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# Existence results for nonlinear boundary value problems with integral boundary conditions on an infinite interval

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## Abstract

In this paper, by using fixed point theorems in a cone, the existence of one positive solution and three positive solutions for nonlinear boundary value problems with integral boundary conditions on an infinite interval are established.

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**Keywords:** positive solutions; fixed point theorems; integral boundary conditions; infinite interval

## 1 Introduction

Consider the following boundary value problem with integral boundary conditions on the half-line of an infinite interval of the form

$$\frac{1}{p(t)}(p(t)z'(t))' + f(t, z(t), z'(t)) = 0, \quad t \in (0, \infty), \quad (1.1)$$

$$a_1 z(0) - b_1 \lim_{t \rightarrow 0^+} p(t)z'(t) = \int_0^\infty g_1(z(s))\psi(s)ds,$$

$$a_2 \lim_{t \rightarrow +\infty} z(t) + b_2 \lim_{t \rightarrow +\infty} p(t)z'(t) = \int_0^\infty g_2(z(s))\psi(s)ds, \quad (1.2)$$

where  $f \in C((0, \infty) \times [0, \infty) \times \mathbb{R}, [0, \infty))$ ,  $f$  may be singular at  $t = 0$ ;  $g_1, g_2 : [0, \infty) \rightarrow [0, \infty)$  are continuous, nondecreasing functions and for  $0 \leq t < \infty$ ,  $z$  in a bounded set,  $g_1(z)$ ,  $g_2(z)$  are bounded;  $\psi : [0, \infty) \rightarrow (0, \infty)$  is a continuous function with  $\int_0^\infty \psi(s)ds < +\infty$ ;  $p \in C[0, \infty) \cap C^1(0, \infty)$  with  $p(t) > 0$  on  $(0, +\infty)$  and  $\int_0^{+\infty} \frac{ds}{p(s)} < +\infty$ ;  $a_1 + a_2 > 0$ ,  $b_i > 0$  for  $i = 1, 2$  with  $D = a_2 b_1 + a_1 b_2 + a_1 a_2 B(0, \infty) > 0$  in which  $B(t, s) = \int_t^s \frac{ds}{p(s)}$ .

Boundary value problems on an infinite interval appear often in applied mathematics and physics. There are many papers concerning the existence of solutions on the half-line for boundary value problems; see [1–5] and the references therein.

At the same time, boundary value problems with integral boundary conditions are of great importance and are an interesting class of problems. They constitute two, three, multi-point, and nonlocal boundary value problems as special cases. For an overview of the literature on integral boundary value problems, see [6–11] and the references therein.

Yan Sun *et al.* [4] studied the existence of positive solutions for singular boundary value problems on the half-line for the following Sturm-Liouville boundary value problem:

$$\begin{aligned} \frac{1}{p(t)}(p(t)z'(t))' + \mu f(t, z(t), z'(t)) &= 0, \quad t \in (0, \infty), \\ a_1 z(0) - b_1 \lim_{t \rightarrow 0^+} p(t)z'(t) &= 0, \\ a_2 \lim_{t \rightarrow +\infty} z(t) + b_2 \lim_{t \rightarrow +\infty} p(t)z'(t) &= 0, \end{aligned}$$

where  $\mu$  is a positive parameter;  $f$  is a continuous, non-negative function and may be singular at  $t = 0$ ;  $p \in C[0, \infty) \cap C^1(0, \infty)$  with  $p(t) > 0$  on  $(0, +\infty)$  and  $\int_0^{+\infty} \frac{ds}{p(s)} < +\infty$ ;  $a_i, b_i \geq 0$  for  $i = 1, 2$ . Wang *et al.* [5] investigated the existence theorems for the boundary value problem given by

$$\begin{aligned} (p(t)z'(t))' + \lambda(f(t, z(t)) - k^2 z(t)) &= 0, \quad t \in (0, \infty), \\ \alpha_1 z(0) - \beta_1 \lim_{t \rightarrow 0^+} p(t)z'(t) &= 0, \\ \alpha_2 \lim_{t \rightarrow +\infty} z(t) + \beta_2 \lim_{t \rightarrow +\infty} p(t)z'(t) &= 0, \end{aligned}$$

where  $f$  is a continuous, non-negative function and may be singular at  $t = 0$ ;  $p \in C[0, \infty) \cap C^1(0, \infty)$  with  $p(t) > 0$  on  $(0, +\infty)$  and  $\int_0^{+\infty} \frac{ds}{p(s)} < +\infty$ ;  $\alpha_i \beta_i \geq 0$  for  $i = 1, 2$ . Also, Feng [11] considered the following boundary value problem with integral boundary conditions on a finite interval:

$$\begin{aligned} (g(t)z'(t))' + w(t)f(t, z(t)) &= 0, \quad t \in (0, 1), \\ az(0) - b \lim_{t \rightarrow 0^+} g(t)z'(t) &= \int_0^1 h(s)z(s) ds, \\ az(1) - b \lim_{t \rightarrow 1^-} g(t)z'(t) &= \int_0^1 h(s)z(s) ds, \end{aligned}$$

where  $a, b > 0$ ;  $g \in C^1([0, 1], [0, \infty))$ ,  $w \in L_p[0, 1]$ ,  $1 \leq p \leq \infty$ , and  $h \in L_1[0, 1]$  are symmetric functions;  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous. The author obtained the existence of symmetric positive solutions by using the fixed point index theory in cones.

Motivated by the above works, we consider the existence of one and three positive solutions for the BVP (1.1), (1.2). However, to our knowledge, although various existence theorems are obtained for Sturm-Liouville boundary value problems with homogeneous boundary conditions, problems with nonhomogeneous boundary conditions, especially integral boundary conditions on an infinite interval have rarely been considered. Therefore, our boundary conditions are more general.

The rest of the paper is organized as follows. In Section 2, we present some necessary lemmas that will be used to prove our main results. In Section 3, we apply the Schauder fixed point theorem to get the existence of at least one positive solution for the nonlinear boundary value problem (1.1) and (1.2). In Section 4, we use the Leggett-Williams fixed point theorem [12] to get the existence of at least three positive solutions for the nonlinear boundary value problem (1.1) and (1.2).

