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Existence and multiplicity of solutions for some second-order systems on time scales with impulsive effects

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Abstract

In this paper, we present a recent approach via variational methods and critical point theory to obtain the existence of solutions for the nonautonomous second-order system on time scales with impulsive effects

$$\begin{cases} u^{\Delta^2}(t) + A(\sigma(t))u(\sigma(t)) + \nabla F(\sigma(t), u(\sigma(t))) = 0, & \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}^{\kappa}; \\ u(0) - u(T) = u^{\Delta}(0) - u^{\Delta}(T) = 0, \\ (u^i)^{\Delta}(t_j^+) - (u^i)^{\Delta}(t_j^-) = I_{ij}(u^i(t_j)), & i = 1,2,\ldots,N, j = 1,2,\ldots,p, \end{cases}$$

where $t_0 = 0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $t_j \in [0,T]_{\mathbb{T}}$ $(j = 0,1,2,\ldots,p+1)$, $u(t) = (u^1(t), u^2(t),\ldots,u^N(t)) \in \mathbb{R}^N$, $A(t) = [d_{lm}(t)]$ is a symmetric $N \times N$ matrix-valued function defined on $[0,T]_{\mathbb{T}}$ with $d_{lm} \in L^{\infty}([0,T]_{\mathbb{T}}, \mathbb{R})$ for all $l,m = 1,2,\ldots,N$, $I_{ij} : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\ldots,N, j = 1,2,\ldots,p)$ are continuous and $F : [0,T]_{\mathbb{T}} \times \mathbb{R}^N \to \mathbb{R}$. Finally, two examples are presented to illustrate the feasibility and effectiveness of our results.

MSC: 34B37; 34N05

Keywords: nonautonomous second-order systems; time scales; impulse; variational approach

1 Introduction

Consider the nonautonomous second-order system on time scales with impulsive effects

$$\begin{cases} u^{\Delta^2}(t) + A(\sigma(t))u(\sigma(t)) + \nabla F(\sigma(t), u(\sigma(t))) = 0, & \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}^{\kappa}; \\ u(0) - u(T) = u^{\Delta}(0) - u^{\Delta}(T) = 0, \\ (u^i)^{\Delta}(t_j^+) - (u^i)^{\Delta}(t_j^-) = I_{ij}(u^i(t_j)), & i = 1,2,\ldots,N, j = 1,2,\ldots,p, \end{cases} \tag{1.1}$$

where $t_0 = 0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $t_j \in [0,T]_{\mathbb{T}}$ $(j = 0,1,2,\ldots,p+1)$,

$$(u^i)^{\Delta}(t_j^+) = \begin{cases} \lim_{t \to t_j^+} (u^i)^{\Delta}(t), & t \text{ is right-dense;} \\ (u^i)^{\Delta}(\sigma(t_j)), & t \text{ is right-scattered,} \end{cases}$$

$$(u^i)^{\Delta}(t_j^-) = \begin{cases} \lim_{t \to t_j^-} (u^i)^{\Delta}(t), & t \text{ is left-dense;} \\ (u^i)^{\Delta}(\rho(t_j)), & t \text{ is left-scattered,} \end{cases}$$

$u(t) = (u^1(t), u^2(t), \dots, u^N(t))$, $A(t) = [d_{lm}(t)]$ is a symmetric $N \times N$ matrix-valued function defined on $[0,T]_{\mathbb{T}}$ with $d_{lm} \in L^\infty([0,T]_{\mathbb{T}}, \mathbb{R})$ for all $l, m = 1, 2, \dots, N$, $I_{ij} : \mathbb{R} \to \mathbb{R}$ ($i = 1, 2, \dots, N, j = 1, 2, \dots, p$) are continuous and $F : [0,T]_{\mathbb{T}} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t,x)$ is Δ-measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for Δ-a.e. $t \in [0,T]_{\mathbb{T}}$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b^\sigma \in L^1(0,T;\mathbb{R}^+)$ such that

$$\left|F(t,x)\right| \le a\bigl(|x|\bigr)b(t), \qquad \left|\nabla F(t,x)\right| \le a\bigl(|x|\bigr)b(t)$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$, where $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

For the sake of convenience, in the sequel, we denote $\Gamma = \{1, 2, 3, \dots, N\}$, $\Lambda = \{1, 2, 3, \dots, p\}$.

When $I_{ij} \equiv 0$, $i \in A$, $j \in B$ and $A(t)$ is a zero matrix, (1.1) is the Hamiltonian system on time scales

$$\begin{cases} u^{\Delta^2}(t) + \nabla F(\sigma(t), u(\sigma(t))) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = u^\Delta(0) - u^\Delta(T) = 0. \end{cases} \tag{1.2}$$

In [1], the authors study the Sobolev's spaces on time scales and their properties. As applications, they present a recent approach via variational methods and the critical point theory to obtain the existence of solutions for (1.2).

When $I_{ij}(t) \not\equiv 0$, $i \in A$, $j \in B$ and $A(t)$ is not a zero matrix, until now the variational structure of (1.1) has not been studied.

Problem (1.1) covers the second-order Hamiltonian system with impulsive effects (when $\mathbb{T} = \mathbb{R}$)

$$\begin{cases} \ddot{u}(t) + A(t)u(t) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0,T]; \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)), \quad i \in \Gamma, j \in \Lambda, \end{cases} \tag{1.3}$$

as well as the second-order discrete Hamiltonian system (when $\mathbb{T} = \mathbb{Z}$, $T \in \mathbb{N}$, $T \ge 2$)

$$\begin{cases} u(t+2) - 2u(t+1) + u(t) + A(t+1)u(t+1) \\ \quad + \nabla F(t+1, u(t+1)) = 0, \quad t \in [1, T-1] \cap \mathbb{Z}, \\ u(0) - u(T) = 0, u(T) - u(0) = u(T+1) - u(1), \\ u^i(t_j+2) - u^i(t_j+1) - u^i(t_j) + u^i(t_j-1) = I_{ij}(u^i(t_j)), \quad i \in \Gamma, j \in \Lambda. \end{cases}$$

In [2], the authors establish some sufficient conditions on the existence of solutions of (1.3) by means of some critical point theorems when $A(t) \equiv 0$. When $\mathbb{T} \neq \mathbb{R}$, until now, it is unknown whether problem (1.1) has a variational structure or not.

Impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians (see [3–5]). Applications of impulsive differential equations with or without delays occur in biology, medicine, mechanics, engineering, chaos theory and so on (see [6–9]).

For a second-order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft, one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position (see [10]). The impulses only on the velocity occur also in impulsive mechanics (see [11]). An impulsive problem with impulses in the derivative only is considered in [12].

The study of dynamical systems on time scales is now an active area of research. One of the reasons for this is the fact that the study on time scales unifies the study of both discrete and continuous processes, besides many others. The pioneering works in this direction are Refs. [13–17]. The theory of time scales was initiated by Stefan Hilger in his Ph.D. thesis in 1988, providing a rich theory that unifies and extends discrete and continuous analysis [18, 19]. The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks and social sciences (see [16]). For example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

There have been many approaches to study solutions of differential equations on time scales, such as the method of lower and upper solutions, fixed-point theory, coincidence degree theory and so on (see [1, 20–29]). In [24], authors used the fixed point theorem of strict-set-contraction to study the existence of positive periodic solutions for functional differential equations with impulse effects on time scales. However, the study of the existence and multiplicity of solutions for differential equations on time scales using the variational method has received considerably less attention (see, for example, [1, 29]). The variational method is, to the best of our knowledge, novel and it may open a new approach to deal with nonlinear problems, with some type of discontinuities such as impulses.

Motivated by the above, we research the existence of variational construction for problem (1.1) in an appropriate space of functions and study the existence of solutions for (1.1) by some critical point theorems in this paper. All these results are new.

2 Preliminaries and statements

In this section, we present some fundamental definitions and results from the calculus on time scales and Sobolev's spaces on time scales that will be required below. These are a generalization to R^n of definitions and results found in [17].

Definition 2.1 ([17, Definition 1.1]) Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\} \quad \text{for all } t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T}, s < t\} \quad \text{for all } t \in \mathbb{T}$$

(supplemented by $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$, where $\emptyset$ denotes the empty set). A point $t \in \mathbb{T}$ is called right-scattered, left-scattered, if $\sigma(t) > t$, $\rho(t) < t$ hold, respectively. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. The set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

When $a, b \in \mathbb{T}$, $a < b$, we denote the intervals $[a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$ and $(a,b]_{\mathbb{T}}$ in $\mathbb{T}$ by

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}, \qquad [a,b)_{\mathbb{T}} = [a,b) \cap \mathbb{T}, \qquad (a,b]_{\mathbb{T}} = (a,b] \cap \mathbb{T},$$

respectively. Note that $[a,b]_{\mathbb{T}}^\kappa = [a,b]_{\mathbb{T}}$ if b is left-dense and $[a,b]_{\mathbb{T}}^\kappa = \big[a,b\big)_{\mathbb{T}} = [a,\rho(b)]_{\mathbb{T}}$ if b is left-scattered. We denote $[a,b]_{\mathbb{T}}^{\kappa^2} = ([a,b]_{\mathbb{T}}^\kappa)^\kappa$, therefore $[a,b]_{\mathbb{T}}^{\kappa^2} = [a,b]_{\mathbb{T}}$ if b is left-dense and $[a,b]_{\mathbb{T}}^{\kappa^2} = [a,\rho(b)]_{\mathbb{T}}^\kappa$ if b is left-scattered.

Definition 2.2 ([17, Definition 1.10]) Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (*i.e.*, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\big|\big[f\big(\sigma(t)\big) - f(s)\big] - f^\Delta(t)\big[\sigma(t) - s\big]\big| \le \epsilon\big|\sigma(t) - s\big| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t. The function f is delta (or Hilger) differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^\kappa$.

Definition 2.3 ([1, Definition 2.3]) Assume that $f : \mathbb{T} \to \mathbb{R}^N$ is a function,

$$f(t) = \big(f^1(t), f^2(t), \ldots, f^N(t)\big),$$

and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t) = (f^{1^\Delta}(t), f^{2^\Delta}(t), \ldots, f^{N^\Delta}(t))$ (provided it exists). We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t. The function f is delta (or Hilger) differentiable provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}^N$ is then called the delta derivative of f on $\mathbb{T}^\kappa$.

Definition 2.4 ([17, Definition 2.7]) For a function $f : \mathbb{T} \to \mathbb{R}$, we will talk about the second derivative f^{Δ^2} provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta^2} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$.

Definition 2.5 ([1, Definition 2.5]) For a function $f : \mathbb{T} \to \mathbb{R}^N$, we will talk about the second derivative f^{Δ^2} provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta^2} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}^N$.

The Δ-measure μ_Δ and Δ-integration are defined as those in [26].

Definition 2.6 ([1, Definition 2.7]) Assume that $f:\mathbb{T}\to\mathbb{R}^N$ is a function, $f(t)=(f^1(t),f^2(t),\dots,f^N(t))$ and let A be a Δ-measurable subset of $\mathbb{T}$. f is integrable on A if and only if f^i $(i=1,2,\dots,N)$ are integrable on A, and $\int_A f(t)\Delta t=(\int_A f^1(t)\Delta t,\int_A f^2(t)\Delta t,\dots,\int_A f^N(t)\Delta t)$.

