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Three solutions for a class of quasilinear elliptic systems involving the $p(x)$-Laplace operator

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Abstract

The existence of at least three weak solutions is established for a class of quasilinear elliptic systems involving the $p(x)$-Laplace operator with Neumann boundary condition. The technical approach is mainly based on a three critical points theorem due to Ricceri.

MSC: 35D05; 35J60; 58E05.

Keywords: $p(x)$-Laplacian, Sobolev space, three critical points theorem

1 Introduction

In this article, we consider the problem of the type

$$\begin{cases} -\Delta_{p(x)}u + e_p(x)|u|^{p(x)-2}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & x \in \Omega, \\ -\Delta_{q(x)}v + e_p(x)|v|^{q(x)-2}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is a bounded domain with boundary of class C^1. ν is the outer unit normal to $\partial\Omega$, λ, $\mu \geq 0$ are real numbers. $p(x), q(x) \in C^0(\bar{\Omega})$ with $N < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x)$,, $N < q^- \leq q^+$, $F : \Omega \times R \times R \to R$ is a function such that $F(\cdot, s, t)$ is measurable in Ω for all $(s, t) \in R \times R$ and $F(x, \cdot, \cdot)$ is C^1 in $R \times R$ for a.e. $x \in \Omega$, F_s denotes the partial derivative of F with respect to s. We assume $G(x, s,t)$ and $e_p(x)$,$e_q(x)$ satisfy the following conditions:

(G) $G : \Omega \times R \times R \to R$ is a Carathéodory function, $\sup_{\{|s|\leq\theta,|t|\leq\vartheta\}} |G(\cdot,s,t)| \in L^1(\Omega)$ for all θ, $\vartheta > 0$;
(E) $e_p(x)$,$e_q(x) \in L^\infty(\Omega)$ and ess $\inf_\Omega e_p(x)$, ess $\inf_\Omega e_q(x) > 0$, we denote $\|e_p\|_1 = \int_\Omega e_p(x)dx$ and $\|e_q\|_1 = \int_\Omega e_q(x)dx$.

It is well known that the operator $-\Delta_{p(x)} = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian and the corresponding problem is called a variable exponent elliptic systems. The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has been attracting attention of many authors in the last two decades. It arises from nonlinear elasticity theory, electro-rheological fluids, etc. see [1,2], many results have been obtained on this kind of problems, for example [3-9]. For the special case, $p(x) \equiv p$(a constant), (1.1) becomes the well known p-Laplacian problem. There have been many papers on this class of problems, see [10-19] and the reference therein.

Recently, many papers have appeared in which the technical approach adopted is based on the three critical points theorem obtained by Ricceri [16]. We cite papers [20-23], where the authors established the existence of at least three weak solutions to the problems with Dirichlet or Neumann boundary value conditions. Li and Tang in [24] obtained the existence of at least three weak solutions to problem (1) when $p(x) \equiv p$ with Dirichlet boundary value conditions. El Manouni and Kbiri Alaoui [25] obtained the existence of at least three solutions of system (1) when $p(x) \equiv p$ in Ω by the three critical points theorem obtained by Ricceri [26].

The main purpose of the present paper is to prove the existence of at least three solutions of problem (1). We study problem (1) by using the three critical points theorem by Ricceri [26] too. On the basis of [27], we state an equivalent formulation of the three critical points theorem in [26] as follows.

Theorem 1. *Let X be a reflexive real Banach space, $\Phi : X \to R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to R$ a C^1 functional with compact Gâteaux derivative. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\, \Psi(u)) = \infty$ *for all* $\lambda > 0$; *and there are* $r \in R$ *and* $u_0, u_1 \in X$ *such that:*
(ii) $\Phi(u_0) < r < \Phi(u_1)$;
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) > \dfrac{(\Phi(u_1) - r)\,\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$.

