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Nontrivial solutions for a higher fractional differential equation with fractional multi-point boundary conditions

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Abstract
This paper investigates the existence and uniqueness of nontrivial solutions to a class of fractional nonlocal multi-point boundary value problems of higher order fractional differential equation, this kind of problems arise from viscoelasticity, electrochemistry control, porous media, electromagnetic and signal processing of wireless communication system. Some sufficient conditions for the existence and uniqueness of nontrivial solutions are established under certain suitable growth conditions, our proof is based on Leray-Schauder nonlinear alternative and Schauder fixed point theorem.

MSC: 34B15; 34B25

Keywords: fractional differential equation; nontrivial solution; Green function; Leray-Schauder nonlinear alternative

1 Introduction

The purpose of this paper is to establish the existence and uniqueness of nontrivial solutions to the following higher fractional differential equation:

$$\begin{cases} -\mathcal{D}^{\alpha}x(t) = f\big(t, x(t), \mathcal{D}^{\mu_1}x(t), \mathcal{D}^{\mu_2}x(t), \ldots, \mathcal{D}^{\mu_{n-1}}x(t)\big), & 0 < t < 1, \\ x(0) = 0, \qquad \mathcal{D}^{\mu_i}x(0) = 0, \qquad \mathcal{D}^{\mu}x(1) = \sum_{j=1}^{p-2} a_j \mathcal{D}^{\mu}x(\xi_j), & 1 \le i \le n-1, \end{cases} \tag{1.1}$$

where $n \ge 3$, $n \in \mathbb{N}$, $n-1 < \alpha \le n$, $n-l-1 < \alpha - \mu_l < n-l$, for $l = 1, 2, \ldots, n-2$, and $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \le 2$, $\alpha - \mu > 1$, $a_j \in [0, +\infty)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} \neq 1$, $\mathcal{D}^{\alpha}$ is the standard Riemann-Liouville derivative, and $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

Differential equations of fractional order occur more frequently in different research areas such as engineering, physics, chemistry, economics, etc. Indeed, we can find numerous applications in viscoelasticity, electrochemistry control, porous media, electromagnetic and signal processing of wireless communication system, etc. [1–6].

For an extensive collection of results about this type of equations, we refer the reader to the monograph by Kilbas et al. [7], Miller and Ross [8], Podlubny [9], the papers [10–24] and the references therein.

Recently, Salem [10] has investigated the existence of Pseudo solutions for the nonlinear m-point boundary value problem of a fractional type. In particular, he considered the following boundary value problem:

$$\begin{cases} \boldsymbol{\mathcal{D}}^{\alpha}x(t)+q(t)f\big(t,x(t)\big)=0, \quad 0<t<1, \alpha\in(n-1,n], n\geq 2, \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, \qquad x(1)=\sum_{i=1}^{m-2}\xi_i x(\eta_i), \end{cases} \tag{1.2}$$

where x takes values in a reflexive Banach space E, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$ and $\xi_i>0$ with $\sum_{j=1}^{m-2}\xi_j\eta_j^{\alpha-1}<1$. $x^{(k)}$ denotes the kth Pseudo-derivative of x and $\boldsymbol{\mathcal{D}}^{\alpha}$ denotes the Pseudo fractional differential operator of order α. By means of the fixed point theorem attributed to O'Regan, a criterion was established for the existence of at least one Pseudo solution for the problem (1.2).

More recently, Zhang [11] has considered the following problem whose nonlinear term and boundary condition contain integer order derivatives of unknown functions:

$$\begin{cases} \boldsymbol{\mathcal{D}}^{\alpha}x(t)+q(t)f\big(x,x',\ldots,x^{(n-2)}\big)=0, \quad 0<t<1, n-1<\alpha\leq n, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=x^{(n-2)}(1)=0, \end{cases} \tag{1.3}$$

where $\boldsymbol{\mathcal{D}}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α, q may be singular at $t=0$ and f may be singular at $x=0, x'=0,\ldots,x^{(n-2)}=0$. By using the fixed point theorem of a mixed monotone operator, a unique existence result of positive solution to the problem (1.3) was established. And then, Goodrich [12] was concerned with a partial extension of the problem (1.3) by extending boundary conditions

$$\begin{cases} -\boldsymbol{\mathcal{D}}^{\alpha}x(t)=f\big(t,x(t)\big), \quad 0<t<1, n-1<\alpha\leq n, n>3, \\ x^{(i)}(0)=0, \quad 0\leq i\leq n-2, \qquad \boldsymbol{\mathcal{D}}^{\alpha}x(1)=0, \quad 1\leq\alpha\leq n-2. \end{cases} \tag{1.4}$$

The author derived the Green's function for the problem (1.4) and showed that it satisfies certain properties. Then, by using cone theoretic techniques, a general existence theorem for (1.4) was obtained when $f(t,x)$ satisfies some growth conditions.

In recent work [13], Rehman and Khan have investigated the multi-point boundary value problems for fractional differential equations of the form

$$\begin{cases} \boldsymbol{\mathcal{D}}^{\alpha}y(t)=f\big(t,y(t),\boldsymbol{\mathcal{D}}^{\beta}y(t)\big), \quad t\in(0,1), \\ y(0)=0, \qquad \boldsymbol{\mathcal{D}}^{\beta}y(1)-\sum_{i=1}^{m-2}\zeta_i\boldsymbol{\mathcal{D}}^{\beta}y(\xi_i)=y_0, \end{cases} \tag{1.5}$$

where $1<\alpha\leq 2$, $0<\beta<1$, $0<\xi_i<1$, $\zeta_i\in[0,+\infty)$ with $\sum_{i=1}^{m-2}\zeta_i\xi_i^{\alpha-\beta-1}<1$. By using the Schauder fixed point theorem and the contraction mapping principle, the authors established the existence and uniqueness of nontrivial solutions for BVP (1.5) provided that the nonlinear function $f:[0,1]\times\mathbb{R}\times\mathbb{R}$ is continuous and satisfies certain growth conditions. However, Rehman and Khan only considered the case $1<\alpha\leq 2$ and the case of the nonlinear term f was not considered comprehensively.

