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# Dirichlet problem for the Schrödinger operator on a cone

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**Abstract**

In this article, a solution of the Dirichlet problem for the Schrödinger operator on a cone is constructed by the generalized Poisson integral with a slowly growing continuous boundary function. A solution of the Poisson integral for any continuous boundary function is also given explicitly by the Poisson integral with the generalized Poisson kernel depending on this boundary function.

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**Keywords:** Dirichlet problem; stationary Schrödinger equation; cone

## 1 Introduction and results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers respectively. We denote the $n$-dimensional Euclidean space by $\mathbf{R}^n$ ($n \geq 2$). A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, where $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points $P$ and $Q$ in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin $O$ of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$ respectively.

We introduce a system of spherical coordinates $(r, \Theta)$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to Cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega$, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote an open ball with a center at $P$ and radius $r$ in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain $\Omega$ on $\mathbf{S}^{n-1}$. We call it a cone. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$. We denote the $(n-1)$-dimensional volume elements induced by the Euclidean metric on $S_r$ by $dS_r$.

Let $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e., $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^b_{\mathrm{loc}}(C_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$\mathrm{Sch}_a\, u(P) = -\Delta u(P) + a(P)u(P) = 0, \tag{1.1}$$

where $P \in C_n(\Omega)$, $\Delta$ is the Laplace operator and $a \in \mathcal{A}_a$. These solutions called $a$-harmonic functions or generalized harmonic functions are associated with the operator $\mathrm{Sch}_a$. Note that they are (classical) harmonic functions in the case $a = 0$. Under these assumptions, the operator $\mathrm{Sch}_a$ can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [1–3]). We will denote it $\mathrm{Sch}_a$ as well. This last one has a Green's function $G(\Omega, a)(P, Q)$. Here $G(\Omega, a)(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G(\Omega, a)(P, Q)/\partial n_Q \geq 0$. We denote this derivative by $\mathbb{P}(\Omega, a)(P, Q)$, which is called the Poisson $a$-kernel with respect to $C_n(\Omega)$. We remark that $G(\Omega, 0)(P, Q)$ and $\mathbb{P}(\Omega, 0)(P, Q)$ are the Green's function and Poisson kernel of the Laplacian in $C_n(\Omega)$ respectively.

Given a domain $D \subset \mathbf{R}^n$ and a continuous function $u$ on $\partial(D)$, we say that $h$ is a solution of the Dirichlet problem for the Schrödinger operator on $D$ with $u$ if $\mathrm{Sch}_a h = 0$ in $D$ and

$$\lim_{P \in D, P \to Q} h(P) = u(Q)$$

for every $Q \in \partial(D)$. Note that $h$ is a solution of the classical Dirichlet problem for the Laplacian in the case $a = 0$.

Let $\Delta^*$ be a Laplace-Beltrami operator (the spherical part of the Laplace) on $\Omega \subset \mathbf{S}^{n-1}$ and $\lambda_j$ ($j = 1, 2, 3, \ldots, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$) be the eigenvalues of the eigenvalue problem for $\Delta^*$ on $\Omega$ (see, e.g., [4, p. 41])

$$\begin{aligned} &\Delta^* \varphi(\Theta) + \lambda \varphi(\Theta) = 0 \quad \text{in } \Omega, \\ &\varphi(\Theta) = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Corresponding eigenfunctions are denoted by $\varphi_{j\nu}$ ($1 \leq \nu \leq \nu_j$), where $\nu_j$ is the multiplicity of $\lambda_j$. We set $\lambda_0 = 0$, norm the eigenfunctions in $L^2(\Omega)$ and $\varphi_1 = \varphi_{11} > 0$. Then there exist two positive constants $d_1$ and $d_2$ such that

$$d_1 \delta(P) \leq \varphi_1(\Theta) \leq d_2 \delta(P) \tag{1.2}$$

for $P = (1, \Theta) \in \Omega$ (see Courant and Hilbert [5]), where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$.