Then there exists a non-empty open set $\Lambda \subseteq [0, \infty)$ and a positive real number ρ with the following property: for each $\lambda \in \Lambda$ and every C^1 functional $J : X \to R$ with compact Gâteaux derivative, there exists $\sigma > 0$ such that for each $\mu \in [0, \sigma]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0 \tag{2}$$

has at least three solutions in X whose norms are less than ρ.

The paper is organized as follows. In section 2, we recall some facts that will be needed in the paper. In section 3, we establish our main result.

2 Notations and preliminaries

In order to deal with $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [1,5,28,29]).

We denote

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real - valued function on } \Omega, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 | \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, and we call it variable exponent Lebesgue space.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) | \, |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and it can be equipped with the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega),$$

and we call it variable exponent Sobolev space. From [5], we know that spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces.

When $e_p(x)$ satisfy (E), we define

$$L^{p(x)}_{e_p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real - valued function, } \int_\Omega e_p(x) |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{(p(x), e_p(x))} = \inf \left\{ \lambda > 0 | \int_\Omega e_p(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

then $L^{p(x)}_{e_p(x)}(\Omega)$ is a Banach space. For any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_{e_p} = \inf \left\{ \lambda > 0 | \int_\Omega \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + e_p(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Then it is easy to see that $\|u\|_{e_p}$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{p(x)}$. In the following, we will use $\|\cdot\|_{e_p}$ to instead of $\|\cdot\|_{p(x)}$ on $W^{1,p(x)}(\Omega)$. Similarly, we use $\|\cdot\|_{e_p}$ to instead of $\|\cdot\|_{q(x)}$ on $W^{1,q(x)}(\Omega)$.

Proposition 1. (see [1,5]) *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, we have*

$$\int_\Omega |uv| \, dx \le \left(\frac{1}{p^-} + \frac{1}{(p^0)^-} \right) |u|_{p(x)} |v|_{p^0(x)} \le 2 |u|_{p(x)} |v|_{p^0(x)}.$$

Proposition 2. (see [1,5])*If we denote $\rho(u) = \int_\Omega |u|^{p(x)} dx$, $\forall u \in L^{p(x)}(\Omega)$, then*

(i) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$; $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;

(iii) $|u|_{p(x)} \to 0(\infty) \Leftrightarrow \rho(u) \to 0(\infty)$.

From Proposition 2, the following inequalities hold:

$$\|u\|^{p^-}_{e_p} \le \int_\Omega |\nabla u(x)|^{p(x)} + e_p(x) |u(x)|^{p(x)} dx \le \|u\|^{p^+}_{e_p}, \text{ if } \|u\|_{e_p} \ge 1; \tag{3}$$

$$\|u\|^{p^+}_{e_p} \le \int_\Omega |\nabla u(x)|^{p(x)} + e_p(x) |u(x)|^{p(x)} dx \le \|u\|^{p^-}_{e_p}, \text{ if } \|u\|_{e_p} \le 1. \tag{4}$$

Proposition 3.*If* $\Omega \subset \mathbf{R}^N$ *is a bounded domain, then the imbedding* $W^{1,p(x)}(\Omega) \hookrightarrow C^0(\bar{\Omega})$*is compact whenever* $N < p^-$.

Proof. It is well know that $W^{1,p(x)}(\Omega) \hookrightarrow W^{1,p^-}(\bar{\Omega})$ is a continuous embedding, and the embedding $W^{1,p-}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact when $N < p^-$ and Ω is bounded. So we obtain the embedding $W^{1,p(x)}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact whenever $N < p^-$.

From now on, we denote X by $W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$ with the norm

$$\|z\| = \|u\|_{e_p} + \|v\|_{e_p} \text{ for any } z = (u, v) \in X.$$

Then X is a separable and reflexive Banach spaces. Naturally, we denote X^* by the space $(W^{1,p(x)})^*(\Omega) \times (W^{1,q(x)})^*(\Omega)$, the dual space of X.

