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Fractional exponential operators and time-fractional telegraph equation

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Abstract

In this paper, the Bromwich integral for the inverse Mellin transform is used for finding an integral representation for a fractional exponential operator. This operator can be considered as an approach for solving partial fractional differential equations. Also, application of this operator for obtaining a formal solution of the time-fractional telegraph equation is discussed.

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1 Introduction and problem

We consider the exponential operator

$$e^{\lambda[q(x)\frac{d}{dx} + \nu(x)]}f(x) = f(x(\lambda))g(\lambda), \quad (1.1)$$

where $x(\lambda)$, $g(\lambda)$ are specified by the system of first-order differential equations [1]

$$\begin{cases} \frac{d}{d\lambda}x(\lambda) = q(x(\lambda)), & x(0) = x \\ \frac{d}{d\lambda}g(\lambda) = \nu(x(\lambda))g(\lambda), & g(0) = 1. \end{cases} \quad (1.2)$$

By the above exponential operator, Dattoli *et al.* found solutions of some boundary value problems arising in mathematical physics in terms of integral transforms type; see [2, 3] and references therein. Also, they used this operational technique to describe properties of some special polynomials and functions [4–6]; also see [4].

When we encounter an exponential operator of higher order $e^{\lambda^\alpha s^\alpha}$, where α is integer or non-integer and $s = q(x)\frac{d}{dx} + \nu(x)$, it is of interest to have an integral representation to reduce the order and apply the relation (1.1). For example, for exponential operators of orders two and three, we can write the Gauss-Weierstrass and the Airy integrals [2, 7]

$$e^{\lambda^2 s^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2\lambda s \xi} d\xi, \quad (1.3)$$

$$e^{\lambda^3 s^3} = \int_{-\infty}^{\infty} e^{\frac{3}{\sqrt{3}}\lambda s \xi} \text{Ai}(\xi) d\xi, \quad (1.4)$$

where $\text{Ai}(\xi)$ is the Airy function of the first kind given by

$$\text{Ai}(\xi) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + \xi t\right) dt. \tag{1.5}$$

For the fractional exponential operator $e^{-\lambda^\alpha s^\alpha}$, $0 < \alpha < 1$, it may occur that this operator can be written as the Laplace transform of the Wright function [8–10]

$$e^{-\lambda^\alpha s^\alpha} = \int_0^\infty e^{-st} \frac{1}{t} W(-\alpha, 0; -\lambda^\alpha t^{-\alpha}) dt, \tag{1.6}$$

where the Wright function is presented by the following relation [11]:

$$W(\alpha, \beta; t) = \sum_{k=0}^\infty \frac{t^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha > -1, \beta \in \mathbb{C}, t \in \mathbb{C}. \tag{1.7}$$

In this paper, in a general case we obtain an integral representation for $e^{\lambda^\alpha s^\alpha}$, $\alpha > 0$, with order one for s , and then we show how this operator can be applied to find the formal solutions of partial fractional differential equations (PFDEs).

This problem for integral representation is referred to as the inverse of the Mellin transform of $e^{\lambda^\alpha s^\alpha}$, $\alpha > 0$, and in Section 2, we state main theorems and corollaries related to it. In Section 3, as an application of this technique, we find formal solutions of the space-fractional Moshinski's equation and the time-fractional telegraph equation. Finally, in Section 4 the main conclusions are drawn.

2 Main theorems and corollaries

In this section, we establish some theorems on the fractional exponential operator which can be useful for solving PFDEs. First, we derive an integral representation for the operator $e^{\lambda^n s^n}$, which can be considered as a generalized representation of the relations (1.3) and (1.4).

Theorem 2.1 *The following identity holds true for $c_1 < \Re s < c_2$:*

$$e^{\lambda^n s^n} = \frac{1}{\pi} \int_{-\infty}^\infty e^{s\xi} \mathcal{A}_n(\xi, \lambda) d\xi, \quad n = 2, 3, \dots, \tag{2.1}$$

where the function $\mathcal{A}_n(\xi, \lambda)$ is presented by

$$\mathcal{A}_n(\xi, \lambda) = \begin{cases} \int_0^\infty \cos(r\xi + (-1)^{\frac{n+1}{2}} \lambda^n r^n) dr, & n = 2k + 1, \\ \frac{1}{n\lambda} \int_0^\infty e^{-r} \frac{\cos(\frac{1}{\lambda} r^{\frac{1}{n}} \xi)}{r^{1-\frac{1}{n}}} dr, & n = 4k + 2, \\ \frac{1}{n\lambda} \int_0^\infty e^r \frac{\cos(\frac{1}{\lambda} r^{\frac{1}{n}} \xi)}{r^{1-\frac{1}{n}}} dr, & n = 4k. \end{cases} \tag{2.2}$$