In order to ensure the existences of $\lambda_j$ ($j = 1, 2, 3, \ldots$). We put a rather strong assumption on $\Omega$: if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see [6, pp. 88-89] for the definition of $C^{2,\alpha}$-domain). Then $\varphi_{j\nu} \in C^2(\overline{\Omega})$ ($j = 1, 2, 3, \ldots, 1 \leq \nu \leq \nu_j$) and $\partial \varphi_1 / \partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

Hence well-known estimates (see, e.g., [7, p. 14]) imply the following inequality:

$$\sum_{\nu=1}^{\nu_j} \varphi_{j\nu}(\Theta) \frac{\partial \varphi_{j\nu}(\Phi)}{\partial n_\Phi} \leq M(n) j^{2n-1}, \tag{1.3}$$

where the symbol $M(n)$ denotes a constant depending only on $n$.

Let $V_j(r)$ and $W_j(r)$ stand, respectively, for the increasing and nonincreasing, as $r \to +\infty$, solutions of the equation

$$-Q''(r) - \frac{n-1}{r} Q'(r) + \left( \frac{\lambda_j}{r^2} + a(r) \right) Q(r) = 0, \quad 0 < r < \infty, \tag{1.4}$$

normalized under the condition $V_j(1) = W_j(1) = 1$.

We shall also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists a finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$; moreover, $r^{-1}|r^2 a(r) - k| \in L(1,\infty)$. If $a \in \mathcal{B}_a$, then the solutions of Equation (1.1) are continuous (see [8]).

In the rest of the article, we assume that $a \in \mathcal{B}_a$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max(u,0)$, $u^- = -\min(u,0)$, $[d]$ is the integer part of $d$ and $d = [d] + \{d\}$, where $d$ is a positive real number.

Denote

$$\iota^{\pm}_{j,k} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda_j)}}{2} \quad (j = 0,1,2,3,\ldots).$$

It is known (see [9]) that in the case under consideration the solutions to Equation (1.4) have the asymptotics

$$V_j(r) \sim d_3 r^{\iota^+_{j,k}}, \qquad W_j(r) \sim d_4 r^{\iota^-_{j,k}}, \quad \text{as } r \to \infty, \tag{1.5}$$

where $d_3$ and $d_4$ are some positive constants.

If $a \in \mathcal{A}_a$, it is known that the following expansion for the Green function $G(\Omega,a)(P,Q)$ (see [10, Ch. 11], [1, 11])

$$G(\Omega,a)(P,Q) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} V_j\bigl(\min(r,t)\bigr) W_j\bigl(\max(r,t)\bigr) \left( \sum_{\nu=1}^{\nu_j} \varphi_{j\nu}(\Theta)\varphi_{j\nu}(\Phi) \right),$$

where $P = (r,\Theta)$, $Q = (t,\Phi)$, $r \neq t$ and $\chi'(s) = w(W_1(r), V_1(r))|_{r=s}$, is their Wronskian. The series converges uniformly if either $r \le st$ or $t \le sr$ $(0 < s < 1)$.

For a nonnegative integer $m$ and two points $P = (r,\Theta)$, $Q = (t,\Phi) \in C_n(\Omega)$, we put

$$K(\Omega,a,m)(P,Q) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ \widetilde{K}(\Omega,a,m)(P,Q) & \text{if } 1 \le t < \infty, \end{cases}$$

where

$$\widetilde{K}(\Omega,a,m)(P,Q) = \sum_{j=0}^{m} \frac{1}{\chi'(1)} V_j(r) W_j(t) \left( \sum_{\nu=1}^{\nu_j} \varphi_{j\nu}(\Theta)\varphi_{j\nu}(\Phi) \right).$$

We introduce another function of $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in C_n(\Omega)$

$$G(\Omega,a,m)(P,Q) = G(\Omega,a)(P,Q) - K(\Omega,a,m)(P,Q).$$

The generalized Poisson kernel $\mathbb{P}(\Omega,a,m)(P,Q)$ $(P = (r,\Theta) \in C_n(\Omega)$, $Q = (t,\Phi) \in S_n(\Omega))$ with respect to $C_n(\Omega)$ is defined by

$$\mathbb{P}(\Omega,a,m)(P,Q) = \frac{\partial G(\Omega,a,m)(P,Q)}{\partial n_Q}.$$

In fact,

$$\mathbb{P}(\Omega,a,0)(P,Q) = \mathbb{P}(\Omega,a)(P,Q).$$

We remark that the kernel function $\mathbb{P}(\Omega, 0, m)(P, Q)$ coincides with the one in Yoshida and Miyamoto [12] (see [10, Ch. 11]).

