{\sigma\beta}{2^*} v|v|^{\beta-2}|u|^{\alpha} + \lambda v|v|^{2^*-2} + a_2 u + a_3 v, & \Omega, \\ u = v = 0, & \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain such that $\xi_i \in \Omega$, $i = 1, 2, \ldots, k$, $k \geq 2$, are different points, $0 \leq \mu_i < \bar{\mu} := (\frac{N-2}{2})^2$, $L := -\Delta \cdot - \sum_{i=1}^{k} \mu_i \frac{\cdot}{|x-\xi_i|^2}$, $\eta, \lambda, \sigma \geq 0$, $a_1, a_2, a_3 \in \mathbb{R}$, $1 < \alpha, \beta < 2^* - 1$, $\alpha + \beta = 2^*$.

We work in the product space $H \times H$, where the space $H := H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla \cdot|^2\, dx)^{\frac{1}{2}}$.

In resent years many publications [1–3] concerning semilinear elliptic equations involving singular points and the critical Sobolev exponent have appeared. Particularly in the last decade or so, many authors used the variational method and analytic techniques to study the existence of positive solutions of systems of the form of (1.1) or its variations; see, for example, [4–8].

Before stating the main result, we clarify some terminology. Since our method is variational in nature, we need to define the energy functional of (1.1) on $H \times H$

$$\begin{aligned} J(u,v) = {} & \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - \sum_{i=1}^{k} \frac{\mu_i (u^2 + v^2)}{|x-\xi_i|^2} \right) dx \\ & - \frac{1}{2} \int_\Omega \left(a_1 u^2 + 2a_2 uv + a_3 v^2 \right) dx - \sigma \frac{\sigma}{2^*} \int_\Omega |u|^\alpha |u|^\beta\, dx \\ & - \frac{\eta}{2^*} \int_\Omega |u|^{2^*}\, dx - \frac{\lambda}{2^*} \int_\Omega |v|^{2^*}\, dx. \end{aligned}$$

Then $J(u,v)$ belongs to $C^1(H\times H,\mathbb{R})$. A pair of functions $(u_0,v_0)\in H\times H$ is said to be a solution of (1.1) if $(u_0,v_0)\neq(0,0)$, and for all $(\varphi,\phi)\in H\times H$, we have

$$\begin{aligned}&\int_\Omega\Bigg(\nabla u_0\cdot\nabla\varphi+\nabla v_0\cdot\nabla\phi-\sum_{i=1}^k\frac{\mu_i(u_0\varphi+v_0\phi)}{|x-\xi_i|^2}-(a_1u_0\varphi+a_2\varphi v_0+a_2\phi u_0+a_3v_0\phi)\\&\quad-\sigma\big(\alpha\varphi u_0|u_0|^{\alpha-2}|v_0|^\beta+\beta\phi v_0|v_0|^{\beta-2}|u_0|^\alpha\big)\\&\quad-\eta|u_0|^{2^*-2}u_0\varphi-\lambda|v_0|^{2^*-2}v_0\phi\Bigg)\,dx=\big\langle J'(u_0,v_0),(\varphi,\phi)\big\rangle=0.\end{aligned}$$

Standard elliptic arguments show that

$$u,v\in C^2\big(\Omega\backslash\{\xi_1,\ldots,\xi_k\}\big)\cap C^1\big(\overline{\Omega}\backslash\{\xi_1,\ldots,\xi_k\}\big).$$

The following assumptions are needed:

($\mathcal{H}_1$) $\eta+\lambda+\sigma>0$, $0\le\mu_1\le\mu_2\le\cdots\le\mu_k<\bar{\mu}-1$ and $\sum_{i=1}^k\mu_i<\bar{\mu}$, $\alpha+\beta>1$, $\alpha+\beta=2^*$, $a_1,a_2,a_3>0$, $a_1a_3-a_2^2>0$,

($\mathcal{H}_2$) $0<\lambda_1\le\lambda_2<\Lambda_1(\mu)$, where $\Lambda_1(\mu)$ is the first eigenvalue of L, λ_1, λ_2 are the eigenvalues of the matrix $A=\left(\begin{smallmatrix}a_1&a_2\\a_2&a_3\end{smallmatrix}\right)$.

The quadratic from $Q(u,v):=(u,v)A(u,v)^T=a_1u^2+2a_2uv+a_3v^2$ is positively defined and satisfies

$$\lambda_1\big(u^2+v^2\big)\le a_1u^2+2a_2uv+a_3v^2\le\lambda_2\big(u^2+v^2\big)\quad\forall u,v\in H.\tag{1.2}$$

Our main results are as follows.

Theorem 1.1 *Suppose* ($\mathcal{H}_1$) *holds. Then for any solution* $(u,v)\in H\times H$ *of problem* (1.1), *there exists a positive constant* C_1 *such that*

$$\max\big\{\big|u(x)\big|,\big|v(x)\big|\big\}\le C_1|x-\xi_i|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu_i})},\quad\forall x\in B_{\rho_1}(\xi_i)\backslash\{\xi_i\},$$

where $\rho_1>0$ *and* $B_{\rho_1}(\xi_i)\subset\Omega$.

Theorem 1.2 *Suppose* ($\mathcal{H}_1$) *holds. Then for any positive solution* $(u,v)\in H\times H$ *of problem* (1.1), *there exists a positive constant* C_2 *such that* $B_{\rho_2}(\xi_i)\subset\Omega$ *and*

$$\min\big\{u(x),v(x)\big\}\ge C_2|x-\xi_i|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu_i})},\quad\forall x\in B_{\rho_2}(\xi_i)\backslash\{\xi_i\},$$

where $\rho_2>0$.

Theorem 1.3 *Suppose* ($\mathcal{H}_1$), ($\mathcal{H}_2$) *hold. Then the problem* (1.1) *has a positive solution.*

2 Preliminaries

On $H\times H$, we use the norm

$$\big\|(u,v)\big\|_{H\times H}=\left(\int_\Omega\big(|\nabla u|^2+|\nabla v|^2\big)\,dx\right)^{\frac{1}{2}}.$$

Using the Young inequality, the following best constant is well defined:

$$S_{\mu_i} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 - \mu_i \frac{u^2}{|x-\xi_i|^2})\,dx}{(\int_{\mathbb{R}^N}|u|^{2^*}\,dx)^{2/2^*}}, \tag{2.1}$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N}|\nabla\cdot|^2\,dx)^{\frac{1}{2}}$.