From Proposition 3, we know that when $p^-,q^- > N$, the embedding $x \hookrightarrow C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ is compact, there exist a positive constant c such that

$$\|z\|_\infty = \|u\|_\infty + \|v\|_\infty = \sup_{x\in\bar{\Omega}} |u(x)| + \sup_{x\in\bar{\Omega}} |v(x)| \leq c\,\|z\|. \tag{5}$$

3 Existence of three solutions

We define Φ, Ψ, $J : X \to R$ by

$$\begin{aligned}\Phi(z) = &\int_\Omega \frac{1}{p(x)} \left(|\nabla u(x)|^{p(x)} + e_p(x)|u(x)|^{p(x)}\right) dx \\ &+ \int_\Omega \frac{1}{q(x)} \left(|\nabla v(x)|^{q(x)} + e_p(x)|v(x)|^{q(x)}\right) dx,\end{aligned} \tag{6}$$

$$\Psi(z) = -\int_\Omega F(x, u, v)dx, \tag{7}$$

$$J(z) = \int_\Omega G(x, u, v)dx. \tag{8}$$

Then for any $(\zeta,\eta) \in X$,

$$\begin{aligned}\left(\Phi'(z), (\zeta, \eta)\right) = &\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\zeta + e_p(x)|u|^{p(x)-2}u\zeta\, dx \\ &+ \int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\eta + e_q(x)|v|^{q(x)-2}v\eta dx \quad \forall z \in X,\end{aligned}$$

$$\left(\Psi'(z), (\zeta, \eta)\right) = \int_\Omega F_u(x, u, v)\zeta\, dx - \int_\Omega F_v(x, u, v)\eta dx, \quad \forall z \in X.$$

$$\left(J'(z), (\zeta, \eta)\right) = \int_\Omega G_u(x, u, v)\zeta\, dx - \int_\Omega G_v(x, u, v)\eta dx, \quad \forall z \in X.$$

We say that $z = (u, v) \in X$ is a weak solution of problem (1) if for any $(\zeta, \eta) \in X$

$$\left(\Psi'(z), (\zeta, \eta)\right) + \lambda\left(\Psi'(z), (\zeta, \eta)\right) + \mu(J'(z), (\zeta, \eta)) = 0.$$

Thus, we deduce that $z \in X$ is a weak solution of (1) if z is a solution of (2). It follows that we can seek for weak solutions of (1) by applying Theorem 1.

We first give the following result.

Lemma 1. *If* Φ *is defined in (6), then* $(\Phi')^{-1} : X^* \to X$ *exists and it is continuous.*

Proof. First, we show that Φ' is uniformly monotone. In fact, for any $\zeta, \eta \in R^N$, we have the following inequality (see [30]):

$$\left(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta\right)(\zeta - \eta) \geq \frac{1}{2^p}|\zeta - \eta|, \; p \geq 2.$$

Thus, we deduce that

$$\left(\Phi'(z_1) - \Phi'(z_2), z_1 - z_2\right) \geq \min\left\{\frac{1}{2^{p^+}}, \frac{1}{2^{q^+}}\right\}\left(\min\left\{\|u_1 - u_2\|_{e_p}^{p^+}, \|u_1 - u_2\|_{e_p}^{p^-}\right\} + \min\left\{\|v_1 - v_2\|_{e_q}^{q^+}, \|v_1 - v_2\|_{e_q}^{q^-}\right\}\right),$$

for any $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in X$, i.e.,Φ' is uniformly monotone.

From (3), (4), we can see that for any $z \in X$, we have that

$$\frac{\left(\Phi'(z), z\right)}{\|z\|} \geq \frac{\min\left\{\|u\|_{e_p}^{p^+}, |u|\|_{e_p}^{p^-}\right\} + \min\left\{\|v\|_{e_p}^{p^+}, \|v\|_{e_p}^{p^-}\right\}}{\|u\|_{e_p} + \|v\|_{e_p}}.$$

That's meaning Φ' is coercive on X.

By a standard argument, we know that Φ' is hemicontinuous. Therefore, the conclusion follows immediately by applying Theorem 26.A [31].