Definition 2.7 ([17, Definition 2.3]) Let $B\subset\mathbb{T}$. B is called a Δ-null set if $\mu_\Delta(B)=0$. Say that a property P holds Δ-almost everywhere (Δ-a.e.) on B, or for Δ-almost all (Δ-a.a.) $t\in B$ if there is a Δ-null set $E_0\subset B$ such that P holds for all $t\in B\backslash E_0$.

For $p\in\mathbb{R}$, $p\geq 1$, we set the space

$$L^p_\Delta([0,T)_\mathbb{T},\mathbb{R}^N)=\left\{u:[0,T)_\mathbb{T}\to\mathbb{R}^N:\int_{[0,T)_\mathbb{T}}|f(t)|^p\Delta t<+\infty\right\}$$

with the norm

$$\|f\|_{L^p_\Delta}=\left(\int_{[0,T)_\mathbb{T}}|f(t)|^p\Delta t\right)^{\frac{1}{p}}.$$

We have the following theorem.

Theorem 2.1 ([1, Theorem 2.1]) *Let $p\in\mathbb{R}$ be such that $p\geq 1$. Then the space $L^p_\Delta([0,T)_\mathbb{T},\mathbb{R}^N)$ is a Banach space together with the norm $\|\cdot\|_{L^p_\Delta}$. Moreover, $L^2_\Delta([a,b)_\mathbb{T},\mathbb{R}^N)$ is a Hilbert space together with the inner product given for every $(f,g)\in L^p_\Delta([a,b)_\mathbb{T},\mathbb{R}^N)\times L^p_\Delta([a,b)_\mathbb{T},\mathbb{R}^N)$ by*

$$\langle f,g\rangle_{L^2_\Delta}=\int_{[a,b)_\mathbb{T}}\big(f(t),g(t)\big)\Delta t,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.

Definition 2.8 ([1, Definition 2.11]) A function $f:[a,b]_\mathbb{T}\to\mathbb{R}^N, f(t)=(f^1(t),f^2(t),\dots,f^N(t))$. We say that f is absolutely continuous on $[a,b]_\mathbb{T}$ (*i.e.*, $f\in AC([a,b]_\mathbb{T},\mathbb{R}^N)$) if for every $\epsilon>0$, there exists $\delta>0$ such that if $\{[a_k,b_k)_\mathbb{T}\}_{k=1}^n$ is a finite pairwise disjoint family of subintervals of $[a,b]_\mathbb{T}$ satisfying $\sum_{k=1}^n(b_k-a_k)<\delta$, then $\sum_{k=1}^n|f(b_k)-f(a_k)|<\epsilon$.

Now, we recall the Sobolev space $W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N)$ on $[0,T]_\mathbb{T}$ defined in [1]. For the sake of convenience, in the sequel we let $u^\sigma(t)=u(\sigma(t))$.

Definition 2.9 ([1, Definition 2.12]) Let $p\in\mathbb{R}$ be such that $p>1$ and $u:[0,T]_\mathbb{T}\to\mathbb{R}^N$. We say that $u\in W^{1,p}_{\Delta,T}([0,T]_\mathbb{T},\mathbb{R}^N)$ if and only if $u\in L^p_\Delta([0,T)_\mathbb{T},\mathbb{R}^N)$ and there exists $g:[0,T]^\kappa_\mathbb{T}\to\mathbb{R}^N$ such $g\in L^p_\Delta([0,T)_\mathbb{T},\mathbb{R}^N)$ and

$$\int_{[0,T)_\mathbb{T}}\big(u(t),\phi^\Delta(t)\big)\Delta t=-\int_{[0,T)_\mathbb{T}}\big(g(t),\phi^\sigma(t)\big)\Delta t,\quad\forall\phi\in C^1_{T,rd}\big([0,T]_\mathbb{T},\mathbb{R}^N\big).\tag{2.1}$$

For $p\in\mathbb{R}$, $p>1$, we denote

$$V^{1,p}_{\Delta,T}\big([0,T]_\mathbb{T},\mathbb{R}^N\big)=\big\{x\in AC\big([0,T]_\mathbb{T},\mathbb{R}^N\big):x^\Delta\in L^p_\Delta\big([0,T)_\mathbb{T},\mathbb{R}^N\big),x(0)=x(T)\big\}.$$

It follows from Remark 2.2 in [1] that

$$V_{\Delta}^{1,p}\big([0,T]_{\mathbb{T}},\mathbb{R}_N\big)\subset W_{\Delta}^{1,p}\big([0,T]_{\mathbb{T}},\mathbb{R}^N\big)$$

is true for every $p\in\mathbb{R}$ with $p>1$. These two sets are, as a class of functions, equivalent. It is the characterization of functions in $W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ in terms of functions in $V_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ too. That is the following theorem.

Theorem 2.2 ([1, Theorem 2.5]) *Suppose that* $u\in W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *for some* $p\in\mathbb{R}$ *with* $p>1$, *and that* (2.1) *holds for* $g\in L_{\Delta}^{p}([0,T)_{\mathbb{T}},\mathbb{R}^N)$. *Then there exists a unique function* $x\in V_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *such that the equalities*

$$x=u,\qquad x^{\Delta}=g\quad \Delta\text{-a.e. on }[0,T)_{\mathbb{T}}\tag{2.2}$$

are satisfied and

$$\int_{[0,T)_{\mathbb{T}}}g(t)\Delta t=0.\tag{2.3}$$

By identifying $u\in W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ with its absolutely continuous representative $x\in V_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ for which (2.2) holds, the set $W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ can be endowed with the structure of a Banach space. That is the following theorem.

Theorem 2.3 ([25, Theorem 2.21]) *Assume* $p\in\mathbb{R}$ *and* $p>1$. *The set* $W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *is a Banach space together with the norm defined as*

$$\|u\|_{W_{\Delta,T}^{1,p}}=\left(\int_{[0,T)_{\mathbb{T}}}\big|u^{\sigma}(t)\big|^{p}\Delta t+\int_{[0,T)_{\mathbb{T}}}\big|u^{\Delta}(t)\big|^{p}\Delta t\right)^{\frac{1}{p}}$$
$$\forall u\in W_{\Delta,T}^{1,p}\big([0,T]_{\mathbb{T}},\mathbb{R}^N\big).\tag{2.4}$$

Moreover, the set $H_{\Delta,T}^{1}=W_{\Delta,T}^{1,2}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ *is a Hilbert space together with the inner product*

$$\langle u,v\rangle_{H_{\Delta,T}^{1}}=\int_{[0,T)_{\mathbb{T}}}\big(u^{\sigma}(t),v^{\sigma}(t)\big)\Delta t+\int_{[0,T)_{\mathbb{T}}}\big(u^{\Delta}(t),v^{\Delta}(t)\big)\Delta t\quad\forall u,v\in H_{\Delta,T}^{1}.$$

The Banach space $W_{\Delta,T}^{1,p}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ has some important properties.

Theorem 2.4 ([25, Theorem 2.23]) *There exists* $C_1>0$ *such that the inequality*

$$\|u\|_{\infty}\le C_1\|u\|_{H_{\Delta,T}^{1}}\tag{2.5}$$

holds for all $u\in H_{\Delta,T}^{1}$, *where* $\|u\|_{\infty}=\max_{t\in[0,T]_{\mathbb{T}}}|u(t)|$.

Moreover, if $\int_{[0,T)_{\mathbb{T}}}u(t)\Delta t=0$, *then*

$$\|u\|_{\infty}\le C_1\big\|u^{\Delta}\big\|_{L_{\Delta}^{2}}.$$

Theorem 2.5 ([25, Theorem 2.25]) *If the sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ converges weakly to u in $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$, then $\{u_k\}_{k\in\mathbb{N}}$ converges strongly in $C([0,T]_{\mathbb{T}},\mathbb{R}^N)$ to u.*

Theorem 2.6 ([25, Theorem 2.27]) *Let $L:[0,T]_{\mathbb{T}}\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}, (t,x,y)\to L(t,x,y)$ be Δ-measurable in t for each $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and continuously differentiable in (x,y) for Δ-almost every $t\in[0,T]_{\mathbb{T}}$. If there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+), b\in L^1_{\Delta}([0,T]_{\mathbb{T}},\mathbb{R}^+)$ and $c\in L^q_{\Delta}([0,T]_{\mathbb{T}},\mathbb{R}^+)$ $(1<q<+\infty)$ such that for Δ-almost $t\in[0,T]_{\mathbb{T}}$ and every $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, one has*

$$\begin{aligned}&\left|L(t,x,y)\right|\le a\big(|x|\big)\big(b(t)+|y|^p\big),\\&\left|L_x(t,x,y)\right|\le a\big(|x|\big)\big(b(t)+|y|^p\big),\\&\left|L_y(t,x,y)\right|\le a\big(|x|\big)\big(c(t)+|y|^{p-1}\big),\end{aligned}\tag{2.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$, then the functional $\Phi: W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)\to\mathbb{R}$ defined as

$$\Phi(u)=\int_{[0,T)_{\mathbb{T}}} L\big(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\big)\Delta t$$

is continuously differentiable on $W^{1,p}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ and

$$\begin{aligned}\big\langle\Phi'(u),v\big\rangle=\int_{[0,T)_{\mathbb{T}}}&\Big[\big(L_x\big(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\big),v^{\sigma}(t)\big)\\&+\big(L_y\big(\sigma(t),u^{\sigma}(t),u^{\Delta}(t)\big),v^{\Delta}(t)\big)\Big]\Delta t.\end{aligned}\tag{2.7}$$

3 Variational setting

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure. From this variational structure, we can reduce the problem of finding solutions of (1.1) to the one of seeking the critical points of a corresponding functional.

If $u\in H^1_{\Delta,T}$, by identifying $u\in H^1_{\Delta,T}$ with its absolutely continuous representative $x\in V^{1,2}_{\Delta,T}([0,T]_{\mathbb{T}},\mathbb{R}^N)$ for which (2.2) holds, then u is absolutely continuous and $\dot{u}\in L^2([0,T)_{\mathbb{T}};\mathbb{R}^N)$. In this case, $u^{\Delta}(t^+)-u^{\Delta}(t^-)=0$ may not hold for some $t\in(0,T)_{\mathbb{T}}$. This leads to impulsive effects.