Put

$$U(\Omega, a, m; u)(P) = \int_{S_n(\Omega)} \mathbb{P}(\Omega, a, m)(P, Q)u(Q)\, d\sigma_Q,$$

where $u(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is a surface area element on $S_n(\Omega)$.

With regard to classical solutions of the Dirichlet problem for the Laplacian, Yoshida and Miyamoto [12, Theorem 1] proved the following result.

**Theorem A** *If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)} \frac{|u(t, \Phi)|}{1 + t^{\iota^+_{m+1,0}+n-1}}\, d\sigma_Q < \infty,$$

*then $U(\Omega, 0, m; u)(P)$ is a classical solution of the Dirichlet problem on $C_n(\Omega)$ with $g$ and satisfies*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,0}} U(\Omega, 0, m; u)(P) = 0.$$

Our first aim is to give growth properties at infinity for $U(\Omega, a, m; u)(P)$.

**Theorem 1** *Let $\gamma \geq 0$ (resp. $\gamma < 0$), $\iota^+_{[\gamma],k} + \{\gamma\} > -\iota^+_{1,k} + 1$ (resp. $-\iota^+_{[-\gamma],k} - \{-\gamma\} > -\iota^+_{1,k} + 1$) and*

$$\iota^+_{[\gamma],k} + \{\gamma\} - n + 1 \leq \iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 2$$
$$\left(\textit{resp. } s - \iota^+_{[-\gamma],k} - \{-\gamma\} - n + 1 \leq \iota^+_{m+1,k} < -\iota^+_{[-\gamma],k} - \{-\gamma\} - n + 2\right).$$

*If $u$ is a measurable function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)} \frac{|u(t, \Phi)|}{1 + t^{\iota^+_{[\gamma],k}+\{\gamma\}}}\, d\sigma_Q < \infty \quad \left(\textit{resp. } \int_{S_n(\Omega)} |u(t, \Phi)|\big(1 + t^{\iota^+_{[-\gamma],k}+\{-\gamma\}}]\big)\, d\sigma_Q < \infty\right), \tag{1.6}$$

*then*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{[\gamma],k}-\{\gamma\}+n-1} U(\Omega, a, m; u)(P) = 0 \tag{1.7}$$

$$\left(\textit{resp. } \lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{\iota^+_{[-\gamma],k}+\{-\gamma\}+n-1} U(\Omega, a, m; u)(P) = 0\right). \tag{1.8}$$

Next, we are concerned with solutions of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$.

**Theorem 2** *Let $\gamma$ and $\iota^+_{m+1,k}$ be as in Theorem 1. If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying (1.6), then $U(\Omega, a, m; u)(P)$ is a solution of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$ with $u$ and (1.7) (resp. (1.8)) holds.*

If we take $\iota^+_{[\gamma],k} + \{\gamma\} = \iota^+_{m+1,k} + n - 1$, then we immediately have the following corollary, which is just Theorem A in the case $a = 0$.

**Corollary** *If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|}{1+t^{\iota^+_{m+1,k}+n-1}}\,d\sigma_Q < \infty, \tag{1.9}$$

*then $U(\Omega,a,m;u)(P)$ is a solution of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$ with $u$ and satisfies*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} U(\Omega,a,m;u)(P) = 0. \tag{1.10}$$

By using Corollary, we can give a solution of the Dirichlet problem for any continuous function on $\partial C_n(\Omega)$.

**Theorem 3** *If $u$ is a continuous function on $\partial C_n(\Omega)$ satisfying (1.9) and $h(r,\Theta)$ is a solution of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$ with $u$ satisfying*

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} h^+(P) = 0, \tag{1.11}$$

*then*

$$h(P) = U(\Omega,a,m;u)(P) + \sum_{j=0}^{m}\left(\sum_{\nu=1}^{\nu_j} d_{j\nu}\varphi_{j\nu}(\Theta)\right)V_j(r),$$

*where $P=(r,\Theta)\in C_n(\Omega)$ and $d_{j\nu}$ are constants.*

## 2 Lemmas

Throughout this article, let $M$ denote various constants independent of the variables in questions, which may be different from line to line.