Notice that the results dealing with the existence and uniqueness of solution for multi-point boundary value problems of fractional order differential equations are relatively scarce when the nonlinear term f and the boundary conditions all involve fractional derivatives of unknown functions. Thus, the aim of this paper is to establish the existence and uniqueness of nontrivial solutions for the higher nonlocal fractional differential equations (1.1) where nonlinear term f and the boundary conditions all involve fractional derivatives of unknown functions. In our study, the proof is based on the reduced order method as in [11] and the main tool is the Leray-Schauder nonlinear alternative and the Schauder fixed point theorem.

2 Basic definitions and preliminaries

Definition 2.1 A function x is said to be a solution of BVP (1.1) if $x \in C[0,1]$ and satisfies BVP (1.1). In addition, x is said to be a nontrivial solution if $x \not\equiv 0$ for $t \in (0,1)$ and x is solution of BVP (1.1).

For the convenience of the reader, we present some definitions, lemmas, and basic results that will be used later. These and other related results and their proofs can be found, for example, in [6–9].

Definition 2.2 (see [8]) Let $\alpha > 0$ with $\alpha \in \mathbb{R}$. Suppose that $x : [a,\infty) \to \mathbb{R}$ then the αth Riemann-Liouville fractional integral is defined by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\,ds$$

whenever the right-hand side is defined. Similarly, with $\alpha > 0$ with $\alpha \in \mathbb{R}$, we define the αth Riemann-Liouville fractional derivative to be

$$\boldsymbol{\mathcal{D}}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_a^t (t-s)^{n-\alpha-1}x(s)\,ds,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \le \alpha < n$ and $t > a$.

Remark 2.1 If $x, y : (0,+\infty) \to \mathbb{R}$ with order $\alpha > 0$, then

$$\boldsymbol{\mathcal{D}}^{\alpha}\big(x(t)+y(t)\big) = \boldsymbol{\mathcal{D}}^{\alpha}x(t) + \boldsymbol{\mathcal{D}}^{\alpha}y(t).$$

Lemma 2.1 (see [7])

(1) If $x \in L^1(0,1)$, $\rho > \sigma > 0$, then

$$I^{\rho}I^{\sigma}x(t) = I^{\rho+\sigma}x(t), \qquad \boldsymbol{\mathcal{D}}^{\sigma}I^{\rho}x(t) = I^{\rho-\sigma}x(t), \qquad \boldsymbol{\mathcal{D}}^{\sigma}I^{\sigma}x(t) = x(t).$$

(2) If $\rho > 0$, $\nu > 0$, then

$$\boldsymbol{\mathcal{D}}^{\rho}t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\rho)}t^{\nu-\rho-1}.$$

Lemma 2.2 (see [8]) *Assume that $x \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$, then $I^{\alpha}\boldsymbol{\mathcal{D}}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}$, where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$ ($n = [\alpha]+1$). Here I^{α} stands for the standard Riemann-Liouville fractional integral of order $\alpha > 0$ and $\boldsymbol{\mathcal{D}}^{\alpha}$ denotes the Riemann-Liouville fractional derivative as Definition 2.1.*

Lemma 2.3 *If* $1 < \alpha - \mu_{n-1} \le 2$, $\alpha - \mu > 1$ *and* $h \in L^1[0,1]$, *then the boundary value problem*

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}} w(t) = h(t), \\ w(0) = 0, \qquad \mathcal{D}^{\mu-\mu_{n-1}} w(1) = \sum_{j=1}^{p-2} a_j \mathcal{D}^{\mu-\mu_{n-1}} w(\xi_j), \end{cases} \tag{2.1}$$

has the unique solution

$$w(t) = \int_0^1 K(t,s)h(s)\,ds,$$

where $K(t,s)$ *is the Green function of BVP (2.1), and*

$$K(t,s) = k_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s), \tag{2.2}$$

$$\begin{aligned} k_1(t,s) &= \begin{cases} \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le t \le s \le 1, \end{cases} \\ k_2(t,s) &= \begin{cases} \dfrac{(t(1-s))^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le s \le t \le 1, \\ \dfrac{(t(1-s))^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, & 0 \le t \le s \le 1. \end{cases} \end{aligned} \tag{2.3}$$

Proof By applying Lemma 2.2, we may reduce (2.1) to an equivalent integral equation

$$w(t) = -I^{\alpha-\mu_{n-1}} h(t) + c_1 t^{\alpha-\mu_{n-1}-1} + c_2 t^{\alpha-\mu_{n-1}-2}, \quad c_1, c_2 \in \mathbb{R}. \tag{2.4}$$

Note that $w(0) = 0$ and (2.4), we have $c_2 = 0$. Consequently, a general solution of (2.3) is

$$w(t) = -I^{\alpha-\mu_{n-1}} h(t) + c_1 t^{\alpha-\mu_{n-1}-1}. \tag{2.5}$$

By (2.5) and Lemma 2.1, we have

$$\begin{aligned} \mathcal{D}^{\mu-\mu_{n-1}} w(t) &= -\mathcal{D}^{\mu-\mu_{n-1}} I^{\alpha-\mu_{n-1}} h(t) + c_1 \mathcal{D}^{\mu-\mu_{n-1}} t^{\alpha-\mu_{n-1}-1} \\ &= -I^{\alpha-\mu} h(t) + c_1 \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu)} t^{\alpha-\mu-1} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} h(s)\,ds + c_1 \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu)} t^{\alpha-\mu-1}. \end{aligned} \tag{2.6}$$

So, from (2.6), we have

$$\begin{aligned} \mathcal{D}^{\mu-\mu_{n-1}} w(1) &= -\int_0^1 \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} h(s)\,ds + c_1 \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu)}, \\ \mathcal{D}^{\mu-\mu_{n-1}} w(\xi_j) &= -\int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} h(s)\,ds + c_1 \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu)} \xi_j^{\alpha-\mu-1}, \end{aligned} \tag{2.7}$$

for $j = 1, 2, \ldots, p-2$.