Take $v\in H^1_{\Delta,T}$ and multiply the two sides of the equality

$$u^{\Delta^2}(t)+A\big(\sigma(t)\big)u\big(\sigma(t)\big)+\nabla F\big(\sigma(t),u^{\sigma}(t)\big)=0$$

by v^{σ} and integrate on $[0,T)_{\mathbb{T}}$, then we have

$$\int_{[0,T)_{\mathbb{T}}}\big[u^{\Delta^2}(t)+A\big(\sigma(t)\big)u\big(\sigma(t)\big)+\nabla F\big(\sigma(t),u^{\sigma}(t)\big)\big]v^{\sigma}(t)\Delta t=0.\tag{3.1}$$

Moreover, combining $u^{\Delta}(0)-u^{\Delta}(T)=0$, one has

$$\begin{aligned}&\int_{[0,T)_{\mathbb{T}}}\big(u^{\Delta^2}(t),v^{\sigma}(t)\big)\Delta t\\&\quad=\sum_{j=0}^{p}\int_{[t_j,t_{j+1})_{\mathbb{T}}}\big(u^{\Delta^2}(t),v^{\sigma}(t)\big)\Delta t\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{p}\left[\left(u^{\Delta}\left(t_{j+1}^{-}\right), v\left(t_{j+1}^{-}\right)\right) - \left(u^{\Delta}\left(t_{j}^{+}\right), v\left(t_{j}^{+}\right)\right) - \int_{[t_j,t_{j+1})}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t\right] \\
&= \sum_{j=0}^{p}\left[\sum_{i=1}^{N}\left(\left(u^{i}\right)^{\Delta}\left(t_{j+1}^{-}\right)v^{i}\left(t_{j+1}^{-}\right) - \left(u^{i}\right)^{\Delta}\left(t_{j}^{+}\right)v^{i}\left(t_{j}^{+}\right)\right) - \int_{[t_j,t_{j+1})}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t\right] \\
&= u^{\Delta}(T)v(T) - u^{\Delta}(0)v(0) - \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\left(\left(u^{i}\right)(t_j)\right)\left(v^{i}\right)(t_j) - \int_{[0,T)_{\mathbb{T}}}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t \\
&= -\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\left(u^{i}(t_j)\right)\left(v^{i}\right)(t_j) - \int_{[0,T)_{\mathbb{T}}}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t.
\end{aligned}
$$

Combining (3.1), we have

$$
\begin{aligned}
&\int_{[0,T)_{\mathbb{T}}}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\left(u^{i}(t_j)\right)v^{i}(t_j) \\
&\quad - \int_{[0,T)_{\mathbb{T}}}\left(A^{\sigma}(t)u^{\sigma}(t), v^{\sigma}(t)\right)\Delta t - \int_{[0,T)_{\mathbb{T}}}\left(\nabla F\left(\sigma(t), u^{\sigma}(t)\right), v^{\sigma}(t)\right)\Delta t = 0.
\end{aligned}
$$

Considering the above, we introduce the following concept solution for problem (1.1).

Definition 3.1 We say that a function $u \in H_{\Delta,T}^{1}$ is a weak solution of problem (1.1) if the identity

$$
\begin{aligned}
&\int_{[0,T)_{\mathbb{T}}}\left(u^{\Delta}(t), v^{\Delta}(t)\right)\Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\left(u^{i}(t_j)\right)v^{i}(t_j) \\
&\quad = \int_{[0,T)_{\mathbb{T}}}\left(A^{\sigma}(t)u^{\sigma}(t), v^{\sigma}(t)\right)\Delta t + \int_{[0,T)_{\mathbb{T}}}\left(\nabla F\left(\sigma(t), u^{\sigma}(t)\right), v^{\sigma}(t)\right)\Delta t
\end{aligned}
$$

holds for any $v \in H_{\Delta,T}^{1}$.

Consider the functional $\varphi : H_{\Delta,T}^{1} \to \mathbb{R}$ defined by

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^{2}\Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_{0}^{u^{i}(t_j)} I_{ij}(t)\,\mathrm{d}t \\
&\quad - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\left(A^{\sigma}(t)u^{\sigma}(t), u^{\sigma}(t)\right)\Delta t + J(u) \\
&= \psi(u) + \phi(u),
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
J(u) &= -\int_{[0,T)_{\mathbb{T}}} F\left(\sigma(t), u^{\sigma}(t)\right)\Delta t, \\
\psi(u) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\left|u^{\Delta}(t)\right|^{2}\Delta t - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\left(A^{\sigma}(t)u^{\sigma}(t), u^{\sigma}(t)\right)\Delta t + J(u)
\end{aligned}
$$

and

$$\phi(u)=\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t.$$

Lemma 3.1 *The functional φ is continuously differentiable on $H^1_{\Delta,T}$ and*

$$\begin{aligned}\langle\varphi'(u),v\rangle &= \int_{[0,T)_{\mathbb{T}}}\big(u^{\Delta}(t),v^{\Delta}(t)\big)\Delta t+\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\big(u^i(t_j)\big)v^i(t_j)\\ &\quad-\int_{[0,T)_{\mathbb{T}}}\big[\big(A^{\sigma}(t)u^{\sigma}(t),v^{\sigma}(t)\big)-\big(\nabla F\big(\sigma(t),u^{\sigma}(t)\big),v^{\sigma}(t)\big)\big]\Delta t. \end{aligned}\tag{3.3}$$

Proof Set $L(t,x,y)=\frac{1}{2}|y|^2-\frac{1}{2}(A(t)x,x)-F(t,x)$ for all $x,y\in R^N$ and $t\in[0,T]_{\mathbb{T}}$. Then $L(t,x,y)$ satisfies all assumptions of Theorem 2.6. Hence, by Theorem 2.6, we know that the functional ψ is continuously differentiable on $H^1_{\Delta,T}$ and

$$\langle\psi'(u),v\rangle=\int_{[0,T)_{\mathbb{T}}}\big[\big(u^{\Delta}(t),v^{\Delta}(t)\big)-\big(A^{\sigma}(t)u^{\sigma}(t),v^{\sigma}(t)\big)-\big(\nabla F\big(\sigma(t),u^{\sigma}(t)\big),v^{\sigma}(t)\big)\big]\Delta t$$

for all $u,v\in H^1_{\Delta,T}$.

On the other hand, by the continuity of I_{ij}, $i\in\Gamma$, $j\in\Lambda$, one has that $\phi\in C^1(H^1_{\Delta,T},\mathbb{R})$ and

$$\langle\phi'(u),v\rangle=\sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}\big(u^i(t_j)\big)v^i(t_j)$$

for all $u,v\in H^1_{\Delta,T}$. Thus, φ is continuously differentiable on $H^1_{\Delta,T}$ and (3.3) holds. □

By Definition 3.1 and Lemma 3.1, the weak solutions of problem (1.1) correspond to the critical points of φ.

Moreover, we need more preliminaries. For any $u\in H^1_{\Delta,T}$, let

$$q(u)=\frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\big[\big|u^{\sigma}(t)\big|^2-\big(A^{\sigma}(t)u^{\sigma}(t),u^{\sigma}(t)\big)\big]\Delta t.$$

We see that

$$\begin{aligned}q(u)&=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_{[0,T)_{\mathbb{T}}}\big(\big(A^{\sigma}(t)+I_{N\times N}\big)u^{\sigma}(t),u^{\sigma}(t)\big)\Delta t\\ &=\frac{1}{2}\big\langle(I-K)u,u\big\rangle,\end{aligned}$$

where $K:H^1_{\Delta,T}\to H^1_{\Delta,T}$ is the bounded self-adjoint linear operator defined, using the Riesz representation theorem, by

$$\langle Ku,v\rangle=\int_{[0,T)_{\mathbb{T}}}\big(\big(A^{\sigma}(t)+I_{N\times N}\big)u^{\sigma}(t),v^{\sigma}(t)\big)\Delta t,\quad \forall u,v\in H^1_{\Delta,T},$$

$I_{N\times N}$ and I denote an $N\times N$ identity matrix and an identity operator, respectively. By (3.2), $\varphi(u)$ can be rewritten as

$$\begin{aligned}\varphi(u) &= q(u)+\phi(u)+J(u)\\ &= \frac{1}{2}\langle (I-K)u,u\rangle+\phi(u)+J(u).\end{aligned} \tag{3.4}$$

The compact imbedding of $H^1_{\Delta,T}$ into $C([0,T]_{\mathbb{T}},R^N)$ implies that K is compact. By classical spectral theory, we can decompose $H^1_{\Delta,T}$ into the orthogonal sum of invariant subspaces for $I-K$

$$H^1_{\Delta,T}=H^-\oplus H^0\oplus H^+,$$

where $H^0=\ker(I-K)$ and H^-, H^+ are such that, for some $\delta>0$,

$$q(u)\le -\delta\|u\|^2 \quad \text{if } u\in H^-, \tag{3.5}$$

$$q(u)\ge \delta\|u\|^2 \quad \text{if } u\in H^+. \tag{3.6}$$

Remark 3.1 K has only finitely many eigenvalues λ_i with $\lambda_i>1$ since K is compact on H^1_T. Hence H^- is finite dimensional. Notice that $I-K$ is a compact perturbation of the self-adjoint operator I. By a well-known theorem, we know that 0 is not in the essential spectrum of $I-K$. Hence, H^0 is a finite dimensional space too.

To prove our main results, we need the following definitions and theorems.

Definition 3.2 ([30, P_{81}]) Let X be a real Banach space and $I\in C^1(X,\mathbb{R})$. I is said to be satisfying (PS) condition on X if any sequence $\{x_n\}\subseteq X$ for which $I(x_n)$ is bounded and $I'(x_n)\to 0$ as $n\to\infty$, possesses a convergent subsequence in X.

Firstly, we state the local linking theorem.

Let X be a real Banach space with a direct decomposition $X=X^1\oplus X^2$. Consider two sequences of a subspace

$$X^1_0\subset X^1_1\subset\cdots\subset X^1, \qquad X^2_0\subset X^2_1\subset\cdots\subset X^2$$

such that

$$\dim X^1_n<+\infty, \qquad \dim X^2_n<+\infty, \quad n\in\mathbb{N}$$

and

$$X^1=\overline{\bigcup_{n\in\mathbb{N}}X^1_n}, \qquad X^2=\overline{\bigcup_{n\in\mathbb{N}}X^2_n}.$$

For every multi-index $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$, we denote by X_α the space $X_{\alpha_1}\oplus X_{\alpha_2}$. We say $\alpha\le\beta\Leftrightarrow\alpha_1\le\beta_1,\alpha_2\le\beta_2$. A sequence $(\alpha_n)\subset\mathbb{N}^2$ is admissible if, for every $\alpha\in\mathbb{N}^2$, there is $m_0\in\mathbb{N}$ such that $n\ge m_0\Rightarrow\alpha_n\ge\alpha$.

Definition 3.3 ([31, Definition 2.2]) Let $I \in C^1(X, \mathbb{R})$. The functional I satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that α_n is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \sup\left|I(u_{\alpha_n})\right| < \infty, \quad \left(1 + \|u_{\alpha_n}\|\right)I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

Theorem 3.1 [31, Theorem 2.2] *Suppose that $I \in C^1(X, \mathbb{R})$ satisfies the following assumptions*:

(I_1) *$X^1 \neq \{0\}$ and I has a local linking at* 0 *with respect to (X^1, X^2); that is, for some $M > 0$,*

$$I(u) \geq 0, \quad u \in X^1, \|u\| \leq M,$$
$$I(u) \leq 0, \quad u \in X^2, \|u\| \leq M.$$

(I_2) *I satisfies $(C)^*$ condition.*

(I_3) *I maps bounded sets into bounded sets.*

(I_4) *For every $n \in \mathbb{N}$, $I(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_n^1 \oplus X^2$.*

Then I has at least two critical points.

Remark 3.2 Since $I \in C^1(X, \mathbb{R})$, by the condition (I_1) of Theorem 3.1, 0 is the critical point of I. Thus, under the conditions of Theorem 3.1, I has at least one nontrivial critical point.

Secondly, we state another three critical point theorems.

Theorem 3.2 ([32, Theorem 5.29]) *Let E be a Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^{\perp}$. Suppose $I \in C^1(E, R)$, satisfies* (PS) *condition, and*

(I_5) *$I(u) = \frac{1}{2}\langle Lu, u\rangle + b(u)$, where $Lu = L_1P_1u + L_2P_2u$ and $L_\kappa : E_\kappa \to E_\kappa$ is bounded and self-adjoint, $\kappa = 1, 2$,*

(I_6) *b' is compact, and*

(I_7) *there exist a subspace $\widetilde{E} \subset E$ and sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\alpha > \omega$ such that*
   (i) *$S \subset E_1$ and $I|_S \geq \alpha$,*
   (ii) *Q is bounded and $I|_{\partial Q} \leq \omega$,*
   (iii) *S and ∂Q link.*

Then I possesses a critical value $c \geq \alpha$.