We infer that S_{μ_i} is attained in $\mathbb{R}^N$ by the functions

$$V^{\xi_i}_{\mu_i,\varepsilon}(x) = \varepsilon^{\frac{2-N}{2}} U_{\mu_i}\left(\frac{x-\xi_i}{\varepsilon}\right), \quad \forall\varepsilon > 0,$$

where

$$U_{\mu_i}(x-\xi_i) = \frac{(\frac{2N(\bar\mu-\mu_i)}{\sqrt{\bar\mu}})^{\sqrt{\bar\mu}/2}}{|x-\xi_i|^{(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu_i})}(1+|x-\xi_i|^{\frac{2\sqrt{\bar\mu-\mu_i}}{\sqrt{\bar\mu}}})^{\sqrt{\bar\mu}}}.$$

For all $\eta,\lambda,\sigma \ge 0$, $\eta+\lambda+\sigma > 0$, $\alpha+\beta > 1$, $\alpha+\beta = 2^*$, by the Young and Hardy-Sobolev inequalities, the following constant is well defined on $\mathcal{D} := (D^{1,2}(\mathbb{R}^N)\setminus\{0\})^2$:

$$S_{\eta,\lambda,\sigma}(\mu_i) := \inf_{(u,v)\in\mathcal{D}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2 - \mu_i\frac{u^2+v^2}{|x-\xi_i|^2})\,dx}{(\int_{\mathbb{R}^N}(\eta|u|^{2^*} + \lambda|v|^{2^*} + \sigma|u|^\alpha|v|^\beta)\,dx)^{2/2^*}}. \tag{2.2}$$

Set

$$u^\xi_{\mu,\varepsilon}(x) = \psi(x)V^\xi_{\mu,\varepsilon}(x) = \varepsilon^{\frac{2-N}{2}}\psi(x)U_\mu\left(\frac{x-\xi}{\varepsilon}\right),$$

where $\xi\in\Omega$, $0\le\mu<\bar\mu$, $\psi\in C_0^\infty(B_\rho(\xi))$ satisfies $0\le\psi\le 1$ and $\psi\equiv 1$, $\forall x\in B_{\frac{\rho}{2}(\xi)}$, for all $\rho > 0$ small. Then for any $0<\mu<\bar\mu$, by [9] we have the following estimates:

$$\int_\Omega\left(|\nabla u^\xi_{\mu,\varepsilon}|^2 - \mu\frac{(u^\xi_{\mu,\varepsilon})^2}{|x-\xi|^2}\right)dx = S_\mu^{\frac{N}{2}} + o(\varepsilon^{2\sqrt{\bar\mu-\mu}}),$$
$$\int_\Omega|u^\xi_{\mu,\varepsilon}|^{2^*}\,dx = S_\mu^{\frac{N}{2}} + o(\varepsilon^{\frac{2N}{N-2}\sqrt{\bar\mu-\mu}}),$$

and for any $a\in\mathbb{R}^N\setminus\{0\}$,

$$\int_\Omega\frac{|u^0_{\mu,\varepsilon}|^2}{|x+\xi|^2}\,dx = \begin{cases}\frac{\varepsilon^2}{|\xi|^2}\int_{\mathbb{R}^N}U_\mu^2(x)\,dx + o(\varepsilon^2) & \text{if } \mu<\bar\mu-1,\\ \frac{C_\mu^2\omega_N}{|\xi|^2}\varepsilon^2|\log\varepsilon| + o(\varepsilon^2) & \text{if } \mu=\bar\mu-1,\end{cases}$$
$$\int_\Omega|u^\xi_{\mu,\varepsilon}|^2\,dx = \begin{cases}o_1(\varepsilon^2) & \text{if } 0\le\mu<\bar\mu-1,\\ o_1(\varepsilon^2|\log\varepsilon|) & \text{if } \mu=\bar\mu-1,\end{cases}$$

where $C_\mu = (\frac{4N(\bar\mu-\mu)}{N-2})^{\frac{N-2}{4}}$, ω_N is the volume of the unit ball in $\mathbb{R}^N$.

3 Asymptotic behavior of solutions

Proof of Theorem 1.1 Suppose $(u_0, v_0) \in H \times H$ is a nontrivial solution to problem (1.1). For all $0 \le \mu_i \le \bar{\mu}$ define

$$u(x) = |x - \xi_i|^{\gamma} u_0(x) \quad \text{and} \quad v(x) = |x - \xi_i|^{\gamma} v_0(x), \quad \text{where } \gamma = (\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu_i}).$$

It is not difficult to verify that $u, v \in H_0^1(\Omega, |x - \xi_i|^{-2\gamma})$ and satisfy

$$\begin{cases} -\operatorname{div}(|x-\xi_i|^{-2\gamma}\nabla u) \\ \quad = \frac{\sigma\alpha}{2^*}|x-\xi_i|^{-2^*\gamma}u|u|^{\alpha-2}|v|^{\beta} + \eta|x-\xi_i|^{-2^*\gamma}u|u|^{2^*-2} \\ \qquad + a_1|x-\xi_i|^{-2\gamma}u + a_2|x-\xi_i|^{-2\gamma}v + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\mu_j}{|x-\xi_j|^2}|x-\xi_i|^{-2\gamma}u, \\ -\operatorname{div}(|x-\xi_i|^{-2\gamma}\nabla v) \\ \quad = \frac{\sigma\beta}{2^*}|x-\xi_i|^{-2^*\gamma}v|v|^{\beta-2}|u|^{\alpha} + \lambda|x-\xi_i|^{-2^*\gamma}v|v|^{2^*-2} \\ \qquad + a_2|x-\xi_i|^{-2\gamma}u + a_3|x-\xi_i|^{-2\gamma}v + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\mu_j}{|x-\xi_j|^2}|x-\xi_i|^{-2\gamma}v. \end{cases} \tag{3.1}$$

Let $R > 0$ small enough such that $B_R(\xi_i) \subset \Omega$ and $\xi_j \notin B_R(\xi_i)$ for $j \neq i$. Also, let $\varphi_i \in C_0^\infty(B_R(\xi_i))$ be a cut-off function. Set

$$u_n := \min\{|u|, n\}; \qquad v_n := \min\{|v|, n\}; \qquad \phi_{1_i} := \varphi_i^2 u u_n^{2(s-1)}, \qquad \phi_{2_i} := \varphi_i^2 v v_n^{2(s-1)},$$

where $s, n > 1$. Multiplying the first equation of (3.1) by ϕ_{1_i} and the second one by ϕ_{2_i} respectively and integrating, we have

$$\begin{aligned} \int_\Omega |x-\xi_i|^{-2\gamma}\nabla u \nabla \phi_{1_i} = {} & \frac{\sigma\alpha}{2^*}\int_\Omega |x-\xi_i|^{-2^*\gamma}u|u|^{\alpha-2}|v|^{\beta}\phi_{1_i} + \eta\int_\Omega |x-\xi_i|^{-2^*\gamma} \\ & \times u|u|^{2^*-2}\phi_{1_i} + a_1\int_\Omega |x-\xi_i|^{-2\gamma}u\phi_{1_i} + a_2\int_\Omega |x-\xi_i|^{-2\gamma} \\ & \times v\phi_{1_i} + \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\mu_j}{|x-\xi_j|^2}|x-\xi_i|^{-2\gamma}u\phi_{1_i}. \end{aligned}$$

Note that $\nabla\phi_{1_i} = 2\varphi_i u u_n^{2(s-1)}\nabla\varphi_i + \varphi_i^2 u_n^{2(s-1)}\nabla u + 2(s-1)\varphi_i^2 u_n^{2(s-1)}\nabla u_n$.