## 2 Preliminaries

In this section, we will employ several lemmas to prove the main results in this paper. These lemmas are based on the following BVP for  $h \in \mathcal{C}((0, \infty))$ :

$$\frac{1}{p(t)}(p(t)z'(t))' + h(t) = 0, \quad t \in (0, \infty), \quad (2.1)$$

$$\begin{aligned} a_1 z(0) - b_1 \lim_{t \rightarrow 0^+} p(t)z'(t) &= \int_0^\infty g_1(z(s))\psi(s) ds, \\ a_2 \lim_{t \rightarrow +\infty} z(t) + b_2 \lim_{t \rightarrow +\infty} p(t)z'(t) &= \int_0^\infty g_2(z(s))\psi(s) ds. \end{aligned} \quad (2.2)$$

Define  $\varphi(t)$  and  $\theta(t)$  to be the solutions of the corresponding homogeneous equation

$$\frac{1}{p(t)}(p(t)z'(t))' = 0, \quad t \in (0, \infty), \quad (2.3)$$

under the initial conditions,

$$\begin{aligned} \theta(0) &= b_1, \quad \lim_{t \rightarrow 0^+} p(t)\theta'(t) = a_1, \\ \lim_{t \rightarrow +\infty} \varphi(t) &= b_2, \quad \lim_{t \rightarrow +\infty} p(t)\varphi'(t) = -a_2. \end{aligned} \quad (2.4)$$

Using the initial conditions (2.4), we can deduce, from equation (2.3) for  $\theta(t)$  and  $\varphi(t)$ , the following equations:

$$\theta(t) = b_1 + a_1 \int_0^t \frac{d\tau}{p(\tau)}, \quad (2.5)$$

$$\varphi(t) = b_2 + a_2 \int_t^\infty \frac{d\tau}{p(\tau)}. \quad (2.6)$$

Let  $G(t, s)$  be the Green function for (2.1), (2.2) is given by

$$G(t, s) = \frac{1}{D} \begin{cases} \theta(t)\varphi(s), & 0 \leq t \leq s < \infty; \\ \theta(s)\varphi(t), & 0 \leq s \leq t < \infty, \end{cases} \quad (2.7)$$

where  $\theta(t)$  and  $\varphi(t)$  are given in (2.5) and (2.6) respectively.

**Lemma 2.1** Suppose the conditions  $\int_0^{+\infty} \frac{ds}{p(s)} < +\infty$  and  $D > 0$  hold. Then for any  $h \in \mathcal{C}((0, \infty), \mathbb{R})$ , the BVP (2.1), (2.2) has the unique solution

$$z(t) = \int_0^\infty G(t, s)p(s)h(s) ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s) ds + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s) ds,$$

where  $G(t, s)$  is given by (2.7).

Furthermore, it is easy to prove the following properties of  $G(t, s)$ :

- (1)  $G(t, s)$  is continuous on  $[0, +\infty) \times [0, +\infty)$ .
- (2) For each  $s \in [0, +\infty)$ ,  $G(t, s)$  is continuously differentiable on  $[0, +\infty)$  except  $t = s$ .
- (3)  $\frac{\partial G(t, s)}{\partial t} \Big|_{t=s^+} - \frac{\partial G(t, s)}{\partial t} \Big|_{t=s^-} = -\frac{1}{p(s)}$ .

(4)  $|G'(t, s)| \leq \frac{c}{p(t)} G(s, s)$ , for  $t \in [0, \infty)$ , where

$$c = \frac{\max\{a_1, a_2\}}{\min\{b_1, b_2\}}. \quad (2.8)$$

(5) For each  $s \in [0, +\infty)$ ,  $G(t, s)$  satisfies the corresponding homogeneous BVP (i.e.,  $h(t) \equiv 0$  in the BVP (2.1)) on  $[0, +\infty)$  except  $t = s$ .

(6)  $0 \leq G(t, s) \leq G(s, s)$  for  $t, s \in [0, \infty)$  and

$$\overline{G}(s) := \lim_{t \rightarrow +\infty} G(t, s) = \frac{b_2}{D} \theta(s) \leq G(s, s) < +\infty.$$

(7) For any  $t \in [a, b] \subset (0, \infty)$  and  $s \in [0, +\infty)$ , we have

$$G(t, s) \geq \gamma_0 G(s, s),$$

where

$$\gamma_0 = \min \left\{ \frac{b_2 + a_2 B(b, \infty)}{b_2 + a_2 B(0, \infty)}, \frac{b_1 + a_1 B(0, a)}{b_1 + a_1 B(0, \infty)} \right\}.$$

Obviously,  $0 < \gamma_0 < 1$ .

It is convenient to list the following conditions which are to be used in our theorems:

(H1)  $f \in C((0, \infty) \times [0, \infty) \times \mathbb{R})$  and also,  $u(t)h(x, y) \leq f(t, x, y) \leq v(t)h(x, y)$ ,  $0 < t < \infty$ , where  $h \in C([0, \infty) \times \mathbb{R}, (0, \infty))$ ; and for  $0 \leq t < \infty$ ,  $x, y$  in a bounded set,  $h(x, y)$  is bounded and  $u, v : (0, \infty) \rightarrow (0, \infty)$  is continuous and may be singular at  $t = 0$ ; and also, there exists  $0 < k_0 < 1$  such that  $u(t) \geq k_0 v(t)$  for  $t \in (0, \infty)$ .

(H2)  $g_1, g_2 : [0, \infty) \rightarrow [0, \infty)$  are continuous, nondecreasing functions, and for  $0 \leq t < \infty$ ,  $z$  in a bounded set,  $g_1(z), g_2(z)$  are bounded.

(H3)  $\psi : [0, \infty) \rightarrow (0, \infty)$  is a continuous function with  $\int_0^\infty \psi(s) ds < +\infty$ .

(H4)  $\int_0^\infty G(s, s)p(s)v(s) ds < +\infty$  and  $\int_0^\infty G(s, s)p(s)u(s) ds < +\infty$ .

Consider the Banach space

$$B = \left\{ z \in C^1[0, +\infty) : \lim_{t \rightarrow +\infty} z(t) < \infty, \sup_{t \in [0, \infty)} |z'(t)| < \infty \right\}$$

with the norm  $\|z\| = \max\{\sup_{t \in [0, \infty)} |z(t)|, \sup_{t \in [0, \infty)} |z'(t)|\}$ .