Theorem 3.3 ([32, Theorem 9.12]) *Let E be a Banach space. Let $I \in C^1(E, \mathbb{R})$ be an even functional which satisfies the* (PS) *condition and $I(0) = 0$. If $E = V \oplus W$, where V is finite dimensional, and I satisfies*

(I_8) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap W} \geq \alpha$, where $B_\rho = \{x \in E : \|x\| < \rho\}$,*

(I_9) *for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \backslash B_{R(\widetilde{E})}$,*

then I possesses an unbounded sequence of critical values.

In order to state another critical point theorem, we need the following notions. Let X and Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a dense subset. For each $s \in \mathcal{S}$, there is a semi-norm on E defined by

$$p_s : E \to R, \quad p_s(u) = \left|s(x)\right| + \|y\| \quad \text{for } u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the topology on E induced by a semi-norm family $\{p_s\}$, and let w and w^* denote the weak-topology and weak*-topology, respectively.

For a functional $\Phi \in C^1(E,R)$, we write $\Phi_a = \{u \in E : \Phi(u) \geq a\}$. Recall that Φ' is said to be weak sequentially continuous if, for any $u_k \rightharpoonup u$ in E, one has $\lim_{k\to\infty} \Phi'(u_k)v \to \Phi'(u)v$ for each $v \in E$, *i.e.*, $\Phi' : (E,w) \to (E^*,w^*)$ is sequentially continuous. For $c \in R$, we say that Φ satisfies the $(C)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $(1 + \|u_k\|)\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

Suppose that

(Φ_0) for any $c \in R$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous;

(Φ_1) there exists $\rho > 0$ such that $\kappa := \inf \Phi(\partial B_\rho \cap Y) > 0$, where

$$B_\rho = \left\{u \in E : \|u\| < \rho\right\};$$

(Φ_2) there exist a finite dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \backslash S_0) < \inf \Phi(B_\rho \cap Y)$, where

$$E_0 := X \oplus Y_0, \quad \text{and} \quad S_0 = \left\{u \in E_0 : \|u\| \leq R\right\}.$$

Theorem 3.4 ([33]) *Assume that Φ is even and (Φ_0)-(Φ_2) are satisfied. Then Φ has at least $m = \dim Y_0$ pairs of critical points with critical values less than or equal to $\bar{c}$ provided Φ satisfies the $(C)_c$ condition for all $c \in [\kappa, \bar{c}]$.*

Remark 3.3 In our applications, we take $\mathcal{S}$=X^* so that $\mathcal{T}_{\mathcal{S}}$ is the product topology on $E = X \oplus Y$ given by the weak topology on X and the strong topology on Y.

4 Main results

Lemma 4.1 *ϕ' is compact on $H^1_{\Delta,T}$.*

Proof Let $\{u_k\} \subset H^1_{\Delta,T}$ be any bounded sequence. Since $H^1_{\Delta,T}$ is a Hilbert space, we can assume that $u_k \rightharpoonup u$. Theorem 2.5 implies that $\|u_k - u\|_\infty \to 0$. By (2.5), we have

$$\begin{aligned}
&\left\|\phi'(u_k) - \phi'(u)\right\| \\
&\quad = \sup_{v\in H^1_{\Delta,T}, \|v\|\leq 1} \left|\left\langle\phi'(u_k) - \phi'(u), v\right\rangle\right| \\
&\quad = \sup_{v\in H^1_{\Delta,T}, \|v\|\leq 1} \left|\sum_{j=1}^{p}\sum_{i=1}^{N}\left[I_{ij}\left(u_k^i(t_j)\right) - I_{ij}\left(u^i(t_j)\right)\right]v^i(t_j)\right| \\
&\quad \leq \|v\|_\infty \sup_{v\in H^1_{\Delta,T}, \|v\|\leq 1} \sum_{j=1}^{p}\sum_{i=1}^{N}\left|I_{ij}\left(u_k^i(t_j)\right) - I_{ij}\left(u^i(t_j)\right)\right|
\end{aligned}$$

$$\leq C_1\|v\| \sup_{v\in H^1_{\Delta,T},\|v\|\leq 1} \sum_{j=1}^{p}\sum_{i=1}^{N}\left|I_{ij}\big(u_k^i(t_j)\big) - I_{ij}\big(u^i(t_j)\big)\right|$$
$$= C_1 \sup_{v\in H^1_{\Delta,T},\|v\|\leq 1} \sum_{j=1}^{p}\sum_{i=1}^{N}\left|I_{ij}\big(u_k^i(t_j)\big) - I_{ij}\big(u^i(t_j)\big)\right|.$$

The continuity of I_{ij} and this imply that $\phi'(u_k) \to \phi'(u)$ in $H^1_{\Delta,T}$. The proof is complete. $\square$

First of all, we give two existence results.

Theorem 4.1 *Suppose that* (A) *and the following conditions are satisfied.*

(F_1) $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = +\infty$ *uniformly for* Δ*-a.e.* $t \in [0,T]_{\mathbb{T}}$,
(F_2) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0$ *uniformly for* Δ*-a.e.* $t \in [0,T]_{\mathbb{T}}$,
(F_3) *there exist* $\lambda > 2$ *and* $\beta > \lambda - 2$ *such that*

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^\lambda} < \infty \quad \textit{uniformly for } \Delta\textit{-a.e. } t \in [0,T]_{\mathbb{T}}$$

and

$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\beta} > 0 \quad \textit{uniformly for } \Delta\textit{-a.e. } t \in [0,T]_{\mathbb{T}},$$

(F_4) *there exists* $r > 0$ *such that*

$$F(t,x) \geq 0, \quad \forall |x| \leq r, \textit{ and } \Delta \textit{-a.e. } t \in [0,T]_{\mathbb{T}},$$

(F_5) *there exist* $a_{ij}, b_{ij} > 0$ *and* $\xi_{ij} \in [0,1)$ *such that*

$$\left|I_{ij}(t)\right| \leq a_{ij} + b_{ij}|t|^{\xi_{ij}} \quad \textit{for every } t \in \mathbb{R}, i \in \Gamma, j \in \Lambda,$$

(F_6) $\int_0^t I_{ij}(s)\,\mathrm{d}s \leq 0$ *for every* $t \in \mathbb{R}$, $i \in \Gamma$, $j \in \Lambda$,
(F_7) *there exists* $\zeta_{ij} > 0$ *such that*

$$2\int_0^t I_{ij}(s)\,\mathrm{d}s - I_{ij}(t)t \geq 0 \quad \textit{for all } i \in \Gamma, j \in \Lambda \textit{ and } |t| \geq \zeta_{ij},$$

and

$$\lim_{t\to 0} \frac{I_{ij}(t)}{t} = 0 \quad \textit{for all } i \in \Gamma, j \in \Lambda.$$

Then problem (1.1) *has at least two weak solutions. The one is a nontrivial weak solution, the other is a trivial weak solution.*

Proof By Lemma 3.1, $\varphi \in C^1(X,\mathbb{R})$. Set $X = H^1_{\Delta,T}, X^1 = H^+$ with $(e_n)_{n\geq 1}$ being its Hilbertian basis, $X^2 = H^- \oplus H^0$ and define

$$X_n^1 = \operatorname{span}\{e_1, e_2, \ldots, e_n\}, \quad n \in \mathbb{N}, \qquad X_n^2 = X^2, \quad n \in \mathbb{N}.$$

Then we have

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1, \qquad X_0^2 \subset X_1^2 \subset \cdots \subset X^2, \qquad X^1 = \overline{\bigcup_{n\in\mathbb{N}} X_n^1}, \qquad X^2 = \overline{\bigcup_{n\in\mathbb{N}} X_n^2}$$

and

$$\dim X_n^1 < +\infty, \qquad \dim X_n^2 < +\infty, \quad n \in \mathbb{N}.$$

We divide our proof into four parts in order to show Theorem 4.1.

Firstly, we show that φ satisfies the $(C)^*$ condition.

Let $\{u_{\alpha_n}\}$ be a sequence in $H^1_{\Delta,T}$ such that α_n is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \sup\left|\varphi(u_{\alpha_n})\right| < +\infty, \quad \left(1 + \|u_{\alpha_n}\|\right)\varphi'(u_{\alpha_n}) \to 0,$$

then there exists a constant $C_2 > 0$ such that

$$\left|\varphi(u_{\alpha_n})\right| \le C_2, \quad \left(1 + \|u_{\alpha_n}\|\right)\varphi'(u_{\alpha_n}) \le C_2 \tag{4.1}$$

for all large n. On the other hand, by (F_3), there are constants $C_3 > 0$ and $\rho_1 > 0$ such that

$$F(t,x) \le C_3|x|^\lambda \tag{4.2}$$

for all $|x| \ge \rho_1$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. By (A) one has

$$\left|F(t,x)\right| \le \max_{s\in[0,\rho_1]} a(s)b(t) \tag{4.3}$$

for all $|x| \le \rho_1$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. It follows from (4.2) and (4.3) that

$$\left|F(t,x)\right| \le \max_{s\in[0,\rho_1]} a(s)b(t) + C_3|x|^\lambda \tag{4.4}$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. Since $d_{lm} \in L^\infty([0,T])$ for all $l,m = 1,2,\ldots,N$, there exists a constant $C_4 \ge 1$ such that

$$\left|\int_{[0,T)_{\mathbb{T}}} \left(A^\sigma(t)u^\sigma(t), u^\sigma(t)\right)\Delta t\right| \le C_4 \int_{[0,T)_{\mathbb{T}}} \left|u^\sigma(t)\right|^2 \Delta t, \quad \forall u \in H^1_{\Delta,T}. \tag{4.5}$$

From (F_5) and (2.5), we have that

$$\begin{aligned}\left|\phi(u)\right| &\le \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{|u^i(t_j)|}\left(a_{ij} + b_{ij}|t|^{\xi_{ij}}\right)\mathrm{d}t \\ &\le \overline{a}pN\|u\|_\infty + \overline{b}\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|_\infty^{\xi_{ij}+1} \\ &\le \overline{a}pNC_1\|u\| + \overline{b}C_1\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|^{\xi_{ij}+1} \end{aligned} \tag{4.6}$$

for all $u \in H^1_{\Delta,T}$, where $\overline{a} = \max_{i\in\Gamma,j\in\Lambda}\{a_{ij}\}$, $\overline{b} = \max_{i\in\Gamma,j\in\Lambda}\{b_{ij}\}$. Combining (4.4), (4.5), (4.6) and Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{2}\|u_{\alpha_n}\|^2 &= \varphi(u_{\alpha_n}) - \phi(u_{\alpha_n}) + \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \left|u^\sigma_{\alpha_n}(t)\right|^2 \Delta t \\
&\quad + \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \left(A^\sigma(t)u^\sigma_{\alpha_n}(t), u^\sigma_{\alpha_n}(t)\right)\Delta t - J(u_{\alpha_n}) \\
&\le C_2 + \overline{a}pNC_1\|u_{\alpha_n}\| + \overline{b}C_1\sum_{j=1}^{p}\sum_{i=1}^{N}\|u_{\alpha_n}\|^{\xi_{ij}+1} + C_4\int_{[0,T)_{\mathbb{T}}} \left|u^\sigma_{\alpha_n}(t)\right|^2\Delta t \\
&\quad + C_3\int_{[0,T)_{\mathbb{T}}} \left|u^\sigma_{\alpha_n}(t)\right|^\lambda \Delta t + \max_{s\in[0,\rho_1]} a(s)\int_{[0,T)_{\mathbb{T}}} b^\sigma(t)\Delta t \\
&\le C_2 + \overline{a}pNC_1\|u_{\alpha_n}\| + \overline{b}C_1\sum_{j=1}^{p}\sum_{i=1}^{N}\|u_{\alpha_n}\|^{\xi_{ij}+1} \\
&\quad + C_4 T^{\frac{\lambda-2}{\lambda}}\left(\int_{[0,T)_{\mathbb{T}}} \left|u^\sigma_{\alpha_n}(t)\right|^\lambda \Delta t\right)^{\frac{2}{\lambda}} + C_3\int_{[0,T)_{\mathbb{T}}} \left|u^\sigma_{\alpha_n}(t)\right|^\lambda\Delta t + C_5
\end{aligned} \tag{4.7}$$