Then

$$\begin{aligned} \int_\Omega |x-\xi_i|^{-2\gamma}\nabla u\nabla\phi_{1_i} = {} & 2\int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i u u_n^{2(s-1)}\nabla\varphi_i\nabla u + \int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2 \\ & \times u_n^{2(s-1)}|\nabla u|^2 + 2(s-1)\int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2 u_n^{2(s-1)}|\nabla u_n|^2. \end{aligned}$$

By the Cauchy inequality and the Young inequality, we get

$$\begin{aligned} & \left| 2\int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i u u_n^{2(s-1)}\nabla\varphi_i\nabla u \right| \\ & \quad \le \frac{1}{2}\int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2 u_n^{2(s-1)}|\nabla u|^2 + 2\int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2 u^2 u_n^{2(s-1)}. \end{aligned} \tag{3.2}$$

The same result holds for $\int_\Omega |x-\xi_i|^{-2\gamma}\nabla\phi_{2_i}\nabla v$.

By letting $\psi_1(x) = \varphi_i u u_n^{(s-1)}$, $\psi_2(x) = \varphi_i v v_n^{(s-1)}$, we have

$$\begin{aligned}
&\int_\Omega |x-\xi_i|^{-2\gamma}\left(|\nabla\psi_1|^2 + |\nabla\psi_2|^2\right) \\
&\quad\le C\biggl(\int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2 u^2 u_n^{2(s-1)} + \int_\Omega |x-\xi_i|^{-2\gamma}\nabla u\nabla\phi_{1_i} \\
&\qquad + \int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2 v^2 v_n^{2(s-1)} + \int_\Omega |x-\xi_i|^{-2\gamma}\nabla v\nabla\phi_{2_i}\biggr) \\
&\quad\le C\biggl(\int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2\left(u^2 u_n^{2(s-1)} + v^2 v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|u|^\alpha|v|^\beta\left(u_n^{2(s-1)} + v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|u|^{2^*}u_n^{2(s-1)} + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|v|^{2^*}v_n^{2(s-1)} \\
&\qquad + \sum_{\substack{j=1\\ j\neq i}}^k \mu_j\int_\Omega \frac{|x-\xi_i|^{-2\gamma}}{|x-\xi_j|^2}\varphi_i^2\left(u^2 u_n^{2(s-1)} + v^2 v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2\left(u^2+v^2\right)\left(u_n^{2(s-1)} + v_n^{2(s-1)}\right)\biggr).
\end{aligned}\tag{3.3}$$

Using Caffarelli-Kohn-Nirenberg inequality [10], we infer that

$$\begin{aligned}
&\left(\int_\Omega \frac{|\varphi_i u u_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} + \left(\int_\Omega \frac{|\varphi_i v v_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} \\
&\quad\le C\left(\int_\Omega |x-\xi_i|^{-2\gamma}\left|\nabla\left(\varphi_i u u_n^{(s-1)}\right)\right|^2 + \int_\Omega |x-\xi_i|^{-2\gamma}\left|\nabla\left(\varphi_i v v_n^{(s-1)}\right)\right|^2\right) \\
&\quad= C\int_\Omega |x-\xi_i|^{-2\gamma}\left(|\nabla\psi_1|^2 + |\nabla\psi_2|^2\right) \\
&\quad\le C\biggl(\int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2\left(u^2 u_n^{2(s-1)} + v^2 v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|u|^\alpha|v|^\beta\left(u_n^{2(s-1)} + v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|u|^{2^*}u_n^{2(s-1)} + \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|v|^{2^*}v_n^{2(s-1)} \\
&\qquad + \sum_{\substack{j=1\\ j\neq i}}^k \mu_j\int_\Omega \frac{|x-\xi_i|^{-2\gamma}}{|x-\xi_j|^2}\varphi_i^2\left(u^2 u_n^{2(s-1)} + v^2 v_n^{2(s-1)}\right) \\
&\qquad + \int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2\left(u^2+v^2\right)\left(u_n^{2(s-1)} + v_n^{2(s-1)}\right)\biggr).
\end{aligned}\tag{3.4}$$

Define

$$\omega(x) := \max\{u(x), v(x)\}, \qquad \omega_n(x) := \min\{\omega(x), n\}.$$

Then $\omega_n(x) := \max\{u_n(x), v_n(x)\}$. Now, from the Hölder inequality, we deduce that

$$\int_\Omega \frac{\varphi_i^2|\omega|^{2^*}\omega_n^{2(s-1)}}{|x-\xi_i|^{2^*\gamma}} \le \left(\int_\Omega \frac{|\varphi_i\omega\omega_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} \left(\int_{B_R(\xi_i)} \frac{|\omega|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{N}}, \tag{3.5}$$

$$\int_\Omega \frac{\varphi_i^2|\omega|^2\omega_n^{2(s-1)}}{|x-\xi_i|^{2\gamma}} \le \left(\int_\Omega \frac{|\varphi_i\omega\omega_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} \left|B_R(\xi_i)\right|^{\frac{2}{N}}. \tag{3.6}$$

In the sequel, we have

$$\begin{aligned}
&\int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|u|^\alpha|v|^\beta\left(u_n^{2(s-1)}+v_n^{2(s-1)}\right)\\
&\quad\le \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|\omega|^\alpha|\omega|^\beta\left(\omega_n^{2(s-1)}+\omega_n^{2(s-1)}\right)\\
&\quad= \int_\Omega |x-\xi_i|^{-2^*\gamma}\varphi_i^2|\omega|^{2^*}\omega_n^{2(s-1)}\\
&\quad\le 2\left(\int_\Omega \frac{|\varphi_i\omega\omega_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} \left(\int_{B_R(\xi_i)} \frac{|\omega|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{N}}.
\end{aligned} \tag{3.7}$$

