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Oscillating global continua of positive solutions of second order Neumann problem with a set-valued term

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Abstract

In this note, we study the oscillating global continua of the differential inclusion of the form

$$\begin{cases} -u'' + qu \in \lambda F(\cdot, u), \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

where F is a "set-valued representation" of a function with jump discontinuities along the line segment $[0, 1] \times \{0\}$, and $\lambda \in [0, \infty)$ is a parameter. The proof of our main result relies on an approximation procedure.

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1 Introduction

In recent years, nonsmooth analysis has come to play an important role in functional analysis [1], dynamical systems [2], control theory [3], optimization [4], mechanical systems [5], differential equation [6,7] etc. Since many mathematical and physical problems may be reduced to ODES or PDES with discontinuous nonlinearities, the existence of multiple solutions for differential inclusion problems has been widely investigated [8-19].

In this article, we are concerned with the following differential inclusion problem which raises from a Budyko-North type energy balance climate models:

$$\begin{cases} -u''(x) + q(x)u(x) \in \lambda F(x, u(x)), & \text{a.e. } x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = 0 \end{cases} \quad (1.1)$$

(see [20-25] and the references therein). In particular, the set-valued right hand side arise from a jump discontinuity of the albedo at the ice-edge in these models. By filling such a gap, one arrives at the set-valued problem (1.1). As in [25], we are here interested in a considerably simplify version as compared to the situation from climate modeling, e.g. a one-dimensional regular Sturm-Liouville differential operator substitutes for a two-dimensional Laplace-Beltrami operator or a singular Legendre-type operator, and the jump discontinuity is transformed to $u = 0$ in a way, which resembles only locally the climatological problem.

We are concerned with the set-valued problem (1.1) under the following assumptions

$$(H1) \quad q \in C([0, 1], (0, +\infty));$$

$$(H2) \quad f^+ \in C([0, 1] \times [0, +\infty), (0, +\infty)),$$

$$\inf_{x \in [0, 1]} f^+(x, 0) > 0, \quad \lim_{s \rightarrow +\infty} \frac{f^+(x, s)}{s} = b(x) \in C([0, 1], (0, \infty)).$$

Let the set-valued function F in (1.1) is given by

$$F(x, \gamma) = \begin{cases} \{f^+(x, \gamma)\}, & x \in [0, 1], \quad \gamma > 0, \\ [0, f^+(x, 0)], & x \in [0, 1]. \end{cases}$$

Notice that if $f^+(x, 0) \equiv 0$, $x \in [0, 1]$, then the differential inclusion problem (1.1) reduces to the BVP of differential equation

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda f^+(x, u(x)), & x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \quad (1.2)$$

In the last 20 years, the positive solutions of (1.2) have been studied by several authors, see Jiang and Liu [26], Chu et al. [27] and Sun et al. [28].

The purpose of this article is to investigate the oscillating global continua of positive solutions of the differential inclusion problem (1.1). The proof of our main result relies on an approximation procedure. The rest of the article is organized as follows. In Section 2, we state some notations and prove some preliminary results. In Section 3, we state and prove our main result. In Section 4, an example is given to illustrate the application of our main result.

2 Notations and preliminaries

Recall Kuratowski's notion of lower and upper limits of sequence of sets.

Definition 2.1. [29] Let X be a metric space and $\{Z_l\}_{l \in \mathbb{N}}$ be a sequence of subsets of X . The set

$$\limsup_{l \rightarrow \infty} Z_l := \left\{ x \in X : \liminf_{l \rightarrow \infty} \text{dist}(x, Z_l) = 0 \right\}$$

is called the upper limit of the sequence $\{Z_l\}$, whereas

$$\liminf_{l \rightarrow \infty} Z_l := \left\{ x \in X : \lim_{l \rightarrow \infty} \text{dist}(x, Z_l) = 0 \right\}$$

is called the lower limit of the sequence $\{Z_l\}$.

Definition 2.2. [29] A component of a set M is meant a maximal connected subset of M .

Lemma 2.1. [29] Suppose that Y is a compact metric space, A and B are non-intersecting closed subsets of Y , and no component of Y intersects both A and B . Then there exist two disjoint compact subsets Y_A and Y_B , such that $Y = Y_A \cup Y_B$, $A \subset Y_A$, $B \subset Y_B$.