Proof By the definition of the inverse of the Mellin transform for a function $e^{\lambda^n s^n}$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda^n s^n} t^{-s} ds &= \frac{1}{2\pi} \int_0^\infty e^{(-i\lambda r)^n} t^{ir} dr + \frac{1}{2\pi} \int_0^\infty e^{(i\lambda r)^n} t^{-ir} dr \\ &= \frac{1}{2\pi} \int_0^\infty e^{\lambda^n r^n} e^{-\frac{n\pi}{2}i} t^{ir} dr + \frac{1}{2\pi} \int_0^\infty e^{\lambda^n r^n} e^{\frac{n\pi}{2}i} t^{-ir} dr \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_0^\infty e^{\lambda^n r^n \cos(\frac{n\pi}{2})} e^{-i\lambda^n r^n \sin(\frac{n\pi}{2}) + ir \ln(t)} dr \\
 &\quad + \frac{1}{2\pi} \int_0^\infty e^{\lambda^n r^n \cos(\frac{n\pi}{2})} e^{i\lambda^n r^n \sin(\frac{n\pi}{2}) - ir \ln(t)} dr \\
 &= \frac{1}{\pi} \int_0^\infty e^{\lambda^n r^n \cos(\frac{n\pi}{2})} \cos\left(r \ln(t) - \lambda^n r^n \sin\left(\frac{n\pi}{2}\right)\right) dr. \tag{2.3}
 \end{aligned}$$

The above relation implies that the Mellin transform of the last integral is equal to the function $e^{\lambda^n s^n}$, that is,

$$\begin{aligned}
 e^{\lambda^n s^n} &= \frac{1}{\pi} \mathcal{M} \left\{ \int_0^\infty e^{\lambda^n r^n \cos(\frac{n\pi}{2})} \cos\left(r \ln(t) - \lambda^n r^n \sin\left(\frac{n\pi}{2}\right)\right) dr; s \right\} \\
 &= \frac{1}{\pi} \int_0^\infty t^{s-1} \int_0^\infty e^{\lambda^n r^n \cos(\frac{n\pi}{2})} \cos\left(r \ln(t) - \lambda^n r^n \sin\left(\frac{n\pi}{2}\right)\right) dr dt.
 \end{aligned}$$

By setting $\ln(t) = \xi$, we get the relation (2.1). □

Theorem 2.2 (The Schouten-Van der Pol theorem for the Laplace transform [12]) *Let c be a suitable real constant such that $F(s)$ and $\Phi(s)$ are analytic functions in the half-plane $\Re s > c$ and $F(s)$ is the Laplace transform of $f(t)$. Then the inverse of the Laplace transform $F(\Phi(s))$ is given by*

$$g(t) = \mathcal{L}^{-1}\{F(\Phi(s)); t\} = \int_0^\infty f(\tau) \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-\Phi(s)\tau + ts} ds \right] d\tau. \tag{2.4}$$

Proof Using the definition of the Laplace transform for $F(\Phi(s))$

$$F(\Phi(s)) = \int_0^\infty e^{-\Phi(s)\tau} f(\tau) d\tau,$$

replacing in the inverse of the Laplace transform $F(\Phi(s))$

$$g(t) = \mathcal{L}^{-1}\{F(\Phi(s)); t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\Phi(s)) e^{ts} ds$$

and changing the order of integration, we get the relation (2.4). □

Corollary 2.3 *It is obvious that by setting $\Phi(s) = s^\alpha$, $0 < \alpha < 1$, in the relations (2.4) and using the relation (1.6) for the inverse of the Laplace transform e^{-s^α} , the inverse of the Laplace transform $F(s^\alpha)$ can be presented by*

$$\mathcal{L}^{-1}\{F(s^\alpha); t\} = \frac{1}{t} \int_0^\infty f(\tau) W(-\alpha, 0; -\tau t^{-\alpha}) d\tau. \tag{2.5}$$

Corollary 2.4 *By setting $F(s) = e^{\lambda^n s^n}$ and combining the relations (2.1) and (2.5), we get a new integral representation for the fractional exponential equation $e^{\lambda^n s^\gamma}$*

$$e^{\lambda^n s^\gamma} = \frac{1}{\pi} \int_{-\infty}^\infty e^{s\xi} \mathcal{A}_n^{(\alpha)}(\xi, \lambda) d\xi, \quad \gamma = n\alpha, 0 < \alpha \leq 1, \tag{2.6}$$

where the function $\mathcal{A}_n^{(\alpha)}(\xi)$ is given by

$$\mathcal{A}_n^{(\alpha)}(\xi, \lambda) = \frac{1}{\xi} \int_0^\infty \mathcal{A}_n(\tau, \lambda) W(-\alpha, 0; -\tau \xi^{-\alpha}) d\tau. \tag{2.7}$$

In view of the theorems of a fractional exponential operator expressed in this section, we may apply this operator to PFDEs in the next section.