By the choice of φ_i, we obtain

$$\begin{aligned}
&\sum_{\substack{j=1\\ j\neq i}}^{k}\mu_j\int_\Omega \frac{|x-\xi_i|^{-2\gamma}}{|x-\xi_j|^2}\varphi_i^2\left(u^2u_n^{2(s-1)}+v^2v_n^{2(s-1)}\right)\\
&\quad\le C\int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2\left(u^2u_n^{2(s-1)}+v^2v_n^{2(s-1)}\right)\\
&\quad\le \int_\Omega |x-\xi_i|^{-2\gamma}\varphi_i^2\omega^2\omega_n^{2(s-1)}\\
&\quad\le 2C\left(\int_\Omega \frac{|\varphi_i\omega\omega_n^{(s-1)}|^{2^*}}{|x-\xi_i|^{2^*\gamma}}\right)^{\frac{2}{2^*}} \left|B_R(\xi_i)\right|^{\frac{2}{N}}.
\end{aligned} \tag{3.8}$$

So, from (3.4) to (3.8) it follows that

$$\begin{aligned}
&\left(\int_\Omega |x-\xi_i|^{-2^*\gamma}\left|\varphi_i\omega\omega_n^{s-1}\right|^{2^*}\right)^{\frac{2}{2^*}}\\
&\quad\le C\int_\Omega |x-\xi_i|^{-2\gamma}|\nabla\varphi_i|^2\omega^2\omega_n^{2(s-1)}.
\end{aligned} \tag{3.9}$$

Take $s = \frac{2^*}{2}$ and $\varphi_i(x)$ to be a constant near the zero. Letting $n\to\infty$, we infer that $\omega \in L^{\frac{2^{*2}}{2}}(B_R(\xi_i), |x-\xi_i|^{-2^*\gamma})$ and so

$$u, v \in L^{\frac{2^{*2}}{2}}\left(B_R(\xi_i), |x-\xi_i|^{-2^*\gamma}\right). \tag{3.10}$$

Suppose $r > 0$ is sufficiently small such that $r + l < R$ and φ_i is a cut-off function with the properties $|\nabla\varphi_i| < \frac{1}{l}$ and $\varphi_i(x) = 1$ in $B_r(\xi_i)$.

Set $t := \frac{2^{*2}}{2(2^*-2)}$, $\delta := 2\gamma t - 2^*\gamma(t-1)$.

Then we have the following results:

$$\begin{aligned}
&\int_{\Omega} |x-\xi_i|^{-2\gamma} |\nabla\varphi_i|^2 \omega^2 \omega_n^{2(s-1)} \\
&\quad \le Cl^{-2} \int_{\Omega} |x-\xi_i|^{-(2^*\gamma \frac{t-1}{t})} \omega^2 \omega_n^{2(s-1)} |x-\xi_i|^{-\gamma(2-2^* \frac{t-1}{t})} \\
&\quad \le Cl^{-2} \biggl(\int_{B_{r+l}(\xi_i)} \frac{|\omega\omega_n^{(s-1)}|^{\frac{2t}{t-1}}}{|x-\xi_i|^{2^*\gamma}} \biggr)^{\frac{t-1}{t}} \biggl(\int_{B_{r+l}(\xi_i)} \frac{1}{|x-\xi_i|^{\delta}} \biggr)^{\frac{1}{t}} \\
&\quad \le Cl^{-2} \biggl(\int_{B_{r+l}(\xi_i)} \frac{|\omega\omega_n^{(s-1)}|^{\frac{2t}{t-1}}}{|x-\xi_i|^{2^*\gamma}} \biggr)^{\frac{t-1}{t}},
\end{aligned} \tag{3.11}$$

where we used the Hölder inequality. From (3.9) in combination with (3.11), it follows that

$$\biggl(\int_{B_{r+l}(\xi_i)} |x-\xi_i|^{2^*\gamma} |\omega|^{2^*s} \biggr)^{\frac{1}{2^*s}} \le C^{\frac{1}{2s}} l^{-\frac{1}{2s}} \biggl(\int_{B_{r+l}(\xi_i)} |x-\xi_i|^{2^*\gamma} |\omega|^{\bar{p}_0 s} \biggr)^{\frac{1}{\bar{p}_0 s}}, \tag{3.12}$$

where $\bar{p}_0 = \frac{2t}{t-1} < 2^*$.

Denote $s = \chi^j$, $\chi = \frac{p}{\bar{p}_0}$ and $l = \rho^j$, $j \ge 1$, where $\chi \ge 1$, $2^*\gamma < N$ and $\bar{p}_0\chi^j = p\chi^{j-1}$. Using (3.12) recursively, we get

$$\begin{aligned}
\biggl(\int_{B_r(\xi_i)} |\omega|^{2^*\chi^j} \biggr)^{\frac{1}{2^*\chi^j}} &\le r^{\frac{\gamma}{\chi^j}} \biggl(\int_{B_r(\xi_i)} \frac{|\omega|^{2^*\chi^j}}{|x-\xi_i|^{2^*\gamma}} \biggr)^{\frac{1}{2^*\chi^j}} \\
&\le r^{\frac{\gamma}{\chi^j}} C^{\sum_{i=1}^{j} \frac{1}{2\chi^j}} \rho^{-\sum_{i=1}^{j} \frac{i}{\chi^i}} \biggl(\int_{B_{r+\rho}(\xi_i)} |x-\xi_i|^{2^*\gamma} |\omega|^{2^*} \biggr)^{\frac{1}{2^*}},
\end{aligned}$$

we have $\chi^j \to \infty$ as $j \to \infty$. Note that the infinite sums on the right-hand side converge, then we obtain that $\omega \in L^\infty(B_r(\xi_i))$, particularly, we have $u, v \in L^\infty(\Omega)$. Thus,

$$u_0(x) = |x-\xi_i|^{-\gamma} u(x) \le M_1 |x-\xi_i|^{-\gamma} \quad \text{for } x \in B_r(\xi_i) \backslash \{\xi_i\},$$

where $M_1 = \max\{\|u\|_{L^\infty(B_r(\xi_i))}, 1 \le i \le k\}$.

$$v_0(x) = |x-\xi_i|^{-\gamma} v(x) \le M_2 |x-\xi_i|^{-\gamma} \quad \text{for } x \in B_r(\xi_i) \backslash \{\xi_i\},$$

where $M_1 = \max\{\|v\|_{L^\infty(B_r(\xi_i))}, 1 \le i \le k\}$. The proof is complete. □