From the above assumptions, we can define an operator  $A : B \rightarrow B$  by

$$\begin{aligned} Az(t) &= \int_0^\infty G(t, s)p(s)f(s, z(s), z'(s)) ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s) ds \\ &\quad + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s) ds, \quad t \in (0, \infty), \end{aligned} \quad (2.9)$$

where  $G(t, s)$  is given by (2.7).

**Lemma 2.2** ([13]) *Let  $B$  be defined as before and  $M \subset B$ . Then  $M$  is relatively compact in  $B$  if the following conditions hold:*

(a)  $M$  is uniformly bounded in  $B$ ;

- (b) The functions belonging to  $M$  are equicontinuous on any compact interval of  $[0, \infty)$ ;
- (c) The functions from  $M$  are equiconvergent, that is, given  $\epsilon > 0$ , there corresponds a  $T(\epsilon) > 0$  such that  $|f(t) - f(+\infty)| < \epsilon$  for any  $t \geq T(\epsilon)$  and  $f \in M$ .

**Definition 2.1** An operator is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

### 3 Existence of at least one positive solution

In this section, we will apply the following Schauder fixed point theorem to get an existence of one positive solution.

**Theorem 3.1** (Schauder fixed point theorem) *Let  $\mathcal{B}$  be a Banach space and  $S$  be a nonempty bounded, convex, and closed subset of  $\mathcal{B}$ . Assume  $A : \mathcal{B} \rightarrow \mathcal{B}$  is a completely continuous operator. If the operator  $A$  leaves the set  $S$  invariant, i.e., if  $A(S) \subset S$ , then  $A$  has at least one fixed point in  $S$ .*

For convenience, let us set

$$m := \min\{\varphi(\infty), \theta(0)\},$$

$$M := \max\{\varphi(0), \theta(\infty)\},$$

and

$$B(R) = \frac{M}{D} [g_1(R) + g_2(R)] \int_0^\infty \psi(s) ds.$$

**Theorem 3.2** *Assume conditions (H1)-(H4) are satisfied. In addition, let there exist a number  $R > 0$  such that*

$$\max\left\{1, \sup_{t \in [0, \infty)} \frac{c}{p(t)}\right\} \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \left( \int_0^\infty G(s, s) p(s) v(s) ds \right) + B(R) \leq R,$$

where  $c$  is defined by (2.8).

*Then the BVP (1.1), (1.2) has at least one solution  $z$  with*

$$0 \leq z(t) \leq R, \quad t \in [0, \infty).$$

*Proof* Let  $A : \mathcal{B} \rightarrow \mathcal{B}$  be the operator defined by (2.9). We claim that  $A$  is a completely continuous operator. To justify this, we first show that  $A : \mathcal{B} \rightarrow \mathcal{B}$  is well defined. Let  $z \in \mathcal{B}$ , then there exists  $r_0 > 0$  such that  $\|z\| \leq r_0$  and from conditions (H1) and (H2), we have

$$S_{r_0} := \sup\{h(x, y) : 0 \leq x \leq r_0, -r_0 \leq y \leq r_0\} < +\infty,$$

$$T_{r_0} := \sup\{g_1(x) : 0 \leq x \leq r_0\} < +\infty,$$

and

$$T'_{r_0} := \sup\{g_2(x) : 0 \leq x \leq r_0\} < +\infty.$$

Let  $t_1, t_2 \in [0, \infty)$ ,  $t_1 < t_2$ , then

$$\int_0^\infty |G(t_1, s) - G(t_2, s)| p(s) v(s) ds \leq 2 \int_0^\infty G(s, s) p(s) v(s) ds < +\infty. \quad (3.1)$$

Hence, by the Lebesgue dominated convergence theorem and the fact that  $G(t, s)$  is continuous on  $t$ , we have

$$\begin{aligned} |(Az)(t_1) - (Az)(t_2)| &\leq \int_0^\infty |G(t_2, s) - G(t_1, s)| p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{|\theta(t_2) - \theta(t_1)|}{D} \int_0^\infty g_2(z(s)) \psi(s) ds \\ &\quad + \frac{|\varphi(t_2) - \varphi(t_1)|}{D} \int_0^\infty g_1(z(s)) \psi(s) ds \\ &\leq \int_0^\infty |G(t_2, s) - G(t_1, s)| p(s) v(s) h(z(s), z'(s)) ds \\ &\quad + \frac{|\theta(t_2) - \theta(t_1)|}{D} \int_0^\infty g_2(z(s)) \psi(s) ds \\ &\quad + \frac{|\varphi(t_2) - \varphi(t_1)|}{D} \int_0^\infty g_1(z(s)) \psi(s) ds \\ &\leq S_{r_0} \int_0^\infty |G(t_2, s) - G(t_1, s)| p(s) v(s) ds \\ &\quad + \frac{1}{D} \max\{|\varphi(t_2) - \varphi(t_1)|, |\theta(t_2) - \theta(t_1)|\} \\ &\quad \times \int_0^\infty [g_1(z(s)) + g_2(z(s))] \psi(s) ds \\ &\leq S_{r_0} \int_0^\infty |G(t_2, s) - G(t_1, s)| p(s) v(s) ds \\ &\quad + \frac{1}{D} \max\{|\varphi(t_2) - \varphi(t_1)|, |\theta(t_2) - \theta(t_1)|\} (T_{r_0} + T'_{r_0}) \int_0^\infty \psi(s) ds \\ &\rightarrow 0 \quad \text{as } t_1 \rightarrow t_2. \end{aligned} \quad (3.2)$$

Also, by (H1) and (H2), we get

$$\begin{aligned} |(Az)'(t_1) - (Az)'(t_2)| &\leq \frac{a_2}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{t_1} \theta(s) p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{a_1}{D p(t_1)} \int_{t_1}^{t_2} \varphi(s) p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{a_1}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_{t_2}^\infty \varphi(s) p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{a_2}{D p(t_2)} \int_{t_1}^{t_2} \theta(s) p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{a_2}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^\infty g_1(z(s)) \psi(s) ds \\ &\quad + \frac{a_1}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^\infty g_2(z(s)) \psi(s) ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{a_2 S_{r_0}}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{t_1} \theta(s) p(s) v(s) ds \\
&\quad + \frac{a_1 S_{r_0}}{D p(t_1)} \int_{t_1}^{t_2} \varphi(s) p(s) v(s) ds \\
&\quad + \frac{a_1 S_{r_0}}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_{t_2}^{\infty} \varphi(s) p(s) v(s) ds \\
&\quad + \frac{a_2 S_{r_0}}{D p(t_2)} \int_{t_1}^{t_2} \theta(s) p(s) v(s) ds \\
&\quad + \frac{a_2 T_{r_0}}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{\infty} \psi(s) ds \\
&\quad + \frac{a_1 T'_{r_0}}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{\infty} \psi(s) ds \\
&\rightarrow 0 \quad \text{as } t_1 \rightarrow t_2.
\end{aligned} \tag{3.3}$$

So,  $Az \in C^1[0, \infty)$ .