By $\mathcal{D}^{\mu-\mu_{n-1}}w(1) = \sum_{j=1}^{p-2} a_j \mathcal{D}^{\mu-\mu_{n-1}}w(\xi_j)$, combining with (2.7), we obtain

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-\mu-1}h(s)\,ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu-1}h(s)\,ds}{\Gamma(\alpha-\mu_{n-1})(1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1})}.$$

So, substituting c_1 into (2.5), the unique solution of the problem (2.1) is

$$\begin{aligned}
w(t) &= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \\
&\quad \times \left\{ \int_0^1 \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds \right\} \\
&= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds + \frac{(1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu-1} + \sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu-1}) t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \\
&\quad \times \int_0^1 \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds - \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds \\
&= -\int_0^t \frac{(t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds + \int_0^1 \frac{(1-s)^{\alpha-\mu-1} t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds \\
&\quad + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j \int_0^1 \frac{(1-s)^{\alpha-\mu-1}\xi_j^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds \\
&\quad - \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} h(s)\,ds \\
&= \int_0^1 \left(k_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j,s) \right) h(s)\,ds \\
&= \int_0^1 K(t,s)h(s)\,ds.
\end{aligned}$$

The proof is completed. □

Lemma 2.4 $|K(t,s)| \le M(1-s)^{\alpha-\mu-1}$, *for* $t,s \in [0,1]$, *where*

$$M = \frac{1 + \frac{\sum_{j=1}^{p-2} a_j}{|1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}|}}{\Gamma(\alpha-\mu_{n-1})}. \tag{2.8}$$

Proof Obviously, for $t,s \in [0,1]$, we have $k_i(t,s) \le \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}$, $i = 1,2$. Thus

$$\begin{aligned}
|K(t,s)| &= \left| k_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j,s) \right| \\
&\le \frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} + \frac{\sum_{j=1}^{p-2} a_j (1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})|1-\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}|}
\end{aligned}$$

$$\leq \left(1+\frac{\sum_{j=1}^{p-2} a_j}{|1-\sum_{j=1}^{p-2} a_j\xi_j^{\alpha-\mu-1}|}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}.$$

This completes the proof. □

Now let us consider the following modified problem of BVP (1.1)

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}}v(t) = f\big(t, I^{\mu_{n-1}}v(t), I^{\mu_{n-1}-\mu_1}v(t),\ldots, I^{\mu_{n-1}-\mu_{n-2}}v(t), v(t)\big), \\ v(0)=0, \qquad \mathcal{D}^{\mu-\mu_{n-1}}v(1) = \sum_{j=1}^{p-2} a_j \mathcal{D}^{\mu-\mu_{n-1}}v(\xi_j). \end{cases} \tag{2.9}$$

Lemma 2.5 *Let $x(t) = I^{\mu_{n-1}}v(t)$, $v(t)\in C[0,1]$. Then (2.9) can be transformed into (1.1). Moreover, if $v\in C([0,1],\mathbb{R})$ is a solution of the problem (2.9), then the function $x(t) = I^{\mu_{n-1}}v(t)$ is a solution of the problem (1.1).*

Proof Substituting $x(t) = I^{\mu_{n-1}}v(t)$ into (1.1), by Lemmas 2.1 and 2.2, we can obtain that

$$\begin{aligned} \mathcal{D}^{\alpha}x(t) &= \frac{d^n}{dt^n}I^{n-\alpha}x(t) = \frac{d^n}{dt^n}I^{n-\alpha}I^{\mu_{n-1}}v(t) \\ &= \frac{d^n}{dt^n}I^{n-\alpha+\mu_{n-1}}v(t) = \mathcal{D}^{\alpha-\mu_{n-1}}v(t), \\ \mathcal{D}^{\mu_1}x(t) &= \mathcal{D}^{\mu_1}I^{\mu_{n-1}}v(t) = I^{\mu_{n-1}-\mu_1}v(t), \\ \mathcal{D}^{\mu_2}x(t) &= \mathcal{D}^{\mu_2}I^{\mu_{n-1}}v(t) = I^{\mu_{n-1}-\mu_2}v(t), \\ &\vdots \qquad\qquad\qquad \vdots \\ \mathcal{D}^{\mu_{n-2}}x(t) &= \mathcal{D}^{\mu_{n-2}}I^{\mu_{n-1}}v(t) = I^{\mu_{n-1}-\mu_{n-2}}v(t), \\ \mathcal{D}^{\mu_{n-1}}x(t) &= \mathcal{D}^{\mu_{n-1}}I^{\mu_{n-1}}v(t) = v(t), \end{aligned} \tag{2.10}$$

and also $\mathcal{D}^{\mu_{n-1}}x(0) = v(0) = 0$. It follows from $\mathcal{D}^{\mu}x(t) = \mathcal{D}^{\mu}I^{\mu_{n-1}}v(t) = \frac{d^n}{dt^n}I^{n-\mu}I^{\mu_{n-1}}v(t) = \mathcal{D}^{\mu-\mu_{n-1}}v(t)$ that $\mathcal{D}^{\mu-\mu_{n-1}}v(1) = \sum_{j=1}^{p-2} a_j\mathcal{D}^{\mu-\mu_{n-1}}v(\xi_j)$. Using $x(t) = I^{\mu_{n-1}}v(t)$, $v\in C[0,1]$, (2.9) is transformed into (1.1).