Proof of Theorem 1.2 Suppose $(u_0, v_0) \in H \times H$ is a positive solution to problem (1.1). For all $0 \le \mu_i \le \bar{\mu}$, set

$$u(x) = |x-\xi_i|^{\gamma} u_0(x) \quad \text{and} \quad v(x) = |x-\xi_i|^{\gamma} v_0(x).$$

Then

$$\begin{cases} -\operatorname{div}(|x-\xi_i|^{-2\gamma}\nabla u) \\ \quad = \frac{\sigma\alpha}{2^*}|x-\xi_i|^{-2^*\gamma}u^{\alpha-1}v^{\beta} + \eta|x-\xi_i|^{-2^*\gamma}u^{2^*-1} \\ \qquad + a_1|x-\xi_i|^{-2\gamma}u + a_2|x-\xi_i|^{-2\gamma}v + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\mu_j}{|x-\xi_j|^2}|x-\xi_i|^{-2\gamma}u, \\ -\operatorname{div}(|x-\xi_i|^{-2\gamma}\nabla v) \\ \quad = \frac{\sigma\beta}{2^*}|x-\xi_i|^{-2^*\gamma}v^{\beta-1}u^{\alpha} + \lambda|x-\xi_i|^{-2^*\gamma}v^{2^*-1} \\ \qquad + a_2|x-\xi_i|^{-2\gamma}u + a_3|x-\xi_i|^{-2\gamma}v + \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\mu_j}{|x-\xi_j|^2}|x-\xi_i|^{-2\gamma}v. \end{cases} \tag{3.13}$$

Choose $0 < \rho_0 < \rho$ and define $n(t) = \min_{|x-\xi_i|=t} u(x)$ for $\rho_0 \le t \le \rho$. Let

$$n(\rho_0) = A\rho_0^{-2\sqrt{\bar\mu-\mu_i}} + B, \qquad n(\rho) = A\rho^{-2\sqrt{\bar\mu-\mu_i}} + B, \quad \text{where}$$
$$A = \frac{n(\rho)-n(\rho_0)}{\rho^{-2\sqrt{\bar\mu-\mu_i}} - \rho_0^{-2\sqrt{\bar\mu-\mu_i}}}, \qquad B = \frac{n(\rho_0)\rho^{-2\sqrt{\bar\mu-\mu_i}} - n(\rho)\rho_0^{-2\sqrt{\bar\mu-\mu_i}}}{\rho^{-2\sqrt{\bar\mu-\mu_i}} - \rho_0^{-2\sqrt{\bar\mu-\mu_i}}}.$$

It is easy to verify that

$$-\operatorname{div}\big(|x-\xi_i|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu_i})}\nabla\big(A|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} + B\big)\big) = 0 \quad \forall x\in\Omega\backslash\{\xi_i\}. \tag{3.14}$$

Combining (3.13) with (3.14), we get

$$-\operatorname{div}\big(|x-\xi_i|^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu_i})}\nabla\big(u - A|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} + B\big)\big) \ge 0, \quad \forall x\in B_\rho(\xi_i)\backslash B_{\rho_0}(\xi_i),$$
$$u(x) - \big(A|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} + B\big) \ge 0, \quad \forall x\in\partial\big(B_\rho(\xi_i)\backslash B_{\rho_0}(\xi_i)\big).$$

Therefore, by the maximum principle in $B_\rho(\xi_i)\backslash B_{\rho_0}(\xi_i)$, we obtain

$$u(x) - \big(A|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} + B\big) \ge 0, \quad \forall x\in B_\rho(\xi_i)\backslash B_{\rho_0}(\xi_i).$$

Thus, for all $x\in B_\rho(\xi_i)\backslash B_{\rho_0}(\xi_i)$,

$$\begin{aligned} u(x) &\ge A|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} + B \\ &= \frac{|x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}} - \rho^{-2\sqrt{\bar\mu-\mu_i}}}{\rho_0^{-2\sqrt{\bar\mu-\mu_i}} - \rho^{-2\sqrt{\bar\mu-\mu_i}}}n(\rho_0) + \frac{\rho_0^{-2\sqrt{\bar\mu-\mu_i}} - |x-\xi_i|^{-2\sqrt{\bar\mu-\mu_i}}}{\rho_0^{-2\sqrt{\bar\mu-\mu_i}} - \rho^{-2\sqrt{\bar\mu-\mu_i}}}n(\rho) \\ &\ge \frac{|x-\xi_i|^{2\sqrt{\bar\mu-\mu_i}} - \rho_0^{2\sqrt{\bar\mu-\mu_i}}}{|x-\xi_i|^{2\sqrt{\bar\mu-\mu_i}} - \rho_0^{2\sqrt{\bar\mu-\mu_i}}\rho^{-2\sqrt{\bar\mu-\mu_i}}|x-\xi_i|^{2\sqrt{\bar\mu-\mu_i}}}n(\rho). \end{aligned}$$

Taking $\rho_0 \to 0$, we conclude $u(x) \ge n(\rho) = \min_{|x-\xi_i|=\rho} u(x) > 0$ for all $x\in B_\rho(\xi_i)\backslash\{\xi_i\}$.

Similar result also holds for $v(x)$. Therefore, we have

$$\begin{aligned} u_0(x) &= |x-\xi_i|^{-\gamma}u(x) \ge |x-\xi_i|^{-\gamma}\min_{|x-\xi_i|=\rho} u(x) = |x-\xi_i|^{-\gamma}C_i \\ &\ge |x-\xi_i|^{-\gamma}\min_{i=1,2,\ldots,k} C_i = |x-\xi_i|^{-\gamma}N_1. \end{aligned}$$

For any $x \in B_\rho(\xi_i)\backslash\{\xi_i\}$,

$$\begin{aligned} v_0(x) &= |x-\xi_i|^{-\gamma} v(x) \geq |x-\xi_i|^{-\gamma} \min_{|x-\xi_i|=\rho} v(x) = |x-\xi_i|^{-\gamma} \acute{C}_i \\ &\geq |x-\xi_i|^{-\gamma} \min_{i=1,2,\ldots,k} \acute{C}_i = |x-\xi_i|^{-\gamma} N_2. \end{aligned}$$

For any $x \in B_\rho(\xi_i)\backslash\{\xi_i\}$. This proves the theorem. □

4 Local $(PS)_c$-condition and the existence of positive solutions

We first establish a compactness result.