Using the above Whyburn Lemma, Ma and An [30] proved the following

Lemma 2.2. [30, Lemma 2.1] Let Z be a Banach space and let $\{A_n\}$ be a family of closed connected subsets of Z . Assume that

- (i) there exist $z_n \in A_n$, $n = 1, 2, \dots$, and $z^* \in Z$, such that $z_n \rightarrow z^*$;
- (ii) $r_n = \sup \{\|x\| \mid x \in A_n\} = \infty$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty} A_n) \cap B_R$ is a relatively compact set of Z , where $B_R = \{x \in Z \mid \|x\| \leq R\}$. Then there exists an unbounded component C in $\limsup_{l \rightarrow \infty} A_l$ and $z^* \in C$.

Remark 2.1. The limiting processes for sets go back at least to the work of Kuratowski [31]. Lemma 2.2 is a slight generalization of the following well-know result due to Whyburn [29]:

Proposition 2.1. (Whyburn [29, p. 12]) Let Z be a Banach space and $\{A_n\}$ be a family of closed connected subsets of Z . Let $\liminf_{l \rightarrow \infty} A_l \neq \emptyset$ and $\bigcup_{l \in \mathbb{N}} A_l$ is relatively compact. Then $\limsup_{l \rightarrow \infty} A_l$ is nonempty, compact and connected.

Next, we introduce the result of global solution behavior of the bifurcation branches of the equation

$$x = \mu(Lx + Nx), \quad \mu \in R, \quad x \in X, \tag{2.1}$$

to wit the following lemma.

Lemma 2.3. [32] (Dancer (1974)) *Assume that*

(C1) *The operators $L, N: X \rightarrow X$ are compact on the Banach space X over R . Furthermore, L is linear and $\|Nx\|/\|x\| \rightarrow 0$ as $\|x\| \rightarrow 0$;*

(C2) *The real number μ_0 is a characteristic number of L of odd algebraic multiplicity;*

(C1+) *The real Banach space X has an order cone K with $X = K - K$, i.e., every $x \in X$ can be represented as $x = x_1 - x_2$, where $x_1, x_2 \in K$. Furthermore, $L + N$ is positive, i.e., $L + N$ maps K into K ;*

(C2+) *The spectral radius $r(L)$ of L is positive. We set $\mu_0 = (r(L))^{-1}$.*

Then $(\mu_0, 0)$ is a bifurcation point of equation (2.1) and

$$S_+ := \overline{\{(\mu, x) \in R \times X : (\mu, x) \text{ is a solution of (2.1) with } \mu > 0, x > 0\}}$$

contains an unbounded solution component C_1^+ which passes through $(\mu_0, 0)$.

If additionally

(C3+) *The linear operator L is strongly positive, then $(\mu, x) \in C_1^+$ and $\mu \neq \mu_0$ always implies $x > 0$ and $\mu > 0$.*

Remark 2.2. This result is often called the nonlinear Krein-Rutman theorem. It will play an important role in the proof of our main result.

Let ϕ and ψ be the unique solution of the problems

$$\begin{cases} -u''(x) + q(x)u(x) = 0, & x \in (0, 1), \\ u'(0) = 0, & u(0) = 1 \end{cases}$$

and

$$\begin{cases} -u''(x) + q(x)u(x) = 0, & x \in (0, 1), \\ u'(1) = 0, & u(1) = 1 \end{cases}$$

respectively. Then it is easy to check $\phi(\cdot)$ is nondecreasing on $(0,1)$, $\psi(\cdot)$ is nonincreasing on $(0,1)$, and the Green's function $G(x, s)$ of

$$\begin{cases} -u''(x) + q(x)u(x) = 0, & x \in (0, 1), \\ u'(0) = 0, & u'(1) = 0 \end{cases}$$

is explicitly given by

$$G(x, s) = -\frac{1}{\psi'(0)} \begin{cases} \psi(x)\varphi(s), & 0 \leq s \leq x \leq 1, \\ \varphi(x)\psi(s), & 0 \leq x \leq s \leq 1. \end{cases} \quad (2.2)$$

Moreover, we have that

$$\begin{aligned} 0 < G(x, s) \leq G(s, s), \quad (x, s) \in [0, 1] \times [0, 1]; \\ \sigma G(s, s) \leq G(x, s), \quad (x, s) \in [0, 1] \times [0, 1] \end{aligned} \quad (2.3)$$

with $\sigma := \min \left\{ \frac{1}{\psi(0)}, \frac{1}{\varphi(1)} \right\}$.