To obtain our main result, we assume the following conditions on $F(x,s,t)$:

(A1) There exist $d(x) \in L^1(\Omega)$ and $0 < \varsigma < p^-$, $0 < \tau < q^-$ such that

$$F(x, s, t) \leq d(x)\left(1 + |s|^{\zeta} + |t|^{\tau}\right)$$

for a.e.$x \in \Omega$ and $(s,t) \in R \times R$;

(A2) $F(x,0,0) = 0$ for a.e.$x \in \Omega$;

(A3) There exist $s_1,t_1 \in R$ with $|s_1|$, $|t_1| \geq 1$ such that

$$\text{meas}(\Omega) \sup_{(x,|s|,|t|)\in\Omega\times[0,ck_p]\times[0,ck_q]} F(x, s, t) \leq \frac{\left(\frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+}\right)\int_\Omega F(x, s_1, s_1)dx}{\frac{\|e_p\|_1}{p^-}|s_1|^{p^+} + \frac{\|e_q\|_1}{q^-}|t_1|^{q^+}}, \tag{9}$$

where c is given in (5) and

$$k_p = \max\left\{\left(\|e_p\|_1 + \frac{p^+\|e_q\|_1}{q^+}\right)^{\frac{1}{p^+}}, \left(\|e_p\|_1 + \frac{p^+\|e_q\|_1}{q^+}\right)^{\frac{1}{p^-}}\right\},$$

$$k_q = \max\left\{\left(\frac{q^+\|e_p\|_1}{p^+} + \|e_q\|_1\right)^{\frac{1}{q^+}}, \left(\frac{q^+\|e_p\|_1}{p^+} + \|e_q\|_1\right)^{\frac{1}{q^-}}\right\}.$$

(A3)' $F(x,s,t) > 0$ for any $x \in \Omega$ and $|s|$ or $|t|$ large enough, and there exist $M, N > 0$ such that

$$F(x, s, t) \leq 0, \; x \in \Omega, |s| \leq M, |t| \leq N;$$

Then we have the following main theorem.

Theorem 2. *Assume (A1),(A2),(A3)(or (A3)'),(G) and (E) hold. Then there exist an open interval* $\Lambda \subseteq [0, \infty)$ *and a positive real number* ρ *with the following property: for each* $\lambda \in \Lambda$, *there exists* $\sigma > 0$ *such that for each* $\mu \in [0, \sigma]$, *problem (1) has at least three weak solutions whose norms are less than* ρ.

Proof. By the definitions of Φ, Ψ, J, we know that Ψ' is compact, Φ is weakly lower semi-continuous and bounded on each bounded subset of X. From lemma 1 we can see that $(\Phi')^{-1}$ is well defined, from condition (G), J is well defined and continuously Gâteaux differentiable on X, with compact derivative. Then we can use Theorem 1 to obtain the result. Now we show that the hypotheses of Theorem 1 are fulfilled.

Thanks to (A1), for each $\lambda \geq 0$, one has that

$$\lim_{\|z\| \to \infty} \Phi(z) + \lambda\Psi(z) = +\infty,$$

and so the assumption (i) of Theorem 1 holds.

Now we consider in two cases:

Case (i): (A3) holds, i.e., there exist $1 \leq |s_1|, |t_1|$ such that (9) hold.

Now we set $z_0 = (0,0)$, $z_1 = (s_1, s_1)$ and denote $r = \frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+} > 0$, then it is easy to see that

$$\Phi(z_1) > r > 0 = \Phi(z_0).$$

Thus, (ii) of Theorem 1 is satisfied.