Lemma 4.1 *Suppose that* $(\mathcal{H}_1)$ *holds. Then* J *satisfies the* $(PS)_c$*-condition for all*

$$c < c^* := \frac{1}{N} \min\left\{S^{\frac{N}{2}}_{\eta,\lambda,\sigma}(\mu_1), \ldots, S^{\frac{N}{2}}_{\eta,\lambda,\sigma}(\mu_k), (S_0)^{\frac{N}{2}}\right\} = \frac{1}{N} S^{\frac{N}{2}}_{\eta,\lambda,\sigma}(\mu_k).$$

Proof Suppose that $\{(u_n, v_n)\} \subset H \times H$ satisfies $J(u_n, v_n) \to c < c^*$ and $J'(u_n, v_n) \to 0$. The standard argument shows that $\{(u_n, v_n)\}$ is bounded in $H \times H$.

For some $(u, v) \in H \times H$, we have

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } H \times H, \\ &(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } L^2\left(\Omega, |x-\xi_i|^{-2}\right) \times L^2\left(\Omega, |x-\xi_i|^{-2}\right), \\ &(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } L^{2^*}(\Omega) \times L^{2^*}(\Omega), \\ &(u_n, v_n) \to (u, v) \quad \text{strongly in } L^{q_1}(\Omega) \times L^{q_2}(\Omega), q_1, q_2 \in [1, 2^*), \\ &(u_n, v_n) \to (u, v) \quad \text{a.e. in } \Omega. \end{aligned}$$

Therefore, (u, v) is a solution to (1.1). Then by the concentration-compactness principle [11–13] and up to a subsequence, there exist an at most countable set $\mathcal{J}$, a set of different points $\{x_j\}_{j\in\mathcal{J}} \subset \Omega\backslash\xi_{i\,i=1}^{\,k}$, nonnegative real numbers $\tilde{\tau}_{x_j}, \tilde{v}_{x_j}, j \in \mathcal{J}$, and $\tilde{\tau}_{\xi_i}, \tilde{v}_{\xi_i}, \tilde{\gamma}_{\xi_i}$ $(1 \leq i \leq k)$ such that the following convergence holds in the sense of measures:

$$\begin{aligned} &|\nabla u_n|^2 + |\nabla v_n|^2 \rightharpoonup d\tilde{\tau} \geq |\nabla u|^2 + |\nabla v|^2 + \sum_{j\in\mathcal{J}} \tilde{\tau}_{x_j}\delta_{x_j} + \sum_{i=1}^{k} \tilde{\tau}_{\xi_i}\delta_{\xi_i}, \\ &\frac{u_n^2 + v_n^2}{|x-\xi_i|^2} \rightharpoonup d\tilde{\gamma} = \frac{u^2 + v^2}{|x-\xi_i|^2} + \tilde{\gamma}_{\xi_i}\delta_{\xi_i}, \\ &\eta|u_n|^{2^*} + \lambda|v_n|^{2^*} + \sigma|u_n|^\alpha|v_n|^\beta \rightharpoonup d\tilde{v} \\ &\quad = \eta|u|^{2^*} + \lambda|v|^{2^*} \\ &\qquad + \sigma|u|^\alpha|v|^\beta + \sum_{j\in\mathcal{J}} \tilde{v}_{x_j}\delta_{x_j} + \sum_{i=1}^{k} \tilde{v}_{\xi_i}\delta_{\xi_i}. \end{aligned}$$

By the Sobolev inequalities [10], we have

$$S_{\mu_i}\tilde{v}_{\xi_i}^{\frac{2}{2^*}} \leq \tilde{\tau}_{\xi_i} - \mu_i\tilde{\gamma}_{\xi_i}, \quad 1 \leq i \leq k. \tag{4.1}$$

We claim that $\mathcal{J}$ is finite, and for any $j \in \mathcal{J}$, $\tilde{\nu}_{x_j} = 0$ or $\tilde{\nu}_{x_j} \geq S_0^{\frac{N}{2}}$.

In fact, let $\varepsilon > 0$ be small enough for any $1 \leq i \leq k$, $\xi_i \notin B_\varepsilon(x_j)$ and $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \emptyset$ for $i \neq j$, $i, j \in \mathcal{J}$. Let ϕ_ε^j be a smooth cut-off function centered at x_j such that $0 \leq \phi_\varepsilon^j \leq 1$, $\phi_\varepsilon^j = 1$ for $|x - x_j| \leq \frac{\varepsilon}{2}$, $\phi_\varepsilon^j = 0$ for $|x - x_j| \geq \varepsilon$ and $|\nabla \phi_\varepsilon^j| \leq \frac{4}{\varepsilon}$. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \big(|\nabla u_n|^2 + |\nabla v_n|^2\big)\phi_\varepsilon^j &= \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon^j \, d\tilde{\tau} \\
&\geq \lim_{\varepsilon\to 0}\bigg(\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)\phi_\varepsilon^j + \tilde{\tau}_{x_j}\bigg) \\
&= \tilde{\tau}_{x_j},
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{u_n^2 + v_n^2}{|x - \xi_i|^2}\phi_\varepsilon^j = \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon^j \, d\tilde{\gamma} = \int_\Omega \frac{u^2 + v^2}{|x - \xi_i|^2}\phi_\varepsilon^j = 0, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \big(\eta|u_n|^{2^*} + \lambda|v_n|^{2^*} + \sigma|u_n|^\alpha|v_n|^\beta\big)\phi_\varepsilon^j \\
&\quad = \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon^j \, d\tilde{\nu} \\
&\quad = \lim_{\varepsilon\to 0}\bigg(\int_\Omega \big(\eta|u|^{2^*} + \lambda|v|^{2^*} + \sigma|u|^\alpha|v|^\beta\big)\phi_\varepsilon^j + \tilde{\nu}_{x_j}\bigg) \\
&\quad = \tilde{\nu}_{x_j}, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (u_n\nabla u_n + v_n\nabla v_n)\nabla\phi_\varepsilon^j = 0.
\end{aligned}$$

Then we have

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\big\langle J_\lambda'(u_n, v_n), (u_n\phi_\varepsilon^j, v_n\phi_\varepsilon^j)\big\rangle \geq \tilde{\tau}_{x_j} - \tilde{\nu}_{x_j}.$$

By the Sobolev inequality, $S_0\tilde{\nu}_{x_j}^{\frac{2}{2^*}} \leq \tilde{\tau}_{x_j}$; and then we deduce that $\tilde{\nu}_{x_j} = 0$ or $\tilde{\nu}_{x_j} \geq S_0^{\frac{N}{2}}$, which implies that $\mathcal{J}$ is finite.