for all large n, where $C_5 = \max_{s\in[0,\rho_1]} a(s)\int_{[0,T)_{\mathbb{T}}} b^\sigma(t)\Delta t$. On the other hand, by (F_3), there exist $C_6 > 0$ and $\rho_2 > 0$ such that

$$\left(\nabla F(t,x),x\right) - 2F(t,x) \ge C_6|x|^\beta \tag{4.8}$$

for all $|x| \ge \rho_2$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. By (A),

$$\left|\left(\nabla F(t,x),x\right) - 2F(t,x)\right| \le C_7 b(t) \tag{4.9}$$

for all $|x| \le \rho_2$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$, where $C_7 = (2+\rho_2)\max_{s\in[0,\rho_2]} a(s)$. Combining (4.8) and (4.9), one has

$$\left(\nabla F(t,x),x\right) - 2F(t,x) \ge C_6|x|^\beta - C_6\rho_2^\beta - C_7 b(t) \tag{4.10}$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. According to (F_7), there exists $C_8 > 0$ such that

$$2\int_0^t I_{ij}(s)\,\mathrm{d}s - I_{ij}(t)t \ge -C_8 \quad \text{for all } i \in \Gamma, j \in \Lambda \text{ and } t \in \mathbb{R}. \tag{4.11}$$

Thus, by (4.1), (4.10) and (4.11), we obtain

$$\begin{aligned}
3C_2 &\ge 2\varphi(u_{\alpha_n}) - \left\langle \varphi'(u_{\alpha_n}), u_{\alpha_n}\right\rangle \\
&= 2\phi(u_{\alpha_n}) - \left\langle \phi'(u_{\alpha_n}), u_{\alpha_n}\right\rangle \\
&\quad + \int_{[0,T)_{\mathbb{T}}} \left[\left(\nabla F\left(\sigma(t), u^\sigma_{\alpha_n}(t)\right), u^\sigma_{\alpha_n}(t)\right) - 2F\left(\sigma(t), u^\sigma_{\alpha_n}(t)\right)\right]\Delta t \\
&= \sum_{j=1}^{p}\sum_{i=1}^{N}\left(2\int_0^{u^i_{\alpha_n}(t_j)} I_{ij}(t)\,\mathrm{d}t - I_{ij}\left(u^i_{\alpha_n}(t_j)\right)u^i_{\alpha_n}(t_j)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{[0,T)_{\mathbb{T}}} \big[\big(\nabla F\big(\sigma(t), u^{\sigma}_{\alpha_n}(t)\big), u^{\sigma}_{\alpha_n}(t)\big) - 2F\big(\sigma(t), u^{\sigma}_{\alpha_n}(t)\big)\big]\Delta t \\
&\geq -pNC_8 + C_6 \int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}\big|^{\beta}\Delta t - C_6\rho_2^{\beta}T - C_7\int_{[0,T)_{\mathbb{T}}} b^{\sigma}(t)\Delta t
\end{aligned} \tag{4.12}
$$

for all large n. From (4.12), $\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}_{\alpha_n}|^{\beta}\Delta t$ is bounded. If $\beta > \lambda$, by Hölder's inequality, we have

$$
\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\lambda}\Delta t \leq T^{\frac{\beta-\lambda}{\beta}}\left(\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\beta}\Delta t\right)^{\frac{\lambda}{\beta}}. \tag{4.13}
$$

Since $\xi_{ij} \in [0,1)$ for all $i \in \Gamma$, $j \in \Lambda$, by (4.7) and (4.13), $\{u_{\alpha_n}\}$ is bounded in $H^1_{\Delta,T}$. If $\beta \leq \lambda$, by (2.5), we obtain

$$
\begin{aligned}
\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\lambda}\Delta t &= \int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\beta}\big|u^{\sigma}_{\alpha_n}(t)\big|^{\lambda-\beta}\Delta t \\
&\leq \|u_{\alpha_n}\|_{\infty}^{\lambda-\beta}\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\beta}\Delta t \\
&\leq C_1^{\lambda-\beta}\|u_{\alpha_n}\|^{\lambda-\beta}\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}_{\alpha_n}(t)\big|^{\beta}\Delta t.
\end{aligned} \tag{4.14}
$$

Since $\xi_{ij} \in [0,1)$, $\lambda - \beta < 2$, by (4.7) and (4.14), $\{u_{\alpha_n}\}$ is also bounded in $H^1_{\Delta,T}$. Hence, $\{u_{\alpha_n}\}$ is also bounded in $H^1_{\Delta,T}$. Going if necessary to a subsequence, we can assume that $u_{\alpha_n} \rightharpoonup u$ in $H^1_{\Delta,T}$. From Theorem 2.5, we have $\|u_{\alpha_n} - u\|_{\infty} \to 0$ and $\int_{[0,T)_{\mathbb{T}}} |u^{\sigma}_{\alpha_n}(t) - u^{\sigma}(t)|^2\Delta t \to 0$. Since

$$
\begin{aligned}
&\int_{[0,T)_{\mathbb{T}}} \big|u^{\Delta}_{\alpha_n}(t) - u^{\Delta}(t)\big|^2\Delta t \\
&\quad= \big\langle \varphi'(u_{\alpha_n}) - \varphi'(u), u_{\alpha_n} - u\big\rangle + \int_{[0,T)_{\mathbb{T}}} \big(A^{\sigma}(t)\big(u^{\sigma}_{\alpha_n}(t) - u^{\sigma}(t)\big), u^{\sigma}_{\alpha_n}(t) - u^{\sigma}(t)\big)\Delta t \\
&\qquad - \sum_{j=1}^{p}\sum_{i=1}^{N}\big(I_{ij}\big(u^i_{\alpha_n}(t_j)\big) - I_{ij}\big(u^i(t_j)\big)\big)\big(u^i_{\alpha_n}(t_j) - u^i(t_j)\big) \\
&\qquad + \int_{[0,T)_{\mathbb{T}}} \big(\nabla F\big(\sigma(t), u^{\sigma}_{\alpha_n}(t)\big) - \nabla F\big(\sigma(t), u^{\sigma}(t)\big), u^{\sigma}_{\alpha_n}(t) - u^{\sigma}(t)\big)\Delta t.
\end{aligned}
$$

This implies $\int_{[0,T)_{\mathbb{T}}} |u^{\Delta}_{\alpha_n}(t) - u^{\Delta}(t)|^2\Delta t \to 0$, and hence $\|u_{\alpha_n} - u\| \to 0$. Therefore, $u_{\alpha_n} \to u$ in $H^1_{\Delta,T}$. Hence φ satisfies the $(C)^*$ condition.

Secondly, we show that φ maps bounded sets into bounded sets.

It follows from (3.2), (4.4), (4.5) and (4.6) that

$$
\begin{aligned}
\big|\varphi(u)\big| &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big|u^{\Delta}(t)\big|^2\Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t \\
&\quad - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big(A^{\sigma}(t)u^{\sigma}(t), u^{\sigma}(t)\big)\Delta t + J(u) \\
&\leq \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big|u^{\Delta}(t)\big|^2\Delta t + \frac{C_4}{2}\int_{[0,T)_{\mathbb{T}}} \big|u^{\sigma}(t)\big|^2\Delta t + \overline{a}pNC_1\|u\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \overline{b}C_1 \sum_{j=1}^{p} \sum_{i=1}^{N} \|u\|^{\xi_{ij}+1} \\
&+ C_3 \int_{[0,T)_{\mathbb{T}}} \left|u^{\sigma}(t)\right|^{\lambda} \Delta t + \max_{s\in[0,\rho_1]} a(s) \int_{[0,T)_{\mathbb{T}}} b^{\sigma}(t) \Delta t \\
\leq{}& \frac{1}{2} C_4 \|u\|^2 + \overline{a}pNC_1 \|u\| + \overline{b}C_1 \sum_{j=1}^{p} \sum_{i=1}^{N} \|u\|^{\xi_{ij}+1} + C_3 T \|u\|_{\infty}^{\lambda} + C_5 \\
\leq{}& \frac{1}{2} C_4 \|u\|^2 + \overline{a}pNC_1 \|u\| + \overline{b}C_1 \sum_{j=1}^{p} \sum_{i=1}^{N} \|u\|^{\xi_{ij}+1} + C_3 T C_1^{\lambda} \|u\|^{\lambda} + C_5
\end{aligned}
$$

for all $u \in H^1_{\Delta,T}$. Thus, φ maps bounded sets into bounded sets.

Thirdly, we claim that φ has a local linking at 0 with respect to (X^1, X^2).

Applying (F_2), for $\epsilon_1 = \frac{\delta}{4}$, there exists $\rho_3 > 0$ such that

$$
\left|F(t,x)\right| \leq \epsilon_1 |x|^2 \tag{4.15}
$$

for all $|x| \leq \rho_3$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. By (F_7), for $\epsilon_2 = \frac{\delta}{4pNC_1}$, there exists $\rho_4 > 0$ such that

$$
\left|I_{ij}(t)\right| \leq \epsilon_2 |t|, \quad |t| \leq \rho_4, i \in \Gamma, j \in \Lambda. \tag{4.16}
$$

Let $\rho_5 = \min\{\rho_3, \rho_4\}$. For $u \in X^1$ with $\|u\| \leq r_1 \triangleq \frac{\rho_5}{C_1}$, by (2.5), (3.2), (3.6), (4.15) and (4.16), we have

$$
\begin{aligned}
\varphi(u) &= q(u) + \sum_{j=1}^{p} \sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^{\sigma}(t)\big) \Delta t \\
&\geq \delta \|u\|^2 - \sum_{j=1}^{p} \sum_{i=1}^{N} \int_0^{|u^i(t_j)|} \left|I_{ij}(t)\right| \mathrm{d}t - \epsilon_1 \int_{[0,T)_{\mathbb{T}}} \left|u^{\sigma}(t)\right|^2 \Delta t \\
&\geq \delta \|u\|^2 - \sum_{j=1}^{p} \sum_{i=1}^{N} \int_0^{|u^i(t_j)|} \epsilon_2 |t|\,\mathrm{d}t - \epsilon_1 \int_{[0,T)_{\mathbb{T}}} \left|u^{\sigma}(t)\right|^2 \Delta t \\
&\geq \delta \|u\|^2 - \epsilon_2 \sum_{j=1}^{p} \sum_{i=1}^{N} \|u\|_{\infty}^2 - \epsilon_1 \int_{[0,T)_{\mathbb{T}}} \left|u^{\sigma}(t)\right|^2 \Delta t \\
&\geq \delta \|u\|^2 - \epsilon_2 pNC_1 \|u\|^2 - \epsilon_1 \|u\|^2 \\
&\geq \delta \|u\|^2 - \frac{\delta}{4} \|u\|^2 - \frac{\delta}{4} \|u\|^2 \\
&= \frac{\delta}{2} \|u\|^2.
\end{aligned}
$$

This implies that

$$
\varphi(u) \geq 0, \quad \forall u \in X^1 \text{ with } \|u\| \leq r_1.
$$

On the other hand, it follows from (F_6) that

$$
\phi(u) \leq 0 \tag{4.17}
$$

for all $u \in H^1_{\Delta,T}$. Let $u = u^- + u^0 \in X^2$ satisfy $\|u\| \leq r_2 \triangleq \frac{r}{C_1}$. Using (F$_4$), (2.5), (3.2), (3.5) and (4.17), we obtain

$$\begin{aligned}\varphi(u) &= q(u) + \phi(u) - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^\sigma(t)\big)\Delta t \\ &\leq -\delta\|u^-\|^2 - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^\sigma(t)\big)\Delta t \\ &\leq -\delta\|u^-\|^2.\end{aligned}$$

This implies that

$$\varphi(u) \leq 0, \quad \forall u \in X^2 \text{ with } \|u\| \leq r_2.$$

Let $M = \min\{r_1, r_2\}$. Then φ satisfies the condition (I$_1$) of Theorem 3.1.