Now, let $v\in C([0,1],\mathbb{R})$ be a solution for the problem (2.9). Then, from Lemma 2.1, (2.9) and (2.10), one has

$$\begin{aligned} -\mathcal{D}^{\alpha}x(t) &= -\frac{d^n}{dt^n}I^{n-\alpha}x(t) = -\frac{d^n}{dt^n}I^{n-\alpha}I^{\mu_{n-1}}v(t) \\ &= -\frac{d^n}{dt^n}I^{n-\alpha+\mu_{n-1}}v(t) = -\mathcal{D}^{\alpha-\mu_{n-1}}v(t) \\ &= f\big(t, I^{\mu_{n-1}}v(t), I^{\mu_{n-1}-\mu_1}v(t),\ldots, I^{\mu_{n-1}-\mu_{n-2}}v(t), v(t)\big) \\ &= f\big(t, x(t), \mathcal{D}^{\mu_1}x(t), \mathcal{D}^{\mu_2}x(t),\ldots,\mathcal{D}^{\mu_{n-1}}x(t)\big), \quad 0<t<1. \end{aligned}$$

Notice

$$I^{\alpha}v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\,ds,$$

which implies that $I^{\alpha}v(0) = 0$. Thus from (2.10), for $i = 1, 2, \dots, n-1$, we have

$$x(0) = 0, \qquad \boldsymbol{\mathcal{D}}^{\mu_i}x(0) = 0, \qquad \boldsymbol{\mathcal{D}}^{\mu}x(1) = \sum_{j=1}^{p-2} a_j \boldsymbol{\mathcal{D}}^{\mu}x(\xi_j).$$

Moreover, it follows from the monotonicity and property of $I^{\mu_{n-1}}$ that

$$I^{\mu_{n-1}}v \in C\big([0,1], [0, +\infty)\big).$$

Consequently, $x(t) = I^{\mu_{n-1}}v(t)$ is a solution of the problem (1.1). □

Now let us define an operator $T : C[0,1] \to C[0,1]$ by

$$(Tv)(t) = \int_0^1 K(t,s)f\big(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s), \dots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s)\big)\,ds. \tag{2.11}$$

Clearly, the fixed point of the operator T is a solution of BVP (2.9); and consequently is also a solution of BVP (1.1) from Lemma 2.5.

Lemma 2.6 $T : C[0,1] \to C[0,1]$ *is a completely continuous operator.*

Proof Noticing that $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, by using the Ascoli-Arzela theorem and standard arguments, the result can easily be shown. □

Lemma 2.7 (see [25]) *Let X be a real Banach space, Ω be a bounded open subset of X, where $\theta \in \Omega$, $T : \overline{\Omega} \to X$ is a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3 Main results

For the convenience of expression in rest of the paper, we let $\mu_0 = 0$.

Theorem 3.1 *Suppose $f(t, 0, \dots, 0) \not\equiv 0$ for any $t \in [0,1]$. Moreover, there exist nonnegative functions $p_1, p_2, \dots, p_n, q \in L^1[0,1]$ such that*

$$\big|f(t, u_1, u_2, \dots, u_n)\big| \le \sum_{i=1}^{n} p_i(t)|u_i| + q(t), \quad \text{a.e. } (t, u_1, u_2, \dots, u_n) \in [0,1] \times \mathbb{R}^n, \tag{3.1}$$

and

$$M\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n} p_i(s)\,ds < \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}, \tag{3.2}$$

where M is defined by (2.8). Then BVP (1.1) has at least one nontrivial solution.

Proof Since $f(t, 0, \dots, 0) \not\equiv 0$, there exists $[\sigma, \tau] \in [0,1]$ such that

$$\min_{t \in [\sigma,\tau]} |f(t, 0, \dots, 0)| > 0.$$

By condition (3.1), we have $q(t) \geq |f(t,0,\ldots,0)|$, a.e. $t \in [0,1]$, thus

$$\int_0^1 (1-s)^{\alpha-\mu-1} q(s)\,ds > 0.$$

On the other hand, from (3.2), we know

$$\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right) M \int_0^1 (1-s)^{\alpha-\mu-1} \sum_{i=1}^{n} p_i(s)\,ds < 1.$$

Take

$$r = \frac{M\int_0^1 (1-s)^{\alpha-\mu-1} q(s)\,ds}{1 - (1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1}-\mu_i)}) M \int_0^1 (1-s)^{\alpha-\mu-1} \sum_{i=1}^{n} p_i(s)\,ds}$$

then $r > 0$.

Now let $\Omega_r = \{v \in C[0,1] : \|x\| < r\}$, suppose $v \in \partial\Omega_r$, $\lambda > 1$ such that $Tv = \lambda v$. Then

$$\begin{aligned}
\lambda r &= \lambda\|v\| = \|Tv\| = \max_{t\in[0,1]} \bigl|Tv(t)\bigr| \\
&\leq M \int_0^1 (1-s)^{\alpha-\mu-1} f\bigl(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s)\bigr)\,ds.
\end{aligned} \tag{3.3}$$