**Lemma 1**

$$\left|\mathbb{P}(\Omega,a)(P,Q)\right| \le M r^{\iota^-_{1,k}} t^{\iota^+_{1,k}-1} \tag{2.1}$$

$$\left(\textit{resp. } \left|\mathbb{P}(\Omega,a)(P,Q)\right| \le M r^{\iota^+_{1,k}} t^{\iota^-_{1,k}-1}\right) \tag{2.2}$$

*for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega)$ satisfying $0<\frac{t}{r}\le\frac{4}{5}$ (resp. $0<\frac{r}{t}\le\frac{4}{5}$);*

$$\left|\mathbb{P}(\Omega,0)(P,Q)\right| \le M\frac{1}{t^{n-1}} + M\frac{r}{|P-Q|^n} \tag{2.3}$$

*for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$.*

*Proof* (2.1) and (2.2) are obtained by Kheyfits (see [10, Ch. 11]). (2.3) follows from Azarin (see [13, Lemma 4 and Remark]). □

**Lemma 2** (see [1]) *For a nonnegative integer $m$, we have*

$$\left|\mathbb{P}(\Omega,a,m)(P,Q)\right| \le M(n,m,s)V_{m+1}(r)\frac{W_{m+1}(t)}{t}\varphi_1(\Theta)\frac{\partial\varphi_1(\Phi)}{\partial n_\Phi} \tag{2.4}$$

*for any $P = (r,\Theta) \in C_n(\Omega)$ and $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $r \leq st$ $(0 < s < 1)$, where $M(n,m,s)$ is a constant dependent of $n$, $m$ and $s$.*

**Lemma 3** (see [2, Theorem 1]) *If $u(r,\Theta)$ is a solution of Equation (1.1) on $C_n(\Omega)$ satisfying*

$$\int_\Omega u^+(r,\Theta)\,dS_1 = O\bigl(r^{\iota^+_{m,k}}\bigr), \quad \text{as } r \to \infty, \tag{2.5}$$

*then*

$$u(r,\Theta) = \sum_{j=0}^{m}\left(\sum_{\nu=1}^{\nu_j} d_{j\nu}\varphi_{j\nu}(\Theta)\right)V_j(r).$$

**Lemma 4** *Obviously, the conclusion of Lemma 3 holds true if (2.5) is replaced by*

$$\lim_{r\to\infty,(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}}u^+(r,\Theta) = 0. \tag{2.6}$$

*Proof* Since

$$V_{m+1}(r) \sim r^{\iota^+_{m+1,k}} \quad \text{as } r \to \infty$$

from (1.5) and

$$\iota^+_{m+1,k} \geq \iota^+_{m,k},$$

(2.6) gives that (2.5) holds, from which the conclusion immediately follows. □

## 3 Proof of Theorem 1

We only prove the case $\gamma \geq 0$, the remaining case $\gamma < 0$ can be proved similarly.

For any $\epsilon > 0$, there exists $R_\epsilon > 1$ such that

$$\int_{S_n(\Omega;(R_\epsilon,\infty))} \frac{|u(Q)|}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}}\,d\sigma_Q < \epsilon. \tag{3.1}$$

The relation $G(\Omega,a)(P,Q) \leq G(\Omega,0)(P,Q)$ implies this inequality (see [14])

$$\mathbb{P}(\Omega,a)(P,Q) \leq \mathbb{P}(\Omega,0)(P,Q). \tag{3.2}$$

For $0 < s < \frac{4}{5}$ and any fixed point $P = (r,\Theta) \in C_n(\Omega)$ satisfying $r > \frac{5}{4}R_\epsilon$, let $I_1 = S_n(\Omega;(0,1))$, $I_2 = S_n(\Omega;[1,R_\epsilon])$, $I_3 = S_n(\Omega;(R_\epsilon,\frac{4}{5}r])$, $I_4 = S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$, $I_5 = S_n(\Omega;[\frac{5}{4}r,\frac{r}{s}))$, $I_6 = S_n(\Omega;[1,\frac{r}{s}))$ and $I_7 = S_n(\Omega;[\frac{r}{s},\infty))$, we write