At last, by (A2) we know $\Psi(z_0) = 0$, then

$$\begin{aligned} &\frac{\left(\Phi(z_1) - r\right)\Psi(z_0) + (r - \Phi(z_0))\Psi(z_1)}{\Phi(z_1) - \Phi(z_0)} \\ &= r\frac{\Psi(z_1)}{\Phi(z_1)} \leq -r\frac{\int_\Omega F(x, s_1, s_1)dx}{\frac{|s_1|^{p^+}}{p^-}\|e_p\|_1 + \frac{|t_1|^{q^+}}{q^-}\|e_q\|_1}. \end{aligned} \tag{10}$$

On the other way, when $\Phi(z) \leq r$, we have

$$\min\left\{\|u\|_{e_p}^{p^+}, \|u\|_{e_p}^{p^-}\right\} \leq rp^+, \min\left\{\|v\|_{e_q}^{p^+}, \|v\|_{e_q}^{p^-}\right\} \leq rq^+.$$

We deduce that

$$\|u\|_{e_p} \leq \max\left\{(rp^+)^{\frac{1}{p^+}}, (rp^+)^{\frac{1}{p^-}}\right\}$$

and

$$\|v\|_{e_q} \leq \max\left\{(rq^+)^{\frac{1}{q^+}}, (rq^+)^{\frac{1}{q^-}}\right\}$$

For $r = \frac{\|e_p\|_1}{p^+} + \frac{\|e_q\|_1}{q^+}$, then we have

$$\|u\|_{e_p} \leq k_p, \|v\|_{e_q} \leq k_q.$$

By (5), we obtain

$$\|u\|_\infty \leq ck_p,\ \|v\|_\infty \leq ck_q.$$

Thus, from (7), we have

$$\begin{aligned} -\inf_{z\in\Phi^{-1}((-\infty,r])}\Psi(z) &= \sup_{z\in\Phi^{-1}((-\infty,r])} -\Psi(z) \\ &\le \int_\Omega \sup_{(|u|,|v|)\in[0,ck_p]\times[0,ck_q]} F(x,u,v)dx \\ &\le \operatorname{meas}(\Omega) \sup_{(x,|u|,|v|)\in\Omega\times[0,ck_p]\times[0,ck_q]} F(x,u,v) \end{aligned} \tag{11}$$

From (9)-(11) and the definition of r, we can see (iii) of Theorem 1 is hold.

Case (ii): (A3)' holds. Then there exist $|s_2|,|t_2| > 1$ such that $F(x,s_2,t_2) > 0$ for any $x \in \Omega$ and $|s_2|^{p^-}\|e_p\|_1 \ge 1$, $|t_2|^{q^-}\|e_q\|_1 \ge 1$. Set $a = \min\{c,M\}$, $b = \min\{c, N\}$ then we have

$$\int_\Omega \sup_{(|s|,|t|)\in[0,a]\times[0,b]} F(x,s,t)dx \le 0 < \int_\Omega F(x,s_2,t_2)dx. \tag{12}$$

We denote $z_2 = (s_2,t_2)$ and $r = \min\left\{\frac{1}{p^+}\left(\frac{a}{c}\right)^{p^+}, \frac{1}{q^+}\left(\frac{b}{c}\right)^{q^+}\right\}$. Then it is easy to see that

$$\Phi(z_2) > r > \Phi(z_0).$$

So, (ii) of Theorem 1 is satisfied.

When $\Phi(z) \le r$, similar to the above arguments, we obtain that

$$\|u\|_\infty \le a, \|v\|_\infty \le b. \tag{13}$$

At last, we see that

$$\begin{aligned} &\frac{(\Phi(z_2)-r)\,\Psi(z_0) + (r-\Phi(z_0))\Psi(z_2)}{\Phi(z_2)-\Phi(z_0)} \\ &= r\frac{\Psi(z_2)}{\Phi(z_2)} \le -r\frac{\int_\Omega F(x,s_2,t_2)dx}{\frac{|s_2|^{p^+}}{p^-}\|e_p\|_1 + \frac{|t_2|^{q^+}}{q^-}\|e_q\|_1} < 0. \end{aligned} \tag{14}$$

From (7) and (12), we have

$$\begin{aligned} -\inf_{z\in\Phi^{-1}((-\infty,r])}\Psi(z) &= \sup_{z\in\Phi^{-1}((-\infty,r])} -\Psi(z) \\ &\le \int_\Omega \sup_{(|u|,|v|)\in[0,a]\times[0,b]} F(x,u,v)dx \le 0. \end{aligned} \tag{15}$$

From (14) and (15), we can see (iii) of Theorem 1 is still hold.