Moreover, for $i = 0,1,2,\ldots,n-2$,

$$\bigl|I^{\mu_{n-1}-\mu_i} v(t)\bigr| = \left|\int_0^t \frac{(t-s)^{\mu_{n-1}-\mu_i-1} v(s)}{\Gamma(\mu_{n-1}-\mu_i)}\,ds\right| \leq \frac{\|v\|}{\Gamma(\mu_{n-1}-\mu_i)},$$

thus we have, by hypothesis (3.1),

$$\begin{aligned}
&\bigl|f\bigl(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s)\bigr)\bigr| \\
&\quad\leq p_1(s)\bigl|I^{\mu_{n-1}}v(s)\bigr| + p_2(s)\bigl|I^{\mu_{n-1}-\mu_1}v(s)\bigr| + \cdots \\
&\qquad + p_{n-1}(s)\bigl|I^{\mu_{n-1}-\mu_{n-2}}v(s)\bigr| + p_n(s)\bigl|v(s)\bigr| + q(s) \\
&\quad\leq \frac{\|v\|}{\Gamma(\mu_{n-1})} p_1(s) + \frac{\|v\|}{\Gamma(\mu_{n-1}-\mu_1)} p_2(s) + \cdots \\
&\qquad + \frac{\|v\|}{\Gamma(\mu_{n-1}-\mu_{n-2})} p_{n-1}(s) + \|v\| p_n(s) + q(s) \\
&\quad\leq \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right) \|v\| \bigl[p_1(s) + p_2(s) + \cdots + p_{n-1}(s) + p_n(s)\bigr] + q(s).
\end{aligned}$$

Consequently, from (3.3), we have

$$\begin{aligned}
\lambda r &\leq \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right) M \int_0^1 (1-s)^{\alpha-\mu-1} \sum_{i=1}^{n} p_i(s)\,ds \|v\| \\
&\quad + M \int_0^1 (1-s)^{\alpha-\mu-1} q(s)\,ds
\end{aligned}$$

$$= r\left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)M\int_0^1(1-s)^{\alpha-\mu-1}\sum_{i=1}^{n}p_i(s)\,ds + M\int_0^1(1-s)^{\alpha-\mu-1}q(s)\,ds.$$

Therefore,

$$\begin{aligned}\lambda &\le \left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)M\int_0^1(1-s)^{\alpha-\mu-1}\sum_{i=1}^{n}p_i(s)\,ds + \frac{M\int_0^1(1-s)^{\alpha-\mu-1}q(s)\,ds}{r}\\ &= 1.\end{aligned}$$

This contradicts $\lambda > 1$. By Lemma 2.7, T has a fixed point $v^* \in \overline{\Omega}$, since $f(t,0,\ldots,0)\not\equiv 0$; so then, by Lemma 2.5, BVP (1.1) has a nontrivial solution v^*. This completes the proof. □

Theorem 3.2 *Suppose $f(t,0,\ldots,0)\not\equiv 0$ for any $t\in[0,1]$. Moreover, there exist nonnegative functions $p_1,p_2,\ldots,p_n,q\in L^1[0,1]$ such that*

$$\left|f(t,u_1,u_2,\ldots,u_n)\right| \le \sum_{i=1}^{n}p_i(t)|u_i|^{\sigma_i}+q(t),\quad \text{a.e. } (t,u_1,u_2,\ldots,u_n)\in[0,1]\times\mathbb{R}^n, \tag{3.4}$$

where $0<\sigma_1,\sigma_2,\ldots,\sigma_n<1$ are nonnegative constants. Then BVP (1.1) has at least one nontrivial solution.

Proof By Lemma 2.6, we know $T : C[0,1]\to C[0,1]$ is a completely continuous operator.
Let

$$\begin{aligned}a &= \left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma^{\sigma_{i+1}}(\mu_{n-1}-\mu_i)}\right)M\int_0^1(1-s)^{\alpha-\mu-1}\sum_{i=1}^{n}p_i(s)\,ds,\\ b &= M\int_0^1(1-s)^{\alpha-\mu-1}q(s)\,ds.\end{aligned}$$

Choose

$$R \ge \left\{(n+1)b,\left[(n+1)a\right]^{\frac{1}{1-\sigma_1}},\left[(n+1)a\right]^{\frac{1}{1-\sigma_2}},\ldots,\left[(n+1)a\right]^{\frac{1}{1-\sigma_n}}\right\}$$

and define a ball $\mathfrak{M}=\{v\in C[0,1]:\|v\|\le R, t\in[0,1]\}$. For every $v\in\mathfrak{M}$, we have

$$\begin{aligned}\left|Tv(t)\right| &\le \int_0^1 K(t,s)\left|f\left(s,I^{\mu_{n-1}}v(s),I^{\mu_{n-1}-\mu_1}v(s),\ldots,I^{\mu_{n-1}-\mu_{n-2}}v(s),v(s)\right)\right|ds\\ &\le M\int_0^1(1-s)^{\alpha-\mu-1}\left|f\left(s,I^{\mu_{n-1}}v(s),I^{\mu_{n-1}-\mu_1}v(s),\ldots,I^{\mu_{n-1}-\mu_{n-2}}v(s),v(s)\right)\right|ds.\end{aligned}$$

On the other hand, it follows from (3.4) that

$$\begin{aligned}
&\left|f\left(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s), \ldots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s)\right)\right| \\
&\quad\le p_1(s)\left|I^{\mu_{n-1}}v(s)\right|^{\sigma_1} + p_2(s)\left|I^{\mu_{n-1}-\mu_1}v(s)\right|^{\sigma_2} + \cdots \\
&\qquad + p_{n-1}(s)\left|I^{\mu_{n-1}-\mu_{n-2}}v(s)\right|^{\sigma_{n-1}} + p_n(s)\left|v(s)\right|^{\sigma_n} + q(s) \\
&\quad\le \frac{\|v\|^{\sigma_1}}{\Gamma^{\sigma_1}(\mu_{n-1})}p_1(s) + \frac{\|v\|^{\sigma_2}}{\Gamma^{\sigma_2}(\mu_{n-1}-\mu_1)}p_2(s) + \cdots \\
&\qquad + \frac{\|v\|^{\sigma_{n-1}}}{\Gamma^{\sigma_{n-1}}(\mu_{n-1}-\mu_{n-2})}p_{n-1}(s) + \|v\|^{\sigma_n}p_n(s) + q(s) \\
&\quad\le \left(\|v\|^{\sigma_n} + \sum_{i=0}^{n-2}\frac{\|v\|^{\sigma_{i+1}}}{\Gamma^{\sigma_{i+1}}(\mu_{n-1}-\mu_i)}\right)\left[p_1(s) + p_2(s) + \cdots + p_{n-1}(s) + p_n(s)\right] + q(s) \\
&\quad\le \left(1 + \sum_{i=0}^{n-2}\frac{1}{\Gamma^{\sigma_{i+1}}(\mu_{n-1}-\mu_i)}\right)\sum_{i=1}^{n}\|v\|^{\sigma_i}\sum_{i=1}^{n}p_i(s) + q(s).
\end{aligned} \tag{3.5}$$