Now, we consider the possibility of concentration at points ξ_i ($1 \leq i \leq k$), for $\varepsilon > 0$ small enough that $x_j \notin B_\varepsilon(\xi_i)$ for all $j \in \mathcal{J}$ and $B_\varepsilon(\xi_i) \cap B_\varepsilon(\xi_j) = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq k$. Let φ_ε^i be a smooth cut-off function centered at ξ_i such that $0 \leq \varphi_\varepsilon^i \leq 1$, $\varphi_\varepsilon^i = 1$ for $|x - \xi_i| \geq \varepsilon$ and $|\nabla\varphi_\varepsilon^i| \leq \frac{4}{\varepsilon}$. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \big(|\nabla u_n|^2 + |\nabla v_n|^2\big)\varphi_\varepsilon^i &= \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\tilde{\tau} \\
&\geq \lim_{\varepsilon\to 0}\bigg(\int_\Omega \big(|\nabla u|^2 + |\nabla v|^2\big)\varphi_\varepsilon^i + \tilde{\tau}_{\xi_i}\bigg) \\
&= \tilde{\tau}_{\xi_i},
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{u_n^2 + v_n^2}{|x - \xi_i|^2}\varphi_\varepsilon^i = \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\tilde{\gamma} = \lim_{\varepsilon\to 0}\bigg(\int_\Omega \frac{u^2 + v^2}{|x - \xi_i|^2}\varphi_\varepsilon^i + \tilde{\gamma}_{\xi_i}\bigg) = \tilde{\gamma}_{\xi_i}, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \big(\eta|u_n|^{2^*} + \lambda|v_n|^{2^*} + \sigma|u_n|^\alpha|v_n|^\beta\big)\varphi_\varepsilon^i \\
&\quad = \lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon^i \, d\tilde{\nu}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\left(\int_\Omega \left(\eta|u|^{2^*}+\lambda|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\right)\varphi_\varepsilon^i+\tilde{v}_{\xi_i}\right)\\
&= \tilde{v}_{\xi_i},\\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{u_n^2+v_n^2}{|x-\xi_i|^2}\varphi_\varepsilon^i = 0, \quad \text{for } i\neq j,\\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (u_n\nabla u_n+v_n\nabla v_n)\nabla\varphi_\varepsilon^i = 0.
\end{aligned}
$$

Thus, we have

$$
0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\langle J_\lambda'(u_n,v_n),\left(u_n\varphi_\varepsilon^i,v_n\varphi_\varepsilon^i\right)\right\rangle \geq \tilde{\tau}_{\xi_i}-\mu_i\tilde{\gamma}_{\xi_i}-\tilde{v}_{\xi_i}. \tag{4.2}
$$

From (4.1) and (4.2) we derive that $S_{\mu_i}\tilde{v}_{\xi_i}^{\frac{2}{2^*}}\leq \tilde{v}_{\xi_i}$, $1\leq i\leq k$, and then either $\tilde{v}_{\xi_i}=0$ or $\tilde{v}_{\xi_i}\geq S_{\mu_i}^{\frac{N}{2}}$. On the other hand, from the above arguments, we conclude that

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left(J(u_n,v_n)-\frac{1}{2}\left\langle J'(u_n,v_n),(u_n,v_n)\right\rangle\right)\\
&= \frac{1}{N}\lim_{n\to\infty}\int_\Omega\left(\eta|u_n|^{2^*}+\lambda|v_n|^{2^*}+\sigma|u_n|^\alpha|v_n|^\beta\right)dx\\
&= \frac{1}{N}\left(\int_\Omega\left(\eta|u|^{2^*}+\lambda|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\right)dx+\sum_{j\in\mathcal{J}}\tilde{v}_{x_j}+\sum_{i=1}^k\tilde{v}_{\xi_i}\right)\\
&= \frac{1}{N}\left(\sum_{j\in\mathcal{J}}\tilde{v}_{x_j}+\sum_{i=1}^k\tilde{v}_{\xi_i}\right)+J(u,v).
\end{aligned}
$$

If $\tilde{v}_{\xi_i}=\tilde{v}_{x_j}=0$ for all $i\in\{1,\dots,k\}$ and $j\in\mathcal{J}$, then $c=0$, which contradicts the assumption that $c>0$. On the other hand, if there exists an $i\in\{1,\dots,k\}$ such that $\tilde{v}_{\xi_i}\neq 0$ or there exists a $j\in\mathcal{J}$ with $\tilde{v}_{x_j}\neq 0$, then we infer that

$$
c\geq\frac{1}{N}\min\left\{\left(S_{\eta,\lambda,\sigma}(0)\right)^{N/2},\left(S_{\eta,\lambda,\sigma}(\mu_1)\right)^{N/2},\dots,\left(S_{\eta,\lambda,\sigma}(\mu_k)\right)^{N/2}\right\}=\frac{1}{N}\left(S_{\eta,\lambda,\sigma}(\mu_k)\right)^{N/2},
$$

which contradicts our assumptions. Hence, $(u_n,v_n)\to(u,v)$, as $n\to\infty$ in $H\times H$. □

First, under the assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, we have the following notations:

$$
\begin{aligned}
&f_{\eta,\lambda,\sigma}(\tau)=\frac{(1+\tau^2)S_{\mu_k}}{(\eta+\sigma\tau^\beta+\lambda\tau^{2^*})^{\frac{2}{2^*}}}, \quad \tau>0;\\
&f_{\eta,\lambda,\sigma}(\tau_{\min}):=\min_{\tau>0}f_{\eta,\lambda,\sigma}(\tau)>0, \quad \sigma>0,
\end{aligned}
$$

where $\tau_{\min}>0$ is a minimal point of $f_{\eta,\lambda,\sigma}(\tau)$, and therefore a root of the equation

$$
\alpha\sigma\tau^\beta-\sigma\beta\tau^{\beta-2}-2^*\lambda\tau^{2^*-2}+2^*\eta=0, \quad \tau>0.
$$

Lemma 4.2 *Suppose that* $(\mathcal{H}_1)$ *holds. Then we have*

(i) $S_{\eta,\lambda,\sigma}(\mu)=f_{\eta,\lambda,\sigma}(\tau_{\min})$

(ii) $S_{\eta,\lambda,\sigma}(\mu)$ *has the minimizers* $(V^{\xi}_{\mu,\varepsilon}(x), \tau_{\min} V^{\xi}_{\mu,\varepsilon}(x))$, $\forall\varepsilon > 0$, *where* $V^{\xi}_{\mu,\varepsilon}(x)$ *are the extremal functions of* $S_{\eta,\lambda,\sigma}(\mu)$ *defined as in* (2.2).