Then all the hypotheses of Theorem 1 are fulfilled. By Theorem 1, we know that there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0,\sigma]$, problem (1) has at least three weak solutions whose norms are less than ρ.

By Theorem 2, we have the following result.

Corollary 1. *Let f, $g : \Omega \times R \to R$ be Carathéodory functions,* $\sup_{|\zeta|\le s} |g(\cdot, \zeta)| \in L^1(\Omega)$ *for all* $s > 0$, *and define* $F(x,t) = \int_0^t f(x,y)dy$ *for any* $(x,t) \in \Omega \times R$, $e(x) \in L^\infty(\Omega)$ *and ess* $\inf_\Omega e(x) > 0$. *Assume the following conditions hold.*

(B1) There exist $d(x) \in L^1(\Omega)$ *and* $0 < \varsigma < p^-$ *such that*

$$F(x,t) \le d(x)\left(1 + |t|^{\varsigma}\right)$$

for a.e.$x \in \Omega$ *and* $t \in R$*;*

(B2) There exists $t_3 \in R$ *with* $|t_3| \ge 1$ *such that*

$$meas(\Omega) \sup_{(x,|s|)\in\Omega\times[0,ck]} F(x,s) \le \frac{p^-}{p^+}\frac{\int_\Omega F(x,t_3)dx}{|t_3|^{p^+}}, \tag{16}$$

where c is given in (5) and

$$k = \max\left\{(\|e\|_1)^{\frac{1}{p^+}}, (\|e\|_1)^{\frac{1}{p^-}}\right\};$$

or

(B2)' $F(x,t) > 0$ *for any* $x \in \Omega$ *and* $|t|$ *large enough, and there exist* $M > 0$ *such that*

$$F(x,t) \le 0,\ x \in \Omega, |t| \le M.$$

Then there exist an open interval $\Lambda \subseteq [0, \infty)$ *and a positive constant* ρ *such that for any* $\lambda \in \Lambda$*, there exists* $\sigma > 0$ *and for each* $\mu \in [0, \sigma]$*, the problem*

$$\begin{cases} -\Delta_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{cases} \tag{17}$$

has at least three weak solutions whose norms are less than ρ*.*

Remark 1. if $p(x) = p$ in Ω, $\mu = 0$, problem (17) was considered in [21]. If we take $f(x,t) = |t|^{\gamma(x)-2}t - t$ with $\gamma(x) \in C^0(\bar{\Omega})$ satisfies $2 < \gamma^- \le \gamma^+ < p^-$, $\mu = 0$, Corollary 1 becomes a version of Theorem 2 in [23]. Hence our Corollary 1 unifies and generalizes Theorem 2 in [21] and Theorem 2 in [23] and our Theorem 2 generalizes the main results of [21-25] to the system (1).

At last, we give two examples.

Example 1. Let $\Omega = B(0,1)$ be the unit ball of R^N with $N \ge 2$, set $p(x) = N + e^{|x|}$,$q(x) = N + 1 + \ln(1 + x^2)$, $e_p(x) = (1 + x^2) = e_q(x)$, $G(x,u,v) = x^2(u^2 + v^2)$ and

$$F(x,u,v) = \begin{cases} e^{x^2}(e^u + uv - 1), & x \in \Omega, u \le M, v \in R, \\ e^{x^2}\left(ue^M + uv + \frac{1}{2}u^2 - Mu - (M-1)e^M + \frac{1}{2}M^2\right), & x \in \Omega, u \le M, v \in R, \end{cases} \tag{18}$$