$$U(\Omega,a,m;u)(P) \leq \sum_{i=1}^{7} U_{\Omega,a,i}(P),$$

where

$$U_{\Omega,a,i}(P) = \int_{I_i} \bigl|\mathbb{P}(\Omega,a)(P,Q)\bigr|\bigl|u(Q)\bigr|\,d\sigma_Q \quad (i = 1,2,3,4,5),$$

$$U_{\Omega,a,6}(P) = \int_{I_6} \left|\mathbb{P}(\Omega,a,m)(P,Q)\right|\left|u(Q)\right| d\sigma_Q,$$
$$U_{\Omega,a,7}(P) = \int_{I_7} \left|\frac{\partial \widetilde{K}(\Omega,a,m)(P,Q)}{\partial n_Q}\right|\left|u(Q)\right| d\sigma_Q.$$

By $\iota^+_{[\gamma],k} + \{\gamma\} > -\iota^+_{1,k} + 1$, (1.6), (2.1) and (3.1), we have the following growth estimates

$$\begin{aligned} U_{\Omega,a,2}(P) &\le Mr^{\iota^-_{1,k}} \int_{I_2} t^{\iota^+_{1,k}-1}\left|u(Q)\right| d\sigma_Q \\ &\le Mr^{\iota^-_{1,k}} R_\epsilon^{\iota^+_{[\gamma],k}+\{\gamma\}+\iota^+_{1,k}-1}, \end{aligned} \tag{3.3}$$

$$U_{\Omega,a,1}(P) \le Mr^{\iota^-_{1,k}}, \tag{3.4}$$

$$U_{\Omega,a,3}(P) \le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}. \tag{3.5}$$

We obtain by $\iota^+_{m+1,k} \ge \iota^+_{[\gamma],k} + \{\gamma\} - n + 1$, (2.2) and (3.1)

$$\begin{aligned} U_{\Omega,a,5}(P) &\le Mr^{\iota^+_{1,k}} \int_{S_n(\Omega;[(5/4)r,\infty))} t^{\iota^-_{1,k}-1}\left|u(Q)\right| d\sigma_Q \\ &\le Mr^{\iota^+_{1,k}} \int_{S_n(\Omega;[(5/4)r,\infty))} t^{\iota^+_{[\gamma],k}+\{\gamma\}+\iota^-_{1,k}-1} \frac{|u(Q)|}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q \\ &\le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}. \end{aligned} \tag{3.6}$$

By (2.3) and (3.2), we consider the inequality

$$U_{\Omega,a,4}(P) \le U_{\Omega,0,4}(P) \le U'_{\Omega,0,4}(P) + U''_{\Omega,0,4}(P),$$

where

$$U'_{\Omega,0,4}(P) = M\int_{I_4} t^{1-n}\left|u(Q)\right| d\sigma_Q, \qquad U''_{\Omega,0,4}(P) = Mr\int_{I_4} \frac{|u(Q)|}{|P-Q|^n} d\sigma_Q.$$

We first have

$$\begin{aligned} U'_{\Omega,0,4}(P) &= M\int_{I_4} t^{\iota^+_{1,k}+\iota^-_{1,k}-1}\left|u(Q)\right| d\sigma_Q \\ &\le Mr^{\iota^+_{1,k}} \int_{S_n(\Omega;((4/5)r,\infty))} t^{\iota^-_{1,k}-1}\left|u(Q)\right| d\sigma_Q \\ &\le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}, \end{aligned} \tag{3.7}$$

which is similar to the estimate of $U_{\Omega,a,5}(P)$.