3 The main result

Let Σ be the closure of the set of positive solutions of (1.1) in $[0, \infty) \times C^1[0, 1]$, and $\mathbb{N}^* := \{1, 2, \dots, N\}$. The main result of this article is the following theorem.

Theorem 3.1. *Assume that (H1)-(H2) hold. If*

(H3) there is an increasing sequence of positive numbers $\{\xi_j\}_1^N$ and a small enough constant δ such that $\xi_1 < \sigma(\xi_2 - \delta)$ and

$$\begin{aligned} \Phi(\xi_{2j-1}) &< \frac{1}{2} \left(\int_0^1 G(s, s) ds \right)^{-1} (\xi_{2j-1} - \delta), \quad j \in \mathbb{N}^*; \\ \Psi(\xi_{2j}) &> 2 \left(\int_0^1 G\left(\frac{1}{2}, s\right) ds \right)^{-1} (\xi_{2j} + \delta), \quad j \in \mathbb{N}^*, \end{aligned}$$

where

$$\begin{aligned} \Phi(l) &:= \max\{f^+(t, c) : 0 \leq t \leq 1, \quad 0 \leq c \leq l + \delta\}, \\ \Psi(l) &:= \min\{f^+(t, c) : 0 \leq t \leq 1, \quad \sigma(l - \delta) \leq c \leq l + \delta\}, \end{aligned}$$

then there exists an unbounded component C_1^+ in Σ with $(0, 0) \in C_1^+$. Moreover,

(i) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = \xi_{2j-1}$ for some $j \in \mathbb{N}^*$ implies that $\lambda \geq 2$;

(ii) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = \xi_{2j}$ for some $j \in \mathbb{N}^*$ implies that $\lambda \leq \frac{1}{2}$.

Actually, such continua C_1^+ can be obtained as upper limits in the sense of Kuratowski of sequence of solution continua from associated continuous problems. To this end one sets

$$d_g := \inf\{f^+(x, 0) : x \in [0, 1]\}, \quad (3.1)$$

fixes $l_0 \in \mathbb{N}$ such that $\frac{d_g}{l_0} < \xi_1$, and selects an approximation sequence $\{f_l\} \subset C([0, 1] \times \mathbb{R}, \mathbb{R})$ ($l > l_0$) of F satisfying:

$$(A1) \quad f_l(x, y) = ly \text{ for } x \in [0, 1] \text{ and } y \in \left[0, \frac{d_g}{2l}\right];$$

$$(A2) \quad \frac{d_g}{2} \leq f_l(x, y) \leq f^+(x, y) \text{ for } x \in [0, 1] \text{ and } y \in \left[\frac{d_g}{2l}, \frac{d_g}{l}\right];$$

(A3) $f_l(x, y) = f^*(x, y)$ for $x \in [0, 1]$ and $y \geq \frac{d_g}{l}$;

(A4) $\{f_l(x, y)\}_{l \in \mathbb{N}}$ is nondecreasing in l for $(x, y) \in [0, 1] \times (0, \infty)$.

Next, we show that the continuous problem

$$\begin{cases} -v''(x) + q(x)v(x) = \lambda f_l(x, v(x)), & x \in (0, 1), \\ v'(0) = 0, \quad v'(1) = 0 \end{cases} \quad (3.2l)$$

has an unbounded closed subsets $C_{1,l}^+$ of the positive solutions set of (3.2) with

(a) $\left(\frac{\lambda_1}{l}, 0\right)$ is the bifurcation point contained in $C_{1,l}^+$;

(b) If $(\mu, \vartheta) \in C_{1,l}^+$ and $\vartheta \not\equiv 0$, then ϑ is positive on $(0, 1)$.