where M is a positive constant, i.e., we consider the following problem

$$\begin{cases} -\Delta_{p(x)}u + (1+x^2)|u|^{p(x)-2}u = \lambda f(x,u,v) + \mu^2 x^2 u, & x \in \Omega, \\ -\Delta_{q(x)}v + (1+x^2)|v|^{q(x)-2}v = \lambda u + \mu 2x^2 v, & x \in \Omega, \\ \dfrac{\partial u}{\partial v} = \dfrac{\partial v}{\partial v} = 0 \quad x \in \partial\Omega. \end{cases} \tag{19}$$

where

$$f(x,u,v) = F_u(x,u,v) = \begin{cases} e^{x^2}(e^u + v), & x \in \Omega, u \le M, v \in R, \\ e^{x^2}(e^M + v + u - M), & x \in \Omega, u \le M, v \in R, \end{cases} \tag{20}$$

We can see that $p^+ = N + e, p^- = N + 1, q^+ = N + 1 + \ln 2, q^- = N + 1, \|e\|_1 = \frac{4}{3}$, and it is easy to see that for any $t_1 > 1$, there exists $s_1 > 1$ such that

$$\frac{e^{s_1} + s_1 t_1 - 1}{\frac{s_1^{p^+}}{p^-} + \frac{t_1^{q^+}}{q^-}} \geq \frac{e\left(e^{ck_p} + c^2 k_p k_q - 1\right)}{\frac{1}{p^+} + \frac{1}{q^+}}, \tag{21}$$

were $k_p = \left(\frac{4}{3} + \frac{4(N+1+\ln+2)}{3(N+e)}\right)^{\frac{1}{N+1}}$, $k_q = \left(\frac{4}{3} + \frac{4(N+e)}{3(N+1+\ln 2)}\right)^{\frac{1}{N+1}}$ are positive constants and c is given by (5). Then when $M \geq s_1$, $F(x,u,v)$ defined in (18) satisfies (A1)-(A3) of Theorem 2, and $G(x,u,v)$,$e(x)$ satisfy

(G) and (E) respectively, by Theorem 2, there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0,\sigma]$, system (19) has at least three weak solutions whose norms are less than ρ.

Example 2. Assume Ω,$p(x)$,$q(x)$,$e_p(x)$,$e_q(x)$,$G(x,u,v)$ are the same as in example 1, and suppose $N \geq 8$. Let

$$F(x,u,v) = \left(1 + 2x^2\right)\left(u^4 v^2 + v^4 u^2 - 2u^2 v^2\right), \quad x \in \Omega, u, v \in R. \tag{22}$$

Obviously, $F(x,u,v)$ satisfies (A1) and (A2). By simple computation, we can see that

$$F(x,u,v) > 0, \quad \text{when } |u| > \sqrt{2} \text{ or } |v| > \sqrt{2}$$

and

$$F(x,u,v) < 0, \quad \text{when } |u| < 1 \text{ and } |v| < 1,$$

i.e., (A3)' hold for $F(x,u,v)$ defined in (22).

Thus, there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, there exists $\sigma > 0$ and for each $\mu \in [0, \sigma]$, the system

$$\begin{cases} -\Delta_{p(x)} u + \left(1 + x^2\right)|u|^{p(x)-2} u = \lambda 4u^3 v^2 + 2v^4 u - 4uv^2 + \mu 2x^2 u, & x \in \Omega, \\ -\Delta_{p(x)} v + \left(1 + x^2\right)|v|^{q(x)-2} v = \lambda 4u^3 v^2 + 2u^4 v - 4vu^2 + \mu 2x^2 v, & x \in \Omega, \\ \dfrac{\partial u}{\partial v} = \dfrac{\partial v}{\partial v} = 0 \quad x \in \partial\Omega. \end{cases} \tag{23}$$

has at least three weak solutions whose norms are less than ρ.

Remark 2. We remark that the methods used in this paper are also applicable for the cases of the other boundary value conditions, for example, Dirichlet boundary value conditions.

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Authors' contributions
All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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