Finally, we claim that for every $n \in \mathbb{N}$,

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to \infty, u \in X^1_n \oplus X^2.$$

For given $n \in \mathbb{N}$, since $X^1_n \oplus X^2$ is a finite dimensional space, there exists $C_9 > 0$ such that

$$\|u\| \leq C_9\|u\|_{L^2}, \quad \forall u \in X^1_n \oplus X^2. \tag{4.18}$$

By (F$_1$), there exists $\rho_6 > 0$ such that

$$F(t,x) \geq C_9^2(C_4 + \delta)|x|^2 \tag{4.19}$$

for all $|x| \geq \rho_6$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. From (A), we get

$$\big|F(t,x)\big| \leq \max_{s\in[0,\rho_6]} a(s)b(t) \tag{4.20}$$

for all $|x| \leq \rho_6$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$. Equations (4.19) and (4.20) imply that

$$F(t,x) \geq C_9^2(C_4 + \delta)|x|^2 - C_{10} - \max_{s\in[0,\rho_6]} a(s)b(t) \tag{4.21}$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$, where $C_{10} = C_9^2(C_4 + \delta)\rho_6^2$. Using (3.2), (3.6), (4.5), (4.17), (4.18) and (4.21), we have, for $u = u^+ + u^0 + u^- \in X^1_n \oplus X^2 = X^1_n \oplus H^0 \oplus H^-$,

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big|u^\Delta(t)\big|^2\Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t \\ &\quad - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big(A^\sigma(t)u^\sigma(t), u^\sigma(t)\big)\Delta t - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^\sigma(t)\big)\Delta t \\ &\leq -\delta\|u^-\|^2 + \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big|\big(u^+\big)^\Delta(t)\big|^2\Delta t \\ &\quad - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big(A^\sigma(t)\big(u^+\big)^\sigma(t), \big(u^+\big)^\sigma(t)\big)\Delta t - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^\sigma(t)\big)\Delta t\end{aligned}$$

$$\begin{aligned}
&\leq -\delta\|u^-\|^2 + \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big|(u^+)^{\Delta}(t)\big|^2\Delta t + \frac{C_4}{2}\int_{[0,T)_{\mathbb{T}}} \big|(u^+)^{\sigma}(t)\big|^2\Delta t \\
&\quad - \int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u^{\sigma}(t)\big)\Delta t \\
&\leq -\delta\|u^-\|^2 + \frac{C_4}{2}\|u^+\|^2 - C_9^2(C_4+\delta)\|u^{\sigma}\|_{L^2}^2 + C_{10}T + \max_{s\in[0,\rho_6]} a(s)\int_{[0,T)_{\mathbb{T}}} b^{\sigma}(t)\Delta t \\
&\leq -\delta\|u^-\|^2 + C_4\|u^+\|^2 - (C_4+\delta)\|u\|^2 + C_{10}T + C_{11} \\
&= -\delta\|u^-\|^2 + C_4\|u^+\|^2 - (C_4+\delta)\|u^+ + u^0 + u^-\|^2 + C_{10}T + C_{11} \\
&\leq -\delta\|u^-\|^2 + C_4\|u^+\|^2 - (C_4+\delta)\|u^+\|^2 - \delta\|u^0 + u^-\|^2 + C_{10}T + C_{11} \\
&\leq -\delta\|u^-\|^2 + C_4\|u^+\|^2 - (C_4+\delta)\|u^+\|^2 - \delta\|u^0\|^2 + C_{10}T + C_{11} \\
&= -\delta\|u\|^2 + C_{10}T + C_{11},
\end{aligned}$$

where $C_{11} = \max_{s\in[0,\rho_6]} a(s)\int_{[0,T)_{\mathbb{T}}} b^{\sigma}(t)\Delta t$. Hence, for every $n\in\mathbb{N}$, $\varphi(u)\to-\infty$ as $\|u\|\to\infty$ and $X_n^1\oplus X^2$.

Thus, by Theorem 3.1, problem (1.1) has at least one nontrivial weak solution. The proof is complete. □

Example 4.1 Let $\mathbb{T}=\mathbb{R}$, $T=\frac{\pi}{2}$, $N=1$, $t_1=\frac{\pi}{4}$. Consider the second-order Hamiltonian system with impulsive effects

$$\begin{cases} \ddot{u}(t) + A(t)u(t) + \nabla F(t,x) = 0, & \text{a.e. } t\in[0,\frac{\pi}{2}]; \\ u(0) - u(\frac{\pi}{2}) = \dot{u}(0) - \dot{u}(\frac{\pi}{2}) = 0, \\ \Delta\dot{u}(t_1) = \dot{u}(t_1^+) - \dot{u}(t_1^-) = I(u(t_1)), \end{cases} \tag{4.22}$$

where $A(t)=1$,

$$F(t,x) = \begin{cases} |x|^4, & |x|\geq 5, \\ \frac{625}{5-3\sqrt{2}}x - \frac{1875\sqrt{2}}{5-3\sqrt{2}}, & 3\sqrt{2}<x<5, \\ 0, & |x|\leq 3\sqrt{2}, \\ \frac{625}{3\sqrt{2}-5}x + \frac{1875\sqrt{2}}{3\sqrt{2}-5}, & -5\leq x<-3\sqrt{2} \end{cases}$$

for all $x\in\mathbb{R}$ and $t\in[0,\frac{\pi}{2}]$,

$$I(t) = \begin{cases} 0, & t\geq 4, \\ -6(t-4)^3, & 3\leq t<4, \\ 6t-12, & 1<t<3, \\ -6t^3, & |t|\leq 1, \\ 6t+12, & -3<t<-1, \\ -6(t+4)^3, & -4<t\leq -3, \\ 0, & t\leq -4, \end{cases}$$

then all conditions of Theorem 4.1 hold. According to Theorem 4.1, problem (4.22) has at least one nontrivial weak solution. In fact,

$$u(t) = \begin{cases} 3\sqrt{2}\cos t, & t \in [0, \frac{\pi}{4}]; \\ 3\sqrt{2}\sin t, & t \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$$

is the solution of problem (4.22).

Theorem 4.2 *Assume that* (A), (F_5), (F_6), (F_7) *and the following conditions are satisfied.*

(F_8) $\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^2} \le 0$ *uniformly for* Δ*-a.e.* $t \in [0,T]_{\mathbb{T}}$,
(F_9) *there exist constants* $\mu > 2$ *and* $r_3 \ge 0$ *such that* $(\nabla F(t,x),x) \ge \mu F(t,x) > 0$ *for all* $t \in [0,T]_{\mathbb{T}}$ *and* $|x| \ge r_3$,
(F_{10}) $F(t,x) \ge 0$ *for all* $x \in \mathbb{R}^N$ *and* Δ*-a.e.* $t \in [0,T]_{\mathbb{T}}$.

Then problem (1.1) *has at least one nontrivial weak solution.*

Proof Set $E_1 = H^+$, $E_2 = H^- \oplus H^0$ and $E = H^1_{\Delta,T}$. Then E is a real Hilbert space, $E = E_1 \oplus E_2$, $E_2 = E_1^\perp$ and $\dim(E_2) < +\infty$.

Firstly, we prove that φ satisfies the (PS) condition. Indeed, let $\{u_k\} \subset H^1_{\Delta,T}$ be a sequence such that $|\varphi(u_k)| \le C_{12}$ and $\varphi'(u_k) \to 0$ as $k \to \infty$. As the proof of Theorem 4.1, it suffices to show that $\{u_k\}$ is bounded in $H^1_{\Delta,T}$. By (F_9) there exist positive constants C_{13}, C_{14} such that

$$F(t,x) \ge C_{13}|x|^\mu - C_{14}, \quad \forall t \in [0,T]_{\mathbb{T}}, \forall x \in \mathbb{R}^n \tag{4.23}$$

(see [34]). By (F_9), (4.11) and (4.23), we have

$$\begin{aligned}
&2C_{12} + \|u_k\| \\
&\quad \ge 2\varphi(u_k) - \langle \varphi'(u_k), u_k \rangle \\
&\quad = 2\phi(u_k) - \langle \phi'(u_k), u_k \rangle \\
&\qquad + \int_{[0,T)_{\mathbb{T}}} \big[\big(\nabla F\big(\sigma(t), u_k^\sigma(t)\big), u_k^\sigma(t)\big) - 2F\big(\sigma(t), u_k^\sigma(t)\big)\big] \Delta t \\
&\quad = \sum_{j=1}^{p} \sum_{i=1}^{N} \left(2\int_0^{u_k^i(t_j)} I_{ij}(t)\,\mathrm{d}t - I_{ij}\big(u_k^i(t_j)\big) u_k^i(t_j) \right) \\
&\qquad + \int_{[0,T)_{\mathbb{T}}} \big[\big(\nabla F\big(\sigma(t), u_k^\sigma(t)\big), u_k^\sigma(t)\big) - 2F\big(\sigma(t), u_k^\sigma(t)\big)\big] \Delta t \\
&\quad = -pNC_8 + \int_{[0,T)_{\mathbb{T}}} \big[\big(\nabla F\big(\sigma(t), u_k^\sigma(t)\big), u_k^\sigma(t)\big) - 2F\big(\sigma(t), u_k^\sigma(t)\big)\big] \Delta t \\
&\quad = -pNC_8 + (\mu - 2)\int_{[0,T)_{\mathbb{T}}} F\big(\sigma(t), u_k^\sigma(t)\big) \Delta t \\
&\qquad + \int_{[0,T)_{\mathbb{T}}} \big[\big(\nabla F\big(\sigma(t), u_k^\sigma(t)\big), u_k^\sigma(t)\big) - \mu F\big(\sigma(t), u_k^\sigma(t)\big)\big] \Delta t \\
&\quad \ge -pNC_8 + (\mu - 2)\int_{[0,T)_{\mathbb{T}}} \big(C_{13}\big|u_k^\sigma(t)\big|^\mu - C_{14}\big) \Delta t
\end{aligned}$$

$$\begin{aligned}&+\int_{[0,T)_{\mathbb{T}}}\big[\big(\nabla F\big(\sigma(t),u_k^\sigma(t)\big),u_k^\sigma(t)\big)-\mu F\big(\sigma(t),u_k^\sigma(t)\big)\big]\Delta t\\ &\geq -pNC_8+(\mu-2)C_{13}\int_{[0,T)_{\mathbb{T}}}|u_k^\sigma|(t)^\mu\,\Delta t-(\mu-2)C_{14}T-C_{15}\end{aligned}\tag{4.24}$$

for large k, where $C_{15}=(r_3+\mu)\max_{s\in[0,r_3]}a(s)\int_{[0,T)_{\mathbb{T}}}b^\sigma(t)\Delta t$. Equation (4.24) implies that there exists $C_{16}>0$ such that

$$\int_{[0,T)_{\mathbb{T}}}\big|u_k^\sigma(t)\big|^\mu\,\Delta t\leq C_{16}\big(1+\|u_k\|\big).\tag{4.25}$$