In view of (3.5), we have the following estimate:

$$\begin{aligned}
\left|Tv(t)\right| &\le \left(1 + \sum_{i=0}^{n-2}\frac{1}{\Gamma^{\sigma_{i+1}}(\mu_{n-1}-\mu_i)}\right)M\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n}p_i(s)\,ds\sum_{i=1}^{n}\|v\|^{\sigma_i} \\
&\quad + M\int_0^1 (1-s)^{\alpha-\mu-1}q(s)\,ds \\
&= a\sum_{i=1}^{n}\|v\|^{\sigma_i} + b \le \frac{nR}{n+1} + \frac{R}{n+1} = R.
\end{aligned}$$

Therefore, $\|Tv\| \le R$. Thus we have $T : \mathfrak{M} \to \mathfrak{M}$. Hence the Schauder fixed point theorem implies the existence of a solution in $\mathfrak{M}$ for BVP (2.9). Since $f(t,0,\ldots,0) \not\equiv 0$, then by Lemma 2.5, BVP (1.1) has a nontrivial solution v^*. This completes the proof. □

Theorem 3.3 *Suppose $f(t,0,\ldots,0) \not\equiv 0$ for any $t \in [0,1]$. Moreover, there exist nonnegative functions $p_1, p_2, \ldots, p_n, q \in L^1[0,1]$ such that*

$$\left|f(t,u_1,u_2,\ldots,u_n)\right| \le \sum_{i=1}^{n}p_i(t)|u_i|^{\sigma_i} + q(t), \quad \text{a.e. } (t,u_1,u_2,\ldots,u_n) \in [0,1]\times\mathbb{R}^n, \tag{3.6}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n > 1$ are nonnegative constants. Then BVP (1.1) has at least one nontrivial solution.

Proof The proof is similar to that of Theorem 3.2, so it is omitted. □

Remark 3.1 In [13], the authors studied the cases $1 < \alpha \le 2$, $\mu_1 = \mu_2 = \cdots = \mu_{n-1} = \beta$, $0 < \beta < 1$, but the case of $\sigma_i = 1$, $i = 1,2,\ldots,n$ was not considered. Here we extend the results of [13] and fill the case $\sigma_i = 1$, $i = 1,2,\ldots,n$.

Theorem 3.4 *Suppose $f(t,0,\dots,0) \not\equiv 0$ for any $t \in [0,1]$. Moreover, there exist nonnegative functions $p_1, p_2, \dots, p_n \in L^1[0,1]$ such that*

$$\begin{aligned} &\bigl|f(t,u_1,u_2,\dots,u_n) - f(t,v_1,v_2,\dots,v_n)\bigr| \le \sum_{i=1}^{n} p_i(t)|u_i - v_i|, \\ &\qquad \text{a.e. } (t,u_1,u_2,\dots,u_n),(t,v_1,v_2,\dots,v_n) \in [0,1]\times\mathbb{R}^n, \end{aligned} \tag{3.7}$$

and (3.2) holds. Then BVP (1.1) has a unique nontrivial solution.

Proof In fact, if $v_1 = v_2 = \cdots = v_n \equiv 0$, then we have

$$\bigl|f(t,u_1,u_2,\dots,u_n)\bigr| \le \sum_{i=1}^{n} p_i(t)|u_i| + \bigl|f(t,0,0,\dots,0)\bigr|.$$

From Theorem 3.1, we know BVP (1.1) has a nontrivial solution.

But in this case, we prefer to concentrate on the uniqueness of a nontrivial solution for BVP (1.1). Let T be given in (2.11), we shall show that T is a contraction. In fact, by (3.7), a similar method to Theorem 3.1, we have

$$\begin{aligned} &\bigl|f\bigl(s, I^{\mu_{n-1}}u(s), I^{\mu_{n-1}-\mu_1}u(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\bigr) \\ &\quad - f\bigl(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}v(s),v(s)\bigr)\bigr| \\ &\quad \le \left(1 + \sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)\|u-v\|\sum_{i=1}^{n} p_i(s). \end{aligned}$$

And then

$$\begin{aligned} \|Tu - Tv\| &\le M\int_0^1 (1-s)^{\alpha-\mu-1}\bigl|f\bigl(s, I^{\mu_{n-1}}u(s), I^{\mu_{n-1}-\mu_1}u(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}u(s),u(s)\bigr) \\ &\qquad - f\bigl(s, I^{\mu_{n-1}}v(s), I^{\mu_{n-1}-\mu_1}v(s),\dots,I^{\mu_{n-1}-\mu_{n-2}}v(s),v(s)\bigr)\bigr|\,ds \\ &\le \left(1 + \sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)M\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n} p_i(s)\,ds\|u-v\|. \end{aligned}$$

Then (3.2) implies that T is indeed a contraction. Finally, we use the Banach fixed point theorem to deduce the existence of a unique nontrivial solution to BVP (1.1). □

Corollary 3.1 *Suppose $f(t,0,\dots,0) \not\equiv 0$ for any $t \in [0,1]$, and (3.1) holds. Then BVP (1.1) has at least one nontrivial solution if one of the following conditions holds*