We can show that  $Az \in \mathcal{B}$ . Notice that

$$\begin{aligned}
\lim_{t \rightarrow +\infty} (Az)(t) &= \int_0^{\infty} \overline{G}(s) p(s) f(s, z(s), z'(s)) ds + \frac{\varphi(\infty)}{D} \int_0^{\infty} g_1(z(s)) \psi(s) ds \\
&\quad + \frac{\theta(\infty)}{D} \int_0^{\infty} g_2(z(s)) \psi(s) ds < +\infty.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
|(Az)'(t)| &\leq \int_0^{\infty} |G'(t, s)| p(s) v(s) h(z(s), z'(s)) ds \\
&\quad + \frac{a_2}{D p(t)} \int_0^{\infty} g_1(z(s)) \psi(s) ds + \frac{a_1}{D p(t)} \int_0^{\infty} g_2(z(s)) \psi(s) ds \\
&\leq \frac{c}{p(t)} \int_0^{\infty} G(s, s) p(s) v(s) h(z(s), z'(s)) ds \\
&\quad + \frac{\max\{a_1, a_2\}}{D p(t)} \int_0^{\infty} [g_1(z(s)) + g_2(z(s))] \psi(s) ds \\
&\leq S_{r_0} \frac{c}{p(t)} \int_0^{\infty} G(s, s) p(s) v(s) ds \\
&\quad + (T_{r_0} + T'_{r_0}) \frac{\max\{a_1, a_2\}}{D p(t)} \int_0^{\infty} \psi(s) ds.
\end{aligned}$$

Therefore,  $\sup |(Az)'(t)| < \infty$ .

Hence,  $A : \mathcal{B} \rightarrow \mathcal{B}$  is well defined.

Next, for any positive integer  $m$ , we denote the operator  $A_m : \mathcal{B} \rightarrow \mathcal{B}$  by

$$\begin{aligned}
(A_m z)(t) &= \int_{\frac{1}{m}}^{\infty} G(t, s) p(s) f(s, z(s), z'(s)) ds + \frac{\varphi(t)}{D} \int_0^{\infty} g_1(z(s)) \psi(s) ds \\
&\quad + \frac{\theta(t)}{D} \int_0^{\infty} g_2(z(s)) \psi(s) ds, \quad t \in [0, \infty),
\end{aligned} \tag{3.4}$$

and prove that  $A_m : \mathcal{B} \rightarrow \mathcal{B}$  is completely continuous for each  $m \geq 1$ . Let  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow +\infty$ . We will show that  $\|A_m z_n - A_m z\| \rightarrow 0$  as  $n \rightarrow \infty$  in  $\mathcal{B}$ . We know that

$$\begin{aligned} & \int_{\frac{1}{m}}^{\infty} G(t, s) p(s) |f(s, z_n(s), z'_n(s)) - f(s, z(s), z'(s))| ds \\ & + \frac{\varphi(t)}{D} \int_0^{\infty} |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ & + \frac{\theta(t)}{D} \int_0^{\infty} |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds \\ & \leq \int_{\frac{1}{m}}^{\infty} G(s, s) p(s) (|f(s, z_n(s), z'_n(s))| + |f(s, z(s), z'(s))|) ds \\ & + \frac{\varphi(0)}{D} \int_0^{\infty} |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ & + \frac{\theta(\infty)}{D} \int_0^{\infty} |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds \\ & \leq \int_{\frac{1}{m}}^{\infty} G(s, s) p(s) (v(s) h(z_n(s), z'_n(s)) + v(s) h(z(s), z'(s))) ds \\ & + \frac{M}{D} \int_0^{\infty} |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ & + \frac{M}{D} \int_0^{\infty} |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds \\ & \leq 2S_{r^*} \int_{\frac{1}{m}}^{\infty} G(s, s) p(s) v(s) ds \\ & + \frac{M}{D} \int_0^{\infty} |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ & + \frac{M}{D} \int_0^{\infty} |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds, \end{aligned}$$

where  $r^* > 0$  is a real number such that  $r^* \geq \max_{n \in N} \{\|z\|, \|z_n\|\}$ ,  $N$  is a natural number set,  $S_{r^*} := \sup\{h(x, y) : 0 \leq x \leq r^*, -r^* \leq y \leq r^*\} < +\infty$ .

Therefore, for any  $\varepsilon > 0$ , there exists a sufficiently large  $K_0$  ( $K_0 > \frac{1}{m}$ ) such that

$$2S_{r^*} \int_{K_0}^{\infty} G(s, s) p(s) v(s) ds < \frac{\varepsilon}{4}. \quad (3.5)$$

From the fact that  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow \infty$ , we can see that for the above  $\varepsilon > 0$ , there exists a sufficiently large natural number  $N_0 \in N$  such that if  $n > N_0$ , for any  $s \in [0, \infty)$ , we have

$$|z_n(s) - z(s)| \leq \|z_n - z\| < \frac{\varepsilon}{4} \left( \frac{M}{D} \int_{\frac{1}{m}}^{K_0} G(k, k) p(k) dk \right)^{-1} \left( \int_0^{\infty} \psi(s) ds \right)^{-1}$$

and

$$|z'_n(s) - z'(s)| \leq \|z_n - z\| < \frac{\varepsilon}{4} \left( \frac{M}{D} \int_{\frac{1}{m}}^{K_0} G(k, k) p(k) dk \right)^{-1} \left( \int_0^{\infty} \psi(s) ds \right)^{-1}.$$