Next, we shall estimate $U''_{\Omega,0,4}(P)$. Take a sufficiently small positive number $d_5$ such that $I_4 \subset B(P, \frac{1}{2}r)$ for any $P = (r,\Theta) \in \Pi(d_5)$, where

$$\Pi(d_5) = \left\{P = (r,\Theta) \in C_n(\Omega); \inf_{z\in\partial\Omega}\left|(1,\Theta) - (1,z)\right| < d_5, 0 < r < \infty\right\}$$

and divide $C_n(\Omega)$ into two sets $\Pi(d_5)$ and $C_n(\Omega) - \Pi(d_5)$.

If $P = (r, \Theta) \in C_n(\Omega) - \Pi(d_5)$, then there exists a positive $d_5'$ such that $|P - Q| \geq d_5' r$ for any $Q \in S_n(\Omega)$, and hence

$$\begin{aligned} U''_{\Omega,0,4}(P) &\leq M \int_{I_4} t^{1-n} \bigl|u(Q)\bigr| \, d\sigma_Q \\ &\leq M\epsilon r^{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}, \end{aligned} \tag{3.8}$$

which is similar to the estimate of $U'_{\Omega,0,4}(P)$.

We shall consider the case $P = (r, \Theta) \in \Pi(d_5)$. Now put

$$H_i(P) = \bigl\{Q \in I_4; 2^{i-1}\delta(P) \leq |P - Q| < 2^i \delta(P)\bigr\}.$$

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P - Q| < \delta(P)\} = \varnothing$, we have

$$U''_{\Omega,0,4}(P) = M \sum_{i=1}^{i(P)} \int_{H_i(P)} r \frac{|u(Q)|}{|P - Q|^n} \, d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$.

Since we see from (1.2)

$$r\varphi_1(\Theta) \leq M\delta(P)$$

for $P = (r, \Theta) \in C_n(\Omega)$. Similar to the estimate of $U'_{\Omega,0,4}(P)$, we obtain

$$\begin{aligned} &\int_{H_i(P)} r \frac{|u(Q)|}{|P - Q|^n} \, d\sigma_Q \\ &\quad \leq \int_{H_i(P)} r \frac{|u(Q)|}{(2^{i-1}\delta(P))^n} \, d\sigma_Q \\ &\quad \leq M 2^{(1-i)n} \int_{H_i(P)} t^{1-n} |u(Q)| \, d\sigma_Q \\ &\quad \leq M\epsilon r^{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1} \end{aligned}$$

for $i = 0, 1, 2, \ldots, i(P)$.

So

$$U''_{\Omega,0,4}(P) \leq M\epsilon r^{\iota^+_{[\gamma],k} + \{\gamma\} - n + 1}. \tag{3.9}$$

We only consider $U_{\Omega,a,6}(P)$ in the case $m \geq 1$, since $U_{\Omega,a,6}(P) \equiv 0$ for $m = 0$. By the definition of $\widetilde{K}(\Omega, a, m)$, (1.3) and Lemma 2, we see

$$U_{\Omega,a,6}(P) \leq \frac{M}{\chi'(1)} \sum_{j=0}^{m} j^{2n-1} q_j(r),$$

where

$$q_j(r) = V_j(r) \int_{I_6} \frac{W_j(t)|u(Q)|}{t} \, d\sigma_Q.$$

To estimate $q_j(r)$, we write

$$q_j(r) \le q_j'(r) + q_j''(r),$$

where

$$q_j'(r) = V_j(r)\int_{I_2} \frac{W_j(t)|u(Q)|}{t}\,d\sigma_Q, \qquad q_j''(r) = V_j(r)\int_{S_n(\Omega;(R_\epsilon,r/s))} \frac{W_j(t)|u(Q)|}{t}\,d\sigma_Q.$$

Notice that

$$V_j(r)\frac{V_{m+1}(t)}{V_j(t)t} \le M\frac{V_{m+1}(r)}{r} \le Mr^{\iota^+_{m+1,k}-1} \quad \left(t \ge 1, R_\epsilon < \frac{r}{s}\right).$$