Combining (3.2), (4.6), (4.11) and (4.25), we obtain

$$\begin{aligned}&\mu c+\|u_k\|\\ &\quad\geq\mu\varphi(u_k)-\big\langle\varphi'(u_k),u_k\big\rangle\\ &\quad=\left(\frac{\mu}{2}-1\right)\int_{[0,T)_{\mathbb{T}}}\big[\big|u_k^\Delta(t)\big|^2-\big(A^\sigma(t)u_k^\sigma(t),u_k^\sigma(t)\big)\big]\Delta t\\ &\qquad+\int_{[0,T)_{\mathbb{T}}}\big[\big(\nabla F\big(\sigma(t),u_k^\sigma(t)\big),u_k^\sigma(t)\big)-\mu F\big(\sigma(t),u_k^\sigma(t)\big)\big]\Delta t+\mu\phi(u_k)-\big\langle\phi'(u_k),u_k\big\rangle\\ &\quad=\left(\frac{\mu}{2}-1\right)\int_{[0,T)_{\mathbb{T}}}\big[\big|u_k^\Delta(t)\big|^2-\big(A^\sigma(t)u_k^\sigma(t),u_k^\sigma(t)\big)\big]\Delta t+(\mu-2)\phi(u_k)+2\phi(u_k)\\ &\qquad+\int_{[0,T)_{\mathbb{T}}}\big[\big(\nabla F\big(\sigma(t),u_k^\sigma(t)\big),u_k^\sigma(t)\big)-\mu F\big(\sigma(t),u_k^\sigma(t)\big)\big]\Delta t-\big\langle\phi'(u_k),u_k\big\rangle\\ &\quad\geq\left(\frac{\mu}{2}-1\right)\|u_k\|^2-\left[\frac{\mu}{2}-1-\left(\frac{\mu}{2}-1\right)C_4\right]\int_{[0,T)_{\mathbb{T}}}\big|u_k^\sigma(t)\big|^2\Delta t-C_{15}-pNC_8\\ &\qquad-(\mu-2)\bar{a}pNC_1\|u_k\|-(\mu-2)\bar{b}C_1\sum_{j=1}^p\sum_{i=1}^N\|u_k\|^{\xi_{ij}+1}\\ &\quad\geq\left(\frac{\mu}{2}-1\right)\|u_k\|^2-\left[\frac{\mu}{2}-1-\left(\frac{\mu}{2}-1\right)C_4\right]T^{\frac{\mu-2}{\mu}}\left(\int_{[0,T)_{\mathbb{T}}}\big|u_k^\sigma(t)\big|^\mu\Delta t\right)^{\frac{2}{\mu}}\\ &\qquad-C_{15}-pNC_8-(\mu-2)\bar{a}pNC_1\|u_k\|-(\mu-2)\bar{b}C_1\sum_{j=1}^p\sum_{i=1}^N\|u_k\|^{\xi_{ij}+1}\\ &\quad\geq\left(\frac{\mu}{2}-1\right)\|u_k\|^2-\left[\frac{\mu}{2}-1-\left(\frac{\mu}{2}-1\right)C_4\right]T^{\frac{\mu-2}{\mu}}\big[C_{16}\big(1+\|u_k\|\big)\big]^{\frac{2}{\mu}}\\ &\qquad-C_{15}-pNC_8-(\mu-2)\bar{a}pNC_1\|u_k\|-(\mu-2)\bar{b}C_1\sum_{j=1}^p\sum_{i=1}^N\|u_k\|^{\xi_{ij}+1}\end{aligned}\tag{4.26}$$

for large k. Since $\mu>2$, $\xi_{ij}\in[0,1)$, by (4.26), $\{u_k\}$ is bounded in $H^1_{\Delta,T}$.

For any small $\epsilon_3=\frac{\delta}{2}$, by ($F_8$) we know that there is a $\rho_7>0$ such that

$$F(t,x)\leq\epsilon_3|x|^2,\quad\text{for }|x|<\rho_7\ \Delta\text{-a.e. }t\in[0,T]_{\mathbb{T}}.\tag{4.27}$$

By (F_7), for $\epsilon_4=\frac{\delta}{8pNC_1}$, there exists $\rho_8>0$ such that

$$\big|I_{ij}(t)\big|\leq\epsilon_4|t|,\quad|t|\leq\rho_8,i\in\Gamma,j\in\Lambda.\tag{4.28}$$

Let $\rho_9 = \frac{1}{2}\min\{\rho_7, \rho_8\}$. For $u \in E^1$ with $\|u\| \le r_1 \triangleq \frac{\rho_9}{C_1}$, by (2.5), (3.2), (3.6), (4.27) and (4.28), we have

$$\begin{aligned}
\varphi(u) &= q(u) + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t - \int_{[0,T)_{\mathbb{T}}} F\bigl(\sigma(t), u^\sigma(t)\bigr)\Delta t \\
&\ge \delta\|u\|^2 - \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{|u^i(t_j)|} \bigl|I_{ij}(t)\bigr|\,\mathrm{d}t - \epsilon_3\int_{[0,T)_{\mathbb{T}}} \bigl|u^\sigma(t)\bigr|^2\Delta t \\
&\ge \delta\|u\|^2 - \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{|u^i(t_j)|} \epsilon_4|t|\,\mathrm{d}t - \epsilon_3\int_{[0,T)_{\mathbb{T}}} \bigl|u^\sigma(t)\bigr|^2\Delta t \\
&\ge \delta\|u\|^2 - \epsilon_4\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|_\infty^2 - \epsilon_3\int_{[0,T)_{\mathbb{T}}} \bigl|u^\sigma(t)\bigr|^2\Delta t \\
&\ge \delta\|u\|^2 - \epsilon_4 pNC_1\|u\|^2 - \epsilon_3\|u\|^2 \\
&\ge \delta\|u\|^2 - \frac{\delta}{8}\|u\|^2 - \frac{\delta}{2}|u\|^2 \\
&= \frac{3\delta}{8}\|u\|^2.
\end{aligned}$$

Consequently,

$$\varphi(u) \ge \frac{3\delta\rho_9}{8} \triangleq \sigma > 0, \quad \forall u \in E^1 \text{ with } \|u\| = \rho_9. \tag{4.29}$$

Moreover, we can prove that J' is compact (see [35, p.1437]). It follows from (3.4), (4.29) and Lemma 4.1 that φ satisfies the conditions (I_5), (I_6) and (I_7)(i) with $S = \partial B_\rho \cap E_1$ of Theorem 3.2.

Set $e \in E_1 \cap \partial B_1$, $r_4 > \rho_9$, $r_5 > 0$, $Q = \{se : s \in (0, r_4)\} \oplus (B_{r_5} \cap E_2)$ and $\widetilde{E} = \operatorname{span}\{e\} \oplus E_2$. Then S and ∂Q link, where $B_{r_5} = \{u \in E : \|u\| \le r_5\}$. Set

$$Q_1 = \bigl\{u \in E_2 : \|u\| \le r_5\bigr\}, \qquad Q_2 = \bigl\{r_4 e + u : u \in E_2, \text{ and } \|u\| \le r_5\bigr\}$$

and

$$Q_3 = \bigl\{se + u : s \in [0, r_4], u \in E_2 \text{ and } \|u\| = r_5\bigr\}.$$

Then $\partial Q = Q_1 \cup Q_2 \cup Q_3$.

By (F_{10}), (3.4), (3.5) and (4.17), we know $\varphi|_{Q_1} \le 0$. For each $r_4 e + u \in Q_2$, one has $u = u^0 + u^- \in E_2$ and $\|u\| \le r_5$. By the equivalence of a finite dimensional space and (4.23), there exists $C_{17} > 0$ such that

$$\begin{aligned}
\int_{[0,T)_{\mathbb{T}}} F\bigl(t, r_4 e(t) + u(t)\bigr)\Delta t &\ge C_{13}\int_{[0,T)_{\mathbb{T}}} \bigl|r_4 e(t) + u(t)\bigr|^\mu\Delta t - C_{14}T \\
&\ge C_{17}\|r_4 e + u\|^\mu - C_{14}T \\
&= C_{17}\bigl(r_4^2 + \|u\|^2\bigr)^{\frac{\mu}{2}} - C_{14}T.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\varphi(r_4e+u) &= \frac{r_4^2}{2}\big\langle(I-K)e,e\big\rangle+\frac{1}{2}\big\langle(I-K)u,u\big\rangle+\phi(r_4e+u)\\&\quad-\int_{[0,T)_{\mathbb{T}}}F\big(t,r_4e(t)+u(t)\big)\Delta t\\&\le\frac{r_4^2}{2}\|I-K\|-\delta\big\|u^-\big\|^2-C_{17}\big(r_4^2+\|u\|^2\big)^{\frac{\mu}{2}}+C_{14}T\\&\le\frac{r_4^2}{2}\|I-K\|-C_{17}r_4^{\mu}+C_{14}T\\&\le 0\end{aligned}$$

for large $r_4>\rho_9$ due to $\mu>2$.

Moreover, for each $se+u\in Q_3$, one has $s\in[0,r_4]$, $u\in E_2$ and $\|u\|=r_5$. By the equivalence of a finite dimensional space and (4.23), one has

$$\begin{aligned}\int_{[0,T)_{\mathbb{T}}}F\big(t,se(t)+u(t)\big)\Delta t &\ge C_{13}\int_{[0,T)_{\mathbb{T}}}\big|se(t)+u(t)\big|^{\mu}\Delta t-C_{14}T\\&\ge C_{17}\|se+u\|^{\mu}-C_{14}T\\&= C_{17}\big(s^2+r_5^2\big)^{\frac{\mu}{2}}-C_{14}T.\end{aligned}$$

Hence

$$\begin{aligned}\varphi(se+u) &= \frac{s^2}{2}\big\langle(I-K)e,e\big\rangle+\frac{1}{2}\big\langle(I-K)u,u\big\rangle+\phi(se+u)-\int_{[0,T)_{\mathbb{T}}}F\big(t,se(t)+u(t)\big)\Delta t\\&\le\frac{s^2}{2}\|I-K\|-\delta\big\|u^-\big\|^2-C_{17}\big(s^2+r_5^2\big)^{\frac{\mu}{2}}+C_{14}T\\&\le\frac{r_4^2}{2}\|I-K\|-C_{17}r_5^{\mu}+C_{14}T\\&\le 0\end{aligned}$$

for large $r_5>r_4$.

Summing up the above, φ satisfies all conditions of Theorem 3.2. Hence, φ possesses a critical value $c\ge\sigma>0$, and hence problem (1.1) has at least one nontrivial weak solution. The proof is complete. □

Remark 4.1 There are a number of functions satisfying (A), (F_8), (F_9) and (F_{10}), for example, $F(t,x)=|x|^4$.

Next, we given two multiplicity results.