(1) There exists a constant $p > 1$ such that

$$\int_0^1 \left[\sum_{i=1}^{n} p_i(s)\right]^p ds < \left(\frac{p(\alpha-\mu-1)}{p-1} + 1\right)^{p-1}\left(M + \sum_{i=0}^{n-2}\frac{M}{\Gamma(\mu_{n-1}-\mu_i)}\right)^{-p}. \tag{3.8}$$

(2) There exists a constant $\lambda > -1$ such that

$$\sum_{i=1}^{n} p_i(s) < \frac{\Gamma(\alpha+\lambda-\mu-1)}{\Gamma(\alpha-\mu)\Gamma(\lambda+1)}\left(M + \sum_{i=0}^{n-2}\frac{M}{\Gamma(\mu_{n-1}-\mu_i)}\right)^{-1} s^{\lambda}. \tag{3.9}$$

(3) *There exists a constant* $\lambda > -1$ *such that*

$$\sum_{i=1}^{n} p_i(s) < (\alpha + \lambda - \mu)\left(M + \sum_{i=0}^{n-2} \frac{M}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} (1-s)^{\lambda}. \tag{3.10}$$

(4) $p_i(s)$ $(i = 1, 2, \ldots, n)$ *satisfy*

$$\sum_{i=1}^{n} p_i(s) < (\alpha - \mu)\left(M + \sum_{i=0}^{n-2} \frac{M}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}. \tag{3.11}$$

Proof Let

$$R = M \int_0^1 (1-s)^{\alpha-\mu-1} \sum_{i=1}^{n} p_i(s)\, ds.$$

From the proof of Theorem 3.1, we only need to prove

$$R < \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}.$$

(1) If (3.8) holds, let $\frac{1}{p} + \frac{1}{q} = 1$, and by using Hölder inequality,

$$\begin{aligned}
R &\le M\left(\int_0^1 \left[\sum_{i=1}^{n} p_i(s)\right]^p ds\right)^{\frac{1}{p}} \left(\int_0^1 (1-s)^{q(\alpha-\mu-1)}\, ds\right)^{\frac{1}{q}} \\
&= M\big[q(\alpha - \mu - 1) + 1\big]^{-\frac{1}{q}} \left(\int_0^1 \left[\sum_{i=1}^{n} p_i(s)\right]^p ds\right)^{\frac{1}{p}} \\
&= M\left[\frac{p(\alpha - \mu - 1)}{p-1} + 1\right]^{-\frac{p-1}{p}} \left(\int_0^1 \left[\sum_{i=1}^{n} p_i(s)\right]^p ds\right)^{\frac{1}{p}} \\
&< \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}.
\end{aligned}$$

(2) In this case, it follows from (3.9) that

$$\begin{aligned}
R &< M\frac{\Gamma(\alpha + \lambda - \mu - 1)}{\Gamma(\alpha - \mu)\Gamma(\lambda + 1)}\left(M + \sum_{i=0}^{n-2} \frac{M}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} \int_0^1 (1-s)^{\alpha-\mu-1} s^{\lambda}\, ds \\
&= \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}.
\end{aligned}$$

(3) In this case, it follows from (3.10) that

$$R < M(\alpha + \lambda - \mu)\left(M + \sum_{i=0}^{n-2} \frac{M}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} \int_0^1 (1-s)^{\alpha-\mu-1} (1-s)^{\lambda}\, ds$$

$$= \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}.$$

(4) If (3.11) is satisfied, we have

$$\begin{aligned} R &< M(\alpha - \mu)\left(M + \sum_{i=0}^{n-2} \frac{M}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} \int_0^1 (1-s)^{\alpha-\mu-1}\,ds \\ &= \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}. \end{aligned}$$

This completes the proof of Corollary 3.1. □

Corollary 3.2 *Suppose* $f(t, 0, \ldots, 0) \not\equiv 0$ *for any* $t \in [0,1]$. *Moreover,*

$$0 \le \limsup_{\sum_{i=1}^n |u_i| \to +\infty} \max_{t\in[0,1]} \frac{|f(t, u_1, u_2, \ldots, u_n)|}{\sum_{i=1}^n |u_i|} < \frac{\alpha - \mu}{M}\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}. \tag{3.12}$$

Then BVP (1.1) has at least one nontrivial solution.

Proof Take $\epsilon > 0$ such that

$$\frac{\alpha - \mu}{M}\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} - \epsilon > 0,$$

by (3.12), there exists a large enough constant $R_0 > 0$ such that for any $t \in [0,1]$, $\sum_{i=1}^n |u_i| \ge R_0$, one has

$$|f(t, u_1, u_2, \ldots, u_n)| \le \left(\frac{\alpha - \mu}{M}\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} - \epsilon\right) \sum_{i=1}^n |u_i|.$$

Let

$$\hbar = \max_{t\in[0,1], \sum_{i=1}^n |u_i| \le R_0} \left|f(t, u_1, u_2, \ldots, u_n)\right|.$$

Then for any $(t, u_1, u_2, \ldots, u_n) \in [0,1] \times \mathbb{R}^n$, we have

$$\left|f(t, u_1, u_2, \ldots, u_n)\right| \le \hbar + \left(\frac{\alpha - \mu}{M}\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} - \epsilon\right) \sum_{i=1}^n |u_i|.$$

Let

$$\sum_{i=1}^n p_i(s) = \left(\frac{\alpha - \mu}{M}\left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1} - \epsilon\right), \quad q(s) = \hbar,$$

we prove

$$R < \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(\mu_{n-1} - \mu_i)}\right)^{-1}.$$

In fact,

$$\begin{aligned}
R &= M\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n} p_i(s)\,ds \\
&\le M\left(\frac{\alpha-\mu}{M}\left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)^{-1}-\epsilon\right)\int_0^1 (1-s)^{\alpha-\mu-1}\,ds \\
&< \left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)^{-1}.
\end{aligned}$$

Then it follows from Theorem 3.1 that BVP (1.1) has at least one nontrivial solution. □