It is easy to see that (3.2_l) equivalent to

$$v(x) = \lambda \int_0^1 G(x, s) f_l(s, v(s)) ds. \quad (3.3)$$

Let

$$\begin{aligned} (Lv)(x) &:= l \int_0^1 G(x, s) v(s) ds, \\ (Nv)(x) &:= \int_0^1 G(x, s) (f_l(s, v(s)) - lv(s)) ds, \quad v \in C[0, 1]. \end{aligned}$$

Then according to (3.3), (3.2_l) can be written as the following operator equation

$$v = \lambda(Lv + Nv).$$

Clearly, the operators $L, N : C[0, 1] \rightarrow C[0, 1]$ are compact on the Banach space $C[0, 1]$. Furthermore, L is linear and thanks to (2.3)(A1) that

$$\begin{aligned} \frac{\|Nv\|_\infty}{\|v\|_\infty} &= \left\| \int_0^1 G(x, s) \frac{f_l(s, v(s)) - lv(s)}{\|v\|_\infty} ds \right\|_\infty \\ &\leq \int_0^1 G(s, s) \frac{|f_l(s, v(s)) - lv(s)|}{\|v\|_\infty} ds \rightarrow 0, \quad \text{as } \|v\|_\infty \rightarrow 0, \end{aligned}$$

which implies that the condition (C1) of Lemma 2.3 is satisfied.

Denote the principal eigenvalue of

$$\begin{cases} -\omega''(x) + q(x)\omega(x) = \lambda\omega(x), & x \in (0, 1), \\ \omega'(0) = 0, \quad \omega'(1) = 0, \end{cases} \quad (3.4)$$

by λ_1 , then we know that $\lambda_1 > 0$ (see [33]). Since (3.4) is equivalent to operator equation

$$\omega = \frac{\lambda}{l} L\omega,$$

we have that $(r(L))^{-1} = \frac{\lambda_1}{l}$. Therefore, the conditions (C2)(C2+) of Lemma 2.3 are satisfied.

Let the cone K in $C[0, 1]$ is given by

$$K = \{u \in C[0, 1] \mid u(x) \geq 0, 0 \leq x \leq 1\}.$$

It is easy to see thanks to (A1)-(A4) and (2.3) that the (C1+)(C3+) conditions of Lemma 2.3 are satisfied.

According to Lemma 2.3, we obtain that $\left(\frac{\lambda_1}{l}, 0\right)$ is a bifurcation point of the positive solutions set of (3.2) for every $l \in \{l_0 + 1, l_0 + 2, \dots\} =: \mathbb{N}_0$, and for each $l \in \mathbb{N}_0$ there exists an unbounded closed subsets $C_{1,l}^+$ of the positive solutions set of (3.2) with (a) and (b).

Combining the above with the fact

$$\lim_{l \rightarrow \infty} \left(\frac{\lambda_1}{l}, 0\right) = (0, 0)$$

and utilizing Lemma 2.2, it concludes that there exists an unbounded component C_1^+ with

$$(0, 0) \in C_1^+ \tag{3.5}$$

and

$$C_1^+ \subset \limsup_{l \rightarrow \infty} C_{1,l}^+. \tag{3.6}$$

Denote the cone P in $C[0, 1]$ by

$$P = \left\{ u \in C[0, 1] \mid \min_{0 \leq x \leq 1} u(x) \geq \sigma \|u\|_\infty \right\}.$$

Define an operator $T_\lambda : P \rightarrow C[0, 1]$ by

$$(T_\lambda u)(x) = \lambda \int_0^1 G(x, s) f_1(s, u(s)) ds, \quad x \in [0, 1].$$

It is easy to get the following lemma.

Lemma 3.1. *Assume that (H1), (H2) and (A1)-(A4) hold. Then $T_\lambda: P \rightarrow P$ is completely continuous.*

Lemma 3.2. *Assume that (H1), (H2) and (A1)-(A4) hold. If $0 \leq u(x) \leq r, r > 0$, for $x \in [0, 1]$, then*

$$\|T_\lambda u\|_\infty \leq \lambda M_r \int_0^1 G(s, s) ds,$$

where $M_r = \max_{0 \leq x \leq 1, 0 \leq s \leq r} \{f_1(x, s)\}$.

Proof. Since $f_i(x, u(x)) \leq M_r$ for $x \in [0, 1]$, it follows from (2.3) that

$$\begin{aligned} \|T_\lambda u\|_\infty &= \left\| \lambda \int_0^1 G(x, s) f_i(s, u(s)) ds \right\|_\infty \\ &\leq \lambda \int_0^1 G(s, s) f_i(s, u(s)) ds \\ &\leq \lambda M_r \int_0^1 G(s, s) ds. \end{aligned}$$

Lemma 3.3. Assume that (H1), (H2) and (A1)-(A4) hold. If $\sigma(r - \delta) \leq u(x) \leq r + \delta$, $r > \delta$, for $x \in [0, 1]$, then

$$\|T_\lambda u\|_\infty \geq \lambda m_r \int_0^1 G\left(\frac{1}{2}, s\right) ds,$$

where $m_r = \min_{0 \leq x \leq 1, \sigma(r-\delta) \leq s \leq r+\delta} \{f_i(x, s)\}$.