On the other hand, by the continuity of  $f(t, x, y)$ , for the above  $\varepsilon > 0$ , there exists a  $\delta > 0$ , for any  $t \in [\frac{1}{m}, K_0]$ ,  $x, x_1 \in [0, r^*]$ ,  $y, y_1 \in [-r^*, r^*]$  such that if  $|x - x_1| < \delta$ ,  $|y - y_1| < \delta$ , we have

$$|f(t, x, y) - f(t, x_1, y_1)| < \frac{\varepsilon}{4} \left( \int_{\frac{1}{m}}^{K_0} G(k, k) p(k) dk \right)^{-1}. \quad (3.6)$$

From the fact that  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a natural number  $N_1 > N_0$  such that when  $n > N_1$ , for any  $s \in [\frac{1}{m}, K_0]$ ,  $z_n(s), z(s) \in [0, r^*]$ ,  $z'_n(s), z'(s) \in [-r^*, r^*]$  if  $|z_n(s) - z(s)| < \delta$ ,  $|z'_n(s) - z'(s)| < \delta$ , we have

$$|f(s, z_n(s), z'_n(s)) - f(s, z(s), z'(s))| < \frac{\varepsilon}{4} \left( \int_{\frac{1}{m}}^{K_0} G(k, k) p(k) dk \right)^{-1}. \quad (3.7)$$

In addition to this, by the continuity of  $g_1(x)$  and  $g_2(x)$  on  $[0, \infty)$ , for the above  $\varepsilon > 0$ , there exists a  $\delta > 0$  for any  $t \in [0, \infty)$ ,  $x, x_1 \in [0, r^*]$ , such that if  $|x - x_1| < \delta$ , we have

$$|g_i(x) - g_i(x_1)| < \frac{\varepsilon}{4} \left( \frac{M}{D} \int_0^\infty \psi(s) ds \right)^{-1}, \quad \text{for } i = 1, 2. \quad (3.8)$$

From  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a natural number  $N_2 > N_0$  such that when  $n > N_2$ , for any  $s \in [0, \infty)$ ,  $z_n(s), z(s) \in [0, r^*]$  if  $|z_n(s) - z(s)| < \delta$ , we have

$$|g_i(z_n(s)) - g_i(z(s))| < \frac{\varepsilon}{4} \left( \frac{M}{D} \int_0^\infty \psi(s) ds \right)^{-1}, \quad \text{for } i = 1, 2. \quad (3.9)$$

Hence, if  $N = \max\{N_1, N_2\}$ , then

$$\begin{aligned} |(A_m z_n)(t) - (A_m z)(t)| &= \left| \int_{\frac{1}{m}}^\infty G(t, s) p(s) f(s, z_n(s), z'_n(s)) ds \right. \\ &\quad - \int_{\frac{1}{m}}^\infty G(t, s) p(s) f(s, z(s), z'(s)) ds \\ &\quad + \frac{\varphi(t)}{D} \int_0^\infty [g_1(z_n(s)) - g_1(z(s))] \psi(s) ds \\ &\quad \left. + \frac{\theta(t)}{D} \int_0^\infty [g_2(z_n(s)) - g_2(z(s))] \psi(s) ds \right| \\ &\leq \int_{\frac{1}{m}}^{K_0} G(s, s) p(s) |f(s, z_n(s), z'_n(s)) - f(s, z(s), z'(s))| ds \\ &\quad + \int_{K_0}^\infty G(s, s) p(s) (f(s, z_n(s), z'_n(s)) + f(s, z(s), z'(s))) ds \\ &\quad + \frac{\varphi(0)}{D} \int_0^\infty |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ &\quad + \frac{\theta(\infty)}{D} \int_0^\infty |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds \\ &\leq \int_{\frac{1}{m}}^{K_0} G(s, s) p(s) |f(s, z_n(s), z'_n(s)) - f(s, z(s), z'(s))| ds \\ &\quad + 2S_{r^*} \int_{K_0}^\infty G(s, s) p(s) v(s) ds \end{aligned}$$

$$\begin{aligned} & + \frac{M}{D} \int_0^\infty |g_1(z_n(s)) - g_1(z(s))| \psi(s) ds \\ & + \frac{M}{D} \int_0^\infty |g_2(z_n(s)) - g_2(z(s))| \psi(s) ds \\ & = \varepsilon. \end{aligned}$$

Similarly, we can see that when  $\|z_n - z\| \rightarrow 0$  as  $n \rightarrow +\infty$ ,  $|(A_m z_n)'(t) - (A_m z)'(t)| \rightarrow 0$  as  $n \rightarrow +\infty$ . This implies that  $A_m : \mathcal{B} \rightarrow \mathcal{B}$  is a continuous operator for each natural number  $m$ .

Choose  $P_R$  to be a bounded, convex, and closed set by

$$\mathcal{P}_R = \{z \in \mathcal{B} : \|z\| \leq R, z(t) \geq 0 \text{ for each } t \in [0, \infty)\}.$$

We must show that there exists a positive constant  $R$  such that for each  $z \in P_R$ , one has  $\|Az\| \leq R$ .

Let  $z \in \mathcal{P}_R$ . Then for each  $t \in [0, \infty)$ , we have  $G(t, s) \geq 0$ . Since  $f, g_1, g_2$  are positive functions,  $A_m z(t) \geq 0, t \in [0, \infty)$ . Furthermore, for  $t \in [0, \infty)$