Theorem 4.3 *Assume that* (A), (F_5), (F_7), (F_8), (F_9) *and the following conditions are satisfied.*

(F_{11}) *I_{ij} ($i\in A$, $j\in B$) are odd.*
(F_{12}) *$F(t,x)$ is even in x and* $F(t,0)=0$.

Then problem (1.1) *has an unbounded sequence of weak solutions.*

Proof Set $W = H^+$, $V = H^- \oplus H^0$ and $E = H_T^1$. Then $E = V \oplus W$, $\dim V < +\infty$ and $\varphi \in C^1(E,R)$. From the proof of Theorem 4.2, we know that φ satisfies the (PS) condition, and there exist $\rho_9 > 0$ and $\sigma > 0$ such that

$$\varphi(u) \geq \sigma, \quad \forall u \in W \text{ with } \|u\| = \rho_9.$$

For each finite dimensional subspace $\widetilde{E} \subset E$, combining (3.2), (4.5), (4.6), (4.23) and the equivalence of a finite dimensional space, there exists $C_{18} > 0$ such that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \left|\dot{u}(t)\right|^2 \Delta t + \sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)\,\mathrm{d}t - \frac{1}{2}\int_{[0,T)_{\mathbb{T}}} \big(A(t)u(t),u(t)\big)\Delta t + J(u) \\
&\leq \frac{1}{2}\|u\|^2 + \overline{a}pNC_1\|u\| + \overline{b}C_1\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|^{\xi_{ij}+1} + \frac{1}{2}C_4\int_{[0,T)_{\mathbb{T}}} \left|u(t)\right|^2 \Delta t \\
&\quad - C_{13}\int_{[0,T)_{\mathbb{T}}} \left|u(t)\right|^{\mu} \Delta t + C_{14}T \\
&\leq \frac{1}{2}(1 + 2\overline{a}pNC_1 + C_4)\|u\|^2 + \overline{b}C_1\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|^{\xi_{ij}+1} - C_{18}\|u\|^{\mu} + C_{14}T.
\end{aligned}$$

Thus,

$$\varphi(u) \to -\infty \quad \text{as } u \in \widetilde{E} \text{ and } \|u\| \to \infty. \tag{4.30}$$

This implies that there is an $R = R_{(\widetilde{E})} > 0$ such that $\varphi \leq 0$ on $\widetilde{E}\backslash B_R$.

Moreover, by (F_{10}) and (F_{12}), we know that φ is even and $\varphi(0) = 0$. In view of Theorem 3.3, φ has a sequence of critical points $\{u_n\} \subset E$ such that $|\varphi(u_n)| \to \infty$. If $\{u_n\}$ is bounded in E, then by the definition of φ, one knows that $\{|\varphi(u_n)|\}$ is also bounded, a contradiction. Hence, $\{u_n\}$ is unbounded in E. The proof is completed. □

Example 4.2 Let $\mathbb{T} = \{\sqrt{m}, m \in \mathbb{N}_0\}$, $T = 3$, $N = 4$, $t_1 = 1$, $t_2 = 2$. Consider the second-order Hamiltonian system with impulsive effects

$$\begin{cases}
u^{\Delta^2}(t) + A(\sigma(t))u(\sigma(t)) + \nabla F(\sigma(t), u^{\sigma}(t)) = 0, & \Delta\text{-a.e. } t \in [0,16]_{\mathbb{T}}; \\
u(0) - u(16) = u^{\Delta}(0) - u^{\Delta}(16) = 0, \\
(u^i)^{\Delta}(t_j^+) - (u^i)^{\Delta}(t_j^-) = I_{ij}(u^i(t_j)), & i = 1,2,3,4, j = 1,2,
\end{cases} \tag{4.31}$$

where $A(t)$ is the unit matrix and

$$F(t,x) = |x|^4, \quad \text{for all } x \in \mathbb{R}^4 \text{ and } t \in [0,16]_{\mathbb{T}},$$

$$I_{ij}(t) = \begin{cases}
0, & |t| \geq 4, \\
(t-4)^3, & 3 \leq t < 4, \\
-t+2, & 1 < t < 3, \\
t^3, & |t| \leq 1, \\
-t-2, & -3 < t < -1, \\
(t+4)^3, & -4 < t \leq -3,
\end{cases}$$

for all $i = 1, 2, 3, 4$, $j = 1, 2$. All conditions of Theorem 4.3 hold. According to Theorem 4.3, problem (4.31) has an unbounded sequence of weak solutions.

Remark 4.2 In Theorem 4.3, if we delete the condition '$F(t, 0) = 0$', we have the following theorem.

Theorem 4.4 *Assume that* (A), (F_5), (F_7), (F_8), (F_9), (F_{11}) *and the following condition are satisfied.*

(F_{13}) *$F(t,x)$ is even in x.*

Then problem (1.1) *has an infinite sequence of distinct weak solutions.*

Proof Set $Y = H^+$, $X = H^- \oplus H^0$ and $E = H_T^1$ in Theorem 3.4. Then, from the proof of Theorem 4.3, we know that $E = X \oplus Y$, $\dim(X) < +\infty$, φ is even, $\varphi \in C^1(E, R)$ satisfies the (PS) condition, and there are constants $\rho_9, \sigma > 0$ such that $\varphi|_{\partial B_{\rho_9} \cap Y} \geq \sigma$ and $\inf \varphi(B_{\rho_9} \cap Y) > 0$, where $\partial B_{\rho_9} = \{u \in E : \|u\| = \rho_9\}$.

For each finite dimensional subspace $\widetilde{E} \subset E$, by (4.30), we know that

$$\varphi(u) \to -\infty \quad \text{as } u \in \widetilde{E} \text{ and } \|u\| \to \infty.$$

Consequently, for each finite dimensional subspace $Y_0 \subset Y$, the condition (Φ_2) holds. Moreover, by $\dim(X) < +\infty$ and $\varphi \in C^1(E, R)$, we know that (Φ_0) holds too. Therefore, the conclusion follows from Theorem 2.6. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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References
1. Zhou, JW, Li, YK: Sobolev's spaces on time scales and its applications to a class of second order Hamiltonian systems on time scales. Nonlinear Anal. **73**, 1375-1388 (2010)
2. Zhou, JW, Li, YK: Existence of solutions for a class of second-order Hamiltonian systems with impulsive effects. Nonlinear Anal. **72**, 1594-1603 (2010)
3. Nieto, JJ, O'Regan, D: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. **10**, 680-690 (2009)
4. Nieto, JJ: Impulsive resonance periodic problems of first order. Appl. Math. Lett. **15**, 489-493 (2002)
5. Nieto, JJ, Rodriguez-Lopez, R: Boundary value problems for a class of impulsive functional equations. Comput. Math. Appl. **55**, 2715-2731 (2008)
6. Chen, LJ, Chen, FD: Dynamic behaviors of the periodic predator-prey system with distributed time delays and impulsive effect. Nonlinear Anal., Real World Appl. **12**, 2467-2473 (2011)
7. Liu, ZS, Chen, HB, Zhou, TJ: Variational methods to the second-order impulsive differential equation with Dirichlet boundary value problem. Comput. Math. Appl. **61**, 1687-1699 (2011)
8. Cichoń, M, Satco, B, Sikorska-Nowak, A: Impulsive nonlocal differential equations through differential equations on time scales. Appl. Math. Comput. **218**, 2449-2458 (2011)

9. Xiao, J, Nieto, JJ, Luo, ZG: Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods. Commun. Nonlinear Sci. Numer. Simul. **17**, 426-432 (2012)
10. Carter, TE: Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion. Dyn. Control **10**, 219-227 (2000)
11. Pasquero, S: On the simultaneous presence of unilateral and kinetic constraints in time-dependent impulsive mechanics. J. Math. Phys. **47**, Article ID 082903 (2006)
12. Tian, Y, Ge, WG: Applications of variational methods to boundary value problem for impulsive differential equations. Proc. Edinb. Math. Soc. **51**, 509-527 (2008)
13. Aulbach, B, Hilger, S: Linear dynamic processes with inhomogeneous time scale. In: Nonlinear Dynamics and Quantum Dynamical Systems. Mathematical Research, vol. 59, pp. 9-20. Akademie Verlag, Berlin (1990)
14. Erbe, L, Hilger, S: Sturmanian theory on measure chains. Differ. Equ. Dyn. Syst. **1**(3), 223-244 (1993)
15. Lakshmikantham, V, Sivasundaram, S, Kaymakcalan, B: Dynamic Systems on Measure Chains. Kluwer Academic, Dordrecht (1996)
16. Agarwal, RP, Bohner, M: Basic calculus on time scales and some of its applications. Results Math. **35**(1-2), 3-22 (1999)
17. Agarwal, RP, Otero-Espinar, V, Perera, K, Vivero, DR: Basic properties of Sobolev's spaces on time scales. Adv. Differ. Equ. **2006**, Article ID 38121 (2006)
18. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: an Introduction with Applications. Birkhäuser, Boston (2001)
19. Bohner, M, Peterson, A: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
20. Benchohra, M, Henderson, J, Ntouyas, SK, Ouahab, A: On first order impulsive dynamic equations on time scales. J. Differ. Equ. Appl. **6**, 541-548 (2004)
21. Geng, F, Xu, Y, Zhu, D: Periodic boundary value problems for first-order impulsive dynamic equations on time scales. Nonlinear Anal. **69**, 4074-4087 (2008)
22. Graef, JR, Ouahab, A: Extremal solutions for nonresonance impulsive functional dynamic equations on time scales. Appl. Math. Comput. **196**, 333-339 (2008)
23. Wang, DB: Positive solutions for nonlinear first-order periodic boundary value problems of impulsive dynamic equations on time scales. Comput. Math. Appl. **56**, 1496-1504 (2008)
24. Zhang, HT, Li, YK: Existence of positive periodic solutions for functional differential equations with impulse effects on time scales. Commun. Nonlinear Sci. Numer. Simul. **14**, 19-26 (2009)
25. Li, YK, Zhou, JW: Existence of solutions for a class of damped vibration problems on time scales. Adv. Differ. Equ. **2010**, Article ID 727486 (2010)
26. Li, YK, Dong, YS: Multiple positive solutions for a fourth-order integral boundary value problem on time scales. Bound. Value Probl. **2011**, Article ID 59 (2011)
27. Li, YK, Shu, JY: Multiple positive solutions for first-order impulsive integral boundary value problems on time scales. Bound. Value Probl. **2011**, Article ID 12 (2011)
28. Li, YK, Zhang, TW: Multiple positive solutions for second-order -Laplacian dynamic equations with integral boundary conditions. Bound. Value Probl. **2011**, Article ID 867615 (2011)
29. Su, YH, Feng, Z: A non-autonomous Hamiltonian system on time scales. Nonlinear Anal. **75**, 4126-4136 (2012)
30. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer, Berlin (1989)
31. Luan, SX, Mao, AM: Periodic solutions for a class of non-autonomous Hamiltonian systems. Nonlinear Anal. **61**, 1413-1426 (2005)
32. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Application to Differetial Equations. CBMS Regional Conf. Ser. in Math., vol. 65. Am. Math. Soc., Providence (1986)
33. Bartsch, T, Ding, YH: Deformation theorems on non-metrizable vector spaces and applications to critical point theory. Math. Nachr. **279**, 1267-1288 (2006)
34. Rabinowitz, PH: Periodic solutions of Hamiltonian systems. Commun. Pure Appl. Math. **31**, 157-184 (1978)
35. Wu, X, Chen, SX: Teng, KM: On variational methods for a class of damped vibration problems. Nonlinear Anal. **68**, 1432-1441 (2008)

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