Proof. Since $f_i(x, u(x)) \geq m_r$ for $x \in [0, 1]$, it follows that

$$\begin{aligned} \|T_\lambda u\|_\infty &\geq \lambda \int_0^1 G\left(\frac{1}{2}, s\right) f_i(s, u(s)) ds \\ &\geq \lambda m_r \int_0^1 G\left(\frac{1}{2}, s\right) ds. \end{aligned}$$

Lemma 3.4. Assume that (H1), (H2), (H3) and (A1)-(A4) hold. then

(i) $(\lambda, u) \in C_{1,l}^+$ with $\|u\|_\infty \in (\xi_{2j-1} - \delta, \xi_{2j-1} + \delta)$ for some $j \in \mathbb{N}^*$ implies that $\lambda > 2$;

(ii) $(\lambda, u) \in C_{1,l}^+$, with $\|u\|_\infty \in (\zeta_{2j} - \delta, \zeta_{2j} + \delta)$ for some $j \in \mathbb{N}^*$ implies that $\lambda < \frac{1}{2}$.

Proof. (i) Assume that $(\lambda, u) \in C_{1,l}^+$ with $\|u\|_\infty \in (\xi_{2j-1} - \delta, \xi_{2j-1} + \delta)$ for some $j \in \mathbb{N}^*$, then $u = T_\lambda u$ and

$$0 \leq u(x) \leq \xi_{2j-1} + \delta \quad \text{for } x \in [0, 1].$$

By Lemma 3.2 and (H3), it follows that

$$\begin{aligned} \|u\|_\infty &= \|T_\lambda u\|_\infty \\ &\leq \lambda M_{\xi_{2j-1} + \delta} \int_0^1 G(s, s) ds \\ &= \lambda \max_{0 \leq x \leq 1, 0 \leq s \leq \xi_{2j-1} + \delta} \{f_i(x, s)\} \int_0^1 G(s, s) ds \\ &\leq \lambda \Phi(\xi_{2j-1}) \int_0^1 G(s, s) ds \\ &< \lambda \left[\frac{1}{2} \left(\int_0^1 G(s, s) ds \right)^{-1} (\xi_{2j-1} - \delta) \right] \int_0^1 G(s, s) ds. \\ &= \frac{1}{2} \lambda (\xi_{2j-1} - \delta). \end{aligned}$$

Thus $\lambda > 2$.

(ii) Assume that $(\lambda, u) \in C_{1,l}^+$ with $\|u\|_\infty \in (\xi_{2j} - \delta, \xi_{2j} + \delta)$ for some $j \in \mathbb{N}^*$, then $u = T_\lambda u$ and

$$\sigma(\xi_{2j} - \delta) \leq u(x) \leq \xi_{2j} + \delta \quad \text{for } x \in [0, 1].$$

By Lemma 3.3 and the assumption (H3), it follows that

$$\begin{aligned} \|u\|_\infty &= \|T_\lambda u\|_\infty \\ &\geq \lambda m_{\xi_{2j}} \int_0^1 G\left(\frac{1}{2}, s\right) ds \\ &= \lambda \min_{0 \leq x \leq 1, \sigma(\xi_{2j} - \delta) \leq s \leq \xi_{2j} + \delta} \{f_l(x, s)\} \int_0^1 G\left(\frac{1}{2}, s\right) ds \\ &= \lambda \Psi(\xi_{2j}) \int_0^1 G\left(\frac{1}{2}, s\right) ds \\ &> \lambda \left[2 \left(\int_0^1 G\left(\frac{1}{2}, s\right) ds \right)^{-1} (\xi_{2j} + \delta) \right] \int_0^1 G\left(\frac{1}{2}, s\right) ds. \\ &= 2\lambda(\xi_{2j} + \delta). \end{aligned}$$

Thus $\lambda < \frac{1}{2}$.