3 Application to partial fractional differential equations

Example 3.1 In connection with initial-value diffusions, we consider the space-fractional Moshinskii's equation of order γ in the Riemann-Liouville sense [12]

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{1+x^2} \left((1+x^2) \frac{\partial}{\partial x} \right)^\gamma u(x, t), \quad \gamma > 0, x > 0, t > 0 \tag{3.1}$$

with the Cauchy-type initial condition as $u(x, 0) = f(x)$.

In order to obtain the solution of (3.1), by solving the first-order partial differential with respect to t and applying the initial condition, the formal solution in the form of fractional exponential operator gives rise to

$$u(x, t) = e^{\frac{t}{1+x^2} \left((1+x^2) \frac{\partial}{\partial x} \right)^\gamma} f(x). \tag{3.2}$$

Now, by setting $\left(\frac{t}{1+x^2}\right)^{\frac{1}{n}} = \lambda$, $s = (1+x^2) \frac{\partial}{\partial x}$ and applying Corollary 2.4 for the integral representation of $e^{\lambda^n s^\gamma}$, we can write the solution in terms of the integral transform as

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^\infty e^{(1+x^2) \frac{\partial}{\partial x} \xi} \mathcal{A}_n^{(\alpha)}(\xi, \lambda) f(x) d\xi, \quad \gamma = n\alpha, 0 < \alpha \leq 1,$$

where the function $\mathcal{A}_n^{(\alpha)}(\xi, \lambda)$ is given by the relation (2.7). The above relation can be simplified in the following form:

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^\infty \mathcal{A}_n^{(\alpha)} \left(\xi, \left(\frac{t}{1 + (\tan(\xi + \tan^{-1}(x)))^2} \right)^{\frac{1}{n}} \right) f(\tan(\xi + \tan^{-1}(x))) d\xi, \tag{3.3}$$

where we used the relations (1.1) and (1.2) by choosing the functions $q(x) = 1 + x^2$ and $v(x) = 0$.

Example 3.2 As another application, we consider the time-fractional telegraph equation [13, 14]

$$\frac{\partial^2}{\partial x^2} u(x, t) = \left[a \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} + b \frac{\partial^\alpha}{\partial t^\alpha} + c \right] u(x, t), \quad 0 < \alpha \leq 1, x, t > 0, a, b, c \in \mathbb{R} \tag{3.4}$$

with initial and asymptotic conditions $u(0, t) = f(t)$, $\lim_{x \rightarrow \infty} u(x, t) = 0$.

Similar to the previous problem by solving the equation with respect to x and applying the initial and asymptotic conditions, the formal solution takes the form:

$$u(x, t) = e^{-x \sqrt{a \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} + b \frac{\partial^\alpha}{\partial t^\alpha} + c}} f(t). \tag{3.5}$$

Now, by setting $\frac{\partial}{\partial t} = s$ and writing an integral representation for $e^{-x\sqrt{as^2+bs+c}}$ in terms of the Bessel function of order one, we get [15]

$$e^{-x\sqrt{as^2+bs+c}} = e^{-x(\frac{b}{2\sqrt{a}}+\sqrt{as})} - x\sqrt{\frac{\Delta}{a}} \int_{x\sqrt{a}}^{\infty} e^{-su-\frac{b}{2a}} \frac{J_1(\sqrt{\frac{\Delta}{a}}\sqrt{u^2-ax^2})}{\sqrt{u^2-ax^2}} du, \quad \Delta = ac - \frac{b^2}{4} \neq 0. \quad (3.6)$$

We can rewrite the relation (3.5) in the following form:

$$u(x, t) = e^{-\frac{xb}{2\sqrt{a}}} \int_0^{\infty} \frac{1}{\tau} W(-\alpha, 0; -x\sqrt{a}\tau^{-\alpha}) f(t - \tau) d\tau - x\sqrt{\frac{\Delta}{a}} e^{-\frac{b}{2a}} \int_0^{\infty} \frac{1}{\tau} f(t - \tau) \int_{x\sqrt{a}}^{\infty} \frac{J_1(\sqrt{\frac{\Delta}{a}}\sqrt{u^2-ax^2})}{\sqrt{u^2-ax^2}} \times W(-\alpha, 0; -u\tau^{-\alpha}) du d\tau, \quad (3.7)$$

where we used the relation (2.6) for the linearization of a fractional exponential operator e^{-xs^α} , and then we applied the relations (1.1) and (1.2) by substituting $q(x) = 1$ and $v(x) = 0$.

4 Conclusions

This paper provides some new results in the theory of fractional derivative. These results show the flexible operational technique can be used in a fairly wide context beside the integral transforms for obtaining the formal solutions of PFDEs.

Also, this technique can be considered as a promising approach for many applications in applied sciences.

Competing interests

The author declares that he has no competing interests.

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