$$\begin{aligned} |(A_m z)(t)| & \leq \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \left( \int_{\frac{1}{m}}^\infty G(s, s) p(s) v(s) ds \right) \\ & \quad + \frac{\varphi(0)}{D} \int_0^\infty g_1(z(s)) \psi(s) ds + \frac{\theta(\infty)}{D} \int_0^\infty g_2(z(s)) \psi(s) ds \\ & \leq \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \left( \int_{\frac{1}{m}}^\infty G(s, s) p(s) v(s) ds \right) \\ & \quad + \frac{M}{D} [g_1(R) + g_2(R)] \int_0^\infty \psi(s) ds \\ & \leq \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \left( \int_0^\infty G(s, s) p(s) v(s) ds \right) + B(R) \\ & \leq R \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |(A_m z)'(t)| & \leq \int_{\frac{1}{m}}^\infty |G'(t, s)| p(s) v(s) h(z(s), z'(s)) ds \\ & \quad + \frac{a_2}{D p(t)} \int_0^\infty g_1(z(s)) \psi(s) ds + \frac{a_1}{D p(t)} \int_0^\infty g_2(z(s)) \psi(s) ds \\ & \leq \frac{c}{p(t)} \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \int_0^\infty G(s, s) p(s) v(s) ds \\ & \quad + \frac{\max\{a_1, a_2\}}{D} \sup_{t \in [0, \infty)} \frac{1}{p(t)} [g_1(R) + g_2(R)] \int_0^\infty \psi(s) ds \\ & \leq \sup_{t \in [0, \infty)} \frac{c}{p(t)} \left[ \sup_{z_1 \in [0, R], z_2 \in [-R, R]} h(z_1, z_2) \int_0^\infty G(s, s) p(s) v(s) ds \right. \\ & \quad \left. + \frac{M}{D} [g_1(R) + g_2(R)] \int_0^\infty \psi(s) ds \right] \\ & \leq R. \end{aligned} \tag{3.11}$$

Inequalities (3.10) and (3.11) yield that  $\|A_m z\| \leq R$ . Hence,  $A_m$  is uniformly bounded. Using the similar proof as (3.2) and (3.3), we can obtain that for any  $t, t_1 \in [0, \infty)$ ,  $z \in P_R$ ,

$$\|A_m z(t) - A_m z(t_1)\| \rightarrow 0 \quad \text{as } t \rightarrow t_1.$$

Thus,  $A_m P_R$  is equicontinuous. It follows from

$$\begin{aligned} & |(A_m z)(t) - (A_m z)(\infty)| \\ & \leq \int_{\frac{1}{m}}^{\infty} |G(t, s) - \overline{G}(s)| p(s) v(s) h(z(s), z'(s)) ds \\ & \quad + \frac{|\varphi(t) - \varphi(\infty)|}{D} \int_0^{\infty} |g_1(z(s))| \psi(s) ds \\ & \quad + \frac{|\theta(t) - \theta(\infty)|}{D} \int_0^{\infty} |g_2(z(s))| \psi(s) ds \\ & \leq S_R \int_{\frac{1}{m}}^{\infty} |G(t, s) - \overline{G}(s)| p(s) v(s) ds \\ & \quad + \frac{1}{D} \max\{|\varphi(t) - \varphi(\infty)|, |\theta(t) - \theta(\infty)|\} (T_R + T'_R) \int_0^{\infty} \psi(s) ds \\ & \rightarrow 0 \quad \text{as } t \rightarrow \infty \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} & |(A_m z)'(t) - (A_m z)'(\infty)| \\ & \leq \frac{1}{D} \left[ \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_{\frac{1}{m}}^t a_2 \theta(s) p(s) v(s) h(z(s), z'(s)) ds \right. \\ & \quad + \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_t^{\infty} a_1 \varphi(s) p(s) v(s) h(z(s), z'(s)) ds \\ & \quad + \max\{a_1, a_2\} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^{\infty} [g_1(z(s)) + g_2(z(s))] \psi(s) ds \Big] \\ & \leq \frac{1}{D} \left[ \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| S_R \int_{\frac{1}{m}}^t a_2 \theta(s) p(s) v(s) ds \right. \\ & \quad + \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| S_R \int_t^{\infty} a_1 \varphi(s) p(s) v(s) ds \\ & \quad + \max\{a_1, a_2\} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| (T_R + T'_R) \int_0^{\infty} \psi(s) ds \Big] \\ & \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \tag{3.13}$$

Therefore,  $A_m P_R$  is equiconvergent. Hence, by Lemma 2.2 and the above discussion, we conclude that for each natural number  $m$ ,  $A_m : P \rightarrow P$  is completely continuous.

Finally, observe that

$$\begin{aligned} |(Az)(t) - (A_m z)(t)| & = \left| \int_0^{\frac{1}{m}} G(t, s) p(s) f(s, z(s), z'(s)) ds \right| \\ & \leq S_{r_0} \int_0^{\frac{1}{m}} G(s, s) p(s) v(s) ds < \infty \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} |(Az)'(t) - (A_m z)'(t)| &= \left| \int_0^{\frac{1}{m}} G'(t,s)p(s)f(s,z(s),z'(s)) ds \right| \\ &\leq S_{r_0} \sup_{t \in [0,\infty)} \frac{c}{p(t)} \int_0^{\frac{1}{m}} G(s,s)p(s)v(s) ds < \infty. \end{aligned} \quad (3.15)$$

Hence, inequalities (3.14) and (3.15) imply that  $\sup_{t \in [0,\infty)} |(Az)(t) - (A_m z)(t)| < \infty$  and  $\sup_{t \in [0,\infty)} |(Az)'(t) - (A_m z)'(t)| < \infty$ . Then by the assumption (H4) and the absolute continuity of the integral, we get

$$\lim_{m \rightarrow \infty} \int_0^{\frac{1}{m}} G(t,s)p(s)v(s) ds = 0.$$

Therefore, the operator  $A : \mathcal{B} \rightarrow \mathcal{B}$  is completely continuous and maps the set  $P_R$  into itself. Hence, the Schauder fixed point theorem can be applied to obtain a solution of the BVP (1.1), (1.2). The theorem is proved.  $\square$

**Example 3.1** Consider the following boundary value problem:

$$\frac{1}{e^t} (e^t z'(t))' + f(t, z(t), z'(t)) = 0, \quad t \in (0, \infty), \quad (3.16)$$

$$z(0) - \lim_{t \rightarrow 0^+} e^t z'(t) = \int_0^\infty \frac{z(s)}{24} \frac{ds}{1+s^2}, \quad (3.17)$$

$$\lim_{t \rightarrow +\infty} e^t z'(t) = \int_0^\infty \frac{z(s)}{24} \frac{ds}{1+s^2},$$

where  $f(t, z(t), z'(t)) = \frac{e^{-2t}(1+t)}{12\sqrt{t}}(z + |z'|)$ ,  $a_1 = 1$ ,  $a_2 = 0$ ,  $b_1 = 1$ ,  $b_2 = 1$ ,  $p(t) = e^t$ ,  $\psi(s) = \frac{1}{1+s^2}$ ,  $g_1(z(s)) = g_2(z(s)) = \frac{z(s)}{24}$ .