Lemma 3.5. *If $(\lambda, u) \in C_1^+$, then (λ, u) is a solution of (1.1) and $u \in W^{2,\infty}(0, 1)$.*

Proof. Let $(\lambda, u) \in C_1^+$. By the definition of C_1^+ there exists a sequence $\{l_k\} \in \mathbb{N}_0$ strictly increasing, and $(\lambda_{l_k}, v_{l_k}) \in [0, \infty) \times C^1[0, 1]$ with $(\lambda_{l_k}, v_{l_k}) \in C_{1,l_k}^+$ for $k \in \mathbb{N}$ and

$$(\lambda_{l_k}, v_{l_k}) \rightarrow (\lambda, u).$$

Since $\{f_{l_k}(\cdot, v_{l_k}(\cdot))\}$ is uniformly bounded, i.e.

$$\|f_{l_k}\|_{L^2} \leq M, \tag{3.7}$$

we can assume after passing to a subsequence, if necessary, that it converges weakly in $L^2(0, 1)$ to some φ . We claim that $\varphi(x) \in F(x, u(x))$ a.e. on $(0, 1)$.

Let $x_0 \in (0, 1)$ with $u(x_0) > 0$. Then there exist $\rho > 0$ and $\tau \in (0, \min\{x_0, 1-x_0\})$ with $u(x) > \rho$ for all $x \in (x_0 - \tau, x_0 + \tau)$, hence there is a $k_0 \in \mathbb{N}$ with $v_{l_k}(x) > \frac{\rho}{2}$ for all $k > k_0$ and $x \in (x_0 - \tau, x_0 + \tau)$. Choose $k_1 > k_0$ with $\frac{d_k}{l_{k_1}} < \frac{\rho}{2}$. Then $f_{l_k}(x, v_{l_k}(x)) = f^+(x, v_{l_k}(x))$ for all $k \geq k_1$ and $x \in (x_0 - \tau, x_0 + \tau)$, which yields $\varphi(x) = f^+(x, u(x))$ for $x \in (x_0 - \tau, x_0 + \tau)$ a.e.

Next, if $u \equiv 0$, let $K := \{x \in (0, 1) : \varphi(x) > f^+(x, 0)\}$. We claim that $\text{meas}(K) = 0$. Suppose that $\text{meas}(K) > 0$. Then $\varepsilon := \int_K [\varphi(x) - f^+(x, 0)] dx > 0$, and one finds $\eta \in (0, \infty)$ with $\text{meas}(K) |f^+(x, y) - f^+(x, 0)| \leq \frac{\varepsilon}{2}$ for $x \in [0, 1]$ and $y \in [0, \eta]$. Choosing $k_2 \in \mathbb{N}$ with $\|v_{l_k} - u\|_\infty < \eta$ for $k \geq k_2$. One obtains for $k \geq k_2$:

$$\begin{aligned}
 & \int_K [\phi(x) - f_{i_k}(x, v_{i_k}(x))] dx \\
 &= \int_K [\phi(x) - f^+(x, 0)] dx + \int_K [f^+(x, 0) - f_{i_k}(x, v_{i_k}(x))] dx \\
 &= \varepsilon + \int_K [f^+(x, 0) - f^+(x, v_{i_k}(x))] dx \\
 &\geq \frac{\varepsilon}{2},
 \end{aligned}$$

which contradicts $f_{i_k}(\cdot, v_{i_k}(\cdot)) \rightarrow \phi$. Thus, $\text{meas}(K) = 0$.

Now, let A be the closed linear operator in $L^2(0, 1)$ defined by

$$\text{dom}(A) := \{\varphi \in W^{2,2}[0, 1] : \varphi'(0) = 0 = \varphi'(1)\}$$

and $A\phi := -\phi'' + q\phi$. Clearly,

$$f_{i_k}(x, v_{i_k}(x)) \rightarrow \phi(x), \tag{3.8}$$

hence $v_{i_k} \rightarrow u$ and the fact that A is weakly closed yields

$$Au = \lambda\phi, \tag{3.9}$$

i.e.