Thus, by $\iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 2$, (1.5) and (1.6) we conclude

$$\begin{aligned} q_j'(r) &= V_j(r)\int_{I_2} \frac{|u(Q)|}{V_j(t)t^{n-1}}\,d\sigma_Q \\ &\le MV_j(r)\int_{I_2} \frac{V_{m+1}(t)}{t^{\iota^+_{m+1,k}}}\frac{|u(Q)|}{V_j(t)t^{n-1}}\,d\sigma_Q \\ &\le Mr^{\iota^+_{m+1,k}-1}R_\epsilon^{\iota^+_{[\gamma],k}+\{\gamma\}-\iota^+_{m+1,k}-n+2}. \end{aligned}$$

Analogous to the estimate of $q_j'(r)$, we have

$$q_j''(r) \le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}.$$

Thus we can conclude that

$$q_j(r) \le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1},$$

which yields

$$U_{\Omega,a,6}(P) \le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}. \tag{3.10}$$

By $\iota^+_{m+1,k} \ge \iota^+_{[\gamma],k} + \{\gamma\} - n + 1$, (1.5), (2.4) and (3.1) we have

$$\begin{aligned} U_{\Omega,0,7}(P) &\le MV_{m+1}(r)\int_{I_7} \frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}\,d\sigma_Q \\ &\le M\epsilon r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}. \end{aligned} \tag{3.11}$$

Combining (3.3)–(3.11), we obtain that if $R_\epsilon$ is sufficiently large and $\epsilon$ is sufficiently small, then $U(\Omega,a,m;u)(P) = o(r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1})$ as $r \to \infty$, where $P = (r,\Theta) \in C_n(\Omega)$. Then we complete the proof of Theorem 1.

## 4 Proof of Theorem 2

For any fixed $P=(r,\Theta)\in C_n(\Omega)$, take a number satisfying $R>\max(1,\frac{r}{s})$ $(0<s<\frac{4}{5})$. By $\iota^+_{m+1,k}\geq \iota^+_{[\gamma],k}+\{\gamma\}-n+1$, (1.4), (1.6) and (2.4), we have

$$\begin{aligned}
&\int_{S_n(\Omega;(R,\infty))}\bigl|\mathbb{P}(\Omega,a,m)(P,Q)\bigr|\bigl|u(Q)\bigr|\,d\sigma_Q\\
&\leq MV_{m+1}(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R,\infty))}\frac{|u(Q)|}{t^{\iota^+_{m+1,k}+n-1}}\,d\sigma_Q\\
&\leq Mr^{\iota^+_{m+1,k}}\varphi_1(\Theta)\int_{S_n(\Omega;(r/s,\infty))}t^{\iota^+_{[\gamma],k}+\{\gamma\}-\iota^+_{m+1,k}-n+1}\frac{|u(Q)|}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}\,d\sigma_Q\\
&\leq Mr^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta)\int_{S_n(\Omega;(r/s,\infty))}\frac{|u(Q)|}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}\,d\sigma_Q\\
&\leq Mr^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta)\\
&<\infty.
\end{aligned}$$

Thus $U(\Omega,a,m;u)(P)$ is finite for any $P\in C_n(\Omega)$. Since $\mathbb{P}(\Omega,a,m)(P,Q)$ is a generalized harmonic function of $P\in C_n(\Omega)$ for any fixed $Q\in S_n(\Omega)$, $U(\Omega,a,m;u)(P)$ is also a generalized harmonic function of $P\in C_n(\Omega)$. That is to say, $U(\Omega,a,m;u)(P)$ is a solution of Equation (1.1) on $C_n(\Omega)$.

Now we study the boundary behavior of $U(\Omega,a,m;u)(P)$. Let $Q'=(t',\Phi')\in\partial C_n(\Omega)$ be any fixed point and $l$ be any positive number satisfying $l>\max(t'+1,\frac{4}{5}R)$.