It is clear that  $f : (0, \infty) \times [0, +\infty) \times \mathbb{R} \rightarrow (0, +\infty)$  is continuous and singular at  $t = 0$ . Set  $v(t) = \frac{e^{-2t}(1+t)}{\sqrt{t}}$  and  $h(z, z') = \frac{z+|z'|}{12}$ , it follows from a direct calculation that  $M = 2$ ,  $c = 1$ , and there exists  $R = 1$  such that the following inequality holds:

$$\max \left\{ 1, \sup_{t \in [0,\infty)} \frac{1}{e^t} \right\} \sup_{z_1 \in [0,1], z_2 \in [-1,1]} h(z_1, z_2) \left( \int_0^\infty G(s,s)p(s)v(s) ds \right) + B(1) \leq 1.$$

Then by Theorem 3.2, the boundary value problem (3.16)-(3.17) has at least one positive solution.

#### 4 Existence of at least three positive solutions

**Definition 4.1** Let  $\mathcal{B}$  be a Banach space,  $\mathcal{P} \subset \mathcal{B}$  be a cone in  $\mathcal{B}$ . By a concave nonnegative continuous functional  $\varphi$  on  $\mathcal{P}$ , we mean a continuous functional  $\varphi : \mathcal{P} \rightarrow [0, \infty)$  with

$$\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y) \quad \text{for all } x, y \in \mathcal{P} \text{ and } t \in [0, 1].$$

For  $K, L, R > 0$  being constants with  $\mathcal{P}$  and  $\varphi$  as above, let

$$\mathcal{P}_K = \{x \in \mathcal{P} : \|x\| < K\}$$

and

$$\mathcal{P}(\varphi, L, K) = \{x \in \mathcal{P} : L \leq \varphi(x), \|x\| \leq K\}.$$

**Theorem 4.1** (Leggett-Williams fixed point theorem [12]) *Let  $\mathcal{B}$  be a Banach space,  $\mathcal{P} \subset \mathcal{B}$  be a cone of  $\mathcal{B}$ , and  $R > 0$  be a constant. Suppose  $A : \overline{\mathcal{P}}_R \rightarrow \overline{\mathcal{P}}_R$  is a completely continuous operator and  $\varphi$  is a nonnegative, continuous, concave functional on  $\mathcal{P}$  with  $\varphi(y) \leq \|y\|$  for all  $y \in \overline{\mathcal{P}}_R$ . If there exist  $r, L$ , and  $K$  with  $0 < r < L < K \leq R$  such that the following conditions hold:*

- (i)  $\{y \in \mathcal{P}(\varphi, L, K) : \varphi(y) > L\} \neq \emptyset$  and  $\varphi(Ay) > L$  for all  $y \in \mathcal{P}(\varphi, L, K)$ ;
- (ii)  $\|Ay\| < r$  for all  $\|y\| \leq r$ ;
- (iii)  $\varphi(Ay) > L$  for all  $y \in \mathcal{P}(\varphi, L, R)$  with  $\|Ay\| > K$ .

*Then  $A$  has at least three positive solutions  $y_1, y_2$ , and  $y_3$  in  $\overline{\mathcal{P}}_R$  satisfying*

$$\|y_1\| < r, \quad y_2 \in \{y \in \mathcal{P}(\varphi, L, R) : \varphi(y) > L\},$$

and

$$y_3 \in \overline{\mathcal{P}}_R - \{\mathcal{P}(\varphi, L, R) \cup \overline{\mathcal{P}}_r\}.$$

**Theorem 4.2** *Assume that (H1)-(H4) are satisfied and there exists  $c_1 \in (0, 1)$  such that  $m \geq c_1 M$  holds. Then the boundary value problem (1.1), (1.2) has at least three positive solutions if the following conditions hold:*

(H5) *There exists a constant  $r > 0$  such that*

$$h(z_1, z_2) < \min \left\{ 1, \frac{1}{\sup_{t \in [0, \infty)} \frac{c}{p(t)}} \right\} \frac{r - B(r)}{\int_0^\infty G(s, s)p(s)v(s) ds},$$

*for  $t \in [0, \infty)$  and  $z_1 \in [0, r], z_2 \in [-r, r]$ ;*

(H6) *There exist  $L > r$  and an interval  $[a, b] \subset [0, \infty)$  such that*

$$h(z_1, z_2) > \frac{L}{\gamma_0 k_0 \int_0^\infty G(s, s)p(s)v(s) ds},$$

*for  $t \in [a, b]$  and  $z_1 \in [L, K], z_2 \in [-K, K]$ ;*

(H7) *There exist  $0 < r < L < K < \gamma_0 k_0 \min\{1, \sup_{t \in [0, \infty)} \frac{c}{p(t)}\}[R - B(R)], N^{-1}L \leq \max\{1, \sup_{t \in [0, \infty)} \frac{c}{p(t)}\}K$ , where  $N = \min\{c_1, \gamma_0 k_0\}$  such that*

$$h(z_1, z_2) < \min \left\{ 1, \frac{1}{\sup_{t \in [0, \infty)} \frac{c}{p(t)}} \right\} \frac{R - B(R)}{\int_0^\infty G(s, s)p(s)v(s) ds},$$

*for  $t \in [0, \infty)$  and  $z_1 \in [0, R], z_2 \in [-R, R]$ .*

*Proof* The conditions of the Leggett-Williams fixed point theorem will be shown to be satisfied. Define the cone  $\mathcal{P} \subset B$  by  $\mathcal{P} = \{z \in B : z(t) \geq 0 \text{ for each } t \in [0, \infty)\}$  and the non-negative, continuous, concave functional  $\varphi : \mathcal{P} \rightarrow [0, \infty)$  by  $\varphi(z) = \min_{t \in [a, b]} |z(t)|$ .