$$Au \in \lambda F(\cdot, u(\cdot)) \text{ a.e.}$$

Finally, we show that $u \in W^{2,\infty}(0,1)$. In fact, from (3.9) we have

$$u''(x) = q(x)u(x) - \lambda\phi(x). \tag{3.10}$$

According to (H1) and the boundedness of u we have

$$qu \in L^\infty(0, 1). \tag{3.11}$$

We claim that $\phi \in L^\infty(0,1)$. Suppose on the contrary that there exists a set $E \subset [0, 1]$, $\text{meas}(E) > 0$ such that $|\phi|$ is unbounded on E . Without loss of generality, we assume that

$$|\phi(s)| > \frac{M\|w\|_{L^2} + 1 - \int_{[0,1] \setminus E} |\phi(x)| |w(x)| dx}{\int_E |w(x)| dx}, \quad s \in E, \tag{3.12}$$

where M is given by (3.7) and $w \in L^2(0,1)$. On the one hand, for k larger enough from (3.7), (3.8) and (H2) we have

$$\int_0^1 |\phi(x)| |w(x)| dx \leq \int_0^1 |f_{i_k}(x, v_{i_k}(x))w(x)| dx + 1 \leq M\|w\|_{L^2} + 1. \tag{3.13}$$

On the other hand, from (3.12) we have

$$\begin{aligned}
 & \int_0^1 |\phi(x)| |w(x)| dx \\
 &= \int_E |\phi(x)| |w(x)| dx + \int_{[0,1] \setminus E} |\phi(x)| |w(x)| dx \\
 &> \frac{M\|w\|_{L^2} + 1 - \int_{[0,1] \setminus E} |\phi(x)| |w(x)| dx}{\int_E |w(x)| dx} \int_E |w(x)| dx + \int_{[0,1] \setminus E} |\phi(x)| |w(x)| dx \\
 &= M\|w\|_{L^2} + 1,
 \end{aligned}$$

which contradicts (3.13). Thus,

$$\phi \in L^\infty(0, 1). \tag{3.14}$$

Therefore, from (3.10), (3.11) and (3.14) we obtain $u \in W^{2,\infty}(0,1)$.

Now we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1.

Assume that $(\lambda, u) \in C_1^+$. We divide the proof into two cases.

Case 1. If $\|u\|_\infty = \xi_{2j-1}$ for some $j \in \mathbb{N}^*$, then $\lambda \geq 2$.

Since $(\lambda, u) \in C_1^+$, there exists a sequence $(\lambda_{k_i}, z_{k_i}) \in C_{1,k_i}^+$, such that

$$\lim_{i \rightarrow \infty} \lambda_{k_i} = \lambda, \quad \lim_{i \rightarrow \infty} z_{k_i} = u.$$

Hence, for $\delta > 0$ there exists $i_0 \in \mathbb{N}$, such that

$$\|z_{k_i} - u\|_\infty < \delta, \quad i > i_0,$$

i.e.

$$\xi_{2j-1} - \delta < \|z_{k_i}\|_\infty < \xi_{2j-1} + \delta, \quad i > i_0.$$

By using Lemma 3.4, we obtain that

$$\lambda_{k_i} > 2, \quad i > i_0.$$

Hence, we get

$$\lambda = \lim_{i \rightarrow \infty} \lambda_{k_i} \geq 2.$$

Case 2. If $\|u\|_\infty = \xi_{2j}$ for some $j \in \mathbb{N}^*$, then $\lambda \leq \frac{1}{2}$.

Since $(\lambda, u) \in C_1^+$, there exists a sequence $(\lambda_{k_i}, z_{k_i}) \in C_{1,k_i}^+$, such that

$$\lim_{i \rightarrow \infty} \lambda_{k_i} = \lambda, \quad \lim_{i \rightarrow \infty} z_{k_i} = u.$$

Hence, for $\delta > 0$ there exists $i_0 \in \mathbb{N}$, such that

$$\|z_{k_i} - u\|_\infty < \delta, \quad i > i_0,$$

i.e.

$$\xi_{2j} - \delta < \|z_{k_i}\|_\infty < \xi_{2j} + \delta, \quad i > i_0.$$

By using lemma 3.4, we obtain that

$$\lambda_{k_i} < \frac{1}{2}, \quad i > i_0.$$

Hence, we get

$$\lambda = \lim_{i \rightarrow \infty} \lambda_{k_i} \leq \frac{1}{2}.$$

Corollary 3.1. *Assume that (H1)-(H3) hold. Then*

(i) *for each $\lambda \in (0, \frac{1}{2})$, (1.1) has at least one positive solution: $u_0 \in C_1^+$;*

(ii) for each $\lambda \in [\frac{1}{2}, 2]$, (1.1) has N positive solutions:

$$u_k, \quad k = 1, 2, \dots, N,$$

which satisfy that $u_k \in C_1^+$, $k = 1, 2, \dots, N$.