Set $\chi_{S(l)}$ is a characteristic function of $S(l)=\{Q=(t,\Phi)\in\partial C_n(\Omega),t\leq l\}$ and write

$$U(\Omega,a,m;u)(P)=U'(P)-U''(P)+U'''(P),$$

where

$$\begin{aligned}
U'(P)&=\int_{S_n(\Omega;(0,(5/4)l])}\mathbb{P}(\Omega,a)(P,Q)u(Q)\,d\sigma_Q,\\
U''(P)&=\int_{S_n(\Omega;(1,(5/4)l])}\frac{\partial K(\Omega,a,m)(P,Q)}{\partial n_Q}u(Q)\,d\sigma_Q,\\
U'''(P)&=\int_{S_n(\Omega;((5/4)l,\infty))}\mathbb{P}(\Omega,a,m)(P,Q)u(Q)\,d\sigma_Q.
\end{aligned}$$

Notice that $U'(P)$ is the Poisson $a$-integral of $u(Q)\chi_{S((5/4)l)}$, we have $\lim_{P\to Q',P\in C_n(\Omega)}U'(P)=u(Q')$. Since $\lim_{\Theta\to\Phi'}\varphi_{j\nu}(\Theta)=0$ $(j=1,2,3,\ldots;\ 1\leq\nu\leq\nu_j)$ as $P=(r,\Theta)\to Q'=(t',\Phi')\in S_n(\Omega)$, we have $\lim_{P\to Q',P\in C_n(\Omega)}U''(P)=0$ from the definition of the kernel function $K(\Omega,a,m)(P,Q)$. $U'''(P)=O(r^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta))$, and therefore tends to zero.

So the function $U(\Omega,a,m;u)(P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q',P\in C_n(\Omega)}U(\Omega,a,m;u)(P)=u\bigl(Q'\bigr)$$

for any $Q'=(t',\Phi')\in\partial C_n(\Omega)$ from the arbitrariness of $l$. Thus we complete the proof of Theorem 2 from Theorem 1.

## 5 Proof of Theorem 3

From Corollary, we have the solution $U(\Omega, a, m; u)(P)$ of the Dirichlet problem on $C_n(\Omega)$ with $u$ satisfying (1.9). Consider the function $h(P) - U(\Omega, a, m; u)(P)$. Then it follows that this is the solution of Equation (1.1) in $C_n(\Omega)$ and vanishes continuously on $\partial C_n(\Omega)$.

Since

$$0 \le \left(h - U(\Omega, a, m; u)\right)^+(P) \le h^+(P) + \left(U(\Omega, a, m; u)\right)^-(P)$$

for any $P \in C_n(\Omega)$, we have

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} \left(h - U(\Omega, a, m; u)\right)^+(P) = 0$$

from (1.10) and (1.11). Then the conclusions of Theorem 3 follow immediately from Lemma 4.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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**References**

1. Kheyfits, A: Dirichlet problem for the Schrödinger operator in a half-space with boundary data of arbitrary growth at infinity. Diff. Integr. Equ. **10**, 153–164 (1997)
2. Kheyfits, A: Liouville theorems for generalized harmonic functions. Potential Anal. **16**, 93–101 (2002)
3. Reed, M, Simon, B: Methods of Modern Mathematical Physics, vol. 3. Academic Press, London (1970)
4. Rosenblum, G, Solomyak, M, Shubin, M: Spectral Theory of Differential Operators. VINITI, Moscow (1989)
5. Courant, R, Hilbert, D: Methods of Mathematical Physics, 1st English edn. Interscience, New York (1953)
6. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1977)
7. Muller, C: Spherical Harmonics. Lect. Notes in Math., vol. 17. Springer, Berlin (1966)
8. Simon, B: Schrödinger semigroups. Bull. Am. Math. Soc. **7**, 447–526 (1982)
9. Hartman, P: Ordinary Differential Equations. Wiley, New York (1964)
10. Escassut, A, Tutschke, W, Yang, CC: Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis. Science Press, Beijing (2008)
11. Kheyfits, A: Representation of the analytic functions of infinite order in a half-plane. Izv. Akad. Nauk Armjan SSR Ser. Mat. **6**(6), 472–476 (1971)
12. Yoshida, H, Miyamoto, I: Solutions of the Dirichlet problem on a cone with continuous data. J. Math. Soc. Jpn. **50**(1), 71–93 (1998)
13. Azarin, VS: Generalization of a theorem of Hayman on subharmonic functions in an *m*-dimensional cone. Trans. Am. Math. Soc. **80**(2), 119–138 (1969)
14. Ancona, A: First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains. J. Anal. Math. **72**, 45–92 (1997)

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