Proof The argument is similar to that of [6]. □

Lemma 4.3 *Under the assumptions of* $(\mathcal{H}_1)$, *we have*

$$\sup J\big(tu_{\varepsilon,\mu_k}, t(\tau_{\min} u_{\varepsilon,\mu_k})\big) < c^* = \frac{1}{N}\big(S_{\eta,\lambda,\sigma}(\mu_k)\big)^{N/2}.$$

Proof Suppose $(\mathcal{H}_1)$ holds. Define the function

$$g(t) := J\big(tu_{\varepsilon,\mu_k}, t(\tau_{\min} u_{\varepsilon,\mu_k})\big), \quad t \geq 0.$$

Note that $\lim_{t\to+\infty} g(t) = -\infty$ and $g(t) > 0$ as t is close to 0. Thus, $\sup_{t\geq 0} g(t)$ is attained at some finite $t_\varepsilon > 0$ with $g'(t_\varepsilon) = 0$. Furthermore, $c' < t_\varepsilon < c''$, where c' and c'' are the positive constants independent of ε. By using (1.2), we have

$$\begin{aligned} g(t) \leq{} & \frac{t^2}{2}\big(1 + \tau^2_{\min}\big)\left(\int_\Omega \left(|\nabla u_{\varepsilon,\mu_k}|^2 - \mu_k \frac{u^2_{\varepsilon,\mu_k}}{|x-\xi_i|^2} - \lambda_1 u^2_{\varepsilon,\mu_k}\right) dx\right) \\ & - \frac{t^{2^*}}{2^*}\big(\sigma\tau^{\beta}_{\min} + \eta + \lambda\tau^{2^*}_{\min}\big)\int_\Omega |u_{\varepsilon,\mu_k}|^{2^*}\, dx. \end{aligned}$$

Note that

$$\max\left(\frac{t^2}{2}B_1 - \frac{t^{2^*}}{2^*}B_2\right) = \frac{1}{N}\big(B_1 B_2^{-2/2^*}\big)^{N/2}, \quad B_1 > 0, B_2 > 0, \tag{4.3}$$

and $0 \leq \mu \leq \bar{\mu} - 1$ and so $2 < 2\sqrt{\bar{\mu} - \mu}$.

From (4.3), Lemma 4.2 and Lemma 4.3, it follows that

$$\begin{aligned} g(t_\varepsilon) &\leq \frac{1}{N}\left(\frac{(1+\tau^2_{\min})\int_\Omega (|\nabla u_{\varepsilon,\mu_k}|^2 - \mu_k \frac{u^2_{\varepsilon,\mu_k}}{|x-\xi_k|^2} - \lambda_1 u^2_{\varepsilon,\mu_k})\, dx}{((\sigma\tau^{\beta}_{\min} + \eta + \lambda\tau^{2^*}_{\min})\int_\Omega |u_{\varepsilon,\mu_k}|^{2^*}\, dx)^{2/2^*}}\right)^{N/2} \\ &\leq \frac{1}{N}\left(\frac{f(\tau_{\min})}{S(\mu_k)} \times \frac{(S(\mu_k))^{N/2} + O(\varepsilon^{2\sqrt{\bar{\mu}-\mu}}) - C\varepsilon^2}{(S(\mu_k))^{(N-2)/2} + O(\varepsilon^{2\sqrt{\bar{\mu}-\mu}})}\right) \\ &\leq \frac{1}{N}\big(f(\tau_{\min})\big)^{N/2} + O\big(\varepsilon^{2\sqrt{\bar{\mu}-\mu}}\big) - C\varepsilon^2 \\ &< \frac{1}{N}\big(S_{\eta,\lambda,\sigma}(\mu_k)\big)^{N/2} = c^*, \end{aligned}$$

so $g(t_\varepsilon) < c^*$. Hence, $g(t_\varepsilon) < c^*$, $\forall t \geq 0$ and

$$\sup_{t\geq 0} g(t) = \sup_{t\geq 0} J\big(tu_{\varepsilon,\mu_k}, t(\tau_{\min} u_{\varepsilon,\mu_k})\big) < c^*, \quad \text{if } \mu < \bar{\mu} - 1. \tag{4.4}$$

□

Proof of Theorem 1.3 Set $c := \inf_{h\in\Gamma} \max_{t\in[0,1]} J(h(t))$, where

$$\Gamma = \big\{h \in C\big([0,1], H \times H\big) | h(0) = (0,0), J\big(h(1)\big) < 0\big\}.$$

Suppose that $(\mathcal{H}_1)$ holds. For all $(u,v) \in H \times H\backslash\{(0,0)\}$, from the Young and Hardy-Sobolev inequalities, it follows that

$$\begin{aligned} J(u,v) &\geq C\left(\|u\|^2 + \|v\|^2\right) - C\left(\|u\|^{2^*} + \|v\|^{2^*}\right) \\ &\geq C\left\|(u,v)\right\|^2 - C\left\|(u,v)\right\|^{2^*}, \end{aligned}$$

and there exists a constant $\rho > 0$ small such that

$$b := \inf_{\|(u,v)\|=\rho} J(u,v) > 0 = J(0,0).$$

Since $J(tu,tv) \to -\infty$ as $t \to \infty$, there exists $t_0 > 0$ such that $\|(t_0u,t_0v)\| > \rho$ and $J(t_0u,t_0v) < 0$. By the mountain-pass theorem [14], there exists a sequence $\{(u_n,v_n)\} \subset H \times H$ such that $J(u_n,v_n) \to c$ and $J'(u_n,v_n) \to 0$, as $n \to \infty$.

From Lemma 4.2 it follows that

$$\begin{aligned} 0 < c &\leq \sup_{t\in[0,1]} J(tt_0u_{\varepsilon,\mu_k}, tt_0\tau_{\min}u_{\varepsilon,\mu_k}) \\ &\leq \sup_{t\geq 0} J(tu_{\varepsilon,\mu_k}, t\tau_{\min}u_{\varepsilon,\mu_k}) \\ &< c^*. \end{aligned}$$

By Lemma 4.1 there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $(u_n,v_n) \to (u,v)$ strongly in $H \times H$. Thus, we get a critical point (u,v) of J satisfying (1.1), and c is a critical value. Set $u^+ = \max\{u,0\}$.

Replacing respectively u, v with u^+ and v^+ in terms of the right-hand side of (1.1) and repeating the above process, we can get a nonnegative nontrivial solution (u,v) of (1.1). If $u \equiv 0$, we get $v \equiv 0$ by (1.1) and the assumption $a_2 > 0$. Similarly, if $v \equiv 0$, we also have $u \equiv 0$. There, $u, v \not\equiv 0$. From the maximum principle, it follows that $u, v > 0$ in Ω. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, SK, MF and OKK contributed to each part of this work equally and read and approved the final version of the manuscript.

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