Proof. According to Theorem 3.1, the boundary value problem (1.1) has an unbounded component C_1^+ in Σ with $(0, \mathbf{0}) \in C_1^+$. Moreover,

(i) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = \xi_{2j-1}$ for some $j \in \mathbb{N}^*$ implies that $\lambda \geq 2$;

(ii) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = \xi_{2j}$ for some $j \in \mathbb{N}^*$ implies that $\lambda \leq \frac{1}{2}$.

From the facts $(0, \mathbf{0}) \in C_1^+$, and $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = \xi_1$ implies that $\lambda \geq 2$ and the connectivity of C_1^+ , we obtain

$$C_1^+ \cap \lambda \times C^1[0, 1] \neq \emptyset, \quad \forall \lambda \in \left(0, \frac{1}{2}\right),$$

which implies for each $\lambda \in (0, \frac{1}{2})$, (1.1) has at least one positive solution: $u_0 \in C_1^+$.

Let

$$C_1^{+,k} := \{(\lambda, u) \in C_1^+ | \xi_{k-1} \leq \|u\| < \xi_k\}, \quad k = 1, 2, \dots, N,$$

where $\xi_0 = 0$, ξ_k ($k = 1, 2, \dots, N$) is given by (H3). Then according to (i)(ii) and the connectivity of C_1^+ , we obtain

$$C_1^{+,k} \cap \lambda \times C^1[0, 1] \neq \emptyset, \quad \forall \lambda \in \left(\frac{1}{2}, 2\right), \quad k = 1, 2, \dots, N,$$

which implies for each $\lambda \in (\frac{1}{2}, 2)$, (1.1) has N positive solutions:

$$u_k, \quad k = 1, 2, \dots, N,$$

and $u_k \in C_1^{+,k} \subset C_1^+$, $k = 1, 2, \dots, N$.

4 Example

In this section, an example is given to illustrate the application of our main result (Theorem 3.1). Consider second order Neumann differential inclusion problem

$$\begin{cases} -u''(x) + u(x) \in \lambda F(u(x)), & \text{a.e. } x \in (0, 1), \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \quad (4.1)$$

where the set-valued function F in (4.1) is given by

$$F(y) = \begin{cases} \frac{49}{2}y - \frac{195}{2}, & y \geq 4, \\ \frac{1}{16}y + \frac{1}{4}, & 0 < y < 4, \\ [0, \frac{1}{4}], & y = 0. \end{cases}$$

Obviously, (H1), (H2) conditions of Theorem 3.1 are satisfied. Moreover, Green's function of the associated linear problem

$$\begin{cases} -u'' + u = 0, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

can be explicitly expressed by

$$G(x, s) = \frac{1}{2(e - e^{-1})} \begin{cases} (e^{x-1} + e^{1-x})(e^s + e^{-s}), & 0 \leq s \leq x \leq 1, \\ (e^x + e^{-x})(e^{s-1} + e^{1-s}), & 0 \leq x \leq s \leq 1. \end{cases}$$

By calculation we can get $\int_0^1 G(s, s) ds = \frac{e}{e-1}$, $\int_0^1 G(\frac{1}{2}, s) ds = 1$ and $\sigma = \frac{2}{e+e^{-1}}$.

Let $\zeta_1 = 3$, $\zeta_2 = 11$, $\delta = 1$, then we can check that $\zeta_1 = 3 < 5 < \sigma(\zeta_2 - \delta)$, and

$$\Phi(\xi_1) = \frac{1}{2} < \frac{3}{4} < 1 - e^{-2} = \frac{1}{2} \left(\int_0^1 G(s, s) ds \right)^{-1} (\xi_1 - \delta),$$

$$\Psi(\xi_2) \geq 25 > 24 = 2 \left(\int_0^1 G\left(\frac{1}{2}, s\right) ds \right)^{-1} (\xi_2 + \delta).$$

So that (H3) condition of Theorem 3.1 is satisfied. Therefore, according to Theorem 3.1 the differential inclusion problem (4.1) has an unbounded component C_1^+ in Σ with $(0, 0) \in C_1^+$. Moreover,

- (i) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = 3$ implies that $\lambda \geq 2$;
- (ii) $(\lambda, u) \in C_1^+$ with $\|u\|_\infty = 11$ implies that $\lambda \leq \frac{1}{2}$.

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Competing interests

The authors declare that they have no competing interests.

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