Then we have  $\varphi(z) \leq \|z\|$  for all  $z \in \mathcal{P}$ . If  $z \in \overline{\mathcal{P}}_R$ , then  $\|z\| \leq R$  and from (H7) we have

$$\begin{aligned} |Az(t)| &= \left| \int_0^\infty G(t,s)p(s)f(s,z(s),z'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \right. \\ &\quad \left. + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \right| \\ &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{\varphi(0)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \\ &\quad + \frac{\theta(\infty)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \\ &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{M}{D} [g_1(R) + g_2(R)] \int_0^\infty \psi(s)ds \\ &\leq R. \end{aligned}$$

Furthermore,

$$\begin{aligned} |(Az)'(t)| &\leq \int_0^\infty |G'(t,s)|p(s)v(s)h(z(s),z'(s))ds \\ &\quad + \frac{a_2}{Dp(t)} \int_0^\infty g_1(z(s))\psi(s)ds + \frac{a_1}{Dp(t)} \int_0^\infty g_2(z(s))\psi(s)ds \\ &\leq \frac{c}{p(t)} \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds \\ &\quad + \frac{\max\{a_1, a_2\}}{Dp(t)} [g_1(R) + g_2(R)] \int_0^\infty \psi(s)ds \\ &\leq \sup_{t \in [0, \infty)} \frac{c}{p(t)} \left[ \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + B(R) \right] \\ &\leq R. \end{aligned} \tag{4.1}$$

Therefore, we get  $\|Az\| \leq R$ , and this implies that  $A : \overline{\mathcal{P}}_R \rightarrow \overline{\mathcal{P}}_R$ .

Now we show that condition (i) of Theorem 4.1 is satisfied. Let  $z(t) = \frac{L+K}{2}$  for  $t \in [0, \infty)$ . By the definition of  $\mathcal{P}(\varphi, L, K)$ ,  $z \in \mathcal{P}(\varphi, L, K)$ . Then  $\{z \in \mathcal{P}(\varphi, L, K) : \varphi(z) > L\} \neq \emptyset$ . If  $z \in \mathcal{P}(\varphi, L, K)$ , then by (H6) we get

$$\begin{aligned} \varphi(Az) &= \min_{t \in [a, b]} \left( \int_0^\infty G(t,s)p(s)f(s,z(s),z'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \right. \\ &\quad \left. + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \right) \\ &\geq \min_{t \in [a, b]} \left( \int_0^\infty G(t,s)p(s)f(s,z(s),z'(s))ds \right) \\ &\geq \gamma_0 \int_0^\infty G(s,s)p(s)u(s)h(z(s),z'(s))ds \\ &\geq \gamma_0 k_0 \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds \\ &> L. \end{aligned}$$

Therefore, condition (i) of Theorem 4.1 is satisfied.

If  $\|z\| \leq r$ , then by (H5) we have

$$\begin{aligned} |Az(t)| &= \left| \int_0^\infty G(t,s)p(s)f(s,z(s),z'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \right. \\ &\quad \left. + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \right| \\ &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{\varphi(0)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \\ &\quad + \frac{\theta(\infty)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \\ &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{M}{D} [g_1(r) + g_2(r)] \int_0^\infty \psi(s)ds \\ &< r. \end{aligned}$$

In a similar way as (4.1), we can see that for each  $t \in [0, \infty)$ ,

$$|(Az)'(t)| \leq \sup \frac{c}{p(t)} \left[ \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + B(r) \right] < r.$$

Hence, condition (ii) of Theorem 4.1 holds.

Finally, we show that condition (iii) of Theorem 4.1 is also satisfied. If  $z \in \mathcal{P}(\varphi, L, R)$ , we get

$$\begin{aligned} |Az(t)| &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{\varphi(0)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \\ &\quad + \frac{\theta(\infty)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \\ &\leq \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds + \frac{M}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s)ds \end{aligned}$$

and

$$\begin{aligned} |(Az)'(t)| &\leq \sup \frac{c}{p(t)} \left[ \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds \right. \\ &\quad \left. + \frac{M}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s)ds \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|Az\| &\leq \max \left\{ 1, \sup \frac{c}{p(t)} \right\} \left[ \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s))ds \right. \\ &\quad \left. + \frac{M}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s)ds \right]. \end{aligned}$$

Therefore, for  $z \in \mathcal{P}(\varphi, L, R)$  and  $\|Az\| > K$ , we have

$$\begin{aligned} \varphi(Az) &= \min_{t \in [a,b]} \left( \int_0^\infty G(t,s)p(s)f(s,z(s),z'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(z(s))\psi(s)ds \right. \\ &\quad \left. + \frac{\theta(t)}{D} \int_0^\infty g_2(z(s))\psi(s)ds \right) \end{aligned}$$

$$\begin{aligned}
&\geq \gamma_0 \int_0^\infty G(s,s)p(s)f(s,z(s),z'(s)) ds + \frac{\varphi(\infty)}{D} \int_0^\infty g_1(z(s))\psi(s) ds \\
&\quad + \frac{\theta(0)}{D} \int_0^\infty g_2(z(s))\psi(s) ds \\
&\geq \gamma_0 \int_0^\infty G(s,s)p(s)u(s)h(z(s),z'(s)) ds + \frac{m}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s) ds \\
&\geq \gamma_0 k_0 \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s)) ds + \frac{c_1 M}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s) ds \\
&\geq N \left[ \int_0^\infty G(s,s)p(s)v(s)h(z(s),z'(s)) ds + \frac{M}{D} \int_0^\infty [g_1(z(s)) + g_2(z(s))]\psi(s) ds \right] \\
&\geq N \frac{1}{\max\{1, \sup_{t \in [0, \infty)} \frac{c}{p(t)}\}} \|Az\| > \frac{NK}{\max\{1, \sup_{t \in [0, \infty)} \frac{c}{p(t)}\}} \geq L.
\end{aligned}$$

Therefore, condition (iii) is also satisfied. Then the Leggett-Williams fixed point theorem implies that  $A$  has at least three positive solutions  $z_1$ ,  $z_2$ , and  $z_3$  which are solutions to the problem (1.1)-(1.2). Furthermore, we have

$$\|z_1\| < r, \quad z_2 \in \{z \in \mathcal{P}(\varphi, L, R) : \varphi(z) > L\},$$

and

$$z_3 \in \overline{\mathcal{P}_R} - \{\mathcal{P}(\varphi, L, R) \cup \overline{\mathcal{P}_r}\}.$$

□

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Both authors typed, read and approved the final manuscript.

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