4 Examples

Example 4.1 Consider the boundary value problem

$$\begin{cases}
-\boldsymbol{\mathcal{D}}^{\frac{5}{2}}x(t) = \dfrac{t\sin x(t)}{100\sqrt{\pi+|x(t)|}} - \dfrac{\boldsymbol{\mathcal{D}}^{\frac{9}{8}}x(t)}{10\sqrt{2+|\boldsymbol{\mathcal{D}}^{\frac{5}{4}}x(t)|}} \\
\qquad\qquad + \dfrac{(1+t^2)\boldsymbol{\mathcal{D}}^{\frac{5}{4}}x(t)}{100} + t^{\frac{3}{2}} + \cos t, \quad t\in(0,1), \\
x(0) = \boldsymbol{\mathcal{D}}^{\frac{9}{8}}x(0) = \boldsymbol{\mathcal{D}}^{\frac{5}{4}}x(0) = 0, \qquad \boldsymbol{\mathcal{D}}^{\frac{11}{8}}x(1) = \dfrac{\sqrt{2}}{2}\boldsymbol{\mathcal{D}}^{\frac{11}{8}}\left(\dfrac{1}{4}\right) + \dfrac{1}{5}\boldsymbol{\mathcal{D}}^{\frac{11}{8}}\left(\dfrac{1}{2}\right).
\end{cases} \tag{4.1}$$

Proof Let $\alpha = \frac{5}{2}$, $\mu_1 = \frac{9}{8}$, $\mu_2 = \frac{5}{4}$, $\mu = \frac{11}{8}$, and set

$$\begin{aligned}
&f(t,u_1,u_2,u_3) = \frac{t\sin u_1}{100\sqrt{\pi+|u_1|}} - \frac{u_2}{10\sqrt{2+|u_3|}} + \frac{(1+t^2)u_3}{100} + t^{\frac{3}{2}} + \cos t, \\
&p_1(t) = \frac{t}{100\sqrt{\pi}}, \qquad p_2(t) = \frac{1}{10\sqrt{2}}, \qquad p_3(t) = \frac{1+t^2}{100}, \qquad q(t) = t^{\frac{3}{2}} + \cos t.
\end{aligned}$$

Then

$$\bigl|f(t,u_1,u_2,u_3)\bigr| \le p_1(t)|u_1| + p_2(t)|u_2| + p_3(t)|u_3| + q(t),$$

and

$$\begin{aligned}
&\left(1+\sum_{i=0}^{n-2}\frac{1}{\Gamma(\mu_{n-1}-\mu_i)}\right)^{-1} = \left(\frac{1}{\Gamma(\frac{5}{4})}+\frac{1}{\Gamma(\frac{1}{8})}+1\right)^{-1} \approx 0.4472, \\
&M = \frac{1+\frac{\sum_{j=1}^{p-2}a_j}{|1-\sum_{j=1}^{p-2}a_j\xi_j^{\alpha-\mu-1}|}}{\Gamma(\alpha-\mu_{n-1})} \approx 2.3934.
\end{aligned}$$

Thus we have

$$\begin{aligned}
M\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n} p_i(s)\,ds &= 2.3934\int_0^1 (1-s)^{\alpha-\mu-1}\sum_{i=1}^{n} p_i(s)\,ds \\
&\approx 0.07679 \times 2.3934 \approx 0.1838 < 0.4472.
\end{aligned}$$

Thus the condition (3.2) in Theorem 3.1 is satisfied, and from Theorem 3.1, BVP (4.1) has a nontrivial solution. □

Example 4.2 Consider the boundary value problem

$$\begin{cases} -\boldsymbol{\mathcal{D}}^{\frac{8}{3}}x(t) = \frac{1}{2}\big(t-t^2\big)x^5(t) - \big(\sin t + e^t\big)\big[\boldsymbol{\mathcal{D}}^{\frac{7}{6}}x(t)\big]^{\frac{9}{8}} \\ \qquad\qquad + 2t^3\big[\boldsymbol{\mathcal{D}}^{\frac{4}{3}}x(t)\big]^3 + e^t + \sqrt{t}, \quad t \in (0,1), \\ x(0) = \boldsymbol{\mathcal{D}}^{\frac{7}{6}}x(0) = \boldsymbol{\mathcal{D}}^{\frac{4}{3}}x(0) = 0, \\ \boldsymbol{\mathcal{D}}^{\frac{3}{2}}x(1) = \frac{1}{\pi}\boldsymbol{\mathcal{D}}^{\frac{3}{2}}\left(\frac{1}{3}\right) - 2\boldsymbol{\mathcal{D}}^{\frac{3}{2}}\left(\frac{2}{3}\right) + \frac{1}{2}\boldsymbol{\mathcal{D}}^{\frac{3}{2}}\left(\frac{3}{4}\right). \end{cases} \tag{4.2}$$

Proof Let

$$f(t,u_1,u_2,u_3) = \frac{1}{2}\big(t-t^2\big)|u_1|^5 + \big(\sin t + e^t\big)|u_2|^{\frac{9}{8}} + 2t^3|u_3|^3 + e^t + \sqrt{t},$$

$$p_1(t) = \frac{1}{2}\big(t-t^2\big), \qquad p_2(t) = \sin t + e^t, \qquad p_3(t) = 2t^3, \qquad q(t) = e^t + \sqrt{t}.$$

Then

$$\big|f(t,u_1,u_2,u_3)\big| \le p_1(t)|u_1|^5 + p_2(t)|u_2|^{\frac{9}{8}} + p_3(t)|u_3|^3 + q(t), \quad t \in [0,1].$$

Thus Theorem 3.4 guarantees a nontrivial solution for BVP (4.2). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration between all authors. Each of the authors contributed to every part of this study equally and read and approved the final version of the manuscript.

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